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Partial Differential Equations — *Some results on weighted subquadratic Lane–Emden Elliptic Systems in unbounded domains*, by Sara Barile and Addolorata Salvatore, communicated on 13 November 2015.[1]

Abstract. — We study a nonlinear elliptic system of Lane–Emden type in $\mathbb{R}^N$, $N \geq 3$, which is equivalent to a fourth order quasilinear elliptic equation involving a suitable "sublinear" term. Thanks to some compact imbeddings in weighted Sobolev spaces, existence and multiplicity results are proved by means of a generalized Weierstrass Theorem and a variant of the Symmetric Mountain Pass Theorem. These results apply in particular to a biharmonic equation under Navier conditions in $\mathbb{R}^N$.

Key words: Nonlinear elliptic system of Lane–Emden type, subquadratic growth, fourth order elliptic equation, variational tools, compact imbeddings

Mathematics Subject Classification: 35J35, 35J50, 35J58, 35J60

1. Introduction

In the last years many authors have studied elliptic systems of two coupled semilinear Poisson equations

$$\begin{cases} -\Delta u = g(v) & \text{in } \Omega, \\ -\Delta v = f(u) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 3$ and $f, g : \mathbb{R} \to \mathbb{R}$ are given functions.

In the model case $g(s) = s^{p-1}$ and $f(s) = s^{q-1}$, $p, q > 1$, (here and in the following $s^\alpha = sgn(s)|s|^\alpha$ denotes the odd extension of the power function) the previous problem is referred to as the Lane–Emden system because it is a natural extension of the classical Lane–Emden equation

$$-\Delta u = u^{p-1} \quad \text{in } \Omega,$$

arising in Astronomy. It has been proved that Lane–Emden type systems have non-trivial solutions for all $p, q > 1$ either in the so-called superquadratic but

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subcritical case, i.e. $1 - \frac{2}{N} < \frac{1}{p} + \frac{1}{q} < 1$ (see [2], [11], [13], [14], [19], [23] and [24]) and in the subquadratic case, i.e. $\frac{1}{p} + \frac{1}{q} > 1$ (see [6], [8] and [16]). On the contrary, if p and q belong to the critical hyperbola $\frac{1}{p} + \frac{1}{q} = 1 - \frac{2}{N}$, because of the lack of compactness of the problem, non-existence of solutions has been stated in [21] and [25] by using Pohozaev type arguments.

In the superquadratic case, existence and multiplicity results have been stated also in unbounded domains by adding in the second equation a suitable term of the form $\rho(x)u^{\frac{1}{p-1}}$ (see [3], [4], [5] and [10]).

Aim of the paper is to study nonlinear elliptic systems like (1.1) in the whole space $\mathbb{R}^N$ but in the subquadratic case. More precisely we consider the following nonlinear elliptic system

$$\begin{cases} -\Delta u = v^{p-1} & \text{in } \mathbb{R}^N, \\ -\Delta v = -\rho(x)u^{\frac{1}{p-1}} + f(x,u) & \text{in } \mathbb{R}^N, \\ u, v \to 0 & \text{as } |x| \to +\infty, \end{cases} \tag{1.2}$$

where $\rho : \mathbb{R}^N \to \mathbb{R}$ and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfy the following assumptions:

(ρ_1) ρ is a Lebesgue measurable function verifying $\inf\operatorname{ess}_{\mathbb{R}^N} \rho(x) > 0$;
(ρ_2) $\displaystyle\int_{B_1(x)} \frac{1}{\rho(y)} dy \to 0$ as $|x| \to +\infty$

where $B_1(x)$ is the unit sphere centered in x;

(f_1) f is a Carathéodory function (i.e., $f(\cdot, s)$ is measurable on $\mathbb{R}^N$ for all $s \in \mathbb{R}$ and $f(x, \cdot)$ is continuous on $\mathbb{R}$ for a.e. $x \in \mathbb{R}^N$);
(f_2) there exist $q \in \left(1, \frac{p}{p-1}\right)$ and a positive function $b \in L^\mu(\mathbb{R}^N)$ with $\mu = \left(\frac{p}{q(p-1)}\right)' = \frac{p}{p+q-pq}$ such that

$$|f(x,s)| \le b(x)|s|^{q-1} \quad \text{for a.e. } x \in \mathbb{R}^N \text{ and for all } s \in \mathbb{R};$$

(f_3) there exist $\delta > 0$ and a positive function $b_1 \in L^\mu(\mathbb{R}^N)$ such that

$$f(x,s)s \ge b_1(x)|s|^q \quad \text{for a.e. } x \in \mathbb{R}^N \text{ and for all } s \in \mathbb{R},\ |s| \le \delta,$$

with q and μ the same as in (f_2);
(f_4) $f(x, -s) = -f(x, s)$ for a.e. $x \in \mathbb{R}^N$ and for all $s \in \mathbb{R}$.

We prove the following results. For the definition of the functional space E_ρ and of the energy functional I which appear in next theorems see Section 2.

THEOREM 1.1. *Suppose that* (ρ_1), (ρ_2), (f_1)–(f_3) *hold. Then, system* (1.2) *admits a non-trivial weak solution* $(\bar{u}, (-\Delta\bar{u})^{\frac{1}{p-1}})$. *Moreover, if f satisfies also* (f_3) *globally* (*i.e. with* $\delta = +\infty$) *and* (f_4), *system* (1.2) *has a sequence* $\{(\bar{u}_k, (-\Delta\bar{u}_k)^{\frac{1}{p-1}})\}$ *of non-trivial weak solutions such that* $\bar{u}_k \to 0$ *in* E_ρ *and* $I(\bar{u}_k) \to 0$ *as* $k \to +\infty$.

The multiplicity result contained in Theorem 1.1 can be weakened as follows in the case $1 < p < \frac{N}{N-2}$ by replacing (f_2) and (f_4) with the weaker assumptions

(f_5) there exist $q \in \left(1, \frac{p}{p-1}\right)$, $\delta > 0$ and a positive function $b \in L^\mu(\mathbb{R}^N)$ with $\mu = \left(\frac{p}{q(p-1)}\right)'$ such that

$$f(x,s)s \leq b(x)|s|^q \quad \text{for a.e. } x \in \mathbb{R}^N \text{ and for all } s \in \mathbb{R},\ |s| \leq \delta;$$

(f_6) $f(x,-s) = -f(x,s)$ for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$, $|s| \leq \delta$.

THEOREM 1.2. *Let* $1 < p < \frac{N}{N-2}$. *Assume that* (ρ_1), (ρ_2), (f_1), (f_3), (f_5) *and* (f_6) *hold. Then, system* (1.2) *has a sequence* $\{(\bar{u}_k, (-\Delta\bar{u}_k)^{\frac{1}{p-1}})\}$ *of non-trivial weak solutions such that* $\bar{u}_k \to 0$ *uniformly in* $\mathbb{R}^N$ *and* $I(\bar{u}_k) \to 0$ *as* $k \to +\infty$.

REMARK 1.3. Let us observe that, denoting $F(x,s) = \displaystyle\int_0^s f(x,t)\,dt$, from (f_2), (f_3) and (f_5) by integration it follows that

$$|F(x,s)| \leq \frac{b(x)}{q}|s|^q \quad \text{for a.e. } x \in \mathbb{R}^N \text{ and for all } s \in \mathbb{R}; \tag{1.3}$$

$$F(x,s) \geq \frac{b_1(x)}{q}|s|^q \quad \text{for a.e. } x \in \mathbb{R}^N \text{ and for all } s \in \mathbb{R},\ |s| \leq \delta; \tag{1.4}$$

$$F(x,s) \leq \frac{b(x)}{q}|s|^q \quad \text{for a.e. } x \in \mathbb{R}^N \text{ and for all } s \in \mathbb{R},\ |s| \leq \delta. \tag{1.5}$$

We emphasize that, really, in order to obtain the multiplicity result in the case $1 < p < \frac{N}{N-2}$ we need only local assumptions since Theorem 1.2 still holds if we consider $f : \mathbb{R}^N \times [-\delta, \delta] \to \mathbb{R}$ an odd Carathéodory function verifying (f_3) and (f_5).

REMARK 1.4. Clearly, (f_2) and (f_3) imply that $b_1(x) \leq b(x)$ for a.e. $x \in \Omega$. In particular, the function $f(x,s) = b(x)s^{q-1}$ verifies (f_1)–(f_4) if $b(x)$ is a positive function in $L^\mu(\mathbb{R}^N)$ with $\mu = \frac{p}{p+q-pq}$.

Following the variational formulation in [12] (see also [13]), we shall prove the above results by studying an equivalent fourth order quasilinear elliptic equation under Navier boundary conditions. Really, this equation is very interesting from a mathematical point of view, independently of its equivalence with system (1.2). In the particular case $p = 2$, we obtain the biharmonic equation with Navier boundary conditions

$$\begin{cases} \Delta^2 u = -\rho(x)u + f(x,u) & \text{in } \mathbb{R}^N, \\ u, \Delta u \to 0 & \text{as } |x| \to +\infty. \end{cases} \tag{1.6}$$

We recall that the biharmonic case has been widely studied in literature since it seems to be special and intermediate between the second order and the general polyharmonic case (see e.g. [7, 15] and [17, Section 7.2 and page 358 of Section 7.12] and references within). Here, by Theorems 1.1 and 1.2 we deduce the following results which, to our knowledge, are new within the framework of biharmonic equations with "sublinear" terms.

Corollary 1.5. *Suppose that* (ρ_1), (ρ_2), (f_1)–(f_3) *hold. Then, problem* (1.6) *admits a non-trivial weak solution* $\bar{u}$. *Moreover, if* f *satisfies also* (f_3) *globally (i.e. with* $\delta = +\infty$*) and* (f_4), *problem* (1.6) *has a sequence* $\{\bar{u}_k\}$ *of non-trivial weak solutions such that* $\bar{u}_k \to 0$ *in* E_ρ *and* $I(\bar{u}_k) \to 0$ *as* $k \to +\infty$.

Corollary 1.6. *Let* $N = 3$. *Assume that* (ρ_1), (ρ_2), (f_1), (f_3), (f_5) *and* (f_6) *hold. Then, problem* (1.6) *has a sequence* $\{\bar{u}_k\}$ *of non-trivial weak solutions such that* $\bar{u}_k \to 0$ *uniformly in* $\mathbb{R}^N$ *and* $I(\bar{u}_k) \to 0$ *as* $k \to +\infty$.

The paper is organized as follows: in Section 2 we introduce the variational formulation of the problem and we recall a variant of the Symmetric Mountain Pass Theorem for "sublinear" problems stated in [20]. In Section 3 we prove Theorem 1.1 and Theorem 1.2. In particular, in order to prove the multiplicity result stated in Theorem 1.2, we introduce a new modified problem which admits a sequence of solutions uniformly converging to zero. Finally, we prove that these solutions provide solutions to the original system (1.2).

Notations.

- $L^t(\mathbb{R}^N)$, with $1 \le t \le +\infty$, denotes the Lebesgue space with the usual norm $|\cdot|_t$;
- $W^{k,\sigma}(\mathbb{R}^N)$, with $k \in \mathbb{N}$, $\sigma \in \mathbb{R}$, $1 \le k, \sigma \le \infty$, is the usual Sobolev space equipped with the norm

$$\|u\|_{W^{k,\sigma}} = \left(\sum_{|\alpha|=k} \int_{\mathbb{R}^N} |D^\alpha u|^\sigma \, dx + \int_{\mathbb{R}^N} |u|^\sigma \, dx \right)^{\frac{1}{\sigma}};$$

- $C_B(\mathbb{R}^N)$ is the space of the continuous bounded functions on $\mathbb{R}^N$ equipped with the usual norm $|\cdot|_\infty$;
- c denotes a real positive constant changing line from line.

2. Variational tools

Let $N \ge 3$ and $p > 1$. Arguing as in [12] (see also [13], it is possible to transform system (1.2) in an equivalent quasilinear scalar problem. Indeed, the system (1.2) can be rewritten as

$$\begin{cases} (-\Delta u)^{\frac{1}{p-1}} = v & \text{in } \mathbb{R}^N, \\ -\Delta v = -\rho(x) u^{\frac{1}{p-1}} + f(x,u) & \text{in } \mathbb{R}^N, \\ u, v \to 0 & \text{as } |x| \to +\infty, \end{cases}$$

that is equivalent to the fourth order quasilinear elliptic equation

$$\begin{cases} -\Delta(-\Delta u)^{\frac{1}{p-1}} = -\rho(x)u^{\frac{1}{p-1}} + f(x,u) & \text{in } \mathbb{R}^N, \\ u, \Delta u \to 0 & \text{as } |x| \to +\infty. \end{cases} \tag{2.1}$$

Clearly, if u is a weak solution of (2.1), we define weak solution of system (1.2) the couple $(u, (-\Delta u)^{\frac{1}{p-1}})$. In order to prove that problem (2.1) has a variational structure, let E be the space $W^{2,\frac{p}{p-1}}(\mathbb{R}^N)$ endowed with the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} |\Delta u|^{\frac{p}{p-1}}\,dx + \int_{\mathbb{R}^N} |u|^{\frac{p}{p-1}}\,dx\right)^{\frac{p-1}{p}}$$

and with dual space $(E', \|\cdot\|_{E'})$. As $N \geq 3$, the above norm is equivalent to the norm $\|\cdot\|_{W^{2,p/(p-1)}}$ (see Corollary 9.10 and its previous remark in [18, pp. 235]). As (ρ_1) holds, we can consider the space $E_\rho = W_\rho^{2,\frac{p}{p-1}}(\mathbb{R}^N)$, namely

$$\left\{u \in W^{2,\frac{p}{p-1}}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \rho(x)|u|^{\frac{p}{p-1}}\,dx < \infty\right\}$$

equipped with the norm

$$\|u\|_\rho = \left(\int_{\mathbb{R}^N} |\Delta u|^{\frac{p}{p-1}}\,dx + \int_{\mathbb{R}^N} \rho(x)|u|^{\frac{p}{p-1}}\,dx\right)^{\frac{p-1}{p}}$$

equivalent to

$$\left(\sum_{|\alpha|=2} \int_{\mathbb{R}^N} |D^\alpha u|^{\frac{p}{p-1}}\,dx + \int_{\mathbb{R}^N} \rho(x)|u|^{\frac{p}{p-1}}\,dx\right)^{\frac{p-1}{p}}.$$

From now on, let $1 \leq t < \infty$ and

$$L_\rho^t(\mathbb{R}^N) = \left\{u \in L^t(\mathbb{R}^N) : \int_{\mathbb{R}^N} \rho(x)|u|^t\,dx < \infty\right\}$$

endowed with the norm

$$|u|_{t,\rho} = \left(\int_{\mathbb{R}^N} \rho(x)|u|^t\,dx\right)^{\frac{1}{t}}.$$

Clearly, $E_\rho = E \cap L_\rho^{\frac{p}{p-1}}(\mathbb{R}^N)$ and we have that $E_\rho \hookrightarrow E$. Then, if we set

$$\left(\frac{p}{p-1}\right)^{**} = \begin{cases} \frac{Np}{(N-2)p-N} & \text{if } p > \frac{N}{N-2}, \\ +\infty & \text{if } 1 < p \leq \frac{N}{N-2}, \end{cases} \tag{2.2}$$

the Sobolev imbedding Theorems give the following result (see e.g. [9, Corollary 9.13]).

Proposition 2.1. *Assume that ρ satisfies (ρ_1). The following continuous imbeddings hold:*

(A) *if $p > \frac{N}{N-2}$, i.e. $\frac{p}{p-1} < \frac{N}{2}$, then*

$$E_\rho \hookrightarrow L^t(\mathbb{R}^N) \quad \text{if } \frac{p}{p-1} \le t \le \left(\frac{p}{p-1}\right)^{**};$$

(B) *if $p = \frac{N}{N-2}$, i.e. $\frac{p}{p-1} = \frac{N}{2}$, then*

$$E_\rho \hookrightarrow L^t(\mathbb{R}^N) \quad \text{if } \frac{p}{p-1} \le t < \left(\frac{p}{p-1}\right)^{**};$$

(C) *if $1 < p < \frac{N}{N-2}$, i.e. $\frac{p}{p-1} > \frac{N}{2}$, then*

$$E_\rho \hookrightarrow L^t(\mathbb{R}^N) \quad \text{if } \frac{p}{p-1} \le t \le \left(\frac{p}{p-1}\right)^{**}$$

and, for every $u \in E_\rho$,

$$|u(x) - u(x')| \le C\|u\|\,|x - x'|^\alpha \quad \text{for a.e. } x, x' \in \mathbb{R}^N,$$

where α and C are suitable constants depending on p and N.
Moreover,

$$E_\rho \hookrightarrow C_B(\mathbb{R}^N). \tag{2.3}$$

As ensured by the following result, the presence of the weight $\rho(x)$ allows us to overcome the lack of compactness of the problem.

Proposition 2.2. *Under assumptions (ρ_1) and (ρ_2), it follows that the imbeddings*

$$E_\rho \hookrightarrow L^t(\mathbb{R}^N) \quad \text{for all } \frac{p}{p-1} \le t < \left(\frac{p}{p-1}\right)^{**}$$

are compact. Moreover, if $\frac{p}{p-1} > \frac{N}{2}$, the imbedding is compact also for $t = \left(\frac{p}{p-1}\right)^{**}$, i.e.,

$$E_\rho \hookrightarrow\hookrightarrow C_B(\mathbb{R}^N). \tag{2.4}$$

Proof. For the proof, we refer to [3, Proof of Proposition 3.1]. □

Remark 2.3. Since b is a positive function belonging to $L^\mu(\mathbb{R}^N)$, by Hölder inequality and Proposition 2.2 it follows that

$$E_\rho \hookrightarrow\hookrightarrow L_b^q(\mathbb{R}^N)$$

where

$$L_b^q(\mathbb{R}^N) = \left\{ u \in L^q(\mathbb{R}^N) : \int_{\mathbb{R}^N} b(x)|u|^q \, dx < \infty \right\}$$

endowed with the norm

$$|u|_{q,b} = \left(\int_{\mathbb{R}^N} b(x)|u|^q \, dx \right)^{\frac{1}{q}}.$$

Now, it is possible to state the following variational principle.

PROPOSITION 2.4. *Assume that* (ρ_1), (ρ_2), (f_1) *and* (f_2) *hold. Then, the weak solutions of problem* (2.1) *are the critical points of the energy functional defined on* E_ρ *by*

$$I(u) = \frac{p-1}{p} \int_{\mathbb{R}^N} (|\Delta u|^{\frac{p}{p-1}} + \rho(x)|u|^{\frac{p}{p-1}}) \, dx - \int_{\mathbb{R}^N} F(x,u) \, dx.$$

More precisely, $I \in C^1(E_\rho)$ *and its differential* $dI : E_\rho \to E_\rho'$ *is defined as*

$$dI(u)[\zeta] = \int_{\mathbb{R}^N} [(-\Delta u)^{\frac{1}{p-1}}(-\Delta \zeta) + \rho(x)|u|^{\frac{1}{p-1}}\zeta - f(x,u)\zeta] \, dx \tag{2.5}$$

for all $u, \zeta \in E_\rho$. *Moreover, the function* $u \mapsto f(\cdot, u(\cdot))$ *is compact from* E_ρ *to* E_ρ'.

PROOF. First, we prove that the functional

$$I(u) = \frac{p-1}{p} \|u\|_\rho^{\frac{p}{p-1}} - \int_{\mathbb{R}^N} F(x,u) \, dx, \quad u \in E_\rho,$$

is well defined and its Fréchet differential given in (2.5) is a continuous operator from E_ρ to E_ρ'. We study separately the two maps

$$\varphi_0(u) = \frac{p-1}{p} \|u\|_\rho^{\frac{p}{p-1}}, \quad \varphi_1(u) = \int_{\mathbb{R}^N} F(x,u) \, dx.$$

Clearly, $\varphi_0 \in C^1(E_\rho)$ since φ_0 is continuous from E_ρ to $\mathbb{R}$ and its Gâteaux differential at u

$$d\varphi_0(u)[\zeta] = \int_{\mathbb{R}^N} (-\Delta u)^{\frac{1}{p-1}}(-\Delta \zeta) \, dx + \int_{\mathbb{R}^N} \rho(x) u^{\frac{1}{p-1}} \zeta \, dx$$

is a linear continuous map on E_ρ. For the details we refer to [3, Proof of Proposition 2.7].

Now, we have to prove that also $\varphi_1 \in C^1(E_\rho)$ with

$$d\varphi_1(u)[\zeta] = \int_{\mathbb{R}^N} f(x,u)\zeta\,dx \quad \text{for all } u, \zeta \in E_\rho. \tag{2.6}$$

First, by (1.3) in Remark 1.3 and Hölder inequality it is

$$|\varphi_1(u)| \le \int_{\mathbb{R}^N} |F(x,u)|\,dx \le \frac{1}{q}\int_{\mathbb{R}^N} b(x)|u|^q\,dx \le \frac{1}{q}|b|_\mu |u|^q_{\frac{p}{p-1}}$$

and similarly, by (f_2) we obtain

$$\int_{\mathbb{R}^N} |f(x,u)|\,|\zeta|\,dx \le \int_{\mathbb{R}^N} b(x)|u|^{q-1}|\zeta|\,dx \le |b|_\mu |u|^{q-1}_{\frac{p}{p-1}} |\zeta|_{\frac{p}{p-1}}.$$

Hence, by Sobolev imbeddings in Proposition 2.1 it follows that $\varphi_1(u) \in \mathbb{R}$ and $d\varphi_1(u)[\zeta] \in \mathbb{R}$ for all $u, \zeta \in E_\rho$. Moreover, standard tools imply that the Gâteaux differential of φ_1 at u is as in (2.6) and it is linear and continuous from E_ρ to $\mathbb{R}$.

At this point, we have to prove that $d\varphi_1$ is continuous from E_ρ to E'_ρ, i.e.

$$\|d\varphi_1(u_n) - d\varphi_1(u)\|_{E'_\rho} \to 0 \quad \text{if } u_n \to u \text{ in } E_\rho. \tag{2.7}$$

Indeed, by Hölder inequality and Sobolev imbeddings,

$$\begin{aligned} |(d\varphi_1(u_n) - d\varphi_1(u))[\zeta]| &\le \int_{\mathbb{R}^N} |f(x,u_n) - f(x,u)|\,|\zeta|\,dx \\ &\le |f(\cdot,u_n(\cdot)) - f(\cdot,u(\cdot))|_p |\zeta|_{\frac{p}{p-1}}. \end{aligned}$$

Now, by (f_2) we get

$$\begin{aligned} |f(x,u_n) - f(x,u)|^p &\le c(|f(x,u_n)|^p + |f(x,u)|^p) \\ &\le c((b(x))^p |u_n|^{p(q-1)} + (b(x))^p |u|^{p(q-1)}) \\ &\le c((b(x))^p |u_n - u|^{p(q-1)} + (b(x))^p |u|^{p(q-1)}). \end{aligned}$$

By Fatou's lemma, it follows that

$$\begin{aligned} &\int_{\mathbb{R}^N} \liminf_{n\to+\infty} (c(b(x))^p |u_n - u|^{p(q-1)} + (b(x))^p |u|^{p(q-1)}) \\ &\qquad - |f(x,u_n) - f(x,u)|^p)\,dx \\ &\le \liminf_{n\to+\infty} \int_{\mathbb{R}^N} (c(b(x))^p |u_n - u|^{p(q-1)} + (b(x))^p |u|^{p(q-1)}) \\ &\qquad - |f(x,u_n) - f(x,u)|^p)\,dx. \end{aligned} \tag{2.8}$$

Now, we observe that, since $u_n \to u$ in E_ρ it is $u_n(x) \to u(x)$ a.e. $x \in \mathbb{R}^N$, therefore

$$(b(x))^p |u_n(x) - u(x)|^{p(q-1)} \to 0 \quad \text{a.e. } x \in \mathbb{R}^N$$

and also by (f_1)

$$|f(x, u_n(x)) - f(x, u(x))|^p \to 0 \quad \text{a.e. } x \in \mathbb{R}^N.$$

On the other hand, by Hölder inequality and Sobolev imbeddings we get

$$\int_{\mathbb{R}^N} (b(x))^p |u_n - u|^{p(q-1)}\, dx \leq |b|_\mu^p |u_n - u|_{\frac{p}{p-1}}^{p(q-1)}$$

and, since $u_n \to u$ in $L^{\frac{p}{p-1}}(\mathbb{R}^N)$ by Proposition 2.1, also the left-hand side term goes to zero as $n \to +\infty$. Consequently, (2.8) involves

$$c \int_{\mathbb{R}^N} (b(x))^p |u|^{p(q-1)}\, dx \leq c \int_{\mathbb{R}^N} (b(x))^p |u|^{p(q-1)}\, dx + \liminf_{n\to+\infty} \left(- \int_{\mathbb{R}^N} |f(x, u_n) - f(x, u)|^p\, dx \right)$$

from which it follows that

$$0 \leq - \limsup_{n\to+\infty} \left(\int_{\mathbb{R}^N} |f(x, u_n) - f(x, u)|^p\, dx \right)$$

and therefore

$$0 \leq \liminf_{n\to+\infty} \left(\int_{\mathbb{R}^N} |f(x, u_n) - f(x, u)|^p\, dx \right) \leq \limsup_{n\to+\infty} \left(\int_{\mathbb{R}^N} |f(x, u_n) - f(x, u)|^p\, dx \right) \leq 0.$$

Hence,

$$|f(\cdot, u_n(\cdot)) - f(\cdot, u(\cdot))|_p \to 0 \quad \text{as } n \to +\infty$$

and (2.7) is proved.

Finally, by exploiting Proposition 2.2 instead of Proposition 2.1 in the previous arguments it follows that $d\varphi_1$ is compact from E_ρ to E'_ρ. □

Now, we recall a suitable version stated by R. Kajikiya in [20] of the classical Symmetric Mountain Pass Theorem (see [1]).

Let X be an infinite dimensional Banach space, X' its dual space and $J : X \to \mathbb{R}$ be a C^1 functional. Let us recall that J satisfies the Palais–Smale, briefly (PS), condition, if any (PS) sequence, i.e. any sequence $\{u_k\}$ in X such that $\{J(u_k)\}$ is bounded and $dJ(u_k) \to 0$ in X' as $k \to +\infty$, has a convergent subsequence.

For all integer k, let

$$\Gamma_k = \{A \subset X - \{0\} \mid A \text{ closed and symmetric, } \gamma(A) \geq k\},$$

where, as usual, $\gamma(A)$ denotes the genus of the set A (for the definition and relative properties see, e.g., [22]).

The following result has been proved in [20, Theorem 1].

THEOREM 2.5. *Let $J \in C^1(X, \mathbb{R})$ satisfying*

(A_1) *J is even, bounded from below, $J(0) = 0$ and J satisfies the (PS) condition;*
(A_2) *for every $k \in \mathbb{N}$ there exists $A_k \in \Gamma_k$ such that $\sup_{A_k} J(u) < 0$.*

Then,

(i) *either there exists a sequence $\{u_k\}$ such that $dJ(u_k) = 0$, $J(u_k) < 0$ and $\{u_k\}$ converges to zero;*
(ii) *or there exist two sequences $\{u_k\}$ and $\{v_k\}$ such that $dJ(u_k) = 0$, $J(u_k) = 0$, $u_k \neq 0$, $\lim_k u_k = 0$, $dJ(v_k) = 0$, $J(v_k) < 0$, $\lim_k J(v_k) = 0$ and $\{v_k\}$ converges to a non-zero limit.*

REMARK 2.6. In any case (i) or (ii), Theorem 2.5 gives the existence of a sequence $\{u_k\}$ of critical points such that $J(u_k) \leq 0$, $u_k \neq 0$, $\lim_k u_k = 0$ and, consequently, $\lim_k J(u_k) = 0$.

3. PROOF OF THE MAIN RESULTS

PROOF OF THEOREM 1.1. From (1.3), Hölder inequality and Sobolev embeddings, we get

$$\begin{aligned} I(u) &= \frac{p-1}{p} \int_{\mathbb{R}^N} (|\Delta u|^{\frac{p}{p-1}} + \rho(x)|u|^{\frac{p}{p-1}})\, dx - \int_{\mathbb{R}^N} F(x,u)\, dx \\ &\geq \frac{p-1}{p} \|u\|_\rho^{\frac{p}{p-1}} - \frac{1}{q} \int_{\mathbb{R}^N} b(x)|u|^q\, dx \\ &\geq \frac{p-1}{p} \|u\|_\rho^{\frac{p}{p-1}} - c|b|_\mu \|u\|_\rho^q. \end{aligned}$$

Then, since $1 < q < \frac{p}{p-1}$, it follows that I is bounded from below and coercive on the reflexive Banach space E_ρ.

Moreover, by using the notations introduced in Proposition 2.4, the functional $I = \varphi_0 - \varphi_1$ is weakly lower semicontinuous on E_ρ since φ_0 is weakly lower semicontinuous by the norm properties while φ_1 is weakly continuous as it is of class C^1 on E_ρ and its derivative $d\varphi_1$ is compact by the second part of Proposition 2.4. Then, by a generalized Weierstrass Theorem there exists $\bar{u} \in E_\rho$ such that $I(\bar{u}) = \min_{u \in E_\rho} I(u)$. Hence, by applying now the first part of Proposition 2.4, $\bar{u}$ is a solution of problem (2.1).

Clearly, (f_2) implies $f(x,0) = 0$ for a.e. $x \in \mathbb{R}^N$, then problem (2.1) admits always the trivial solution $u = 0$ with $I(0) = 0$.

Anyway, by (f_3) condition (1.4) in Remark 1.3 holds, therefore the solution $\bar{u}$ is non trivial since, fixed $u_1 \in E_\rho \cap L^\infty(\mathbb{R}^N)$ with $u_1 \neq 0$, by Hölder inequality and $1 < q < \frac{p}{p-1}$ we get

$$\begin{aligned} I(\varepsilon u_1) &= \frac{p-1}{p}\varepsilon^{\frac{p}{p-1}}\|u_1\|_\rho^{\frac{p}{p-1}} - \int_{\mathbb{R}^N} F(x,\varepsilon u_1)\,dx \\ &\leq \frac{p-1}{p}\varepsilon^{\frac{p}{p-1}}\|u_1\|_\rho^{\frac{p}{p-1}} - \frac{\varepsilon^q}{q}|b_1|_\mu |u_1|^q_{\frac{p}{p-1}} \\ &< 0 = I(0) \end{aligned}$$

for $\varepsilon > 0$ small enough.

Now, in order to prove the multiplicity result, assume that also (f_4) holds. Then, the functional I is even. Let us point out that I satisfies (PS) condition. Indeed, if $\{u_k\}$ is a (PS) sequence, $\{u_k\}$ is bounded by the coerciveness of I. Thus, up to subsequence, there exists $u \in E_\rho$ such that $u_k \rightharpoonup u$. By Proposition 2.4 we have that the function $u \to f(\cdot, u(\cdot))$ is compact from E_ρ to E'_ρ and, reasoning as in [15, Section 3] we conclude that $u_k \to u$ in E_ρ. Hence, I satisfies assumption (A_1) in Theorem 2.5.

Now, let us denote by $\{e_j\}$ a Schauder basis of the separable Banach space E_ρ. For $k \in \mathbb{N}$ fixed, let $E_k = \{e_1, \ldots, e_k\}$ be a k-dimensional subspace of E_ρ. By Remark 2.3 we get that $|u|_{q,b_1} := \left(\int_{\mathbb{R}^N} b_1(x)|u|^q\,dx\right)^{\frac{1}{q}}$ is a norm in E_ρ, therefore, since we are in finite dimension, there exists $c_k > 0$ such that $\|u\|_\rho \leq c_k|u|_{q,b_1}$ for every $u \in E_k$. Clearly,

$$c_k = \sup_{u \in E_k,\, |u|_{q,b_1}=1} \|u\|_\rho,$$

hence the sequence $\{c_k\}$ is increasing. Moreover, $c_k \to +\infty$ if $k \to +\infty$. Indeed, if by contradiction $\{c_k\}$ was bounded, taken $u \in E_\rho$ and u_k the component of u along E_k, it is $u = \lim_k u_k$ in E_ρ and in $L^q_{b_1}(\mathbb{R}^N)$. Since for every k it is $\|u_k\|_\rho \leq c_k|u_k|_{q,b_1}$, passing to the limit we have $\|u\|_\rho \leq c|u|_{q,b_1}$, for c suitable constant independent of u. Hence, $L^q_{b_1}(\mathbb{R}^N) \hookrightarrow E_\rho$ while Remark 2.3 ensures that $E_\rho \hookrightarrow L^q_{b_1}(\mathbb{R}^N)$ which gives the contradiction. Therefore, taken $u \in E_k$ from (1.4) with $\delta = +\infty$ we get

$$\begin{aligned} I(u) &\leq \frac{p-1}{p}\|u\|_\rho^{\frac{p}{p-1}} - \frac{1}{q}\int_{\mathbb{R}^N} b_1(x)|u|^q\,dx \\ &\leq \frac{p-1}{p}\|u\|_\rho^{\frac{p}{p-1}} - \frac{1}{q}c_k^{-q}\|u\|_\rho^q \leq -\frac{p-1}{p}\|u\|_\rho^{\frac{p}{p-1}} \end{aligned}$$

if we choose $2\frac{p-1}{p}\|u\|_\rho^{\frac{p}{p-1}} \leq \frac{1}{q}c_k^{-q}\|u\|_\rho^q$ or equivalently $\|u\|_\rho \leq \left(\frac{p}{2(p-1)qc_k^q}\right)^{\frac{1}{\frac{p}{p-1}-q}}$.

Chosen $0 < d_k \le \left(\frac{p}{2(p-1)qc_k^q}\right)^{\frac{1}{\frac{p}{p-1}-q}} = r_k^{\frac{1}{\frac{p}{p-1}-q}}$, it results that $r_k \to 0$ as $k \to +\infty$ and

$$\{u \in E_k : \|u\|_p = d_k\} \subset \left\{u \in E_p : I(u) \le -\frac{p-1}{p} d_k^{\frac{p}{p-1}}\right\}.$$

So, denoted by

$$A_k = \left\{u \in E_p : I(u) \le -\frac{p-1}{p} d_k^{\frac{p}{p-1}}\right\},$$

as I is even and continuous, A_k is closed and symmetric, i.e. $A_k \in \Gamma_k$ and, by well known properties of the genus, $\gamma(A_k) \ge \gamma(E_k \cap S_{d_k}) = k$, where $S_{d_k} = \{u \in E : \|u\|_p = d_k\}$. Consequently, for every $k \in \mathbb{N}$ there exists $A_k \in \Gamma_k$ such that

$$\sup_{A_k} I \le -\frac{p-1}{p} d_k^{\frac{p}{p-1}} < 0.$$

Hence, (A_2) holds and by Theorem 2.5 (see also Remark 2.6), there exists a sequence $\{\bar{u}_k\}$ in E_p such that $\bar{u}_k \ne 0$, $dI(\bar{u}_k) = 0$, $\lim_k \bar{u}_k = 0$ and $\lim_k I(\bar{u}_k) = 0$. Therefore, by Proposition 2.4, $\{\bar{u}_k\}$ is a sequence of non-trivial solutions to (2.1) such that $I(\bar{u}_k) \le 0$, $\bar{u}_k \to 0$ in E_p and $I(\bar{u}_k) \to 0$ as $k \to +\infty$, hence $(\bar{u}_k, (-\Delta\bar{u}_k)^{\frac{1}{p-1}})$ is a solution to system (1.2) with $\bar{u}_k \to 0$ in E_p and $I(\bar{u}_k) \to 0$ as $k \to +\infty$. □

Now, we have to prove the multiplicity result stated in Theorem 1.2 for the case $1 < p < \frac{N}{N-2}$. First of all, let us observe that, under local assumptions (f_3) and (f_5), problem (2.1) does not admit a variational formulation since the functional I is not well defined on the space E_p. Therefore, we modify the term f by introducing a new function $\bar{f}$ satisfying the same hypotheses of f but globally with respect to $s \in \mathbb{R}$.

First, taken $\delta > 0$ as in assumptions (f_3) and (f_5), let us consider a cut-off function φ such that $0 \le \varphi(s) \le 1$, $\varphi(s) = 1$ if $|s| \le \frac{\delta}{2}$, $\varphi(s) = 0$ if $|s| \ge \delta$ and φ is even, continuous and strictly decreasing on $\frac{\delta}{2} \le |s| \le \delta$. Then, let us define

$$\bar{f}(x,s)s = \varphi(s)f(x,s)s + (1-\varphi(s))b_1(x)|s|^q, \quad \text{for a.e. } x \in \mathbb{R}^N, \text{ for all } s \in \mathbb{R}.$$

It is possible to prove the following Proposition.

Proposition 3.1. *Assume that f verifies assumptions (f_1), (f_3), (f_5) and (f_6). Then $\bar{f}$ is an odd Carathéodory function such that for a.e. $x \in \mathbb{R}^N$ and for all $s \in \mathbb{R}$ it is*

$$\frac{1}{2}b_1(x)|s|^q \le \bar{f}(x,s)s \le (b(x)+b_1(x))|s|^q. \tag{3.1}$$

PROOF. Since f satisfies (f_1) and (f_6), by the definition of φ it is easy to see that $\bar{f}$ is an odd Carathéodory function on $\mathbb{R}^N \times \mathbb{R}$.

Moreover, if $|s| \le \frac{\delta}{2}$, it is $\bar{f}(x,s)s = f(x,s)s$ and then by (f_3) and (f_5) we obtain

$$b_1(x)|s|^q \le \bar{f}(x,s)s \le b(x)|s|^q \quad \text{for a.e. } x \in \mathbb{R}^N,$$

and (3.1) follows by the positivity of b_1.

On the other hand, if $|s| \ge \delta$, it is $\bar{f}(x,s)s = b_1(x)|s|^q$ and (3.1) follows again since b and b_1 are positive functions. Finally, if $\frac{\delta}{2} \le |s| \le \delta$, by (f_3) it is $f(x,s)s \ge 0$. Recalling that $\varphi(s) \ge \frac{1}{2}$ or $1 - \varphi(s) \ge \frac{1}{2}$, we have in any case

$$\bar{f}(x,s)s \ge \frac{1}{2}b_1(x)|s|^q \quad \text{for a.e. } x \in \mathbb{R}^N,$$

while by (f_5)

$$\bar{f}(x,s)s \le f(x,s)s + b_1(x)|s|^q \le (b(x) + b_1(x))|s|^q \quad \text{for a.e. } x \in \mathbb{R}^N.$$

Hence, the proof of (3.1) is complete. □

REMARK 3.2. Clearly, since $b_1(x) > 0$ for a.e. $x \in \mathbb{R}^N$, condition (3.1) implies that $\bar{f}$ verifies (f_2) with $b(x)$ replaced by $b(x) + b_1(x)$ for a.e. $x \in \mathbb{R}^N$.

At this point we can consider the following new problem

$$(3.2) \qquad \begin{cases} -\Delta(-\Delta u)^{\frac{1}{p-1}} = -\rho(x)u^{\frac{1}{p-1}} + \bar{f}(x,u) & \text{in } \mathbb{R}^N, \\ u, \Delta u \to 0 & \text{as } |x| \to +\infty. \end{cases}$$

and the associated energy functional defined on E_ρ by

$$\bar{I}(u) = \frac{p-1}{p}\int_{\mathbb{R}^N} (|\Delta u|^{\frac{p}{p-1}} + \rho(x)|u|^{\frac{p}{p-1}})\,dx - \int_{\mathbb{R}^N} \bar{F}(x,u)\,dx,$$

with $\bar{F}(x,t) = \displaystyle\int_0^t \bar{f}(x,s)\,ds$. By Proposition 3.1 and Remark 3.2, Proposition 2.4 can be applied to $\bar{I}$, hence it follows that $\bar{I} \in C^1(E_\rho)$ and its critical points are the weak solutions to problem (3.2). Let us remark that, by integration, from (3.1) we obtain that, for a.e. $x \in \mathbb{R}^N$ and for all $s \in \mathbb{R}$, it is

$$(3.3) \qquad \frac{1}{2q}b_1(x)|s|^q \le \bar{F}(x,s) \le \frac{1}{q}(b(x) + b_1(x))|s|^q.$$

The following Proposition will be crucial in the statement of our multiplicity result since it allows us to obtain solutions of system (1.2) by studying problem (3.2).

PROPOSITION 3.3. *Let $1 < p < \frac{N}{N-2}$. Assume that (ρ_1), (ρ_2), (f_1) and (f_5) hold. Let $\{u_k\}$ be a sequence in E_ρ of solutions of problem* (3.2) *such that $u_k \to 0$ in*

E_ρ as $k \to +\infty$. Thus, $u_k \to 0$ uniformly in $\mathbb{R}^N$ and therefore u_k solves problem (2.1) *for all k large enough.*

PROOF. Since $1 < p < \frac{N}{N-2}$, from (2.3) it follows that $u_k \to 0$ uniformly in $\mathbb{R}^N$. Therefore, let us point out that, taken $\delta > 0$ as in assumptions (f_3) and (f_5), there exists $\bar{k} \in \mathbb{N}$ such that for every $k \geq \bar{k}$ it is $|u_k|_\infty \leq \frac{\delta}{2}$, namely $|u_k(x)| \leq \frac{\delta}{2}$ for every $x \in \mathbb{R}^N$ and for every $k \geq \bar{k}$. It follows that $\bar{f}(x, u_k(x)) = f(x, u_k(x))$ and $\bar{F}(x, u_k(x)) = F(x, u_k(x))$, therefore we have that $\bar{I}(u_k) = I(u_k)$ and $d\bar{I}(u_k) = dI(u_k)$, hence by Proposition 2.4 u_k is a solution to problem (2.1) for every $k \geq \bar{k}$. $\square$

PROOF OF THEOREM 1.2. Observe that by Proposition 3.1 $\bar{f}$ satisfies (f_1)–(f_4). Therefore, Theorem 1.1 applies to the functional $\bar{I}$. In particular, system (3.2) has a sequence $\{(\bar{u}_k, (-\Delta\bar{u}_k)^{\frac{1}{p-1}})\}$ of non trivial weak solutions with $\bar{u}_k \to 0$ in E_ρ and $\bar{I}(\bar{u}_k) \to 0$ as $k \to +\infty$. Finally, by applying Proposition 3.3, $\bar{u}_k \to 0$ uniformly in $\mathbb{R}^N$ and for k large enough $\bar{u}_k$ is a solution to problem (2.1), hence for k large $(\bar{u}_k, (-\Delta\bar{u}_k)^{\frac{1}{p-1}})$ is a solution to system (1.2) with $\bar{u}_k \to 0$ in E_ρ and $I(\bar{u}_k) \to 0$ as $k \to +\infty$. $\square$

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