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Calculus of Variations — *Local and global minimality issues for a nonlocal isoperimetric problem on* $\mathbb{R}^N$, by Marco Bonacini and Riccardo Cristoferi, communicated on 13 November 2015.[1]

Abstract. — We consider a nonlocal isoperimetric problem defined in the whole space $\mathbb{R}^N$, whose nonlocal part is given by a Riesz potential with exponent $\alpha \in (0, N-1)$. We show that critical configurations with positive second variation are local minimizers and satisfy a quantitative inequality with respect to the L^1-norm. This criterion provides the existence of a (explicitly determined) critical threshold determining the interval of volumes for which the ball is a local minimizer. Finally we deduce that for small masses the ball is also the unique global minimizer, and that for small exponents α in the nonlocal term the ball is the unique minimizer as long as the problem has a solution.

Key words: Nonlocal isoperimetric problem, minimality conditions, second variation, local minimizers, global minimizers

Mathematics Subject Classification: 49Q10, 49Q20, 35R35, 82B24

1. Introduction

In these notes we review the main results and ideas of the paper [3], where the full details and proofs can be found. For a parameter $\alpha \in (0, N-1)$, $N \geq 2$, we consider the following functional defined on measurable sets $E \subset \mathbb{R}^N$:

$$\mathscr{F}_\alpha(E) := \mathscr{P}(E) + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\chi_E(x)\chi_E(y)}{|x-y|^\alpha}\,\mathrm{d}x\,\mathrm{d}y, \tag{1.1}$$

where $\mathscr{P}(E)$ is the perimeter of the set E and the second term, the so called *nonlocal term*, will be hereafter denoted by $\mathscr{NL}_\alpha(E)$. We are interested in the study of the volume constrained minimization problem

$$\min\{\mathscr{F}_\alpha(E) : |E| = m\}, \tag{1.2}$$

and in its dependence on the parameters α and $m > 0$.

The reason why the above problem is interesting lies in the fact that the energy (1.1) appears in the modeling of different physical phenomena. The most

[1] This paper reports the content of a talk given by R. Cristoferi at "XXV Convegno Nazionale di Calcolo delle Variazioni"—Levico, 2–6 February 2015. The purpose of this paper is to review the main results and ideas of the reference [3] in the paper.

physically relevant case is in three dimensions with $\alpha = 1$, where the nonlocal term corresponds to a Coulombic repulsive interaction: one of the first examples is the celebrated Gamow's water-drop model for the constitution of the atomic nucleus (see [14]), and energies of this kind are also related (via Γ-convergence) to the Ohta–Kawasaki model for diblock copolymers (see [7, 8, 27]). For the reader interested in a more specific account on the physical background of this kind of problems, we suggest to read [25].

From a mathematical point of view, functionals of the form (1.1) recently drew the attention of many authors (see for example [1, 10, 12, 16, 17, 18, 21, 22, 20, 24, 26]). The main feature of the energy (1.1) is the presence of two competing terms, the sharp interface energy and the long-range repulsive interaction. Indeed, while the first term is minimized by the ball (by the isoperimetric inequality), the nonlocal term is in fact maximized by the ball, as a consequence of the Riesz's rearrangement inequality (see [23, Theorem 3.7]), and favours scattered configurations. Hence, due to the presence of this competition in the structure of the problem, the minimization of $\mathscr{F}_\alpha$ is highly non trivial.

In order to have an idea of the behaviour we would expect for such a functional, we notice that, calling $\tilde{E} := \left(\frac{|B_1|}{|E|}\right)^{\frac{1}{N}} E$, where B_1 is the unit ball of $\mathbb{R}^N$, the functional reads as

$$\mathscr{F}_\alpha(E) = \left(\frac{|E|}{|B_1|}\right)^{\frac{N-1}{N}} \left[\mathscr{P}(\tilde{E}) + \left(\frac{m}{|B_1|}\right)^{\frac{N-\alpha+1}{N}} \mathscr{N}\mathscr{L}_\alpha(\tilde{E})\right].$$

Hence the parameter m appearing in the volume constraint can be normalized and replaced by a coefficient γ in front of the nonlocal energy: one can study the minimization problem, equivalent to (1.2),

$$\min\{\mathscr{F}_{\alpha,\gamma}(E) : |E| = |B_1|\}, \tag{1.3}$$

where we define $\mathscr{F}_{\alpha,\gamma}(E) := \mathscr{P}(E) + \gamma \mathscr{N}\mathscr{L}_\alpha(E)$. It is clear from this expression that, for small masses (*i.e.* small γ's), the interfacial energy is the leading term and this suggests that in this case the functional should behave like the perimeter, namely we expect the ball to be the unique solution of the minimization problem, as in the isoperimetric problem; on the other hand, for large masses the nonlocal term becomes prevalent and causes the existence of a solution to be not guaranteed. But this is just heuristic!

What was proved, in some particular cases, is that the functional $\mathscr{F}_\alpha$ is uniquely minimized (up to translations) by the ball for every value of the volume below a critical threshold: in the planar case in [21], in the case $3 \leq N \leq 7$ in [22], and in any dimension N with $\alpha = N - 2$ in [18]. Moreover, the existence of a critical mass above which the minimum problem does not admit a solution was established in [21] in dimension $N = 2$, in [22] for every dimension and for exponents $\alpha \in (0, 2)$, and in [24] in the physical interesting case $N = 3$, $\alpha = 1$.

In [3] we provide a contribution to a more detailed picture of the nature of the minimization problem (1.2). In particular, we follow the approach used in [1] for the periodic case with $\alpha = N - 2$, which is based on the positivity of the second variation of the functional, in order to obtain a local minimality criterion. This allows us to show the following new results: first, we prove that the ball is the unique global minimizer for small masses, for every values of the parameters N and α (Theorem 3.2); moreover, for α small we also show that the ball is the unique global minimizer, as long as a minimizer exists (Theorem 3.3), and that in this regime we can write $(0, \infty) = \bigcup_k (m_k, m_{k+1}]$, with $m_{k+1} > m_k$, in such a way that for $m \in [m_{k-1}, m_k]$ a minimizing sequence for the functional is given by a configuration of at most k disjoint balls with diverging mutual distance (Theorem 3.4). Finally, we also investigate the issue of *local minimizers*, that is, sets which minimize the energy with respect to competitors sufficiently close in the L^1-sense (where we measure the distance between two sets by the quantity (2.2), which takes into account the translation invariance of the functional). We show the existence of a volume threshold below which the ball is an isolated local minimizer, determining it explicitly in the three dimensional case with a Newtonian potential (Theorem 3.1). The energy landscape of the functional $\mathscr{F}_\alpha$, including the information coming from our analysis and from previous works, is illustrated in Figure 1.

After our work was completed, a deep analysis comprising also the case $\alpha \in [N-1, N)$, and including the possibility for the perimeter term to be a nonlocal s-perimeter, has been performed in the paper [11].

2. The local minimality criterion

The issue of existence and characterization of *global minimizers* of the problem

$$\min\{\mathscr{F}_\alpha(E) : E \subset \mathbb{R}^N, |E| = m\}, \tag{2.1}$$

for $m > 0$, is not at all an easy task. A principal source of difficulty in applying the direct method of the Calculus of Variations comes from the lack of compactness

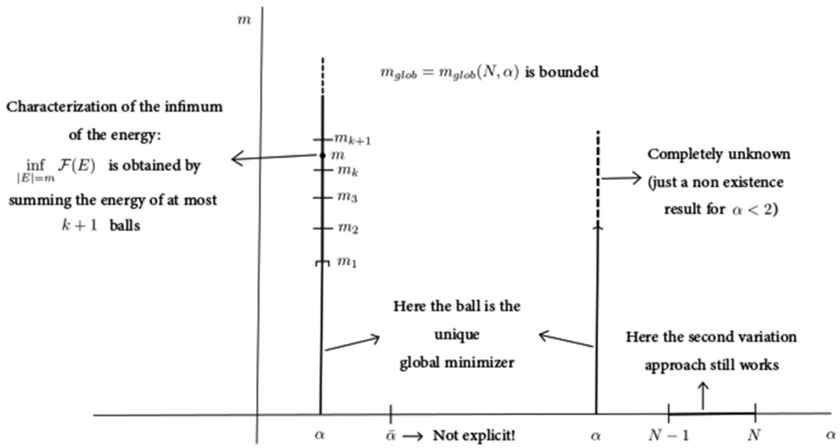

Figure 1. Energy landscape of the functional $\mathscr{F}_{\alpha,\gamma}$.

of the space with respect to L^1 convergence of sets (with respect to which the functional is lower semicontinuous). It is in fact well known that the minimum problem (2.1) does not admit a solution for certain ranges of masses.

Besides the notion of global minimality, we will address also the study of sets which minimize locally the functional with respect to small volume perturbations. Since the functional is translation invariant, we will measure the L^1-distance of two sets modulo translations by the distance

$$\alpha(E,F) := \min_{x\in\mathbb{R}^N} |E\triangle(x+F)|, \tag{2.2}$$

where $\triangle$ denotes the symmetric difference of two sets.

Definition 2.1. We say that $E\subset\mathbb{R}^N$ is a *local minimizer* for the functional (1.1) if there exists $\delta>0$ such that

$$\mathscr{F}_\alpha(E)\le\mathscr{F}_\alpha(F)$$

for every $F\subset\mathbb{R}^N$ such that $|F|=|E|$ and $\alpha(E,F)\le\delta$. We say that E is an *isolated local minimizer* if the previous inequality is strict whenever $\alpha(E,F)>0$.

An important feature of the energy is the Lipschitzianity of the nonlocal term, that allows us to treat it as a bulk perturbation of the area functional (see [3, Proposition 2.3]).

Proposition 2.2 (Lipschitzianity of the nonlocal term). *Given $\bar\alpha\in(0,N-1)$ and $m\in(0,+\infty)$, there exists a constant c_0, depending only on N, $\bar\alpha$ and m such that if $E,F\subset\mathbb{R}^N$ are measurable sets with $|E|,|F|\le m$ then*

$$|\mathscr{N\!L}_\alpha(E)-\mathscr{N\!L}_\alpha(F)|\le c_0|E\triangle F|$$

for every $\alpha\le\bar\alpha$.

The above observation is essential in proving some regularity properties of local and global minimizers, which are mostly known (see, for instance, [22] and [24] for global minimizers, and [1] for local minimizers in a periodic setting). The basic idea is to show that a minimizer solves a suitable penalized minimum problem, where the volume constraint is replaced by a penalization term in the functional, and to deduce that a quasi-minimality property is satisfied. For a proof, see [3, Theorem 2.7].

Theorem 2.3. *Let $E\subset\mathbb{R}^N$ be a global or local minimizer for the functional (1.1) with volume $|E|=m$. Then the reduced boundary ∂^*E is a $C^{3,\beta}$-manifold for all $\beta<N-\alpha-1$, and the Hausdorff dimension of the singular set satisfies $\dim_{\mathscr{H}}(\partial E\setminus\partial^*E)\le N-8$. Moreover, E is (essentially) bounded. Finally, every local minimizer has at most a finite number of connected components and every global minimizer E is connected in a measure theoretic sense,* i.e. *if for a ball B_R we have $|E\cap B_R|>0$ and $|E\setminus B_R|>0$, then $\mathscr{H}^{N-1}(\partial B_R\cap E)\neq 0$.*

As anticipated above, our method follows a second variation approach which has been recently developed and applied to different variational problems, whose common feature is the fact that the energy functionals are characterized by the competition between bulk energies and surface energies (see, for instance, [13] in the context of epitaxially strained elastic films, [5, 4] for the Mumford–Shah functional, [6] for a variational model for cavities in elastic bodies). In particular we stress the attention on [1], which deals with energies in the form (1.1) in a periodic setting (see also [19], where the same problem is considered in an open set with Neumann boundary conditions).

The basic idea is to associate with the second variation of $\mathscr{F}_\alpha$ at a regular critical set E (see Definition 2.6), a quadratic form defined on the functions $\varphi \in H^1(\partial E)$ such that $\int_{\partial E} \varphi = 0$, whose non-negativity is easily seen to be a necessary condition for local minimality. In our main result (Theorem 2.11) we show that the strict positivity of this quadratic form is in fact sufficient for local minimality.

The general strategy for the proof of this minimality criterion is the one developed in [1]. But in our case we have to fix some technical details due to the main differences between the two problems: the fact the α is generic and the non compactness of our domain.

Since we need to use first and second variations, we briefly recall their definitions.

Definition 2.4. Let $X : \mathbb{R}^N \to \mathbb{R}^N$ be a C^2 vector field. We define the *admissible flow* associated to X as the function $\Phi : \mathbb{R}^N \times (-1,1) \to \mathbb{R}^N$ given by the equations

$$\frac{\partial \Phi}{\partial t} = X(\Phi), \quad \Phi(x,0) = x.$$

As for the case of the perimeter, we will use these admissible flows to compute the variations of our functional.

Definition 2.5. Let $E \subset \mathbb{R}^N$ be a set of class C^2, and let Φ be an admissible flow. We define the *first and second variation of $\mathscr{F}_\alpha$ at E with respect to the flow Φ* to be

$$\frac{d}{dt}\mathscr{F}_\alpha(E_t)_{|_{t=0}} \quad \text{and} \quad \frac{d^2}{dt^2}\mathscr{F}_\alpha(E_t)_{|_{t=0}}$$

respectively, where we set $E_t := \Phi_t(E)$.

The first order condition for minimality, coming from the first variation of the functional, requires a C^2-minimizer E (local or global) to satisfy the Euler–Lagrange equation

$$H_{\partial E}(x) + 2v_E(x) = \lambda \quad \text{for every } x \in \partial E, \tag{2.3}$$

for some constant λ which plays the role of a Lagrange multiplier associated with the volume constraint. Here $H_{\partial E} := \mathrm{div}_\tau \, \nu_E(x)$ denotes the sum of the principal curvatures of ∂E (div_τ is the tangential divergence on ∂E and ν_E denotes the exterior unit normal to ∂E), and $v_E(x) := \int_E \frac{1}{|x-y|^\alpha}\,\mathrm{d}y$. Following [1], we define *critical sets* as those satisfying (2.3) in a weak sense, for which further regularity can be gained *a posteriori* (see Remark 2.7).

Definition 2.6. We say that $E \subset \mathbb{R}^N$ is a *regular critical set* for the functional (1.1) if E is of class C^1 and (2.3) holds weakly on ∂E, *i.e.*,

$$\int_{\partial E} \mathrm{div}_\tau \, \zeta \, \mathrm{d}\mathscr{H}^{N-1} = -2 \int_{\partial E} v_E \langle \zeta, \nu_E \rangle \, \mathrm{d}\mathscr{H}^{N-1}$$

for every $\zeta \in C^1(\mathbb{R}^N; \mathbb{R}^N)$ such that $\int_{\partial E} \langle \zeta, \nu_E \rangle \, \mathrm{d}\mathscr{H}^{N-1} = 0$.

Remark 2.7. By standard regularity (see, *e.g.*, [2, Proposition 7.56 and Theorem 7.57]) a critical set E is of class $W^{2,2}$ and $C^{1,\beta}$ for all $\beta \in (0,1)$. In turn it can be proved, by using Schauder estimates (see [15, Theorem 9.19]), that E is of class $C^{3,\beta}$ for all $\beta \in (0, N-\alpha-1)$.

Remark 2.8. Notice that for a ball B we have that v_B is constant on ∂B. Thus every ball is a regular critical set for the functional (1.1).

The second variation of the functional $\mathscr{F}_\alpha$ on a C^2-regular set E, computed in [3, Theorem 3.3], reads as follows:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \mathscr{F}_\alpha(E_t)_{|_{t=0}} = \partial^2 \mathscr{F}_\alpha(E)[\langle X, \nu_E \rangle] + R,$$

where ν_E is the outer normal to ∂E, R is a term that vanishes on regular critical sets and $\partial^2 \mathscr{F}_\alpha(E)$ is the quadratic form defined for $\varphi \in \tilde{H}^1(\partial E)$ by

$$\partial^2 \mathscr{F}_\alpha(E)[\varphi] = \int_{\partial E} (|D_\tau \varphi|^2 - |B_{\partial E}|^2 \varphi^2) \, \mathrm{d}\mathscr{H}^{N-1} + 2 \int_{\partial E} (\partial_{\nu_E} v_E) \varphi^2 \, \mathrm{d}\mathscr{H}^{N-1} + 2 \int_{\partial E} \int_{\partial E} \frac{\varphi(x)\varphi(y)}{|x-y|^\alpha} \, \mathrm{d}\mathscr{H}^{N-1}(x) \, \mathrm{d}\mathscr{H}^{N-1}(y), \tag{2.4}$$

where D_τ denotes the tangential derivative on ∂E, $B_{\partial E} := D_\tau \nu_E$ is the second fundamental form of ∂E and, recalling that we always have to take into account the volume constraint, we introduce the space

$$\tilde{H}^1(\partial E) := \left\{ \varphi \in H^1(\partial E) : \int_{\partial E} \varphi \, \mathrm{d}\mathscr{H}^{N-1} = 0 \right\},$$

endowed with the norm $\|\varphi\|_{\tilde{H}^1(\partial E)} := \|\nabla \varphi\|_{L^2(\partial E)}$.

Notice that if E is a regular critical set and Φ preserves the volume of E, then

$$\partial^2 \mathscr{F}_\alpha(E)[\langle X, \nu\rangle] = \frac{d^2 \mathscr{F}_\alpha(E_t)}{dt^2}\bigg|_{t=0}.$$

This fact suggests that at a regular local minimizer the quadratic form (2.4) must be nonnegative on the space $\tilde{H}^1(\partial E)$. This is the content of the following corollary.

COROLLARY 2.9. *Let E be a local minimizer of $\mathscr{F}_\alpha$ of class C^2. Then*

$$\partial^2 \mathscr{F}_\alpha(E)[\varphi] \geq 0 \quad \textit{for all } \varphi \in \tilde{H}^1(\partial E).$$

Now we want to look for a sufficient condition for local minimality. First of all we notice that, since the functional is translation invariant, if we compute the second variation of $\mathscr{F}_\alpha$ at a regular set E with respect to a flow of the form $\Phi(x, t) := x + t\eta e_i$, where $\eta \in \mathbb{R}$ and e_i is an element of the canonical basis of $\mathbb{R}^N$ and $\nu_i := \langle \nu_E, e_i\rangle$, we obtain that

$$\partial^2 \mathscr{F}_\alpha(E)[\eta \nu_i] = \frac{d^2}{dt^2} \mathscr{F}_\alpha(E_t)_{|_{t=0}} = 0.$$

Following [1], since we aim to prove that the *strict* positivity of the second variation is a sufficient condition for local minimality, we shall exclude the finite dimensional subspace of $\tilde{H}^1(\partial E)$ generated by the functions ν_i, which we denote by $T(\partial E)$. Hence we split

$$\tilde{H}^1(\partial E) = T^\perp(\partial E) \oplus T(\partial E),$$

where $T^\perp(\partial E)$ is the orthogonal complement to $T(\partial E)$ in the L^2-sense, *i.e.*,

$$T^\perp(\partial E) := \left\{\varphi \in \tilde{H}^1(\partial E) : \int_{\partial E} \varphi \nu_i \, d\mathscr{H}^{N-1} = 0 \text{ for each } i = 1, \ldots, N\right\}.$$

It can be shown (see [1]) that there exists an orthonormal frame $(\varepsilon_1, \ldots, \varepsilon_N)$ such that

$$\int_{\partial E} \langle \nu, \varepsilon_i\rangle\langle \nu, \varepsilon_j\rangle \, d\mathscr{H}^{N-1} = 0 \quad \text{for all } i \neq j,$$

so that the projection on $T^\perp(\partial E)$ of a function $\varphi \in \tilde{H}^1(\partial E)$ is

$$\pi_{T^\perp(\partial E)}(\varphi) = \varphi - \sum_{i=1}^N \left(\int_{\partial E} \varphi\langle \nu, \varepsilon_i\rangle \, d\mathscr{H}^{N-1}\right) \frac{\langle \nu, \varepsilon_i\rangle}{\|\langle \nu, \varepsilon_i\rangle\|^2_{L^2(\partial E)}}$$

(notice that $\langle v, \varepsilon_i \rangle \not\equiv 0$ for every i, since on the contrary the set E would be translation invariant in the direction ε_i).

Definition 2.10. We say that $\mathscr{F}_\alpha$ has *positive second variation* at the regular critical set E if

$$\partial^2 \mathscr{F}_\alpha(E)[\varphi] > 0 \quad \text{for all } \varphi \in T^\perp(\partial E)\setminus\{0\}.$$

We are now ready to state the main results of [3], which provides a sufficiency local minimality criterion based on the second variation of the functional.

Theorem 2.11. *Assume that E is a regular critical set for $\mathscr{F}_\alpha$ with compact boundary and with positive second variation, in the sense of Definition* 2.10. *Then there exist $\delta > 0$ and $C > 0$ such that*

$$\mathscr{F}_\alpha(F) \geq \mathscr{F}_\alpha(E) + C(\alpha(E,F))^2 \tag{2.5}$$

for every $F \subset \mathbb{R}^N$ such that $|F| = |E|$ and $\alpha(E,F) < \delta$.

Idea of the proof. The proof consists of two main steps.

Step 1: $W^{2,p}$-local minimality. In this first step we want to prove that the positivity of the second variation allows to show that a critical C^2 set E is a local minimum in the $W^{2,p}$-topology, for $p > \max\{2, N-1\}$, in the sense that it has smaller energy than any set F whose boundary can be written as a normal graph over ∂E with a function with $W^{2,p}$-norm small enough. For, the idea is to exploit the construction provided by [1, Theorem 3.7] in order to connect E and F with a sequence of sets $(E_t)_t \in [0,1]$, given by the evolution of E through the flow generated by a vector field $X \in C^2(\mathbb{R}^N;\mathbb{R}^N)$, in such a way that $E_0 = E$, $E_1 = F$ and $|E_t| = m$. Calling $g(t) := \mathscr{F}_\alpha(E_t)$ and recalling that by criticality of E we have $g'(0) = 0$, we can write

$$\mathscr{F}_\alpha(F) - \mathscr{F}_\alpha(E) = g(1) - g(0) = \int_0^1 (1-t) g''(t)\,\mathrm{d}t,$$

where

$$g''(t) = \left.\frac{\mathrm{d}^2 \mathscr{F}_\alpha(E_s)}{\mathrm{d}s^2}\right|_{s=t} = \partial^2 \mathscr{F}_\alpha(E_t)[\langle X, v_{E_t}\rangle] + R_t.$$

Since by hypothesis $\partial^2 \mathscr{F}_\alpha(E)$ is a positive quadratic form, it can be proved that it is indeed uniformly positive, that is there exists $\delta_1 > 0$ such that if $\partial F = \{x + \psi(x) v_E(x) : x \in \partial E\}$ with $|F| = |E|$ and $\|\psi\|_{W^{2,p}(\partial E)} \leq \delta_1$, then

$$\inf\left\{ \partial^2 \mathscr{F}_\alpha(F)[\varphi] : \varphi \in \tilde{H}^1(\partial F), \|\varphi\|_{\tilde{H}^1(\partial F)} = 1, \left| \int_{\partial F} \varphi v_F \,\mathrm{d}\mathscr{H}^{N-1} \right| \leq \delta_1 \right\} \geq \frac{m_0}{2},$$

for a costant $m_0 > 0$ depending on E. Moreover, given $\varepsilon > 0$, it is possible to find $\delta_2 > 0$ such that if $\|\psi\|_{W^{2,p}(\partial E)} \leq \delta_2$, then we can estimate the reminder as $|R_t| \leq C\varepsilon \|\langle X, \nu_{E_t}\rangle\|^2_{L^2(\partial E_t)}$. Thus, choosing ε sufficiently small we obtain that:

$$\mathscr{F}_\alpha(F) - \mathscr{F}_\alpha(E) \geq C \int_0^1 (1-t)\|\langle X, \nu_{E_t}\rangle\|^2_{L^2(\partial E_t)}\, dt.$$

Finally it is also possible to prove that $\|\langle X, \nu_{E_t}\rangle\|^2_{L^2(\partial E_t)} \geq \|\langle X, \nu_E\rangle\|^2_{L^2(\partial E)}$. We have thus proved an isolated local minmality result in $W^{2,p}$ for a regular critical set with positive second variation.

Step 2: $W^{2,p}$-local minimality implies L^1-local minimality. The idea of this second step goes back to [13], [9], [1] and relies on the following result, observed by White in [28]:

THEOREM 2.12. *Let $E_n \subset \mathbb{R}^N$ be a sequence of Λ-minimizers of the area functional such that*

$$\sup_n \mathscr{P}(E_n) < +\infty \quad \text{and} \quad \chi_{E_n} \to \chi_E \quad \text{in } L^1(\mathbb{R}^N)$$

for some bounded set E of class C^2. Then for n large enough E_n is of class $C^{1,\frac{1}{2}}$ and

$$\partial E_n = \{x + \psi_n(x)\nu_E(x) : x \in \partial E\},$$

with $\psi_n \to 0$ in $C^{1,\beta}(\partial E)$ for all $\beta \in \left(0, \frac{1}{2}\right)$.

What we have is a local minimality result on the $W^{2,p}$-topology, and we would like to extend it to the L^1-topology. We argue by contradiction and we take a sequence of sets $(E_n)_n$ with $|E_n| = |E|$, $E_n \to E$ in L^1,

$$\mathscr{F}_\alpha(E_n) \leq \mathscr{F}_\alpha(E),$$

and we would like to use the previous step to conclude. The problem is that, while the first one is *local* in space, the former one can happen also *at infinity*, namely we can have that $E_n \backslash B_n \neq \emptyset$, where B_n is the ball of radius n. So the fist step seems to be useless. The brillant idea of the above cited papers was to use the sets E_n's to construct a new sequence of sets $(F_n)_n$ such that $|F_n| = |E|$, $F_n \to E$ in L^1 but with the additional property of being a uniform sequence of Λ-minimizers (because they are selected as solution to a penalized minimum problem for the energy (1.3)). In this way it is possible to apply Theorem 2.12 to this sequence to infer that in fact the F_n's are of class $C^{1,\frac{1}{2}}$ and they converge to E in the $C^{1,\beta}$ sense. Now, using the Euler–Lagrange equation for the penalized problem solved by F_n, it is possible to gain the $W^{2,p}$-convergence of the sets F_n's to E. This will give the desired contradiction and thus allows us to conclude the proof of the theorem. □

3. Results

The local minimality criterion in Theorem 2.11 can be applied to obtain information about local and global minimizers of the functional (1.1). We start with the following theorem (see [3, Theorem 2.9]), which shows the existence of a critical mass $m_{\rm loc}$ such that the ball B_R is an isolated local minimizer if $|B_R| < m_{\rm loc}$, but is no longer a local minimizer for larger masses. We also determine explicitly the volume threshold in the three-dimensional case with a Newtonian potential.

Theorem 3.1 (Local minimality of the ball). *Given $N \geq 2$ and $\alpha \in (0, N-1)$, there exists a critical threshold $m_{\rm loc} = m_{\rm loc}(N, \alpha) > 0$ such that the ball B_R is an isolated local minimizer for $\mathscr{F}_\alpha$, in the sense of Definition 2.1, if $0 < |B_R| < m_{\rm loc}$.*

If $|B_R| > m_{\rm loc}$, there exists $E \subset \mathbb{R}^N$ with $|E| = |B_R|$ and $\alpha(E, B_R)$ arbitrarily small such that $\mathscr{F}_\alpha(E) < \mathscr{F}_\alpha(B_R)$.

In particular for dimension $N = 3$ we have that

$$m_{\rm loc}(3, \alpha) = \pi\left(\frac{(6-\alpha)(4-\alpha)}{2^{3-\alpha}\alpha\pi}\right)^{\frac{3}{4-\alpha}}.$$

Finally $m_{\rm loc}(N, \alpha) \to \infty$ as $\alpha \to 0^+$.

Our local minimality criterion allows us to deduce further properties about global minimizers. The first result [3, Theorem 2.10] states that the ball is the unique global minimizer of the functional for small masses. Even if this result was already known in the literature in some particular cases (see [18] for the case $\alpha = N-2$, [21] for the case $N = 2$, and [22] for the case $3 \leq N \leq 7$), we provide an alternative proof which removes the dimensional constraint based on the second variation approach.

Theorem 3.2 (Global minimality of the ball). *Let $m_{\rm glob}(N, \alpha)$ be the supremum of the masses $m > 0$ such that the ball of mass m is a global minimizer of $\mathscr{F}_\alpha$ in dimension N. Then it holds that $0 < m_{\rm glob}(N, \alpha) < \infty$ and that the ball B_R with $|B_R| = m$ is a local minimizer of $\mathscr{F}_\alpha$ if $m \leq m_{\rm glob}(N, \alpha)$. Moreover it is the unique global minimizer of $\mathscr{F}_\alpha$ if $m < m_{\rm glob}(N, \alpha)$.*

Idea of the proof. We need to prove three facts:

- $m_{\rm glob}(N, \alpha) < \infty$,
- $m_{\rm glob}(N, \alpha) > 0$ and the ball B_R with $|B_R| = m$ is a local minimizer of $\mathscr{F}_\alpha$ if $m \leq m_{\rm glob}(N, \alpha, \gamma)$,
- it is the unique global minimizer of $\mathscr{F}_\alpha$ if $m < m_{\rm glob}(N, \alpha)$.

The fact that $m_{\rm glob}(N, \alpha) < \infty$ follows directly from the previous theorem (Theorem 3.1), since the critical threshold $m_{\rm loc}$ of local minimality of the ball is always finite. Here we would like to give an idea of the proof of the second fact, since the proof of the third one is similar.

In the following we will work with the functional $\mathscr{F}_{\alpha,\gamma}$ defined in (1.3). Suppose by contradiction that there exist a sequence $\gamma_n \to 0$ and a sequence of sets $(E_n)_n$ with $|E_n| = |B_1|$ such that

$$\mathscr{F}_{\alpha,\gamma_n}(E_n) \le \mathscr{F}_{\alpha,\gamma_n}(B_1).$$

Since the above inequality can be written as

$$\mathscr{P}(E_n) - \mathscr{P}(B_1) \le \gamma_n(\mathscr{NL}_\alpha(B_1) - \mathscr{NL}_\alpha(E_n)),$$

using the quantitative isoperimetric inequality (recall that $|E_n| = |B_1|$ for all n) to estimate from below the left-hand side and the Lipschitzianity of the nonlocal term (see Proposition 2.2) to estimate the right-hand side from above, we obtain

$$C|E_n \triangle B_1|^2 \le \gamma_n c_0 |E_n \triangle B_1|.$$

Since $\gamma_n \to 0$ we obtain that $E_n \to B_1$ in L^1. But this implies, using Theorem 3.1, that there exists $\bar{n} \in \mathbb{N}$ such that $E_n = B_1$ for all $n \ge \bar{n}$. □

In the following theorems we analyze the global minimality issue for α close to 0, showing in particular that in this case the unique minimizer, as long as a minimizer exists, is the ball [3, Theorem 2.11], and characterizing the infimum of the energy when the problem does not have a solution [3, Theorem 2.12]. These results are completely new (except the first one, proved for the special case of dimension $N = 2$ in [21]).

THEOREM 3.3 (Full characterization of global minimizers for α small). *There exists a critical exponent $\bar{\alpha} = \bar{\alpha}(N) > 0$ such that for every $\alpha < \bar{\alpha}$ the ball with volume m is the unique (up to translations) global minimizer of $\mathscr{F}_\alpha$ if $m \le m_{\mathrm{glob}}(N,\alpha)$, while for $m > m_{\mathrm{glob}}(N,\alpha)$ the minimum problem for $\mathscr{F}_\alpha$ does not have a solution.*

IDEA OF THE PROOF. The proof is similar to the one of Theorem 3.2: indeed suppose by contradiction that there exist sequences $\alpha_n \to 0$, $\gamma_n > 0$ and sets $(E_n)_n$ with $|E_n| = |B_1|$ such that

$$\mathscr{F}_{\alpha_n,\gamma_n}(E_n) \le \mathscr{F}_{\alpha_n,\gamma_n}(B_1),$$

which can be rewritten as before as

$$\mathscr{P}(E_n) - \mathscr{P}(B_1) \le \gamma_n(\mathscr{NL}_{\alpha_n}(B_1) - \mathscr{NL}_{\alpha_n}(E_n)).$$

Using the quantitative isoperimetric inequality to estimate from below the left-hand side, and recalling that we can suppose the γ_n's to be bounded from above by a constant $\bar{\gamma}$ (since it is known that for $\alpha < 2$ there exists a bounded threshold above which the ball is no longer a global minimizer), we infer that

$$C|E_n \triangle B_1|^2 \le \bar{\gamma}(\mathscr{NL}_{\alpha_n}(B_1) - \mathscr{NL}_{\alpha_n}(E_n)).$$

Now it is easy to show that the right-hand side goes to 0 for $\alpha_n \to 0$. Thus $E_n \to B_1$ in L^1. Finally, looking carefully at the proof leading to Theorem 2.11, it can be proved that the result is uniform in α and γ: namely given $\bar{\alpha}, \bar{\gamma} > 0$ there exists $\delta > 0$ such that

$$\mathscr{F}_{\alpha,\gamma}(E) > \mathscr{F}_{\alpha,\gamma}(B_1),$$

for each $\alpha < \bar{\alpha}$, $\gamma < \bar{\gamma}$ and each set $E \subset \mathbb{R}^N$ with $|E| = |B_1|$ and $0 < \alpha(E, B_1) < \delta$. This observation allows us to conclude the proof of the theorem. □

THEOREM 3.4 (Minimizing sequences for α small). *Let $\alpha < \bar{\alpha}$ (where $\bar{\alpha}$ is given by Theorem 3.3) and let*

$$f_k(m) := \min_{\substack{\mu_1,\dots,\mu_k \geq 0 \\ \mu_1+\cdots+\mu_k = m}} \left\{ \sum_{j=1}^{k} \mathscr{F}_\alpha(B^i) : B^i \text{ ball}, |B^i| = \mu_i \right\}.$$

There exists an increasing sequence $(m_k)_k$, with $m_0 = 0$, $m_1 = m_{\mathrm{glob}}$, such that $\lim_k m_k = \infty$ and

$$\inf_{|E|=m} \mathscr{F}_\alpha(E) = f_k(m) \quad \text{for every } m \in [m_{k-1}, m_k], \text{ for all } k \in \mathbb{N}, \tag{3.1}$$

that is, for every $m \in [m_{k-1}, m_k]$ a minimizing sequence for the total energy is obtained by a configuration of at most k disjoint balls with diverging mutual distance. Moreover, the number of non-degenerate balls tends to $+\infty$ as $m \to +\infty$.

REMARK 3.5. Since $m_{\mathrm{loc}}(N, \alpha) \to +\infty$ as $\alpha \to 0^+$ and the non-existence threshold is known to be uniformly bounded for $\alpha \in (0, 1)$, we immediately deduce that we have $m_{\mathrm{glob}}(N, \alpha) < m_{\mathrm{loc}}(N, \alpha)$, for α small. Moreover, by comparing the energy of a ball of volume m with the energy of two disjoint balls of volume $\frac{m}{2}$, and sending to infinity the distance between the balls, we deduce after a straightforward computation (and estimating $\mathscr{N}\mathscr{L}_\alpha(B_1) \geq \omega_N^2 2^{-\alpha}$) that the following upper bound for the global minimality threshold of the ball holds:

$$m_{\mathrm{glob}}(N, \alpha, \gamma) < \omega_N \left(\frac{2^\alpha N(2^{\frac{1}{N}} - 1)}{\omega_N \gamma \left(1 - \left(\frac{1}{2}\right)^{\frac{N-\alpha}{N}}\right)} \right)^{\frac{N}{N+1-\alpha}}.$$

Hence, by comparing this value with the explicit expression of m_{loc} in the physical interesting case $N = 3$, $\alpha = 1$ (see Theorem 3.1), we deduce that $m_{\mathrm{glob}}(3, 1) < m_{\mathrm{loc}}(3, 1)$.

4. OPEN PROBLEMS

There are several open problems regarding this functional. An important issue that remains unsolved is concerned with the structure of the set of masses for which the problem does not have a solution: is it always true that it has the form $(m, +\infty)$ for all the values of α and N? Notice that we provide a positive answer to this question in the case where α is small. Finally, another interesting question asks if there are other global (or local) minimizers different from the ball, and in the affirmative case, to provide some information about these minima.

References

[1] E. Acerbi - N. Fusco - M. Morini, *Minimality via second variation for a nonlocal isoperimetric problem*, Comm. Math. Phys., 322 (2013), pp. 515–557.

[2] L. Ambrosio - N. Fusco - D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 2000.

[3] M. Bonacini - R. Cristoferi, *Local and global minimality results for a nonlocal isoperimetric problem on* $\mathbb{R}^N$, SIAM J. Math. Anal., 46 (2014), pp. 2310–2349.

[4] M. Bonacini - M. Morini, *Stable regular critical points of the Mumford–Shah functional are local minimizers*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 32 (2015), pp. 533–570.

[5] F. Cagnetti - M. G. Mora - M. Morini, *A second order minimality condition for the Mumford–Shah functional*, Calc. Var. Partial Differential Equations, 33 (2008), pp. 37–74.

[6] G. M. Capriani - V. Julin - G. Pisante, *A quantitative second order minimality criterion for cavities in elastic bodies*, SIAM J. Math. Anal., 45 (2013), pp. 1952–1991.

[7] R. Choksi - M. A. Peletier, *Small volume fraction limit of the diblock copolymer problem: I. Sharp-interface functional*, SIAM J. Math. Anal., 42 (2010), pp. 1334–1370.

[8] R. Choksi - M. A. Peletier, *Small volume-fraction limit of the diblock copolymer problem: II. Diffuse-interface functional*, SIAM J. Math. Anal., 43 (2011), pp. 739–763.

[9] M. Cicalese - G. P. Leonardi, *A selection principle for the sharp quantitative isoperimetric inequality*, Arch. Ration. Mech. Anal., 206 (2012), pp. 617–643.

[10] M. Cicalese - E. Spadaro, *Droplet minimizers of an isoperimetric problem with long-range interactions*, Comm. Pure Appl. Math., 66 (2013), pp. 1298–1333.

[11] A. Figalli - N. Fusco - F. Maggi - V. Millot - M. Morini, *Isoperimetry and stability properties of balls with respect to nonlocal energies*, Comm. Math. Phys., 336 (2015), pp. 441–507.

[12] R. L. Frank - E. H. Lieb, *A compactness lemma and its application to the existence of minimizers for the liquid drop model*, Preprint (2015).

[13] N. Fusco - M. Morini, *Equilibrium configurations of epitaxially strained elastic films: second order minimality conditions and qualitative properties of solutions*, Arch. Ration. Mech. Anal., 203 (2012), pp. 247–327.

[14] G. Gamow, *Mass defect curve and nuclear constitution*, Proc. R. Soc. Lond. Ser. A, 126 (1930), pp. 632–644.

[15] D. Gilbarg - N. S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[16] D. Goldman - C. Muratov - S. Serfaty, *The* Γ*-limit of the two-dimensional Ohta–Kawasaki energy. I. Droplet density*, Arch. Rational Mech. Anal., 210 (2013), pp. 581–613.

[17] D. Goldman - C. B. Muratov - S. Serfaty, *The* Γ*-limit of the two-dimensional Ohta–Kawasaki energy. II. Droplet arrangement via the renormalized energy*, Arch. Ration. Mech. Anal., 212 (2014), pp. 445–501.

[18] V. Julin, *Isoperimetric problem with a Coulomb repulsive term*, Indiana Univ. Math. J., 63 (2014), pp. 77–89.

[19] V. Julin - G. Pisante, *Minimality via second variation for microphase separation of diblock copolymer melts*, J. Reine Angew. Math. (to appear).

[20] H. Knüpfer - C. Muratov - M. Novaga, *Low density phases in a uniformly charged liquid*, Preprint (2015).

[21] H. Knüpfer - C. B. Muratov, *On an isoperimetric problem with a competing nonlocal term. I. The planar case*, Comm. Pure Appl. Math., 66 (2013), pp. 1129–1162.

[22] H. Knüpfer - C. B. Muratov, *On an isoperimetric problem with a competing nonlocal term. II. The general case*, Comm. Pure Appl. Math., 67 (2014), pp. 1974–1994.

[23] E. H. Lieb - M. Loss, *Analysis*, vol. 14 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, second ed., 2001.

[24] J. Lu - F. Otto, *Nonexistence of minimizers for Thomas–Fermi–Dirac–von Weizsäcker model*, Comm. Pure Appl. Math., 67 (2014), pp. 1605–1617.

[25] C. B. Muratov, *Theory of domain patterns in systems with long-range interactions of Coulomb type*, Phys. Rev. E (3), 66 (2002), pp. 1–25.

[26] C. B. Muratov, *Droplet phases in non-local Ginzburg–Landau models with Coulomb repulsion in two dimensions*, Comm. Math. Phys., 299 (2010), pp. 45–87.

[27] T. Ohta - K. Kawasaki, *Equilibrium morphology of block copolymer melts*, Macromolecules, 19 (1986), pp. 2621–2632.

[28] B. White, *A strong minimax property of nondegenerate minimal submanifolds*, J. Reine Angew. Math., 457 (1994), pp. 203–218.

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Marco Bonacini
Institut für Angewandte Mathematik
Universität Heidelberg
Im Neuenheimer Feld 294
69120 Heidelberg, Germany
marco.bonacini@uni-heidelberg.de

Riccardo Cristoferi
SISSA
Via Bonomea 265
34136 Trieste, Italy
riccardo.cristoferi@sissa.it