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Calculus of Variations — *Lipschitz continuity for energy integrals with variable exponents*, by Michela Eleuteri, Paolo Marcellini and Elvira Mascolo, communicated on 13 November 2015.[1]

Abstract. — A regularity result for integrals of the Calculus of Variations with variable exponents is presented. Precisely, we prove that any vector-valued minimizer of an energy integral over an open set $\Omega \subset \mathbb{R}^n$, with variable exponent $p(x)$ in the Sobolev class $W^{1,r}_{\mathrm{loc}}(\Omega)$ for some $r > n$, is locally Lipschitz continuous in Ω and an a priori estimate holds.

Key words: Energy integrals, local minimizers, local Lipschitz continuity, $p(x)$-growth, variable exponents

Mathematics Subject Classification (primary; secondary): 35J60, 35B65, 49N60; 35J50, 35B45

1. Introduction

In the past few years the study of energy integrals with variable exponent received a large interest. We refer for instance to the integral functional

$$F(u) = \int_\Omega a(x) h(|Du|)^{p(x)}\, dx, \tag{1.1}$$

where $a(x) > 0$ and $p(x) > 1$ are continuous functions in Ω, h is an increasing convex function and u is a vector-valued map. The variational integral (1.1) exhibits $p(x)$-growth, which is a particular case of the so-called nonstandard growth, with an extensive literature on the subject. While existence of minimizers follows from the direct methods of the Calculus of Variations, the regularity problem is not yet completely settled. We stress that in the vector-valued case, as suggested by the well known counterexamples by De Giorgi [10], Giusti–Miranda [18] and more recently by Sverak–Yan [33], Mooney–Savin [30], some structure conditions on the integrand are required for everywhere regularity.

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Here we consider an open bounded set $\Omega \subset \mathbb{R}^n$, $n \geq 2$, a coefficient $a \in W^{1,r}_{\mathrm{loc}}(\Omega)$ and an exponent $p \in W^{1,r}_{\mathrm{loc}}(\Omega)$ for some $r > n$, an increasing convex function $h = h(t)$, $h : [0, \infty) \to [0, \infty)$, $h \in W^{2,\infty}[0, T]$, for all $T > 0$ and a vector-valued map $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$, with $N \geq 1$. We deal with the local Lipschitz continuity of minimizers, without a prescribed bound on the oscillation of $p(x)$, assuming instead the summability of the weak derivatives of a and p. Precisely we will prove the following result.

Theorem 1.1. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ be a local minimizer of the energy integral* (1.1). *Let $h : [0, +\infty) \to [0, +\infty)$ be an increasing convex function, not identically zero, such that for some $M_0 > 0$, $t_0 \geq 1$ and $0 \leq \delta \leq \frac{1}{n} - \frac{1}{r}$*

$$h'(t) \leq M_0 t^{\delta}, \quad h''(t) \leq M_0 \frac{h'(t)}{t}, \quad \forall t \geq t_0, \tag{1.2}$$

with $a, p \in W^{1,r}_{\mathrm{loc}}(\Omega)$ for $r > n$. Then u is a locally Lipschitz continuous map and there exist constants $C > 0$ and $\beta > 1$ such that, for $0 < \rho < R$,

$$\|Du\|_{L^\infty(B_\rho; \mathbb{R}^{Nn})} \leq C \left[\left(\frac{1 + \|a_x\|_{L^r(B_R)} + \|p_x\|_{L^r(B_R)}}{(R - \rho)} \right)^{\frac{1}{\frac{1}{n} - \frac{1}{r}}} \int_{B_R} \{1 + a(x) h(|Du|)^{p(x)}\} \, dx \right]^{\beta}.$$

As usual B_ρ and B_R are balls in Ω of radii ρ and R with the same center. In particular the constants $C > 0$ and $\beta > 1$ depend on n, r, M_0, t_0 and on the infimum and the supremum of a and p in the ball B_R.

Examples of h functions which satisfy the assumptions of Theorem 1.1 are, of course $h(t) = t$ or

$$h(t) = t \log(1 + t), \quad t \geq 0,$$

more generally $h(t) = t \log^{\beta} t$ for large values of t and for some $\beta > 0$ and also

$$h(t) = \int_1^t \frac{s^{\delta}}{\log s} \, ds, \quad t \geq t_0.$$

As in [9], [16], [23], [34], a different example of a convex function $h(t)$ satisfying the above conditions is given, for large t, by

$$h(t) = t^{a + b \sin \log \log t}, \tag{1.3}$$

when $a > 1$ and $b > 0$ is sufficiently small (see Section 3). The function h has p, q-growth, in the sense that $t^p \leq h(t) \leq t^q$ for large values of t, with $p = a - b$ and $q = a + b$.

The proof of Theorem 1.1 will follow through several steps, the main one being a reduction of the given energy functional in (1.1) to the framework considered by the authors in [13].

The energy variable exponent is nowadays a classical topic in the Calculus of Variations, PDEs and Nonlinear Analysis. The large number of papers studying energies involving variable exponents is motivated by the fact that this type of functionals can be considered as a model in the theory of strongly anisotropic materials (see e.g. Zhikov [35] and Zhikov et al. [36]) and in the theory of electro-rheological fluids (see e.g. Rajagopal–Růžička [31] and Růžička [32]). More recently, functionals as in (1.1) were considered also in the study of image denoising (see e.g. Chen et al. [4]) and in some models for growth of heterogeneous sandpiles (see e.g. Bocea et al. [2]). The regularity of minimizers has been studied by many authors. For the case $h(t) = t$ we mention: Chiadò–Piat–Coscia [5], Coscia–Mingione [7], Acerbi–Mingione [1], Esposito–Leonetti–Mingione [15]. The case $h(t) \sim e^{t^m}$, $m > 0$, was considered by Mascolo-Migliorini [26] and with the x, u dependence by Eleuteri [12]. A further list of references can be found in Hărjulehto–Hăsto–Nuortio [20] and Diening–Hărjulehto–Hăsto–Nuortio [11]. Energies with variable exponents are also studied in the framework of the p, q-growth; indeed if $p(x)$ is a continuous function, on a small ball its minimum and its maximum values behave as p, q, with q arbitrarily close to p. We refer to Marcellini [21], [22], [23], [24] and to Mingione [27] for a survey on this subject. Variable exponents were also considered under different aspects in Nonlinear Analysis, for instance with respect to eigenvalue problems and to the multiplicity of solutions (see e.g. Pucci, Radulescu et al. [3], [28], [29] and, more recently, Colasuonno–Squassina [6]).

2. Proof of Theorem 1.1

The proof of Theorem 1.1 follows in several steps:

2.1. Step 1: localization. For every $x_0 \in \Omega$, there exists $R_0 > 0$ such that the ball $B_{R_0}(x_0)$ is contained in Ω and, if we set

$$\begin{aligned} p &:= \inf\{p(x) : x \in B_{R_0}(x_0)\} > 1 \\ q &:= \sup\{p(x) : x \in B_{R_0}(x_0)\}(1+\delta) + \tau \end{aligned} \tag{2.1}$$

for some

$$\tau < \frac{1}{1+\delta}\left(\frac{1}{n} - \frac{1}{r} - \delta\right), \tag{2.2}$$

then

$$\frac{q}{p} < 1 + \frac{1}{n} - \frac{1}{r}. \tag{2.3}$$

Indeed, for any ε_0 there exists $R_0 > 0$ such that

$$\sup\{p(x) : x \in B_{R_0}(x_0)\} - \inf\{p(x) : x \in B_{R_0}(x_0)\} < \varepsilon_0.$$

We set

$$\varepsilon_0 := \frac{1}{1+\delta}\Big(\frac{1}{n} - \frac{1}{r} - \delta\Big) - \tau.$$

This is possible due to the smallness assumptions on δ and τ we required. At this point

$$\begin{aligned}\frac{\sup\{p(x) : x \in B_{R_0}(x_0)\}}{\inf\{p(x) : x \in B_{R_0}(x_0)\}} - 1 &< \frac{1}{1+\delta}\Big(\frac{1}{n} - \frac{1}{r} - \delta\Big)\frac{1}{\inf\{p(x) : x \in B_{R_0}(x_0)\}} \\ &\quad - \frac{\tau}{\inf\{p(x) : x \in B_{R_0}(x_0)\}} \\ &< \frac{1}{1+\delta}\Big(\frac{1}{n} - \frac{1}{r} - \delta\Big) - \frac{\tau}{\inf\{p(x) : x \in B_{R_0}(x_0)\}}\end{aligned}$$

as long as $p(x) > 1$. This finally entails, taking into account (2.1)

$$\frac{q}{p} < \frac{[\sup\{p(x) : x \in B_{R_0}(x_0)\} + \tau](1+\delta)}{\inf\{p(x) : x \in B_{R_0}(x_0)\}} < 1 + \delta + \frac{1}{n} - \frac{1}{r} - \delta = 1 + \frac{1}{n} - \frac{1}{r}.$$

It is obvious that $\sup\{p(x) : x \in B_{R_0}(x_0)\} < q$.

2.2. Step 2: consequences from the assumptions. Before starting, we need the following elementary properties of convex functions.

Let $h : [0, \infty) \to [0, \infty)$ be an increasing convex function fulfilling (1.2). The functions $h(t)$ and $h'(t)$ are not identically equal to zero, thus there exists $t_0 \geq 1$ such that (1.2) holds and

$$h(t_0) > 0 \quad h'(t_0) > 0.$$

Let us set

$$m_0 := \min\Big\{\frac{h(t_0)}{t_0}, h'(t_0)\Big\}. \tag{2.4}$$

Then we have

$$h'(t) \geq h'(t_0) \geq m_0 \quad \forall t \geq t_0 \tag{2.5}$$

and moreover by convexity and (2.5)

$$\begin{aligned}h(t) &\geq h(t_0) + h'(t_0)(t - t_0) \geq h(t_0) + m_0(t - t_0) \\ &\geq m_0 t_0 + m_0(t - t_0) = m_0 t.\end{aligned} \tag{2.6}$$

On the other hand, we can also show that

$$\frac{1}{\tilde{C}_0} h'(t)t \le h(t) \le C_0 h'(t)t, \tag{2.7}$$

where

$$\tilde{C}_0 := \max\{M_0 + 1, h'(t_0)t_0\} \quad C_0 = 1 + \frac{h(t_0)}{h'(t_0)}. \tag{2.8}$$

Indeed from the Mean Value Theorem and the fact that h is increasing

$$h(t) \le h(0) + h'(t)t \le h(t_0) + h'(t)t.$$

But h is convex, so also h' is increasing; this entails, for all $t \ge t_0 \ge 1$

$$h(t_0) = h'(t_0)\frac{h(t_0)}{h'(t_0)} \le \frac{h(t_0)}{h'(t_0)} h'(t_0)t_0 \le \frac{h(t_0)}{h'(t_0)} h'(t)t, \tag{2.9}$$

thus we get the desired inequality

$$h(t) \le \left(1 + \frac{h(t_0)}{h'(t_0)}\right) h'(t)t \overset{(2.8)}{=} C_0 h'(t)t.$$

We prove now the other inequality in (2.7). Indeed, by the second inequality in (1.2), we have

$$\int_{t_0}^{t} h''(\tau)\tau \, d\tau \le M_0 \int_{t_0}^{t} h'(\tau) \, d\tau$$

and by integration by parts

$$h'(t)t - h'(t_0)t_0 - \int_{t_0}^{t} h'(\tau) \, d\tau \le M_0 \int_{t_0}^{t} h'(\tau) \, d\tau$$

that is

$$h'(t)t \le (M_0 + 1)(h(t) - h(t_0)) + h'(t_0)t_0 \le \tilde{C}_0 h(t).$$

Observe that (2.6) and (2.7) imply

$$m_0 t \le h(t) \le C_0 M_0 t^{1+\delta} \quad \forall t \ge t_0. \tag{2.10}$$

In the sequel it is not restrictive to assume that

$$m_0 \le 1 \le M_0 \quad \inf_{x \in B_{R_0}} a(x) \ge m_0 > 0 \quad \sup_{x \in B_{R_0}} a(x) \le M_0.$$

2.3. Step 3: ellipticity and growth assumptions. Let us denote

$$g(x,t) = a(x)h(t)^{p(x)}.$$

In this step we show that $g(x,t)$ satisfies the following growth conditions, for all $t \geq t_0$ and a.e. $x \in B_{R_0}$

$$\lambda t^{p(x)-2} \leq \frac{g_t(x,t)}{t} \leq \Lambda t^{(1+\delta)p(x)-2} \tag{2.11}$$

$$\lambda t^{p(x)-2} \leq g_{tt}(x,t) \leq \Lambda t^{(1+\delta)p(x)-2} \tag{2.12}$$

$$|g_{tx}(x,t)| \leq \Lambda \ell(x) t^{(1+\delta)p(x)+\tau-1} \tag{2.13}$$

for λ, Λ defined as

$$\lambda := \min\left\{1, \frac{p-1}{C_0^2}\right\} p m_0^{q+1}, \tag{2.14}$$

$$\Lambda := \frac{q(q-1)}{m_0} C_0^q M_0^{q+2} \tilde{C}_0^2 + \frac{2}{m_0} \max\{\log(C_0 M_0), \delta + 1\},$$

with δ and τ as in (1.2) and (2.1) respectively and

$$\ell(x) := |a_x(x)| + |p_x(x)| \quad \text{for a.e. } x \in B_{R_0}. \tag{2.15}$$

Indeed

$$g_t(x,t) = a(x)p(x)h(t)^{p(x)-1}h'(t)$$

so that

$$\frac{g_t(x,t)}{t} = a(x)p(x)h(t)^{p(x)-1}\frac{h'(t)}{t} \overset{(1.2),(2.10)}{\leq} qM_0^{q+1}C_0^{q-1}t^{(1+\delta)p(x)-2} \tag{2.16}$$

$$\overset{(2.14)}{\leq} \Lambda t^{(1+\delta)p(x)-2}.$$

Consider now

$$g_{tt}(x,t) = a(x)p(x)(p(x)-1)h(t)^{p(x)-2}[h'(t)]^2 + a(x)p(x)h(t)^{p(x)-1}h''(t).$$

We deal first with the second term. We have

$$a(x)p(x)h(t)^{p(x)-1}h''(t) \overset{(1.2)}{\leq} M_0 a(x)p(x)h(t)^{p(x)-1}\frac{h'(t)}{t}$$

$$\overset{(2.16)}{\leq} \Lambda t^{(1+\delta)p(x)-2}.$$

On the other hand

$$\begin{aligned} a(x)p(x)(p(x)-1)h(t)^{p(x)-2}[h'(t)]^2 &\overset{(2.7)}{\leq} a(x)p(x)(p(x)-1)\frac{h(t)^{p(x)}}{t^2}\tilde{C}_0^2 \\ &\overset{(2.10)}{\leq} C_0^q M_0^{q+1} q(q-1)\tilde{C}_0^2 t^{(1+\delta)p(x)-2} \\ &\overset{(2.14)}{\leq} \Lambda t^{(1+\delta)p(x)-2}. \end{aligned}$$

We deal now with the lower bounds. First of all we have

$$\frac{g_t(x,t)}{t} = a(x)p(x)h(t)^{p(x)-1}\frac{h'(t)}{t} \overset{(2.5),(2.6)}{\geq} a(x)p(x)[m_0 t]^{p(x)-1}\frac{m_0}{t} \overset{(2.14)}{\geq} \lambda t^{p(x)-2}.$$

We also have, using (2.5)

$$\begin{aligned} g_{tt}(x,t) &\geq a(x)p(x)(p(x)-1)h(t)^{p(x)-2}[h'(t)]^2 \\ &\overset{(2.7)}{\geq} \frac{1}{C_0^2}p(p-1)m_0\frac{h(t)^{p(x)}}{t^2} \overset{(2.10)}{\geq} \frac{1}{C_0^2}m_0^{q+1}p(p-1)t^{p(x)-2} \overset{(2.14)}{\geq} \lambda t^{p(x)-2}, \end{aligned}$$

where we used the fact that $m_0 \leq 1$.

Finally

$$\begin{aligned} g_{tx}(x,t) &= a_x(x)p(x)h(t)^{p(x)-1}h'(t) + a(x)p_x(x)h(t)^{p(x)-1}h'(t) \\ &\quad + a(x)p(x)h(t)^{p(x)-1}h'(t)p_x(x)\log(h(t)) \end{aligned}$$

so that

$$\begin{aligned} |g_{tx}(x,t)| &\leq \frac{|a_x(x)|}{m_0}a(x)p(x)h(t)^{p(x)-1}h'(t) + \frac{|p_x(x)|}{p}a(x)p(x)h(t)^{p(x)-1}h'(t) \\ &\quad + a(x)p(x)h(t)^{p(x)-1}h'(t)|p_x(x)|\log(h(t)) \\ &\leq \frac{\ell(x)}{m_0}g_t(x,t)(1+\log(h(t))), \end{aligned}$$

with ℓ as in (2.15). At this point we observe that, for $t \geq t_0$

$$\log(h(t)) \overset{(2.10)}{\leq} \log(C_0 M_0 t^{\delta+1}) = \log(C_0 M_0) + (\delta+1)\log t \leq Ct^\tau$$

where τ is as in (2.2) and $C := \max\{\log(C_0 M_0), \delta+1\} \geq 1$. This allows us to conclude that

$$|g_{tx}(x,t)| \leq \frac{\ell(x)}{m_0}g_t(x,t)2\max\{\log(C_0 M_0),\delta+1\}t^\tau \overset{(2.14),(2.16)}{\leq} \Lambda\ell(x)t^{(1+\delta)p(x)+\tau-1}.$$

2.4. Step 4: Approximation. We construct a sequence of smooth functions $g^{k\varepsilon}(x,t)$, related to $g(x,|Du|) = a(x)h(|Du|)^{p(x)}$. We will deal with this approximation procedure in two steps. First let us define, for a.e. $x \in B_{R_0}$

$$g_t^k(x,t) := \begin{cases} a(x)p(x)h(t)^{p(x)-1}h'(t) & 0 \le t < k \\ a(x)p(x)h(k)^{p(x)-1}h'(k) + \dfrac{\lambda}{p-1}[t^{p-1} - k^{p-1}] & t \ge k, \end{cases} \tag{2.17}$$

and

$$g^k(x,t) := \int_0^t g_t^k(x,s)\,ds + g(x,0). \tag{2.18}$$

Arguing in a similar way as in [13], it is possible to show that the sequence of functions defined by (2.18) satisfies, for k sufficiently large, the conditions

$$g^k(x,t) \le g(x,t) \tag{2.19}$$

$$g^k(x,t) \le g^{k+1}(x,t) \tag{2.20}$$

for all $t > 0$, a.e. in B_{R_0}.

At this point, let us denote with p_{ε_n} and a_{ε_n} the regularization of the functions p and a respectively

$$p_{\varepsilon_n}(x) = \int_B \rho(y)p(x+\varepsilon_n y)\,dy \quad a_{\varepsilon_n}(x) = \int_B \rho(y)a(x+\varepsilon_n y)\,dy, \tag{2.21}$$

where B denotes the unit ball, ρ is a positive symmetric mollifier such that $\int \rho = 1$ and where ε_n is an infinitesimal sequence of positive numbers. With an abuse of notation in the sequel we denote with $\varepsilon = \varepsilon_n$ and with $p_\varepsilon = p_{\varepsilon_n}$, $a_\varepsilon = a_{\varepsilon_n}$. It is well known that p_ε and a_ε converge in the strong topology of $W^{1,r}(B_{R_0})$ and uniformly in B_{R_0} to the functions p and a respectively. Now let us define

$$\tilde g_t^{k\varepsilon}(x,t) := \begin{cases} a_\varepsilon(x)p_\varepsilon(x)h(t)^{p_\varepsilon(x)-1}h'(t) & 0 \le t < k \\ a_\varepsilon(x)p_\varepsilon(x)h(k)^{p_\varepsilon(x)-1}h'(k) + \dfrac{\lambda}{p-1}[t^{p-1} - k^{p-1}] & t \ge k, \end{cases} \tag{2.22}$$

and

$$\tilde g^{k\varepsilon}(x,t) := \int_0^t \tilde g_t^{k\varepsilon}(x,s)\,ds + g(x,0).$$

Finally consider

$$g^{k\varepsilon}(x,t) := \tilde g^{k\varepsilon}(x,t) + \varepsilon(1+t^2)^{\frac{p}{2}}, \quad \forall t > 0, \text{ a.e. in } B_{R_0}. \tag{2.23}$$

By the properties of function $g(x,t)$ given in the previous step, since $p \le p_\varepsilon(x) \le q$ in B_{R_0} and by proceeding as in Lemmas 4.2 and Lemma 4.3 of [13], we have that the sequence $g^{k\varepsilon}(x,t)$ satisfies the following inequalities a.e. in B_{R_0} and for all $t > 0$, the lower bound in (2.25) and (2.26) ensured by (2.23)

$$g^{k\varepsilon}(x,t) \le C(k)(1+t^2)^{\frac{p}{2}} \tag{2.24}$$

$$\varepsilon(1+t^2)^{\frac{p-2}{2}} \le \frac{g_t^{k\varepsilon}(x,t)}{t} \le C(k)(1+t^2)^{\frac{p-2}{2}} \tag{2.25}$$

$$\min\{p-1,1\}\varepsilon(1+t^2)^{\frac{p-2}{2}} \le g_{tt}^{k\varepsilon}(x,t) \le C(k)(1+t^2)^{\frac{p-2}{2}} \tag{2.26}$$

$$|g_{tx}^{k\varepsilon}(x,t)| \le C(k,\varepsilon,\omega_0)(1+t^2)^{\frac{p-1}{2}} \quad \forall \omega_0 \subset\subset B_{R_0}. \tag{2.27}$$

Moreover the functions $g^{k\varepsilon}$ fulfill for a.e. $x \in B_{R_0}$ and $t \ge t_0$

$$\lambda t^{p-2} \le \frac{g_t^{k\varepsilon}(x,t)}{t} \le \Lambda t^{q-2} \tag{2.28}$$

$$\lambda t^{p-2} \le g_{tt}^{k\varepsilon}(x,t) \le \Lambda t^{q-2} \tag{2.29}$$

$$|g_{tx}^{k\varepsilon}(x,t)| \le \ell_\varepsilon(x)(1+t^2)^{\frac{q-1}{2}}, \tag{2.30}$$

where $\ell_\varepsilon \in \mathscr{C}^\infty(B_{R_0})$ is the regularized function of ℓ is defined as in (2.15) and λ, Λ are as in (2.11), (2.12).

In the sequel, for simplicity of notations, we assume that $t_0 = 1$.

2.5. Step 5: a priori estimates. Let $w \in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ be a local minimizer of the functional (1.1); moreover let us take $B_R \subset\subset B_{R_0}$ to be a ball of radius R compactly contained in B_{R_0}. Consider the following variational problem

$$\inf\{F^{k\varepsilon}(v) : v \in w + W_0^{1,p}(B_R;\mathbb{R}^N)\}. \tag{2.31}$$

where

$$F^{k\varepsilon}(v) = \int_{B_{R_0}} g^{k\varepsilon}(x,|Dv|)\,dx. \tag{2.32}$$

Since $F^{k\varepsilon}$ is lower semicontinuous, there exists $v^{k\varepsilon} \in w + W_0^{1,p}(B_{R_0};\mathbb{R}^N)$ solution to Problem (2.31). The purpose of this Step is to prove an a priori estimate for the L^∞-norm of $Dv^{k\varepsilon}$ independent of k,ε. We claim that, for $0 < \rho < R < R_0$

$$\|Dv^{k\varepsilon}\|_{L^\infty(B_\rho;\mathbb{R}^{Nn})} \le C\left[\frac{(1+\|\ell_\varepsilon\|^2_{L^r(B_{R_0})})^{\frac{1}{2}}}{(R-\rho)}\right]^{\frac{\beta}{\frac{1}{n}-\frac{1}{r}}}\left[\int_{B_R}(1+g^{k\varepsilon}(x,|Dv^{k\varepsilon}|))\,dx\right]^{\beta}, \tag{2.33}$$

with constants C, β independent of k, ε. Once (2.33) is obtained, we observe that, by convolution properties

$$\|\ell_\varepsilon\|_{L^r(B_{R_0})} \le \|\ell\|_{L^r(B_{R_0})},$$

thus we deduce

$$(2.34)\quad \|Dv^{k\varepsilon}\|_{L^\infty(B_\rho;\mathbb{R}^{Nn})} \le C\left[\frac{(1+\|\ell\|^2_{L^r(B_{R_0})})^{\frac{1}{2}}}{(R-\rho)}\right]^{\frac{\beta}{\frac{1}{n}-\frac{1}{r}}}\left[\int_{B_R}(1+g^{k\varepsilon}(x,|Dv^{k\varepsilon}|))\,dx\right]^\beta.$$

The proof of estimate (2.33) turns to be quite similar to the proof of Proposition 3.1 of [13]; here, for sake of clarity, we list the main arguments, referring to [13] for more details. For simplicity of notations, from now on we set

$$f(x,\xi) = f^{k\varepsilon}(x,\xi) = g^{k\varepsilon}(x,|\xi|), \quad v^{k\varepsilon} := u, \quad \ell_\varepsilon := \ell.$$

Observe that, by (2.28), (2.29), (2.30), we have that, for $|\xi| \ge 1$ and a.e. $x \in B_{R_0}$

$$(2.35)\quad \lambda|\xi|^{p-2}|\mu|^2 \le \sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x,\xi)\mu_i^\alpha\mu_j^\beta,$$

$$(2.36)\quad |f_{\xi_i^\alpha\xi_j^\beta}(x,\xi)| \le \Lambda|\xi|^{q-2},$$

$$(2.37)\quad |f_{\xi x}(x,\xi)| \le \ell(x)|\xi|^{q-1},$$

hold, where p and q have been introduced in (2.1). The minimizer u satisfies the Euler's first variation

$$\int_{B_{R_0}}\sum_{i,\alpha} f_{\xi_i^\alpha}(x,Du)\varphi^\alpha_{x_i}(x)\,dx = 0 \quad \forall\varphi = (\varphi^\alpha)_{\alpha=1,\dots,N} \in W_0^{1,p}(B_{R_0};\mathbb{R}^N),$$

and, by using the technique of the different quotients (see for example [14], [17], [19]) we have that

$$(2.38)\quad u \in W^{2,\min(2,p)}_{\rm loc}(B_{R_0};\mathbb{R}^N), \quad (1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2 \in L^1_{\rm loc}(B_{R_0})$$

$u \in W^{1\infty}_{\rm loc}(\Omega,\mathbb{R}^N)$ (see [8] and references there) and the second variation

$$(2.39)\quad \int_{B_{R_0}}\left\{\sum_{i,j,\alpha,\beta,s} f_{\xi_i^\alpha\xi_j^\beta}(x,Du)\varphi^\alpha_{x_i}u^\beta_{x_sx_j} + \sum_{i,\alpha,s} f_{\xi_i^\alpha x_s}(x,Du)\varphi^\alpha_{x_i}\right\}dx = 0$$

$$\forall s = 1,\dots,n, \quad \forall\varphi = (\varphi^\alpha)_{\alpha=1,\dots,N} \in W_0^{1,\min(2,p)}(B_{R_0};\mathbb{R}^N).$$

Let $\eta \in \mathscr{C}_0^1(B_{R_0})$. For any fixed $s \in \{1, \dots, n\}$, we choose

$$\varphi^\alpha = \eta^2 u^\alpha_{x_s} \Phi((|Du| - 1)_+)$$

for $\Phi : [0, +\infty) \to [0, +\infty)$ an increasing, locally Lipschitz continuous function, with Φ and Φ' bounded on $[0, +\infty)$, such that $\Phi(0) = 0$ and satisfying

$$\Phi'(s)s \le c_\Phi \Phi(s) \tag{2.40}$$

for a suitable value of c_Φ. In the following $(a)_+$ denotes the positive part of $a \in \mathbb{R}$ and we write $\Phi((|Du| - 1)_+) = \Phi(|Du| - 1)_+$. Let us compute

$$\begin{aligned}\varphi^\alpha_{x_i} = {} & 2\eta\eta_{x_i} u^\alpha_{x_s} \Phi(|Du| - 1)_+ + \eta^2 u^\alpha_{x_s x_i} \Phi(|Du| - 1)_+ \\ & + \eta^2 u^\alpha_{x_s} \Phi'(|Du| - 1)_+ [(|Du| - 1)_+]_{x_i}.\end{aligned}$$

Here we used the fact that $u \in W^{1,\infty}_{\mathrm{loc}}(B_{R_0}; \mathbb{R}^N)$, see Proposition 3.1 of [8]. Plugging this expression in (2.39) we obtain

$$\begin{aligned} 0 = {} & \int_{B_{R_0}} 2\eta\Phi(|Du| - 1)_+ \sum_{i,j,\alpha,\beta,s} f_{\xi^\alpha_i \xi^\beta_j}(x, Du)\eta_{x_i} u^\alpha_{x_s} u^\beta_{x_s x_j}\, dx \\ & + \int_{B_{R_0}} \eta^2 \Phi(|Du| - 1)_+ \sum_{i,j,\alpha,\beta,s} f_{\xi^\alpha_i \xi^\beta_j}(x, Du) u^\alpha_{x_s x_i} u^\beta_{x_s x_j}\, dx \\ & + \int_{B_{R_0}} \eta^2 \Phi'(|Du| - 1)_+ \sum_{i,j,\alpha,\beta,s} f_{\xi^\alpha_i \xi^\beta_j}(x, Du) u^\alpha_{x_s} u^\beta_{x_s x_j} [(|Du| - 1)_+]_{x_i}\, dx \\ & + \int_{B_{R_0}} 2\eta\Phi(|Du| - 1)_+ \sum_{i,\alpha,s} f_{\xi^\alpha_i x_s}(x, Du)\eta_{x_i} u^\alpha_{x_s}\, dx \\ & + \int_{B_{R_0}} \eta^2 \Phi(|Du| - 1)_+ \sum_{i,\alpha,s} f_{\xi^\alpha_i x_s}(x, Du) u^\alpha_{x_s x_i}\, dx \\ & + \int_{B_{R_0}} \eta^2 \Phi'(|Du| - 1)_+ \sum_{i,\alpha,s} f_{\xi^\alpha_i x_s}(x, |Du|) u^\alpha_{x_s} [(|Du| - 1)_+]_{x_i}\, dx \\ = : {} & I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \end{aligned} \tag{2.41}$$

In the following, constants will be denoted by C, regardless of their actual value. We have that

$$I_1 + I_2 + I_3 = -(I_4 + I_5 + I_6).$$

Consider I_1; by the Cauchy–Schwartz inequality, the Young inequality and (2.36), we have

$$
\begin{aligned}
(2.42)\qquad |I_1| &= \left| \int_{B_{R_0}} 2\eta\Phi(|Du|-1)_+ \sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x,Du)\eta_{x_i} u_{x_s}^\alpha u_{x_s x_j}^\beta \, dx \right| \\
&\le \int_{B_{R_0}} 2\eta\Phi(|Du|-1)_+ \left\{ \sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x,Du)\eta_{x_i} u_{x_s}^\alpha \eta_{x_j} u_{x_s}^\beta \right\}^{\frac{1}{2}} \\
&\quad \times \left\{ \sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x,Du) u_{x_s x_i}^\alpha u_{x_s x_j}^\beta \right\}^{\frac{1}{2}} dx \\
&\le C \int_{B_{R_0}} |D\eta|^2 \Phi(|Du|-1)_+ |Du|^q \, dx \\
&\quad + \frac{1}{2} \int_{B_{R_0}} \eta^2 \Phi(|Du|-1)_+ \sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha \xi_j^\beta}(x,Du) u_{x_s x_i}^\alpha u_{x_s x_j}^\beta \, dx.
\end{aligned}
$$

Therefore

$$
\frac{1}{2} I_2 + I_3 \le C \int_{B_{R_0}} |D\eta|^2 \Phi(|Du|-1)_+ |Du|^q \, dx - (I_4 + I_5 + I_6).
$$

By proceding as in [13], (see also Lemma 4.1 of [25]), we have that

$$
I_3 \ge \int_{B_{R_0}} \eta^2 \Phi'(|Du|-1)_+ \frac{g_{tt}(x,|Du|)}{|Du|} \sum_\alpha \left(\sum_i u_{x_i}^\alpha [(|Du|-1)_+]_{x_i} \right)^2 dx \ge 0,
$$

and (2.35) implies

$$
I_2 \ge \int_{B_{R_0}} \eta^2 \Phi(|Du|-1)_+ |Du|^{p-2} |D^2 u|^2 \, dx.
$$

We now deal with $|I_4|$. We have, by (2.30)

$$
\begin{aligned}
|I_4| &= \left| \int_{B_{R_0}} 2\eta\Phi(|Du|-1)_+ \sum_{i,s,\alpha} f_{\xi_i^\alpha x_s}(x,Du)\eta_{x_i} u_{x_s}^\alpha \, dx \right| \\
&\le \int_{B_{R_0}} 2\eta\Phi(|Du|-1)_+ |g_{tx}(x,|Du|)| \sum_{i,s,\alpha} |\eta_{x_i} u_{x_s}^\alpha| \, dx \\
&\le \int_{B_{R_0}} (\eta^2 + |D\eta|^2)\ell(x)\Phi(|Du|-1)_+ |Du|^q \, dx.
\end{aligned}
$$

Consider now $|I_5|$. We have, again by (2.30)

$$
\begin{aligned}
|I_5| &= \left| \int_{B_{R_0}} \eta^2 \Phi(|Du|-1)_+ |g_{tx}(x,|Du|)|\,|D^2u|\,dx \right| \\
&\le \int_{B_{R_0}} \eta^2 \Phi(|Du|-1)_+ \ell(x)|Du|^{q-1}|D^2u|\,dx \\
&\le \int_{B_{R_0}} [\eta^2 \Phi(|Du|-1)_+ |Du|^{p-2}|D^2u|^2]^{1/2} \\
&\quad \times [\eta^2 \Phi(|Du|-1)_+ |\ell(x)|^2 |Du|^{2q-p}]^{1/2}\,dx \\
&\le \varepsilon \int_{B_{R_0}} \eta^2 \Phi(|Du|-1)_+ |Du|^{p-2}|D^2u|^2\,dx \\
&\quad + C_\varepsilon \int_{B_{R_0}} \eta^2 \Phi(|Du|-1)_+ |\ell(x)|^2 |Du|^{2q-p}\,dx,
\end{aligned}
$$

where in the last line we used the Young inequality. The estimate of I_6 is more delicate. For any $0 < \delta < 1$ (please, excuse the abuse of notation: the δ used here is different from the one in (1.2))

$$
\begin{aligned}
|I_6| &= \left| \int_{B_{R_0}} \eta^2 |g_{tx}(x,|Du|)|\,|Du|\Phi'(|Du|-1)_+[(|Du|-1)_+]_{x_i}\,dx \right| \\
&\le \int_{B_{R_0}} \eta^2 \Phi'(|Du|-1)_+ \ell(x)|Du|^q|D^2u|\,dx \\
&= \int_{B_{R_0}} \eta^2 \Phi'(|Du|-1)_+ \ell(x)[(|Du|-1)_+ + \delta][(|Du|-1)_+ + \delta]^{-1}|Du|^q|D^2u|\,dx \\
&\le \int_{B_{R_0}} \eta^2 \left\{ \frac{1}{c_\Phi} \Phi'(|Du|-1)_+[(|Du|-1)_+ + \delta]|Du|^{p-2}|D^2u|^2 \right\}^{1/2} \\
&\quad \times \{ c_\Phi \Phi'(|Du|-1)_+ |\ell(x)|^2 |Du|^{2q-p+2}[(|Du|-1)_+ + \delta]^{-1} \}^{1/2}\,dx \\
&\le \frac{\varepsilon}{c_\Phi} \int_{B_{R_0}} \eta^2 \Phi'(|Du|-1)_+[(|Du|-1)_+ + \delta]|Du|^{p-2}|D^2u|^2\,dx \\
&\quad + C_\varepsilon c_\Phi \int_{B_{R_0}} \eta^2 \Phi'(|Du|-1)_+ |\ell(x)|^2 |Du|^{2q-p+2}[(|Du|-1)_+ + \delta]^{-1}\,dx.
\end{aligned}
$$

To estimate the first term in the last inequality we split $B_{R_0} = \{x : |Du(x)| \ge 2\} \cup \{x : |Du(x)| < 2\}$ and we observe that in the set $\{x : |Du(x)| \ge 2\}$ we have $(|Du|-1)_+ \ge 1$, since $\delta < 1$

$$
(|Du|-1)_+ + \delta \le 2(|Du|-1)_+. \tag{2.43}
$$

Therefore we have, using (2.40)

$$\int_{|Du|\ge 2} \eta^2\Phi'(|Du|-1)_+[(|Du|-1)_+ + \delta]|Du|^{p-2}|D^2u|^2\,dx$$

$$+\int_{1<|Du|<2} \eta^2\Phi'(|Du|-1)_+[(|Du|-1)_+ + \delta]|Du|^{p-2}|D^2u|^2\,dx$$

$$\le 2\int_{|Du|\ge 2} \eta^2\Phi'(|Du|-1)_+(|Du|-1)_+|Du|^{p-2}|D^2u|^2\,dx$$

$$+\int_{1<|Du|<2} \eta^2\Phi'(|Du|-1)_+(|Du|-1)_+|Du|^{p-2}|D^2u|^2\,dx$$

$$+\delta\int_{1<|Du|<2} \eta^2\Phi'(|Du|-1)_+|Du|^{p-2}|D^2u|^2\,dx$$

$$\le 2c_\Phi\int_{B_{R_0}} \eta^2\Phi(|Du|-1)_+|Du|^{p-2}|D^2u|^2\,dx$$

$$+\delta\int_{1<|Du|<2} \eta^2\Phi'(|Du|-1)_+|Du|^{p-2}|D^2u|^2\,dx.$$

Now, for ε sufficiently small, by collecting the previous estimates, we deduce

$$\begin{aligned}\int_{B_{R_0}} &\eta^2\Phi(|Du|-1)_+|Du|^{p-2}|D^2u|^2\,dx\\ &\le Cc_\Phi\int_{B_{R_0}} (\eta^2+|D\eta|^2)(1+\ell^2(x))|Du|^{2q-p}\\ &\quad\times[\Phi(|Du|-1)_+|Du|^{2q-p}\\ &\qquad+\Phi'(|Du|-1)_+|Du|^2[(|Du|-1)_+ +\delta]^{-1}]\,dx\\ &\quad+\delta\int_{1<|Du|<2} \eta^2\Phi'(|Du|-1)_+|Du|^{p-2}|D^2u|^2\,dx,\end{aligned} \tag{2.44}$$

with a constant C depending on n, r, p, q.

Let define the function

$$\Phi(s) := (1+s)^{\gamma-2}s^2 \quad \gamma\ge 0; \tag{2.45}$$

which satisfies (2.40) with $c_\Phi = 2(1+\gamma)$. We can approximate Φ in (2.45) by a sequence of functions Φ_r, each of them being equal to Φ in the interval $[0,r]$, and then extended to $[r,+\infty)$ with the constant value $\Phi(r)$. Then we insert Φ_r in (2.44) and, passing to the limit as $r\to+\infty$ by the Monotone Convergence Theorem, we obtain for every $0<\delta<1$

$$\begin{aligned}(2.46)\qquad &\int_{B_{R_0}} \eta^2(1+(|Du|-1)_+)^{\gamma-2}(|Du|-1)_+^2|Du|^{p-2}|D^2u|^2\,dx \\ &\le C(1+\gamma)^2\int_{B_{R_0}}(\eta^2+|D\eta|^2)(1+\ell(x)^2)(1+(|Du|-1)_+)^{\gamma+2q-p}\,dx \\ &\quad+\delta c(\gamma)\int_{1<|Du|<2}\eta^2|Du|^{p-2}|D^2u|^2\,dx,\end{aligned}$$

as

$$\frac{(|Du|-1)_+}{(|Du|-1)_++\delta}\le 1\quad\forall\delta>0.$$

By the elementary inequality (for the proof see [13])

$$C_1(1+|\xi|^2)^{\frac{p-2}{2}}\le|\xi|^{p-2}\le C_2(1+|\xi|^2)^{\frac{p-2}{2}},\quad |\xi|\ge t_0,$$

with C_1, C_2 depending on t_0 (here $t_0=1$), we have

$$\int_{1<|Du|<2}\eta^2|Du|^{p-2}|D^2u|^2\,dx\le C\int_{1<|Du|<2}\eta^2(1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\,dx<+\infty$$

and, due to (2.38), the last integral is finite. As $\delta\to 0$, the last term in (2.46) vanishes.

Now, let us define

$$(2.47)\qquad m:=\left(\frac{r}{2}\right)'=\frac{r}{r-2}.$$

Since $\ell\in L^r(B_{R_0})$, by the Hölder inequality

$$\begin{aligned}(2.48)\qquad &\int_{B_{R_0}}\eta^2(1+(|Du|-1)_+)^{\gamma-2}(|Du|-1)_+^2|Du|^{p-2}|D((|Du|-1)_+)|^2\,dx \\ &\le C(1+\gamma)^2H\left[\int_{B_{R_0}}(\eta^2+|D\eta|^2)^m(1+(|Du|-1)_+)^{(\gamma+2q-p)m}\,dx\right]^{\frac{1}{m}},\end{aligned}$$

where C depends also on r and $|B_{R_0}|$ (and so on n) and

$$(2.49)\qquad H:=(1+\|\ell\|^2_{L^r(B_{R_0})}).$$

Let us introduce

$$(2.50)\qquad G(t)=1+\int_0^t\sqrt{\Phi(s)}(1+s)^{\frac{p-2}{2}}\,ds=1+\int_0^t(1+s)^{\frac{\gamma}{2}+\frac{p}{2}-2}s\,ds;$$

since $p \le q \le 2q - p$ we get

$$[G(t)]^2 \le 4(1+t)^{\gamma+p} \le 4(1+t)^{\gamma+2q-p}. \tag{2.51}$$

Moreover

$$G_t(t) = \sqrt{\Phi(t)}(1+t)^{\frac{p-2}{2}} \overset{(2.45)}{=} (1+t)^{\frac{\gamma}{2}+\frac{p}{2}-2}t. \tag{2.52}$$

Set $w = \eta G((|Du|-1)_+)$, we have

$$\begin{aligned} &\int_{B_{R_0}} |D(\eta G((|Du|-1)_+))|^2\,dx \\ &\quad \le 2\int_{B_{R_0}} |D\eta|^2 |G((|Du|-1)_+)|^2\,dx \\ &\qquad + 2\int_{B_{R_0}} \eta^2 [G_t((|Du|-1)_+)]^2 [D((|Du|-1)_+)]^2\,dx \\ &\quad \le C(1+\gamma)^2 H\left[\int_{B_{R_0}} (\eta^2+|D\eta|^2)^m [1+(|Du|-1)_+^{(\gamma+2q-p)m}]\,dx\right]^{\frac{1}{m}}. \end{aligned} \tag{2.53}$$

Now, let $2^* = \frac{2n}{n-2}$ for $n > 2$, while 2^* equal to any fixed real number larger than 2, if $n = 2$. By Sobolev's inequality there exists a constant C such that

$$\left\{\int_{B_{R_0}} [\eta G((|Du|-1)_+)]^{2^*}\,dx\right\}^{\frac{2}{2^*}} \le C\int_{B_{R_0}} |D(\eta G((|Du|-1)_+))|^2\,dx. \tag{2.54}$$

Moreover, since $r > n$, we have

$$1 \le m := \frac{r}{r-2} < \frac{n}{n-2} = \frac{2^*}{2}. \tag{2.55}$$

Observe that

$$(2q-p)m = 2(q-p)m + pm; \tag{2.56}$$

and in view of the strict inequality in (2.3), we infer the existence of $0 < \epsilon < 1$ such that

$$(q-p) + \epsilon\left(\frac{1}{n} - \frac{1}{r}\right) \le p\left(\frac{1}{n} - \frac{1}{r}\right). \tag{2.57}$$

Set now

$$0 < \tilde{M} := 2(q-p)m + p(m-1) + \epsilon \quad 0 < \tilde{N} := p - \epsilon. \tag{2.58}$$

We have

$$\tilde{M} + \tilde{N} = (2q - p)m \quad \text{and} \quad \tilde{M} > (2q - p)m - p. \tag{2.59}$$

By the assumptions on p, q and the definition of $\tilde{M}$ and $\tilde{N}$, we claim that

$$\begin{aligned} &\frac{1}{(\gamma + p)^2}\left[\int_{B_{R_0}} \eta^{2^*}[1 + (|Du| - 1)_+]^{\left(\gamma + \frac{\tilde{M}}{m}\right)\frac{2^*}{2} + \tilde{N}}\, dx\right]^{\frac{2}{2^*}} \\ &\qquad \leq 4\Big(\int_{B_{R_0}} [\eta G((|Du| - 1)_+)]^{2^*}\Big)^{\frac{2}{2^*}}. \end{aligned} \tag{2.60}$$

By the definition of G, if we set $t := (|Du| - 1)_+$, (2.60) is proved once we get

$$\frac{1}{\gamma + p}(1 + t)^{\left(\frac{\gamma}{2} + \frac{\tilde{M}}{2m} + \frac{\tilde{N}}{2^*}\right)} \leq 2G(t) = 2\Big(1 + \int_0^t (1 + s)^{\frac{\gamma}{2} + \frac{p}{2} - 2} s\, ds\Big). \tag{2.61}$$

If $t \leq 1$; (2.61) holds, in fact

$$\frac{1}{\gamma + p}(1 + t)^{\left(\frac{\gamma}{2} + \frac{\tilde{M}}{2m} + \frac{\tilde{N}}{2^*}\right)} \leq \frac{2}{\gamma + p} \leq 2 \leq 2\Big(1 + \int_0^t (1 + s)^{\frac{\gamma}{2} + \frac{p}{2} - 2} s\, ds\Big).$$

Let now $t \geq 1$; then (2.61) becomes, after differentiation

$$\frac{\frac{\gamma}{2} + \frac{\tilde{M}}{2m} + \frac{\tilde{N}}{2^*}}{\gamma + p}(1 + t)^{\frac{\gamma}{2} + \frac{\tilde{M}}{2m} + \frac{\tilde{N}}{2^*} - 1} \leq 2(1 + t)^{\frac{\gamma}{2} + \frac{p}{2} - 2} t. \tag{2.62}$$

Since

$$\frac{1}{2m} - \frac{1}{2^*} = \frac{1}{n} - \frac{1}{r},$$

we have that

$$(q - p) + \frac{p}{2} - \frac{p}{2m} + \frac{\epsilon}{2m} + \frac{p}{2^*} - \frac{\epsilon}{2^*} \leq \frac{p}{2}$$

which implies, by the definition of $\tilde{M}$ and $\tilde{N}$

$$\frac{\tilde{M}}{2m} + \frac{\tilde{N}}{2^*} \leq \frac{p}{2}. \tag{2.63}$$

We then have

$$\frac{\frac{\gamma}{2}+\frac{\tilde{M}}{2m}+\frac{\tilde{N}}{2^*}}{\gamma+p}(1+t)^{\frac{\tilde{M}}{2m}+\frac{\tilde{N}}{2^*}} \le \frac{1}{2}(1+t)^{\frac{p}{2}} = \frac{1}{2}(1+t)^{\frac{p-2}{2}}(1+t) \overset{t\ge 1}{\le} (1+t)^{\frac{p-2}{2}}t$$

and so (2.62) is satisfied.

By collecting (2.53), (2.54) and (2.60), we obtain

$$\left[\int_{B_{R_0}} \eta^{2^*}[1+(|Du|-1)_+]^{\left(\gamma+\frac{\tilde{M}}{m}\right)\frac{2^*}{2}}[1+(|Du|-1)_+]^{\tilde{N}}\,dx\right]^{\frac{2}{2^*}} \tag{2.64}$$
$$\le CH(\gamma+2q-p)^4$$
$$\times\left[\int_{B_{R_0}} (\eta^2+|D\eta|^2)^m[1+(|Du|-1)_+]^{(\gamma+2q-p)m}\,dx\right]^{\frac{1}{m}},$$

where the constant C only depends on n, r, p, q, λ, Λ.

Now, let us choose $0<\rho<R<R_0$ and let η to be equal to 1 in B_ρ, with $\operatorname{supp}\eta\subset B_R$ and such that $|D\eta|\le\frac{1}{(R-\rho)}$. Let us denote by

$$\kappa := \gamma m+\tilde{M} = (\gamma+2q-p)m-\tilde{N}.$$

From (2.64) it follows

$$\left\{\int_{B_\rho}[1+(|Du|-1)_+]^{\kappa\frac{2^*}{2m}}[1+(|Du|-1)_+]^{\tilde{N}}\,dx\right\}^{\frac{2m}{2^*}} \tag{2.65}$$
$$\le CH^m\left(\frac{(\kappa+\tilde{N})^2}{R-\rho}\right)^{2m}\int_{B_R}[1+(|Du|-1)_+]^{\kappa}[1+(|Du|-1)_+]^{\tilde{N}}\,dx.$$

Fixed $\bar{R}$ and $\bar{\rho}$, with $\bar{R}>\bar{\rho}$, we define the decreasing sequence of radii $\{\rho_i\}_{i\ge 0}$

$$\rho_i=\bar{\rho}+\frac{\bar{R}-\bar{\rho}}{2^i},\quad \forall i\ge 0,\quad R-\rho:=\rho_i-\rho_{i+1}=\frac{\bar{R}-\bar{\rho}}{2^{i+1}}$$

with $\rho_0=\bar{R}>\rho_i>\rho_{i+1}>\bar{\rho}$ and the increasing sequence $\{\kappa_i\}_{i\ge 0}$

$$\kappa_0:=\tilde{M}\quad \kappa_{i+1}=\kappa_i\frac{2^*}{2m}\quad i\ge 0.$$

We rewrite (2.65) and we obtain for every $i\ge 0$

$$A_{i+1}\le C_iA_i, \tag{2.66}$$

where

$$A_i := \left(\int_{B_{\rho_i}} [1 + (|Du| - 1)_+]^{\kappa_i} [1 + (|Du| - 1)_+]^{\tilde{N}} \, dx \right)^{\frac{1}{\kappa_i}}$$

$$C_i := \left[CH^m \left(\frac{(\kappa_i + \tilde{N})^{\frac{3}{2}} 2^{i+1}}{\bar{R} - \bar{\rho}} \right)^{2m} \right]^{\frac{1}{\kappa_i}}.$$

By iteration of (2.66), we deduce

$$\text{(2.67)} \qquad \left\{ \int_{B_{\bar{\rho}}} [1 + (|Du| - 1)_+]^{\kappa_0 \left(\frac{2^*}{2m}\right)^{i+1}} [1 + (|Du| - 1)_+]^{\tilde{N}} \, dx \right\}^{\left(\frac{2m}{2^*}\right)^{i+1}} \leq \tilde{C} \int_{B_{\bar{R}}} [1 + (|Du| - 1)_+]^{(2q-p)m} \, dx,$$

where

$$\text{(2.68)} \qquad \tilde{C} \leq \prod_{k=0}^{\infty} \left[CH^m \left(\frac{(\kappa_k + \tilde{N})^{\frac{3}{2}} 2^{k+1}}{\bar{R} - \bar{\rho}} \right)^{2m} \right]^{\left(\frac{2m}{2^*}\right)^k} \leq \frac{CH^{\frac{1}{2\left(\frac{1}{n} - \frac{1}{r}\right)}}}{(\bar{R} - \bar{\rho})^{\frac{22^* m}{2^* - 2m}}},$$

with a constant $C = C(n, r, p, q)$. Let us denote

$$\text{(2.69)} \qquad \tau := \frac{22^* m}{2^* - 2m} = \frac{1}{\frac{1}{n} - \frac{1}{r}};$$

thus (2.67) implies

$$\text{(2.70)} \qquad \left\{ \int_{B_{\bar{\rho}}} [1 + (|Du| - 1)_+]^{\kappa_0 \left(\frac{2^*}{2m}\right)^{i+1}} \, dx \right\}^{\left(\frac{2m}{2^*}\right)^{i+1}} \leq C \left[\frac{\sqrt{H}}{(\bar{R} - \bar{\rho})} \right]^{\tau} \int_{B_{\bar{R}}} [1 + (|Du| - 1)_+]^{(2q-p)m} \, dx.$$

At this point we pass to the limit as $i \to +\infty$, obtaining

$$\text{(2.71)} \qquad \sup_{x \in B_{\bar{\rho}}} [1 + (|Du|(x) - 1)_+]^{\tilde{M}} = \lim_{i \to +\infty} \left\{ \int_{B_{\bar{\rho}}} [1 + (|Du| - 1)_+]^{\tilde{M} \left(\frac{2^*}{2m}\right)^{i+1}} \right\}^{\left(\frac{2m}{2^*}\right)^{i+1}} \leq C \left[\frac{\sqrt{H}}{(\bar{R} - \bar{\rho})} \right]^{\tau} \int_{B_{\bar{R}}} [1 + (|Du| - 1)_+]^{(2q-p)m} \, dx.$$

Set

$$V(x) := 1 + (|Du|(x) - 1)_+ \quad \text{and} \quad s := (2q - p)m; \tag{2.72}$$

the estimate (2.71) becomes, for any $\rho < R < R_0$

$$\sup_{x \in B_\rho} |V(x)| \le C\left(\left[\frac{\sqrt{H}}{(R-\rho)}\right]^{\frac{\tau}{s}} \|V\|_{L^s(B_R)}\right)^{\frac{s}{\tilde{M}}}. \tag{2.73}$$

Our aim is to estimate the essential supremum of $|Du|$ in terms of its L^p-norm. By classical interpolation inequalities

$$\|V\|_{L^s(B_\rho)} \le \|V\|_{L^p(B_\rho)}^{\frac{p}{s}} \|V\|_{L^\infty(B_\rho)}^{1-\frac{p}{s}}, \tag{2.74}$$

and (2.73) and (2.74) give

$$\|V\|_{L^s(B_\rho)} \le C^{1-\frac{p}{s}} \|V\|_{L^p(B_\rho)}^{\frac{p}{s}} \left(\left[\frac{\sqrt{H}}{(\bar{R}-\bar{\rho})}\right]^{\frac{\tau}{s}} \|V\|_{L^s(B_R)}\right)^{\theta} \tag{2.75}$$

where

$$\theta := \frac{s}{\tilde{M}}\left(1 - \frac{p}{s}\right) = \frac{1}{\tilde{M}}(s-p) = \frac{1}{\tilde{M}}[(2q-p)m - p] < 1. \tag{2.76}$$

For $0 < \bar{\rho} < \bar{R}$ and for every $k \ge 0$, let us define

$$\rho_k := \bar{R} - (\bar{R} - \bar{\rho})2^{-k} \quad B_k := \|V\|_{L^s(B_{\rho_k})}.$$

By inserting in (2.75) $\rho = \rho_k$ and $R = \rho_{k+1}$, (so that $R - \rho = (\bar{R} - \bar{\rho})2^{-(k+1)}$), we have for every $k \ge 0$

$$B_k \le C^{1-\frac{p}{s}} \|V\|_{L^p(B_{\bar{R}})}^{\frac{p}{s}} \left(2^{\frac{\tau}{s}(k+1)} \left[\frac{\sqrt{H}}{(\bar{R}-\bar{\rho})}\right]^{\frac{\tau}{s}} B_{k+1}\right)^{\theta}. \tag{2.77}$$

By iteration of (2.77), we deduce for $k \ge 0$

$$B_0 \le \left(C^{1-\frac{p}{s}} \left[\frac{\sqrt{H}}{(\bar{R}-\bar{\rho})}\right]^{\frac{\tau}{s}\theta} \|V\|_{L^p(B_{\bar{R}})}^{\frac{p}{s}}\right)^{\sum_{i=0}^{k} \theta^i} 2^{\frac{\tau}{s}\sum_{i=0}^{k+1} i\theta^i} (B_{k+1})^{\theta^{k+1}}. \tag{2.78}$$

By (2.76), the series appearing in (2.78) are convergent. Since B_k is bounded independently of k, i.e.

$$B_{k+1} \le \|V\|_{L^s(B_{\bar{R}})},$$

we can pass to the limit as $k \to +\infty$ and we obtain for every $0 < \rho < R$ with a constant $C = C(n, r, p, q)$ independent of k

$$\|V\|_{L^s(B_\rho)} \le C\left(\left[\frac{\sqrt{H}}{(R-\rho)}\right]^{\frac{\tau}{s}\theta} \|V\|_{L^p(B_R)}^{\frac{p}{s}}\right)^{\frac{1}{1-\theta}}. \tag{2.79}$$

Combining (2.73) and (2.79), by setting $\rho' = \frac{(R+\rho)}{2}$ we have

$$\|V\|_{L^\infty(B_\rho)} \le C\left(\left[\frac{\sqrt{H}}{(\rho'-\rho)}\right]^{\frac{\tau}{s}(1-\theta)}\left[\frac{\sqrt{H}}{(R-\rho')}\right]^{\frac{\tau}{s}\theta}\|V\|^{\frac{p}{s}}_{L^p(B_R)}\right)^{\frac{s}{M}\frac{1}{1-\theta}}, \tag{2.80}$$

which implies

$$\|Du\|_{L^\infty(B_\rho;\mathbb{R}^{Nn})} \le \|V\|_{L^\infty(B_\rho)} \le C\left[\frac{\sqrt{H}}{(R-\rho)}\right]^{\tilde{\beta}}\left(\int_{B_R}(1+|Du|^p)\,dx\right)^{\beta},$$

with

$$\beta = \frac{1}{\tilde{M}-s+p} = \frac{1}{\epsilon} > 1, \quad \tilde{\beta} = \frac{1}{\frac{1}{n}-\frac{1}{r}}\frac{1}{\epsilon} = \frac{1}{\frac{1}{n}-\frac{1}{r}}\beta. \tag{2.81}$$

Since

$$\frac{\lambda}{p-1}t^p \le g^{k\varepsilon}(x,t) \quad \forall t \ge 1 \text{ a.e. in } B_{R_0},$$

the estimate (2.33) follows.

2.6. Step 6: comparison and conclusion. We go back to Problem (2.31) and we observe that, since

$$t^p \le C(1+g^{k\varepsilon}(x,t)) \quad \forall t > 0, \text{ for a.e. } x \in B_{R_0},$$

by the minimality of $v^{k\varepsilon}$, we have

$$\begin{aligned}\int_{B_R}|Dv^{k\varepsilon}|^p\,dx &\le C\int_{B_R}[1+g^{k\varepsilon}(x,|Dv^{k\varepsilon}|)]\,dx \\ &\le C\int_{B_R}[1+g^{k\varepsilon}(x,|Dw|)]\,dx.\end{aligned} \tag{2.82}$$

Moreover, by the convolution properties, as $\varepsilon \to 0$

$$g^{k\varepsilon}(x,|Dw|) \to g^k(x,|Dw|) \quad \text{a.e. in } B_{R_0},$$

and

$$g^{k\varepsilon}(x,|Dw|) \le C(k)(1+|Dw|^2)^{\frac{p}{2}} \in L^1(B_{R_0}).$$

The Lebesgue Dominated Convergence Theorem and (2.19) imply

$$\lim_{\varepsilon}\int_{B_R} g^{k\varepsilon}(x,|Dw|)\,dx = \int_{B_R} g^k(x,|Dw|)\,dx \overset{(2.19)}{\le} \int_{B_R} a(x)h(|Dw|)^{p(x)}\,dx. \tag{2.83}$$

By collecting (2.82) and (2.83)

$$\sup_{\varepsilon} \int_{B_R} |Dv^{k\varepsilon}|^p \, dx \leq C \int_{B_R} [1 + a(x)h(|Dw|)^{p(x)}] \, dx. \tag{2.84}$$

Therefore there exists $v^k \in w + W_0^{1,p}(B_R; \mathbb{R}^N)$ such that

$$v^{k\varepsilon} \rightharpoonup v^k \quad \text{weakly in } W^{1,p}(B_R; \mathbb{R}^N).$$

Moreover, by (2.82) and (2.83) we get, for all $B_\rho \subset\subset B_R$

$$v^{k\varepsilon} \overset{*}{\rightharpoonup} v^k \quad \text{weakly star in } W^{1,\infty}(B_\rho; \mathbb{R}^N).$$

By the semicontinuity of the norm and (2.84), we obtain

$$\int_{B_R} |Dv^k|^p \, dx \leq \liminf_{\varepsilon} \int_{B_R} |Dv^{k\varepsilon}|^p \, dx \leq C \int_{B_R} [1 + a(x)h(|Dw|)^{p(x)}] \, dx. \tag{2.85}$$

On the other hand, (2.34) and (2.83) imply

$$\begin{aligned} \|Dv^k\|_{L^\infty(B_\rho; \mathbb{R}^{Nn})} &\leq \liminf_{\varepsilon} \|Dv^{k\varepsilon}\|_{L^\infty(B_\rho; \mathbb{R}^{Nn})} \\ &\leq \hat{C} \left[\int_{B_R} [1 + a(x)h(|Dw|)^{p(x)}] \, dx \right]^{\beta} =: M, \end{aligned} \tag{2.86}$$

where we set for brevity

$$\hat{C} := C \left[\frac{(1 + \|\ell\|^2_{L^r(B_{R_0})})^{\frac{1}{2}}}{(R - \rho)} \right]^{\frac{\beta}{\frac{1}{n} - \frac{1}{r}}}.$$

Thus we can deduce that, up to subsequences, there exist $v \in w + W_0^{1,p}(B_R; \mathbb{R}^N)$ such that

$$\begin{aligned} &v^k \rightharpoonup v \quad \text{weakly in } W^{1,p}(B_R; \mathbb{R}^N) \\ &v^k \overset{*}{\rightharpoonup} v \quad \text{weakly star in } W^{1,\infty}(B_\rho; \mathbb{R}^N) \quad \text{for all } B_\rho \subset\subset B_R. \end{aligned}$$

Let us show that v is a solution to the problem

$$\inf \left\{ \int_{B_R} a(x)h(|Dv|)^{p(x)} \, dx : v \in w + W_0^{1,p}(B_R, \mathbb{R}^N) \right\}. \tag{2.87}$$

To this end, using the semicontinuity of the functional $\int_{B_\rho} g^{k_0}(x, |Dw|) \, dx$ and (see (2.20))

$$g^{k_0}(x, t) \leq g^k(x, t) \quad \forall k \geq k_0,$$

we get

$$\int_{B_\rho} g^{k_0}(x,|Dv^k|)\,dx \le \liminf_{\varepsilon} \int_{B_\rho} g^{k_0}(x,|Dv^{k\varepsilon}|)\,dx \tag{2.88}$$

$$\le \liminf_{\varepsilon} \int_{B_\rho} g^{k}(x,|Dv^{k\varepsilon}|)\,dx.$$

Since, up to subsequences, $g^{k\varepsilon}(x,t)$ converges as $\varepsilon \to 0$, a.e. in $B_{R_0} \times [0,+\infty)$ to $g^k(x,t)$, by Egorov theorem, fixed $K = \{\xi \in \mathbb{R}^{Nn} : |\xi| \le M+1\}$, for every $\delta > 0$ there exists A_δ with $|A_\delta| < \delta$ such that $g^{k\varepsilon}$ converges to g^k uniformly in $(B_\rho \backslash A_\delta) \times K$. Thus

$$\limsup_{\varepsilon} \int_{B_\rho \backslash A_\delta} g^{k}(x,|Dv^{k\varepsilon}|)\,dx = \limsup_{\varepsilon} \int_{B_\rho \backslash A_\delta} g^{k\varepsilon}(x,|Dv^{k\varepsilon}|)\,dx$$

and due to (2.86)

$$\limsup_{\varepsilon} \int_{B_\rho \cap A_\delta} g^{k}(x,|Dv^{k\varepsilon}|)\,dx \le C(k)|A_\delta|(1+M^q),$$

with $C(k)$ independent of δ. Thus, putting together the previous inequalities, (2.88) gives

$$\int_{B_\rho} g^{k_0}(x,|Dv^k|)\,dx \le \limsup_{\varepsilon} \int_{B_R} g^{k\varepsilon}(x,|Dv^{k\varepsilon}|)\,dx + C(k)|A_\delta|(1+M^q)$$

so that, letting $\delta \to 0$, by (2.83)

$$\int_{B_\rho} g^{k_0}(x,|Dv^k|)\,dx \le \limsup_{\varepsilon} \int_{B_R} g^{k\varepsilon}(x,|Dw|)\,dx = \int_{B_R} g^{k}(x,|Dw|)\,dx.$$

At this point, by the lower semicontinuity of the functional $\int_{B_\rho} g^{k_0}(x,|Dw|)\,dx$, and the Lebesgue Dominated Convergence Theorem applied to the sequence of functions $g^k(x,|Dw|)$, we obtain

$$\int_{B_\rho} g^{k_0}(x,|Dv|)\,dx \le \liminf_{k} \int_{B_\rho} g^{k_0}(x,|Dv^k|)\,dx \le \int_{B_R} g(x,|Dw|)\,dx.$$

Finally, letting $k_0 \to +\infty$ and $\rho \to R$

$$\int_{B_R} a(x)h(|Dv|)^{p(x)}\,dx \le \int_{B_R} a(x)h(|Dw|)^{p(x)}\,dx, \tag{2.89}$$

and passing to the limit in (2.86), we get

$$\|Dv\|_{L^\infty(B_\rho;\mathbb{R}^{Nn})} \le \hat{C}\left[\int_{B_R} [1 + a(x)h(|Dw|)^{p(x)}]\,dx\right]^\beta. \tag{2.90}$$

Then w and v are two solutions to Problem (2.87), but since g is not strictly convex for all $t > 0$, we may not conclude that $w = v$ in B_R. Let us define

$$E_0 := \left\{x \in B_R : \left|\frac{Dw(x) + Dv(x)}{2}\right| > 1\right\} \quad \text{and} \quad \overline{w} := \frac{w+v}{2}.$$

If E_0 has positive measure, then from the convexity of $g(x,\cdot)$ we have

$$\int_{B_R\setminus E_0} g(x,|D\overline{w}|)\,dx \le \frac{1}{2}\int_{B_R\setminus E_0} g(x,|Dw|)\,dx + \frac{1}{2}\int_{B_R\setminus E_0} g(x,|Dv|)\,dx. \tag{2.91}$$

Now, by the strict convexity of $g(x,t)$ for $t \ge 1$, we have

$$\int_{B_R\cap E_0} g(x,|D\overline{w}|)\,dx < \frac{1}{2}\int_{B_R\cap E_0} g(x,|Dw|)\,dx + \frac{1}{2}\int_{B_R\cap E_0} g(x,|Dv|)\,dx. \tag{2.92}$$

Adding (2.91) and (2.92), we get a contradiction with the minimality of w and v. Therefore the set E_0 has zero measure, which implies that

$$\sup_{B_\rho}|Dw(x)| \le \sup_{B_\rho}|Dw(x) + Dv(x)| + \sup_{B_\rho}|Dv(x)| \le 2 + \sup_{B_\rho}|Dv(x)|$$

and the main estimate follows by (2.90). □

3. An example

In this section we show that $h(t)$, defined in (1.3) with t_0 to be chosen later,

$$h(t) = t^{a+b\sin\log\log t} = e^{(a+b\sin\log\log t)\log t}, \quad \forall t \ge t_0, \tag{3.1}$$

satisfies the assumptions of Theorem 1.1. First let us notice that h satisfies, for large t, the growth condition

$$t^p \le h(t) \le t^q, \quad t \ge t_0$$

with $p = a - b$ and $q = a + b$.

Moreover, the first derivative, when $t \ge t_0$, is

$$\begin{aligned} h'(t) &= t^{a+b\sin\log\log t}\left[b\cos\log\log t \cdot \frac{1}{\log t}\cdot\frac{1}{t}\cdot\log t + (a + b\sin\log\log t)\cdot\frac{1}{t}\right] \\ &= t^{a-1+b\sin\log\log t}[b\cos\log\log t + (a + b\sin\log\log t)]. \end{aligned}$$

Then $h'(t) \ge 0$ for $t \ge t_0$ if $a \ge \sqrt{2}b$ (since $\sin\alpha + \cos\alpha \ge -\sqrt{2}$ for all $\alpha \in \mathbb{R}$).

The second derivative, when $t \geq t_0$, similarly is

$$
\begin{aligned}
h''(t) = t^{a-2+b\sin\log\log t} &\cdot [b\cos\log\log t + (a-1+b\sin\log\log t)] \\
&\cdot [b\cos\log\log t + (a+b\sin\log\log t)] \\
&+ t^{a-1+b\sin\log\log t}\left[b(\cos\log\log t - \sin\log\log t)\cdot\frac{1}{\log t}\cdot\frac{1}{t}\right] \\
= t^{a-2+b\sin\log\log t} &\cdot \Big\{[b\cos\log\log t + (a-1+b\sin\log\log t)] \\
&\qquad \cdot [b\cos\log\log t + (a+b\sin\log\log t)] \\
&\qquad + b(\cos\log\log t - \sin\log\log t)\cdot\frac{1}{\log t}\Big\};
\end{aligned}
$$

since the last addendum converges to zero at $t \to +\infty$, if $a-1 > \sqrt{2}b$, for large values of t $h''(t) > 0$ and we also have

$$0 < \frac{h''(t)\cdot t}{h'(t)} \leq \frac{a-1+\sqrt{2}b}{a-\sqrt{2}b}.$$

Therefore the function $h(t)$ satisfies (1.2) for a given $\delta > 0$, with $a, b > 0$ such that

$$1+\sqrt{2}b < a < a+b < 1+\delta,$$

with $t_0 > 0$ large in dependence of a, b. This is possible if $1 < a < 1+\delta$ and b is sufficiently small, say $b < \min\left\{\frac{a-1}{\sqrt{2}}, 1+\delta-a\right\}$.

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