Rend. Lincei Mat. Appl. 27 (2016), 105–115
DOI 10.4171/RLM/725

Complex variables functions — *Higher order Pizzetti's formulas*, by Grzegorz Łysik, communicated on 13 November 2015.

Abstract. — We introduce integral mean value functions which are averages of integral means over spheres/balls and over their images under the action of a discrete group of complex rotations. In the case of real analytic functions we derive higher order Pizzetti's formulas. As applications we obtain a maximum principle for polyharmonic functions and a characterization of convergent solutions to higher order heat type equations.

Key words: Mean values, Pizzetti formulas, maximum principle, polyharmonic functions, heat type equations

Mathematics Subject Classification: 35B10, 35B05, 35K05

Introduction

The Pizzetti mean-value formula states that the integral mean of a smooth function u over the Euclidean sphere of radius r is expressed as a series of even powers of r whose coefficients are given by iterated Laplacian evaluated on u at the center of the sphere multiplied by numerical factors. In the case of 2-dimensional sphere the formula was derived already in 1909 by Pizzetti [16]. Its extensions to the case of means over Euclidean spheres and balls in arbitrary dimension for polyharmonic functions and the inverse mean-value properties were derived by Nicolesco [15]. Later on the Pizzetti serii were studied from different points of view, see [8, 4, 6] and references therein. In particular, it was proved that real analytic functions can be characterized by convergence of the Pizzetti serii [2, 12]. The Pizzetti formulas were also extended to the case of means on Riemannian manifolds [9] and on the Heisenberg group [5, 10], and to the Dunkl–Laplace operator on $\mathbb{R}^n$ [13, 17]. A generalized mean value theorem with respect to a general Borel measure supported by the unit real ball for solutions of a system of homogeneous partial differential equations was derived by Zalcman [18].

In the paper we introduce integral mean value functions N_k and M_k which are averages of integral means over spheres/balls of radius r over their images under the action of a discrete group W_k generated by a rotation of all variables of $\mathbb{C}^n$ by the angle $2\pi/k$. In the case of real analytic functions we derive higher order Pizzetti's formulas. Due to averaging integral means over W_k our higher order Pizzetti's serii contain only terms of the form $c_{lm}\Delta^{lm}r^{2lm}$, $m \in \mathbb{N}_0$, if $k = 2l$ is even. As applications we obtain a maximum principle for polyharmonic functions, a characterization of functions of Laplacian growth and a characterization of convergent solutions to the initial value problem for the higher order heat type equation $\partial_t u = \Delta^l u$, $u(0,\cdot) = \varphi$ where $l \in \mathbb{N}$ and φ is real analytic.

1. Preliminaries

Let $k \in \mathbb{N}$. Denote by ϵ the transformation of $\mathbb{C}^n$ into $\mathbb{C}^n$ given by $\epsilon(z_1,\dots,z_n) = (e^{2\pi\imath/k}z_1,\dots,e^{2\pi\imath/k}z_n)$ and by W_k the group generated by ϵ. Let u be a continuous function defined on a complex neighborhood U of an open set $\Omega \subset \mathbb{R}^n$. For $x \in \Omega$ and $0 \le |r| < \operatorname{dist}(x, \partial U)$ we define the *W_k-spherical* and *W_k-solid mean value functions*

$$N_k(u;x,r) = \frac{1}{kn\sigma(n)}\sum_{j=0}^{k-1}\int_{S^{n-1}(0,1)} u(x + r\epsilon^j(y))\,dS(y), \tag{1a}$$

$$M_k(u;x,r) = \frac{1}{k\sigma(n)}\sum_{j=0}^{k-1}\int_{B^{n}(0,1)} u(x + r\epsilon^j(y))\,dy, \tag{1b}$$

where $\sigma(n) = \pi^{n/2}/\Gamma(n/2+1)$ is the volume of the unit ball $B^n(0,1)$ in $\mathbb{R}^n$ and $dS = dS^{n-1}$ is the natural measure on the unit sphere $S(0,1) = S^{n-1}(0,1)$.

Note that if k is odd, then $N_k(u;x,r) = N_{2k}(u;x,r)$ and $M_k(u;x,r) = M_{2k}(u;x,r)$. In particular

$$N_1(u;x,r) = N_2(u;x,r) = \frac{1}{n\sigma(n)}\int_{S(0,1)} u(x+ry)\,dS(y)$$

and

$$M_1(u;x,r) = M_2(u;x,r) = \frac{1}{\sigma(n)}\int_{B(0,1)} u(x+ry)\,dy$$

are the classical mean value functions and the same as functions N and M in [12].

2. Higher order Pizzetti's formulas

In this section we assume that $u \in \mathscr{A}(\Omega)$ is a real analytic function on an open set $\Omega \subset \mathbb{R}^n$. Then u extends to a function $\tilde{u}$ holomorphic on a complex neighborhood U of Ω and for any $x \in \Omega$ it holds

$$\tilde{u}(y) = \sum_{\kappa \in \mathbb{N}_0^n} \frac{1}{\kappa!}\frac{\partial^{|\kappa|}u}{\partial x^\kappa}(x)(y-x)^\kappa \quad \text{for } y \in U \text{ with } \|y-x\| < \rho(x), \tag{2}$$

with some function $\rho \in C^0(\Omega, \mathbb{R}_+)$. Hence the functions N_k and M_k are well defined for $x \in \Omega$ and $|r|$ small enough.

THEOREM 1 (Higher order Pizzetti's formulas). *Let $k = 2l$ with $l \in \mathbb{N}$, $u \in \mathscr{A}(\Omega)$ and $x \in \Omega$. Then $N_k(u; x, r)$ and $M_k(u; x, r)$ are real analytic functions at the origin and for $|r|$ small enough it holds*

$$N_k(u; x, r) = \sum_{m=0}^{\infty} \frac{\Delta^{lm} u(x)}{4^{lm} \left(\frac{n}{2}\right)_{lm} (lm)!} r^{2lm}, \tag{3a}$$

$$M_k(u; x, r) = \sum_{m=0}^{\infty} \frac{\Delta^{lm} u(x)}{4^{lm} \left(\frac{n}{2} + 1\right)_{lm} (lm)!} r^{2lm}, \tag{3b}$$

where Δ is the Laplace operator and $(a)_m = a(a+1)\dots(a+m-1)$ for $m \in \mathbb{N}$ is the Pochhammer symbol.

PROOF. For $x \in \Omega$ and $0 < |r|$ small enough we compute

$$\begin{aligned}
N_k(u; x, r) &= \frac{1}{kn\sigma(n)} \sum_{j=0}^{k-1} \int_{S(0,1)} \tilde{u}(x + r\epsilon^j(y))\, dS(y) \\
&\overset{(2)}{=} \frac{1}{kn\sigma(n)} \sum_{\kappa \in \mathbb{N}_0^n} \frac{1}{\kappa!} \frac{\partial^{|\kappa|} u}{\partial x^\kappa}(x) \sum_{j=0}^{k-1} \int_{S(0,1)} (r\epsilon^j(y))^\kappa\, dS(y) \\
&= \frac{1}{kn\sigma(n)} \sum_{\kappa \in \mathbb{N}_0^n} \frac{1}{\kappa!} \frac{\partial^{|\kappa|} u}{\partial x^\kappa}(x) r^{|\kappa|} \sum_{j=0}^{k-1} \int_{S(0,1)} (e^{2j\pi\imath/k} y)^\kappa\, dS(y) \\
&= \frac{1}{kn\sigma(n)} \sum_{\kappa \in \mathbb{N}_0^n} \frac{1}{\kappa!} \frac{\partial^{|\kappa|} u}{\partial x^\kappa}(x) r^{|\kappa|} \sum_{j=0}^{k-1} e^{2j|\kappa|\pi\imath/k} \int_{S(0,1)} y^\kappa\, dS(y) \\
&= \sum_{m=0}^{\infty} \sum_{|\kappa| = km} \frac{1}{\kappa!} \frac{\partial^{km} u}{\partial x^\kappa}(x) r^{km} \frac{1}{n\sigma(n)} \int_{S(0,1)} y^\kappa\, dS(y) \\
&= \sum_{m=0}^{\infty} \sum_{\substack{\ell \in \mathbb{N}_0^n \\ 2|\ell| = km}} \frac{1}{(2\ell)!} \frac{\partial^{km} u}{\partial x^{2\ell}}(x) r^{km} \frac{1}{n\sigma(n)} \int_{S(0,1)} y^{2\ell}\, dS(y)
\end{aligned}$$

since

$$\sum_{j=0}^{k} e^{2j|\kappa|\pi\imath/k} = \begin{cases} k & \text{if } |\kappa| = km \text{ for some } m \in \mathbb{N}_0, \\ 0 & \text{otherwise} \end{cases}$$

and the integral of y^κ over $S(0,1)$ vanishes if at least one of the coordinates κ_i of $\kappa = (\kappa_1, \dots, \kappa_n)$ is odd. Finally using [7, formula 676, 11] we get

$$N_k(u;x,r) = \sum_{m=0}^{\infty} \sum_{\substack{\ell \in \mathbb{N}_0^n \\ 2|\ell|=km}} \frac{1}{(2\ell)!} \frac{\Gamma\left(\ell_1+\frac{1}{2}\right)\ldots\Gamma\left(\ell_n+\frac{1}{2}\right)}{\Gamma\left(|\ell|+\frac{n}{2}\right)} \frac{\partial^{km}u}{\partial x^{2\ell}}(x) r^{km}$$

$$= \sum_{m=0}^{\infty} \sum_{\substack{\ell \in \mathbb{N}_0^n \\ 2|\ell|=km}} \frac{1}{(2\ell)!} \frac{\left(\frac{1}{2}\right)_{\ell_1}\cdots\left(\frac{1}{2}\right)_{\ell_n}}{\left(\frac{n}{2}\right)_{|\ell|}} \frac{\partial^{km}u}{\partial x^{2\ell}}(x) r^{km}$$

$$\overset{k=2l}{=} \sum_{m=0}^{\infty} \frac{r^{2lm}}{4^{lm}\left(\frac{n}{2}\right)_{lm}(lm)!} \sum_{\substack{\ell \in \mathbb{N}_0^n \\ |\ell|=lm}} \frac{(lm)!}{\ell!} \frac{\partial^{2lm}u}{\partial x^{2\ell}}(x)$$

$$= \sum_{m=0}^{\infty} \frac{\Delta^{lm}u(x)}{4^{lm}\left(\frac{n}{2}\right)_{lm}(lm)!} r^{2lm}.$$

In an analogous way we obtain

$$M_k(u;x,r) = \sum_{m=0}^{\infty} \sum_{2|\ell|=km} \frac{1}{(2\ell)!} \frac{\left(\frac{1}{2}\right)_{\ell_1}\cdots\left(\frac{1}{2}\right)_{\ell_n}}{\left(\frac{n}{2}+1\right)_{|\ell|}} \cdot \frac{\partial^{km}u}{\partial x^{2\ell}}(x) r^{km}$$

$$= \sum_{m=0}^{\infty} \frac{\Delta^{lm}u(x)}{4^{lm}\left(\frac{n}{2}+1\right)_{lm}(lm)!} r^{2lm}.$$

The convergence of the serii (3a) and (3b) for $|r| < \rho(x)/\sqrt{n}$ (ρ as in (2)) can be shown as in the proof of [12, Theorem 3.1]. □

Passing to the limit $k \to \infty$ in the formulas (3a) and (3b) we get the following mean value formulas for real analytic functions.

Corollary 1. *Let $u \in \mathscr{A}(\Omega)$ and let $\tilde{u}$ be a holomorphic extension of u. Then for $x \in \Omega$ and $r > 0$ small enough it holds*

$$u(x) = \frac{1}{2n\sigma(n)\pi} \int_{S_\zeta^1(0,1)} \int_{S^{n-1}(0,1)} \tilde{u}(x+r\zeta y)\, dS^{n-1}(y) \cdot dS^1(\zeta)$$

$$= \frac{1}{2\sigma(n)\pi} \int_{S_\zeta^1(0,1)} \int_{B^n(0,1)} \tilde{u}(x+r\zeta y)\, dB^n(y) \cdot dS^1(\zeta),$$

where $S_\zeta^1(0,1) = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$.

Proof. Indeed since $u \in \mathscr{A}(\Omega)$ for $x \in \Omega$ one can find $C < \infty$ and $L < \infty$ such that

$$|\Delta^j u(x)| \leq CL^{2j}(2j)! \quad \text{for } j \in \mathbb{N}_0.$$

Hence for a fixed $\varepsilon > 0$ we can find $l_0 \in \mathbb{N}$ such that for $l \geq l_0$ and $m \geq 1$ we have

$$\frac{|\Delta^{lm} u(x)|}{4^{lm}\left(\frac{n}{2}\right)_{lm}(lm)!} r^{2lm} \leq C \frac{L^{2lm}(2lm)!}{4^{lm}\left(\frac{n}{2}\right)_{lm}(lm)!} r^{2lm} \leq C(Lr)^{2lm} \leq \frac{\varepsilon}{2^m}$$

if r is small enough. So

$$\sum_{m=1}^{\infty} \frac{|\Delta^{lm} u(x)|}{4^{lm}\left(\frac{n}{2}\right)_{lm}(lm)!} r^{2lm} \leq \varepsilon$$

and $\lim_{l\to\infty} N_{2l}(u; x, r) = u(x)$. On the other hand by the definition of the function N_{2l} and the Fubini theorem we get

$$\lim_{l\to\infty} N_{2l}(u; x, r) = \frac{1}{2n\sigma(n)\pi} \int_{S^1_\zeta(0,1)} \int_{S^{n-1}(0,1)} \tilde{u}(x + r\zeta y)\, dS^{n-1}(y) \cdot dS^1(\zeta).$$

The second formula is proved in the same way. □

It appears that real analytic functions can be characterized as those smooth ones for which the higher order Pizzetti's serii converge.

THEOREM 2. *Let $l \in \mathbb{N}$, $\rho \in C^0(\Omega; \mathbb{R}_+)$ and $u \in C^\infty(\Omega)$. If the series*

$$\tilde{N}(x, r) = \sum_{m=0}^{\infty} \frac{\Delta^{lm} u(x)}{4^{lm}\left(\frac{n}{2}\right)_{lm}(lm)!} r^{2lm}$$

is convergent locally uniformly in $\{(x, r) : x \in \Omega, |r| < \rho(x)\}$, then $u \in \mathscr{A}(\Omega)$ and $N_{2l}(u; x, r) = \tilde{N}(x, r)$ for $x \in \Omega$ and $0 \leq |r| < \min(\rho(x), \operatorname{dist}(x, \partial\Omega))$.

PROOF. Fix a compact set $K \Subset \Omega$ and set $\rho = \inf_{x\in K} \rho(x) > 0$. Then the assumption implies that

$$\frac{\Delta^{lm} u(x)}{4^{lm}\left(\frac{n}{2}\right)_{lm}(lm)!} r^{2lm} \to 0 \quad \text{as } m \to \infty$$

uniformly on $K \times \{|r| \leq \rho_1\}$ with any $\rho_1 < \rho$. So for any $\rho_1 < \rho$ there exists a constant $C(\rho_1) < \infty$ such that

$$\sup_{x\in K} |\Delta^{lm} u(x)| \leq C(\rho_1) \cdot 4^{lm}(n/2)_{lm}(lm)!\rho_1^{-2lm} \quad \text{for } m \in \mathbb{N}_0.$$

Applying the inequalities $(n/2)_{lm} \leq (\max\{1, n/2\})^{lm}(lm)!$ and $2^{lm} \times ((lm)!)^2 \leq (2lm)!$ we see that for any compact set $K \Subset \Omega$ one can find $C < \infty$ and $L < \infty$ such that

$$\sup_{x \in K} |\Delta^{lm} u(x)| \leq CL^{2lm}(2lm)! \quad \text{for } m \in \mathbb{N}_0.$$

But by [11, Theorem] this inequality implies that $u \in \mathscr{A}(\Omega)$. Finally, by Theorem 1 we get $\tilde{N}(x, r) = N_{2l}(u; x, r)$. □

Corollary 2. *Under the assumptions of Theorem 2 if the series*

$$\tilde{M}(x, r) = \sum_{m=0}^{\infty} \frac{\Delta^{lm} u(x)}{4^{lm}\left(\frac{n}{2}+1\right)_{lm}(lm)!} r^{2lm}$$

is convergent locally uniformly in $\{(x, r) : x \in \Omega, |r| < \rho(x)\}$*, then* $u \in \mathscr{A}(\Omega)$ *and* $M_{2l}(u; x, r) = \tilde{M}(x, r)$ *for* $x \in \Omega$ *and* $0 \leq |r| < \min(\rho(x), \operatorname{dist}(x, \partial\Omega))$.

3. Maximum principle for polyharmonic functions

It is well known that modulus of a function u harmonic on a connected domain $\Omega \subset \mathbb{R}^n$ cannot attain its maximum at an interior point of Ω unless u is constant. On the other hand this maximum principle does not extend to l-polyharmonic functions, i.e., solutions to $\Delta^l u = 0$ with $l \geq 2$. However due to the real analyticity of such functions by the formula (3b) we obtain the following maximum principle for polyharmonic functions.

Theorem 3. *Let* u *be a real valued,* l*-polyharmonic function on a connected open set* $\Omega \subset \mathbb{R}^n$, $l \in \mathbb{N}$. *Denote by* $\tilde{u}$ *its holomorphic extension to a connected complex neighborhood* U *of* Ω. *If for some* $x_0 \in \Omega$ *and* $r_0 > 0$ *it holds*

$$\pm u(x_0) \geq \pm \operatorname{Re} \tilde{u}(y) \quad \text{for } y \in x_0 + \sum_{j=0}^{l-1} \epsilon^j(B(0, r_0)), \tag{4}$$

where $\epsilon(z) = e^{\pi\imath/l} z$*, then* u *is constant on* Ω.

Proof. Since u is l-polyharmonic the series in (3b) terminates at the first term. So for any $x \in \Omega$ and $0 < r < \rho(x)$ we have

$$M_{2l}(u; x, r) = \frac{1}{l\sigma(n)} \sum_{j=0}^{l-1} \int_{B^n(0,r)} \tilde{u}(x + \epsilon^j(y))\, dy = u(x).$$

So $M_{2l}(u; x_0, r) = u(x_0)$ for $0 < r < \rho(x_0)$ and the assumption (4) implies that $\operatorname{Re} \tilde{u}(y) = u(x_0)$ for $y \in x_0 + \sum_{j=0}^{l-1} \epsilon^j(B(0, r_1))$ with $0 < r_1 < \min(r_0, \rho(x_0))$. In particular $\operatorname{Re} u = u$ is constant on $B(x_0, r_1)$ and since u is real analytic it is constant on Ω. □

Corollary 3. *Let u be a real valued, l-polyharmonic function on a connected open set $\Omega \subset \mathbb{R}^n$, $l \in \mathbb{N}$. Denote by $\tilde{u}$ its holomorphic extension to a connected complex neighborhood U of Ω. If for some $x_0 \in \Omega$ and $r_0 > 0$ it holds*

(5)
$$|u(x_0)| \geq |\tilde{u}(y)| \quad \textit{for } y \in x_0 + \sum_{j=0}^{l-1} \epsilon^j(B(0, r_0)),$$

then u is constant on Ω.

Proof. By possibly multiplying times -1, we can assume that $u(x_0)$ is non-negative. By (5) we have $u(x_0) \geq |\tilde{u}(y)| \geq \operatorname{Re} \tilde{u}(y)$ for $y \in x_0 + \sum_{j=0}^{l-1} \epsilon^j(B(0, r_0))$. Thus by Theorem 3, u is constant on Ω. □

4. Functions of l-Laplacian growth

The notion of functions of Laplacian growth was introduced by Aronszajn et al. [1, Chapter II]. Here we introduce its following generalization.

Definition 1. Let $l \in \mathbb{N}$, $\varrho > 0$ and $\tau \geq 0$. A function u smooth on $\Omega \subset \mathbb{R}^n$ is of *l-Laplacian growth* (ϱ, τ) *on* Ω if for every compact set $K \Subset \Omega$ and $\varepsilon > 0$ one can find $C = C(K, \varepsilon) < \infty$ such that

(6)
$$\sup_{x \in K} |\Delta^{lm} u(x)| \leq C(2lm)!^{1-1/\varrho}(\tau + \varepsilon)^{2lm} \quad \text{for any } m \in \mathbb{N}_0.$$

It follows by [11, Theorem] that a function u of l-Laplacian growth (ϱ, τ) on Ω is real analytic on Ω. Hence (3a) and (3b) hold for any $x \in \Omega$ and r small enough. However due to the estimation (6) both functions N_{2l} and M_{2l} extend to entire functions of r.

Theorem 4. *Let $l \in \mathbb{N}$, $\varrho > 0$, $\tau \geq 0$ and $u \in C^\infty(\Omega)$. If u is of l-Laplacian growth (ϱ, τ) on Ω, then $N_{2l}(u; x, r)$ and $M_{2l}(u; x, r)$ as functions of r extend holomorphically to entire functions of exponential growth $(\varrho, \tau^\varrho/\varrho)$ locally uniformly in Ω.*

Proof. Let u be of l-Laplacian growth (ϱ, τ) on Ω. Set

(7)
$$\tilde{N}_{2l}(u; x, z) = \sum_{m=0}^{\infty} \frac{\Delta^{lm} u(x)}{4^{lm} \left(\frac{n}{2}\right)_{lm} (lm)!} z^{2lm} \quad \text{for } x \in \Omega \text{ and } z \in \mathbb{C}.$$

Then $\tilde{N}_{2l}$ is a holomorphic extension of N_{2l}. Indeed, applying (6) and the inequality $(2lm)! \leq 4^{lm} \left(\frac{n}{2}\right)_{lm} (lm)!$ we get for any $K \Subset \Omega$, $\varepsilon_1 > 0$ and $R < \infty$,

$$\sup_{x\in K}\sup_{|z|\le R}|\tilde N_{2l}(u;x,z)| \le \sup_{x\in K}\sum_{m=0}^{\infty}\frac{|\Delta^{lm}u(x)|}{4^{lm}\left(\frac{n}{2}\right)_{lm}(lm)!}R^{2lm}$$
$$\le C_{K,\varepsilon_1}\sum_{m=0}^{\infty}\frac{(2lm)!^{1-1/\varrho}(\tau+\varepsilon_1)^{2lm}}{4^{lm}\left(\frac{n}{2}\right)_{lm}\cdot(lm)!}R^{2lm}$$
$$\le C_{\varepsilon_1}\sum_{m=0}^{\infty}\frac{1}{(2lm)!^{1/\varrho}}(\tau+\varepsilon_1)^{2lm}R^{2lm}.$$

Clearly, the last series converges for any R and so $\tilde N_{2l}(u;x,z)$ is an entire function of z. Now fix $\varepsilon>0$ and find $\varepsilon_1>0$ such that $e(\tau+\varepsilon_1)^{\varrho}\le e\tau^{\varrho}+\varepsilon$. Then using the inequality $j!\ge(j/e)^j$ for $j\in\mathbb{N}_0$ we estimate for any $m\in\mathbb{N}_0$,

$$\frac{1}{(2lm)!^{1/\varrho}}(\tau+\varepsilon_1)^{2lm}\le\left(\frac{e}{2lm}\right)^{2lm/\varrho}(\tau+\varepsilon_1)^{2lm}\le\left(\frac{e\varrho\tau^{\varrho}/\varrho+\varepsilon}{2lm}\right)^{2lm/\varrho}.$$

Thus by [3, Theorem 2.2.2], $\tilde N_{2l}(u;x,z)$ is an entire function of exponential growth $(\varrho,\tau^{\varrho}/\varrho)$ locally uniformly in Ω.

In the case of $M_{2l}(u;x,r)$ the proof goes along the same lines. □

THEOREM 5. *Let $l\in\mathbb{N}$, $\varrho>0$, $\tau\ge0$ and $u\in\mathscr{A}(\Omega)$. Assume that $M_{2l}(u;x,r)$ (resp. $N_{2l}(u;x,r)$) defined for $x\in\Omega$ and $0\le|r|<\operatorname{dist}(x,\partial\Omega)$ extends holomorphically to an entire function $\tilde M_{2l}(u;x,z)$ (resp. $\tilde N_{2l}(u;x,z)$) of exponential growth (ϱ,τ) locally uniformly in Ω. Then u is of l-Laplacian growth $(\varrho,(\tau\varrho)^{1/\varrho})$ on Ω.*

PROOF. Clearly, for any $x\in\Omega$ the extension $\tilde M_{2l}$ of M_{2l} is given by (3b). Fix $K\Subset\Omega$ and $\varepsilon>0$. Choose $0<\varepsilon_3<\varepsilon_2<\varepsilon_1<\varepsilon$ satisfying

$$(1+\varepsilon_3)^{3/2-1/\varrho}((\varrho\tau)^{1/\varrho}+\varepsilon_1)\le(\varrho\tau)^{1/\varrho}+\varepsilon, \tag{8}$$
$$(\varrho\tau+\varepsilon_2)^{1/\varrho}\le(\varrho\tau)^{1/\varrho}+\varepsilon_1, \tag{9}$$
$$\frac{1+\varepsilon_3}{e}\le\frac{\varrho\tau+\varepsilon_2}{e\varrho\tau+\varepsilon_2}. \tag{10}$$

By [3, Theorem 2.2.10] the assumption that $\tilde M_{2l}$ is an entire function of exponential growth (ϱ,τ) uniformly on K implies that we can find C_{ε_2} such that

$$\sup_{x\in K}\frac{|\Delta^{lm}u(x)|}{4^{lm}\left(\frac{n}{2}+1\right)_{lm}(lm)!}\le C_{\varepsilon_2}\left(\frac{e\varrho\tau+\varepsilon_2}{2lm}\right)^{2lm/\varrho}\quad\text{for } m\in\mathbb{N}.$$

We can also find C_{ε_3} such that

$$\left(\frac{n}{2}+1\right)_{lm}\le C_{\varepsilon_3}(1+\varepsilon_3)^m(lm)!\quad\text{for } m\in\mathbb{N}_0$$

and (by the Stirling formula)

$$(m/e)^m\le m!\le C_{\varepsilon_3}(m/e)^m(1+\varepsilon_3)^m\quad\text{for } m\in\mathbb{N}_0. \tag{11}$$

Hence for $m \in \mathbb{N}$ we get

$$\begin{aligned}(12)\qquad \sup_{x\in K}|\Delta^{lm}u(x)| &\le 4^{lm}\left(\frac{n}{2}+1\right)_{lm}(lm)!\cdot C_{\varepsilon_2}\left(\frac{e\varrho\tau+\varepsilon_2}{2lm}\right)^{2m/\varrho}\\ &\le 4^{lm}C_{\varepsilon_2}C_{\varepsilon_3}(1+\varepsilon_3)^m(m!)^{2l}\left(\frac{e\varrho\tau+\varepsilon_2}{2lm}\right)^{2lm/\varrho}.\end{aligned}$$

Now by (11) and (10),

$$(m!)^{2/\varrho} \le C_{\varepsilon_3}^{2/\varrho}\left(\frac{m(1+\varepsilon_3)}{e}\right)^{2m/\varrho} \le C_{\varepsilon_3}^{2/\varrho}\left(\frac{m(\varrho\tau+\varepsilon_2)}{e\varrho\tau+\varepsilon_2}\right)^{2m/\varrho}.$$

So

$$(m!)^{2/\varrho}\cdot\left(\frac{e\varrho\tau+\varepsilon_2}{2m}\right)^{2m/\varrho} \le C_{\varepsilon_3}^{2/\varrho}\left(\frac{\varrho\tau+\varepsilon_2}{2}\right)^{2m/\varrho}.$$

Thus by (12), (9), (11) and (8) we derive

$$\begin{aligned}\sup_{x\in K}|\Delta^{lm}u(x)| &\le 4^{lm}C_{\varepsilon_2}C_{\varepsilon_3}(1+\varepsilon_3)^m(m!)^{2-2/\varrho}C_{\varepsilon_3}^{2/\varrho}\left(\frac{\varrho\tau+\varepsilon_2}{2}\right)^{2m/\varrho}\\ &\le C_{\varepsilon_2}C_{\varepsilon_3}^{1+2/\varrho}(1+\varepsilon_3)^m 4^{m(1-1/\varrho)}(m!)^{2-2/\varrho}((\varrho\tau)^{1/\varrho}+\varepsilon_1)^{2m}\\ &\le C_{\varepsilon_2}C_{\varepsilon_3}^3(1+\varepsilon_3)^{m(3-2/\varrho)}(2k)!^{1-1/\varrho}((\varrho\tau)^{1/\varrho}+\varepsilon_1)^{2m}\\ &\le C_\varepsilon(2lm)!^{1-1/\varrho}((\varrho\tau)^{1/\varrho}+\varepsilon)^{2lm}.\end{aligned}$$

Since $K \Subset \Omega$ was arbitrary u is of l-Laplacian growth $(\varrho,(\varrho\tau)^{1/\varrho})$ on Ω. In the case of the assumption on N_{2l} the proof goes along the same lines. □

5. Convergent solutions of higher order heat equations

For $l \in \mathbb{N}$ let us consider the initial value problem for the l-th order heat type equation

$$(13)\qquad \begin{cases}\partial_t u - \Delta_x^l u = 0,\\ u_{|t=0} = u_0,\end{cases}$$

where $u_0 \in \mathscr{A}(\Omega)$, $\Omega \subset \mathbb{R}^n$. Clearly, the unique formal power series solution of (13) is given by

$$(14)\qquad \hat{u}(t,x) = \sum_{m=0}^{\infty}\frac{\Delta^{lm}u_0(x)}{m!}t^m.$$

We ask when the solution u is an analytic function of the time variable at $t = 0$.

THEOREM 6. *Let $0 < T \leq \infty$. If the formal power series solution* (14) *of the initial value problem* (13) *is convergent for $|t| < T$ locally uniformly in Ω, then $M_{2l}(u_0; x, r)$ and $N_{2l}(u_0; x, r)$ extend holomorphically to entire functions of exponential growth $\left(\frac{2l}{2l-1}, \frac{2l-1}{2l}(2lT)^{1-2l}\right)$ locally uniformly in Ω.*

Conversely, if $M_{2l}(u_0; x, r)$ or $N_{2l}(u_0; x, r)$ can be holomorphically extended to entire functions of exponential growth $\left(\frac{2l}{2l-1}, \frac{2l-1}{2l}(2lT)^{1-2l}\right)$ locally uniformly in Ω, then the solution (14) *of* (13) *is convergent for $|t| < T$ locally uniformly in Ω.*

PROOF. Assume that $\hat{u}(t,x)$ is convergent for $|t| < T$ locally uniformly in Ω. Then for any compact set $K \Subset \Omega$ and $\varepsilon > 0$ there exists $C_\varepsilon = C(K, \varepsilon) < \infty$ such that

$$\sup_{x \in K} |\Delta^{lm} u_0(x)| \leq C_\varepsilon \Big(\frac{1}{T} + \varepsilon\Big)^m \cdot m! \quad \text{for } m \in \mathbb{N}_0.$$

So for any $m \in \mathbb{N}_0$, we have

$$\begin{aligned}\sup_{x \in K} |\Delta^{lm} u_0(x)| &\leq C_\varepsilon \Big(\frac{1}{T} + \varepsilon\Big)^m \Big(\frac{1}{2l} + \varepsilon\Big)^m \cdot (2lm)!^{1/2l} \\ &\leq C_\varepsilon ((2lT)^{-1/2l} + \varepsilon)^{2lm} \cdot (2lm)!^{1/2l}.\end{aligned}$$

Hence, u_0 is of l-Laplacian growth $\left(\frac{2l}{2l-1}, (2lT)^{-1/(2l)}\right)$ on Ω and by Theorem 4, $M_{2l}(u_0; x, z)$ and $N_{2l}(u_0; x, z)$ extend holomorphically to entire functions of exponential growth $\left(\frac{2l}{2l-1}, \frac{2l-1}{2l}(2lT)^{1-2l}\right)$ locally uniformly in Ω.

Conversely, suppose that $M_{2l}(u_0; x, r)$ or $N_{2l}(u_0; x, r)$ can be holomorphically extended to entire functions of exponential growth $\left(\frac{2l}{2l-1}, \frac{2l-1}{2l}(2lT)^{1-2l}\right)$ locally uniformly in Ω. Then by Theorem 5, u_0 is of l-Laplacian growth $\left(\frac{2l}{2l-1}, (2lT)^{-1/(2l)}\right)$ on Ω. Fix $K \Subset \Omega$ and $|t| < T$. Then for $\varepsilon > 0$ sufficiently small, we get

$$\begin{aligned}\sup_{x \in K} \sum_{m=0}^{\infty} \frac{|\Delta^{lm} u_0(x)|}{m!} |t|^m &\leq C_\varepsilon \sum_{m=0}^{\infty} \frac{((2lT)^{-1/(2l)} + \varepsilon)^{2lm} \cdot (2lm)!^{1/2l} |t|^m}{m!} \\ &\leq C_\varepsilon \sum_{m=0}^{\infty} \Big(\frac{1}{2lT} + \varepsilon\Big)^m (2l + \varepsilon)^m |t|^m \\ &\leq C_\varepsilon \sum_{m=0}^{\infty} \Big[\Big(\frac{1}{T} + \varepsilon\Big)|t|\Big]^m < \infty.\end{aligned}$$

Since $K \Subset \Omega$ was arbitrary, $\hat{u}(t,x)$ is convergent for $|t| < T$ locally uniformly in Ω. □

REMARK 1. The summability of the formal power series solution (14) of the equation (13) is studied by Michalik [14].

References

[1] N. Aronszajn - T. M. Creese - L. J. Lipkin, *Polyharmonic Functions*, Clarendon Press, Oxford, 1983.

[2] D. H. Armitage - U. Kuran, *The convergence of the Pizzetti series in potential theory*, J. Math. Anal. Appl., 171 (1992), 516–531.

[3] R. P. Boas Jr, *Entire Functions*, Academic Press, New York, 1954.

[4] B. Bojanov, *An extension of the Pizzetti formula for polyharmonic functions*, Acta Math. Hungar., 91 (2001), 99–113.

[5] A. Bonfiglioli, *Expansion of the Heisenberg integral mean via iterated Kohn Laplacian: a Pizzetti-type formula*, Potential Analysis, 17 (2002), 165–180.

[6] P. Caramuta - A. Cialdea, *Mean value theorems for polyharmonic functions: A conjecture by Picone*, Analysis, 34 (2014), 51–66.

[7] G. M. Fichtenholz, *Differential- und Integralrechnung*, vol. III, Hochschulbücher für Mathematik 63, Johann Ambrosius Barth Verlag GmbH, Leipzig, 1992.

[8] M. Ghermanescu, *Sur les moyennes successives des fonctions*, Math. Ann., 119 (1943), 288–320.

[9] A. Gray - T. J. Willmore, *Mean value theorems for Riemannian manifolds*, Proc. Roy. Soc. Edinburgh, 92A (1982), 343–364.

[10] M. Pawełko-Grzybowska - A. Strasburger, *On the Pizzetti's formula for the Heisenberg group*, Integral Transf. Spec. Functions, 19 (2008), 665–675.

[11] H. Komatsu, *A characterization of real analytic functions*, Proc. Japan Acad., 36 (1960), 90–93.

[12] G. Łysik, *Mean-value properties of real analytic functions*, Arch. Math. (Basel) 98 (2012), 61–70.

[13] G. Łysik, *On mean-value properties for the Dunkl polyharmonic functions*, Opuscula Math., 35 (2015), 655–664.

[14] S. Michalik, *Summable solutions of some partial differential equations and generalised integral means*, http://arxiv.org/pdf/1502.02462v1.

[15] M. Nicolesco, *Les Fonctions Polyharmoniques*, Hermann, & C^{ie} Éditeurs, Paris, 1936.

[16] P. Pizzetti, *Sulla media dei valori che una funzione dei punti dello spazio assume alla superficie di una sfera*, Rendiconti Lincei, serie V, 18 (1909), 182–185.

[17] N. B. Salem - K. Touahri, *Pizzetti series and polyharmonicity associated with the Dunkl Laplacian*, Mediterr. J. Math., 7 (2010), 455–470.

[18] L. Zalcman, *Mean values and differential equations*, Israel J. Math., 14 (1973), 339–352.

Received 3 June 2015,
and in revised form 28 October 2015.

Grzegorz Łysik
Jan Kochanowski University
Swietokrzyska str. 15
25-406 Kielce, Poland
lysik@impan.pl