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Partial Differential Equations — *Sharp cooling rates in nonlinear friction equations*, by Giulia Furioli, Ada Pulvirenti and Elide Terraneo, communicated on 13 November 2015.

Abstract. — We study some extremal properties of the self-similar solutions of certain one-dimensional kinetic models of granular flows, usually known with the name of nonlinear friction equations. This analysis, inspired by some recent results on nonlinear diffusion equations [6], allows to obtain various sharp inequalities, which can be fruitfully used to better clarify the large-time behavior of the solution density.

Key words: Granular gases, Boltzmann equation, long-time behavior of solution

Mathematics Subject Classification: 76P05, 76T25, 82C40

1. Introduction

This paper is devoted to the study of various properties of the self-similar solutions to the nonlinear evolution equation

$$\partial_t f(v,t) = \partial_v\Big(f(v,t)\int |v-w|(v-w)f(w,t)\,dw\Big), \tag{1}$$

where the unknown $f(v,t)$ is a time-dependent probability density on $\mathbb{R}$. This equation, usually referred as *nonlinear friction equation*, arises in the study of granular flows, and has been first obtained in [9, 10] in connection with the quasi-elastic limit of a model Boltzmann equation for rigid spheres with dissipative collisions. Then, the case of variable coefficient of restitution,

$$\partial_t f(v,t) = \partial_v\Big(f(v,t)\int |v-w|^\gamma(v-w)f(w,t)\,dw\Big), \tag{2}$$

for a general $\gamma < 1$, was discussed in [11]. A review of the state of the art of the mathematical results on kinetic theory of granular media can be found in [13].

Despite its relatively simple (with respect to the full Boltzmann equation) structure, the nonlinear friction equation (1,2) exhibits the main properties of any kinetic model with dissipative collisions, like conservation of mass and mean velocity and decay of the temperature. Likewise, the equilibrium state is given by a Dirac mass located at the mean velocity of particles. In addition, at difference of the full dissipative Boltzmann equation for rigid spheres, this equation possesses explicit similarity solutions, which are in general of noticeable importance for understanding the cooling process of the granular flow, and eventually for constructing reasonable macroscopic equations. These similarity solutions are given by the sum of two delta functions

$$f^{\gamma}_{\infty,A}(v,t) = \frac{1}{2}\delta\Big(v - \frac{1}{2}\alpha(t)\Big) + \frac{1}{2}\delta\Big(v + \frac{1}{2}\alpha(t)\Big) \tag{3}$$

where, for a given positive constant A

$$\alpha(t) = \left[\frac{1}{A^{\gamma} + \gamma t}\right]^{1/\gamma}, \quad 0 < \gamma \le 1. \tag{4}$$

The case $\gamma = 1$ is considered the most relevant from the physical point of view, and the main mathematical results related to equation (1) have been obtained some years ago [2, 3, 4]. Then, the general case has been dealt with in [8]. In particular, Benedetto, Caglioti, Carrillo and Pulvirenti [3] studied the asymptotic behavior of (1) via the study of the free energy, proving convergence to equilibrium in large time, without obtaining any rate. An important step towards the understanding of the role of the self-similar solutions has been achieved by Caglioti and Villani in [4]. The main goal of this paper is to show that the similarity solutions do not represent in a strong sense the intermediate asymptotics of any other solution with the same mass and momentum. Last, in [8] the asymptotic behavior of the solution to the nonlinear friction equation has been studied on the whole allowed range of the parameter γ.

Some related problems linked to dissipative equations containing the nonlinear friction operator in (1) have been also addressed. Indeed, the long-time behavior of these and more complex equations has been deeply investigated by Carrillo, McCann and Villani in [5]. In this paper, by means of a suitable generalization of logarithmic Sobolev inequalities and mass transportation inequalities, the long-time asymptotics of certain nonlocal, diffusive equations with a gradient flow structure has been analyzed. In particular, the results of [5] cover the asymptotic behavior of the equation (2) when $\gamma > 0$.

In what follows, we will study equation (1) from a different point of view. In particular, we will investigate the (eventual) extremal properties of the self-similar solutions to (1). Let us fix $\gamma = 1$, and $A = 0$ in (4) and let us simply denote from now on f_∞ the corresponding self-similar solution introduced in (3). Then, the second moment $T(t)$ of the self-similar (source-type) solution (3) can be explicitly evaluated

$$T(f_\infty(t)) = \int v^2 f_\infty(v,t)\,dv = (2t)^{-2}.$$

Hence, the second moment of the source-type solution, raised to power $-1/2$, grows linearly in time. Hence, the first variation in time of $T(f_\infty(t))^{-1/2}$ is constant, while the second variation is equal to zero. To see how the second moment of any other solution to equation (1) behaves in time, it comes natural to estimate the time variations of $T(f(t))^{-1/2}$, where the *second moment functional* (which represents in this context the temperature or the kinetic energy) is defined by

$$T(\varphi) := \int v^2 \varphi(v)\, dv.$$

This computation will lead to the interesting observation that the second variation of $T(f(t))^{-1/2}$ has a fixed negative sign, which implies concavity in t. Therefore, the first variation is decreasing. It is possible to evaluate its limit for large time thanks to the asymptotic behavior and to the crucial invariance with respect to mass preserving dilations. It follows at once that the first variation satisfies a sharp inequality which connects the second moment to its first variation, and optimality is achieved by the self-similar profiles.

This idea can also be applied to the study of the behavior of the derivative of the second moment of a generic solution, performed by using the nonlinear friction equation. It holds

$$\frac{dT(f(t))}{dt} = -I(f(t))$$

where, the *entropy functional* is defined by

$$I(\varphi) := \iint \varphi(v)\varphi(w)|v-w|^3\, dv\, dw.$$

For the source-type solution (3) one can easily check that $I(f_\infty(t))^{-1/3}$ grows linearly in time. This leads to estimate the time variation of $I(f(t))^{-1/3}$. As in the previous case, we will conclude by showing that the second variation in time of $I(f(t))^{-1/3}$ has a negative sign which implies concavity. In this case also, the first variation is invariant with respect to dilations, which implies a sharp inequality that links entropy to its first variation, and again optimality is achieved by source-type profiles.

This new way of looking at the problem has been recently applied to nonlinear diffusion equations in [6, 7]. There, the study of the time evolution of a suitable power of the second moment of the self-similar Barenblatt solution [12, 14], allowed to discover a delay in the propagation of the second moment of the Barenblatt profile, with respect to any other solution. Also, the study of the time evolution of a suitable power of the entropy functional gived raise to Gagliardo–Nirenberg–Sobolev inequalities in sharp form. In the case of the nonlinear friction equation (1) a similar delay is proved. This shows that a generic solution $f(v,t)$ cools down faster than the homogenous cooling state starting with the same initial temperature. This allows us to get an upper bound for the generic cooling time.

The plan of the paper is the following. In Section 2 we recall the main known results about existence, uniqueness and asymptotic behavior of the solution of a Cauchy problem for equation (1). In Sections 3 and 4 are gathered the new results about the extremal properties of the homogenous cooling states with respect to the temperature and entropy functionals. Moreover in Proposition 6 we collect some sharp functional inequalities which can be interesting by themselves and have been derived as a byproduct of the previous analysis. Some further comments are presented in Section 5. A technical proof of an intermediate result is contained in the Appendix.

2. The nonlinear friction

We will start our analysis by recalling the main results obtained for the nonlinear friction equation. To this aim, let us consider the initial value problem

$$\begin{cases} \partial_t f(v,t) = \partial_v \Big(f(v,t) \displaystyle\int |v-w|(v-w) f(w,t)\, dw \Big) \\ f(v,0) = f_0(v). \end{cases} \tag{5}$$

For any $p \geq 0$, let us introduce the space of functions

$$\Omega_{2p} = \left\{ \varphi \in L^1(\mathbb{R}) : \varphi \geq 0, \|\varphi\|_{L^1} = 1, \int v^{2p} \varphi(v)\, dv < +\infty \right\}. \tag{6}$$

The existence and uniqueness of a solution for any initial data f_0 with bounded second moment have been established by Benedetto, Caglioti and Pulvirenti [2]. They proved the following theorem.

Theorem 1 (Benedetto–Caglioti–Pulvirenti [2]). *Let $f_0 \in \Omega_2$ such that*

$$\int v^2 (1 + \log(1 + |v|)) f_0(v)\, dv < +\infty.$$

Then for any $T > 0$ there exists a unique function $f \in \mathscr{C}([0,T];\Omega_2) \cap \mathscr{C}((0,T];\Omega_{2p})$ which satisfies equation (5) in the sense of weak convergence of measures. Moreover $f(\cdot,t)$ satisfies the following properties for $t > 0$:

i) *supp$(f(\cdot,t)) \subset \left[-\frac{1}{t}, \frac{1}{t}\right]$. In particular $f(\cdot,t) \in \Omega_{2p}$ for all $p \geq 0$ and for t large enough*

$$T_{2p}(t) = \int v^{2p} f(v,t)\, dv \leq \frac{1}{t^{2p}};$$

ii) *$T_{2p}(t)$ is decreasing in time and the kinetic energy $T_2(t)$ satisfies the bound*

$$T_2(t) \leq \frac{T_2(0)}{(1 + t\sqrt{2T_2(0)})^2}. \tag{7}$$

As a consequence

$$\lim_{t\to+\infty} f(\cdot,t) = \delta_0$$

in the sense of weak convergence of measures.

REMARK 2. If the initial data f_0 belong to Ω_{2p}, by using the explicit expression of the solution f (see [2]) it is also possible to prove that $f \in \mathscr{C}([0,T];\Omega_{2p})$ setting in this way the continuity at the origin of the $T_{2p}(t)$ functional.

Note that inequality (7) gives a non optimal upper bound for the cooling process of the dissipative gas. An enlightening deeper analysis of the asymptotic behavior of the solution has been achieved by resorting to a suitable scaling of velocity and time. Benedetto, Caglioti and Pulvirenti [2] considered first the scaling

$$g(v,t) = \frac{1}{t} f\left(\frac{v}{t}, t\right). \tag{8}$$

If we denote by

$$G(\varphi(v)) := \int |v-w|(v-w)\varphi(w)\,dw, \tag{9}$$

it is a simple matter to show that the function g satisfies the equation

$$g(v,t) + t\partial_t g(v,t) + \partial_v g(v,t)v = \partial_v(g(v,t)G(g(v,t))).$$

By changing the time scale through the relation $\tau = \log t$ one shows that the function $\tilde{g}(v,\tau) = g(v,\mathrm{e}^\tau)$ satisfies

$$\partial_\tau \tilde{g}(v,\tau) + \partial_v(\tilde{g}(v,\tau)(v - G(\tilde{g}(v,\tau)))) = 0. \tag{10}$$

The analysis of the large-time behavior of equation (10) carried out in [2] leads to the conclusion that for $\tau \to +\infty$

$$\tilde{g}(v,\tau) \rightharpoonup \frac{1}{2}\delta\left(v+\frac{1}{2}\right) + \frac{1}{2}\delta\left(v-\frac{1}{2}\right) := g_\infty(v). \tag{11}$$

Since $g(v,t) = \tilde{g}(v,\log\tau)$, we also have for $t \to +\infty$

$$g(\cdot,t) \rightharpoonup g_\infty. \tag{12}$$

Therefore, coming back to the original variables, one recovers the source-type self-similar solution measure

$$f_\infty(v,t) = \frac{1}{2}\delta\left(v+\frac{1}{2t}\right) + \frac{1}{2}\delta\left(v-\frac{1}{2t}\right). \tag{13}$$

In a more general setting, for any positive constant A, one can scale the solution as

$$g(v,t) = \frac{1}{t+A} f\Big(\frac{v}{t+A}, t\Big).$$

The function $g(v,t)$ now satisfies

$$g(v,t) + (t+A)\partial_t g(v,t) + \partial_v g(v,t)v = \partial_v(g(v,t)G(g(v,t))).$$

Then, choosing $\tau = \log(t+A)$ and $\tilde{g}(v,\tau) = g(v, \mathrm{e}^\tau - A)$, one obtains again

$$\partial_\tau \tilde{g}(v,\tau) + \partial_v(\tilde{g}(v,\tau)(v - G(\tilde{g}(v,\tau)))) = 0.$$

Since $\tilde{g}(v,\tau) \rightharpoonup \frac{1}{2}\delta\big(v+\frac{1}{2}\big) + \frac{1}{2}\delta\big(v-\frac{1}{2}\big)$, back to the original variables it shows that, for any positive constant A

$$f_{\infty,A}(v,t) = \frac{1}{2}\delta\Big(v + \frac{1}{2(t+A)}\Big) + \frac{1}{2}\delta\Big(v - \frac{1}{2(t+A)}\Big)$$

is also a self-similar solution to equation (5).

3. Extremal properties of self-similar solutions

In what follows, we will consider the time-evolution of various functionals of the solution to the nonlinear friction equation (5). These functionals are closely related each other by equation (5), and the chain starts with the second moment of the solution. We will call them the temperature, the entropy and the entropy production, respectively,

$$T(\varphi) = \int v^2 \varphi(v)\,dv, \tag{14}$$

$$I(\varphi) = \iint \varphi(v)\varphi(w)|v-w|^3\,dv\,dw, \tag{15}$$

$$J(\varphi) = \int \varphi(v)G(\varphi(v))^2\,dv = \int \varphi(v)\Big(\int |v-w|(v-w)\varphi(w)\,dw\Big)^2\,dv. \tag{16}$$

These functionals are well-defined on the space Ω_{2p} introduced in (6), provided $p > 0$ is sufficiently large that all integrals converge. Note that the temperature T had been denoted by T_2 in Theorem 1.

The time-decay of the previous functionals for the self-similar solution (13) can be computed explicitly. One obtains

$$
\begin{aligned}
T_\infty(t) &= T(f_\infty(v,t)) = \frac{1}{4t^2},\\
I_\infty(t) &= I(f_\infty(v,t)) = \frac{1}{2t^3},\\
J_\infty(t) &= J(f_\infty(v,t)) = \frac{1}{4t^4}.
\end{aligned}
\tag{17}
$$

If f is a solution of equation (5), we will denote in short

$$
\begin{aligned}
T(t) &= T(f(v,t))\\
I(t) &= I(f(v,t))\\
J(t) &= J(f(v,t))
\end{aligned}
\tag{18}
$$

and also

$$G(v,t) = G(f(v,t))$$

for $G(f(v,t))$ as in (9). The following relations hold true.

Proposition 3. *Let $T(t)$, $I(t)$ and $J(t)$ defined as in* (14, 15, 16, 18). *Then*

$$
\begin{aligned}
\frac{d}{dt}T(t) &= -I(t)\\
\frac{d}{dt}I(t) &= -6J(t).
\end{aligned}
$$

Proof. The proof is immediate. Indeed

$$
\begin{aligned}
\frac{d}{dt}T(t) &= \int v^2 \partial_v \Big(f(v,t) \int |v-w|(v-w) f(w,t)\, dw \Big)\, dv\\
&= -2 \iint f(v,t) f(w,t) v |v-w|(v-w)\, dv\, dw\\
&= -\iint f(v,t) f(w,t) |v-w|(v-w)^2\, dv\, dw\\
&= -\iint f(v,t) f(w,t) |v-w|^3\, dv\, dw\\
&= -I(t).
\end{aligned}
$$

Likewise

$$\begin{aligned}
\frac{d}{dt}I(t) &= 2\iint \partial_t f(v,t)f(w,t)|v-w|^3\,dv\,dw \\
&= 2\iint \partial_v\Big(f(v,t)\int |v-z|(v-z)f(z,t)\,dz\Big)f(w,t)|v-w|^3\,dv\,dw \\
&= -2\iint \Big(f(v,t)\int |v-z|(v-z)f(z,t)\,dz\Big)f(w,t)3|v-w|(v-w)\,dv\,dw \\
&= -6\iiint f(v,t)|v-w|(v-w)f(w,t)|v-z|(v-z)f(z,t)\,dv\,dw\,dz \\
&= -6J(t). \qquad \square
\end{aligned}$$

Proposition 3 shows that the functionals $T(t)$ and $I(t)$ decay in time at a precise rate. A further insight into their decay can be obtained by looking at some particular powers, that, as discussed in the introduction, are induced by the explicit decay of the functionals when evaluated along the source-type solution (13). Thanks to (17), both $T_\infty(t)^{-1/2}$ and $I_\infty(t)^{-1/3}$ grow linearly in time. We prove

Theorem 4. *Let $f(v,t)$ denote a solution to* (5), *corresponding to an initial value $f_0 \in \Omega_{2p}$, with p large enough. Then, $T(t)^{-\frac{1}{2}}$ and $I(t)^{-\frac{1}{3}}$ are non decreasing, concave functions of time, and*

$$\lim_{t\to+\infty}\frac{T(t)^{-\frac{1}{2}}}{2t} = 1, \tag{19}$$

$$\lim_{t\to+\infty}\frac{I(t)^{-\frac{1}{3}}}{\sqrt[3]{2}t} = 1. \tag{20}$$

Hence, for large times

$$T(t)^{-\frac{1}{2}} \sim T_\infty(t)^{-\frac{1}{2}}, \quad I(t)^{-\frac{1}{3}} \sim I_\infty(t)^{-\frac{1}{3}}.$$

Proof. Let us begin by considering the evolution of $T(t)^{-\frac{1}{2}}$. Of course, by (17) we have $T_\infty(t)^{-\frac{1}{2}} = 2t$. We get

$$\frac{d}{dt}T(t)^{-\frac{1}{2}} = -\frac{1}{2}T(t)^{-\frac{3}{2}}\frac{d}{dt}T(t),$$

which implies by Proposition 3

$$\frac{d}{dt}T(t)^{-\frac{1}{2}} = \frac{1}{2}T(t)^{-\frac{3}{2}}I(t) \geq 0. \tag{21}$$

Consequently $T(t)^{-\frac{1}{2}}$ is a non decreasing function. Using again Proposition 3 we get

$$\frac{d^2}{dt^2}T(t)^{-\frac{1}{2}} = -\frac{3}{4}T(t)^{-\frac{5}{2}}(-I(t))I(t) + \frac{1}{2}T(t)^{-\frac{3}{2}}\frac{d}{dt}I(t)$$
$$= \frac{3}{4}T(t)^{-\frac{5}{2}}I(t)^2 - 3T(t)^{-\frac{3}{2}}J(t)$$
$$= \frac{3}{4}T(t)^{-\frac{5}{2}}(I(t)^2 - 4T(t)J(t)). \tag{22}$$

To prove concavity, one has to verify that

$$I(t)^2 - 4T(t)J(t) \leq 0. \tag{23}$$

However, by Cauchy-Schwarz inequality we get

$$I(t) = \iint f(v,t)f(w,t)(v-w)(v-w)|v-w|\,dv\,dw$$
$$= 2\iint f(v,t)f(w,t)v(v-w)|v-w|\,dv\,dw$$
$$= 2\int f(v,t)^{\frac{1}{2}}vf(v,t)^{\frac{1}{2}}\Big(\int (v-w)|v-w|f(w,t)\,dw\Big)dv$$
$$\leq 2\Big(\int f(v,t)v^2\,dv\Big)^{\frac{1}{2}}\Big(\int f(v,t)\Big(\int (v-w)|v-w|f(w,t)\,dw\Big)^2 dv\Big)^{\frac{1}{2}}$$
$$= 2T(t)^{\frac{1}{2}}J(t)^{\frac{1}{2}}.$$

This shows that $T(t)^{-\frac{1}{2}}$ is concave, and that $\frac{d}{dt}T(t)^{-\frac{1}{2}}$ is a non increasing function. Let us further remark that the functional

$$\frac{d}{dt}T(t)^{-\frac{1}{2}} = \frac{1}{2}T(t)^{-\frac{3}{2}}I(t)$$

is dilation invariant, namely invariant with respect to the mass-preserving scaling $f(v) \mapsto f_a(v) = af(av)$, for any constant $a > 0$. Thus, the value of this functional does not change if at each time t we replace $f(v,t)$ with $g(v,t)$ given by (8)

$$\frac{1}{2}T(t)^{-\frac{3}{2}}I(t) = \frac{1}{2}T(g(v,t))^{-\frac{3}{2}}I(g(v,t)).$$

On the other hand, $g(\cdot,t) \rightharpoonup g_\infty$ in the weak convergence of the measures and $T(g(v,t))$, $I(g(v,t))$ are weakly continuous (see [2] and the proof given in the Appendix). Therefore

$$\lim_{t\to+\infty}\frac{d}{dt}T(t)^{-\frac{1}{2}} = \lim_{t\to+\infty}\frac{1}{2}T(g(t))^{-\frac{3}{2}}I(g(t)) = \frac{1}{2}T(g_\infty)^{-\frac{3}{2}}I(g_\infty) = 2 \tag{24}$$

and this shows (19). Let us now study the evolution in time of $I(t)^{-\frac{1}{3}}$. Note that in this case (17) implies $I_\infty(t)^{-\frac{1}{3}} = \sqrt[3]{2}t$. Thanks to Proposition 3 we obtain

$$\frac{d}{dt}I(t)^{-\frac{1}{3}} = -\frac{1}{3}I(t)^{-\frac{4}{3}}\frac{d}{dt}I(t) = 2I(t)^{-\frac{4}{3}}J(t) \geq 0.$$

Moreover

$$\begin{aligned}\frac{d^2}{dt^2}I(t)^{-\frac{1}{3}} &= -\frac{8}{3}I(t)^{-\frac{7}{3}}(-6J(t))J(t) + 2I(t)^{-\frac{4}{3}}\frac{d}{dt}J(t)\\ &= 16I(t)^{-\frac{7}{3}}J(t)^2 + 2I(t)^{-\frac{4}{3}}\frac{d}{dt}J(t)\\ &= 16I(t)^{-\frac{7}{3}}\Big(J(t)^2 + \frac{1}{8}I(t)\frac{d}{dt}J(t)\Big).\end{aligned}$$

To conclude, we need to show that

$$J(t)^2 + \frac{1}{8}I(t)\frac{d}{dt}J(t) \leq 0. \tag{25}$$

It holds

$$\frac{d}{dt}J(t) = -2\Big(\iint f(v,t)f(w,t)|v-w|(G(v,t)-G(w,t))^2\,dv\,dw\Big) \leq 0.$$

Indeed

$$\begin{aligned}\frac{d}{dt}J(t) &= \int \partial_t f(v,t)G(v,t)^2\,dv + 2\int f(v,t)G(v,t)\partial_t G(v,t)\,dv\\ &= \int \partial_v(f(v,t)G(v,t))G(v,t)^2\,dv + 2\int f(v,t)G(v,t)\partial_t G(v,t)\,dv\\ &= -2\int f(v,t)G(v,t)G(v,t)\partial_v G(v,t)\,dv + 2\int f(v,t)G(v,t)\partial_t G(v,t)\,dv\\ &= -2\int f(v,t)G(v,t)^2\partial_v G(v,t)\,dv + 2\int f(v,t)G(v,t)\partial_t G(v,t)\,dv.\end{aligned}$$

Let us compute separately the two derivatives appearing in the previous integrals. We obtain

$$\begin{aligned}\partial_t G(v,t) = \partial_t \int |v-w|(v-w)f(w,t)\,dw &= \int |v-w|(v-w)\partial_t f(w,t)\,dw\\ &= \int |v-w|(v-w)\partial_w\Big(f(w,t)\int |w-z|(w-z)f(z,t)\,dz\Big)\,dw\end{aligned}$$

$$
\begin{aligned}
&= -\int \Big(f(w,t)\int |w-z|(w-z)f(z,t)\,dz\Big)\partial_w(|v-w|(v-w))\,dw \\
&= -\int \Big(f(w,t)\int |w-z|(w-z)f(z,t)\,dz\Big)(-2|v-w|)\,dw \\
&= 2\int f(w,t)|v-w|G(w,t)\,dw,
\end{aligned}
$$

and

$$
\begin{aligned}
\partial_v G(v,t) = \partial_v \int |v-w|(v-w)f(w,t)\,dw &= \int \partial_v(|v-w|(v-w))f(w,t)\,dw \\
&= 2\int |v-w|f(w,t)\,dw.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{d}{dt}J(t) &= -2\int f(v,t)G(v,t)^2\partial_v G(v,t)\,dv + 2\int f(v,t)G(v,t)\partial_t G(v,t)\,dv \\
&= -4\iint f(v,t)f(w,t)|v-w|G(v,t)^2\,dv\,dw \\
&\quad + 4\iint f(v,t)f(w,t)|v-w|G(w,t)G(v,t)\,dv\,dw \\
&= -4\Big(\frac{1}{2}\iint f(v,t)f(w,t)|v-w|G(v,t)^2\,dv\,dw \\
&\qquad + \frac{1}{2}\iint f(v,t)f(w,t)|v-w|G(w,t)^2\,dv\,dw\Big) \\
&\qquad + 4\iint f(v,t)f(w,t)|v-w|G(w,t)G(v,t)\,dv\,dw \\
&= -2\Big(\iint f(v,t)f(w,t)|v-w|G(v,t)^2\,dv\,dw \\
&\qquad + \iint f(v,t)f(w,t)|v-w|G(w,t)^2\,dv\,dw \\
&\qquad - \iint f(v,t)f(w,t)|v-w|2G(w,t)G(v,t)\,dv\,dw\Big) \\
&= -2\Big(\iint f(v,t)f(w,t)|v-w|(G(v,t)-G(w,t))^2\,dv\,dw\Big) \le 0.
\end{aligned}
$$

It remains to prove (25). By writing $J(t)$ in a symmetric way

$$J(t) = \frac{1}{2}\iint f(v,t)f(w,t)|v-w|(v-w)(G(v,t)-G(w,t))\,dv\,dw$$

and by Cauchy–Schwarz inequality

$$\begin{aligned} J(t) &= \frac{1}{2}\iint f(v,t)f(w,t)|v-w|(v-w)(G(v,t)-G(w,t))\,dv\,dw \\ &\le \frac{1}{2}\Big(\iint f(v,t)f(w,t)|v-w|(v-w)^2\,dv\,dw\Big)^{\frac{1}{2}} \\ &\quad\times\Big(\iint f(v,t)f(w,t)|v-w|(G(v,t)-G(w,t))^2\,dv\,dw\Big)^{\frac{1}{2}}, \end{aligned}$$

we finally get

$$J(t)^2 \le \frac{1}{4}I(t)\Big(-\frac{1}{2}\frac{d}{dt}J(t)\Big) = -\frac{1}{8}I(t)\frac{d}{dt}J(t).$$

In conclusion

$$\frac{d^2}{dt^2}I(t)^{-\frac{1}{3}} = 16I(t)^{-\frac{7}{3}}\Big(J(t)^2 + \frac{1}{8}I(t)\frac{d}{dt}J(t)\Big) \le 0,$$

which shows that $I(t)^{-\frac{1}{3}}$ is a concave function of time, and $\frac{d}{dt}I(t)^{-\frac{1}{3}}$ is a non increasing function. As before, we remark that the functional

$$\frac{d}{dt}I(t)^{-\frac{1}{3}} = 2I(t)^{-\frac{4}{3}}J(t)$$

is dilation invariant. Thus, the value of the functional does not change if at each time t we replace $f(v,t)$ with $g(v,t)$ given by (8)

$$2I(t)^{-\frac{4}{3}}J(t) = 2I(g(v,t))^{-\frac{4}{3}}J(g(v,t)).$$

Since $g(\cdot,t) \rightharpoonup g_\infty$ in the weak convergence of the measures and $I(g(v,t))$, $J(g(v,t))$ are weakly continuous

$$\lim_{t\to+\infty}\frac{d}{dt}I(t)^{-\frac{1}{3}} = \lim_{t\to+\infty} 2I(t)^{-\frac{4}{3}}J(t) = 2I(g_\infty)^{-\frac{4}{3}}J(g_\infty) = \sqrt[3]{2} \tag{26}$$

and this shows (20). □

Remark 5. Since $T(t)^{-\frac{1}{2}}$ is concave and $\lim_{t\to+\infty}\frac{d}{dt}(T(t)^{-\frac{1}{2}}) = 2$ (see (22), (23), (24)), we have proved in particular that for all $t > 0$

$$\frac{d}{dt}(T(t)^{-\frac{1}{2}}) \ge 2. \tag{27}$$

In the same way,

$$\frac{d}{dt}(I(t)^{-\frac{1}{3}}) \geq \sqrt[3]{2}. \tag{28}$$

Theorem 4 enlightens extremal properties of the source-type solutions to nonlinear friction equations, as far as suitable functionals of the solution are concerned. A further consequence of these extremal properties are sharp functional inequalities, which are saturated exactly in the source-type solution. We proved

PROPOSITION 6. *Let* $p > 0$ *and* $\varphi \in \Omega_{2p}$ *with* $\Omega_{2p} = \left\{ \varphi \in L^1(\mathbb{R}) : \varphi \geq 0, \|\varphi\|_{L^1} = 1, \int v^{2p}\varphi(v)\,dv < +\infty \right\}$. *Then, for* $\varphi \in \Omega_3$

$$\left(\int v^2\varphi(v)\,dv \right)^{\frac{3}{2}} \leq \frac{1}{4} \iint |v-w|^3 \varphi(v)\varphi(w)\,dv\,dw. \tag{29}$$

If $\varphi \in \Omega_4$

$$\left(\iint |v-w|^3\varphi(v)\varphi(w)\,dv\,dw \right)^{\frac{4}{3}} \leq 2^{\frac{2}{3}} \int \varphi(v) \left(\int |v-w|(v-w)\varphi(w)\,dw \right)^2 dv. \tag{30}$$

Finally, if $\varphi \in \Omega_5$

$$\left(\int \varphi(v) \left(\int |v-w|(v-w)\varphi(w)\,dw \right)^2 dv \right)^{\frac{5}{4}} \leq \frac{1}{2^{\frac{3}{2}}} \left(\iint \varphi(v)\varphi(w)|v-w|(G(\varphi(v)) - G(\varphi(w)))^2\,dv\,dw \right). \tag{31}$$

PROOF. The proof is a direct consequence of the computations leading to Theorem 4. Indeed, (27) implies, for the solution to equation (5) with intial data φ

$$\frac{1}{2} T(t)^{-\frac{3}{2}} I(t) \geq 2. \tag{32}$$

By using the continuity of $T_{2p}(t)$ for $t \to 0$ of the solution with initial data belonging to Ω_{2p} (see Remark 2) one can prove that $I(t)$ is continuous at the origin and so by letting $t \to 0$ we get that inequality (32) holds true for the initial data. The same argument applies for the second inequality, using (28). Inequalities (30) and (31) can be obtained directly from (29) and from Cauchy–Schwarz inequalities (23) and (25), which in fact hold true in general, and not only for solutions of the nonlinear friction equation (5). Indeed we get from (29) and (23)

$$I(\varphi)^2 \leq 4T(\varphi)J(\varphi) \leq 2^{\frac{2}{3}} I(\varphi)^{\frac{2}{3}} J(\varphi), \tag{33}$$

that is

$$I(\varphi)^{\frac{4}{3}} \le 2^{\frac{2}{3}} J(\varphi)$$

and by (25) and (30)

$$J(\varphi)^2 \le \frac{1}{8} I(\varphi)\Big(-\frac{d}{dt}J(\varphi)\Big) \le \frac{1}{2^{\frac{5}{2}}} J(\varphi)^{\frac{3}{4}}\Big(-\frac{d}{dt}J(\varphi)\Big), \tag{34}$$

namely

$$J(\varphi)^{\frac{5}{4}} \le \frac{1}{2^{\frac{5}{2}}}\Big(-\frac{d}{dt}J(\varphi)\Big). \qquad \square$$

REMARK 7. Inequalities analogous to (29), (30) and (31), with non sharp coefficients could be obtained directly by Cauchy–Schwarz inequality for functions φ satisfying $\varphi \in \Omega_4$, and such that $\displaystyle\int v\varphi(v)\,dv = 0$. Indeed, in this case

$$\int v^2\varphi(v)\,dv = \frac{1}{2}\iint (v-w)^2\varphi(v)\varphi(w)\,dv\,dw$$

which implies

$$\begin{aligned}
&\Big(\frac{1}{2}\iint (v-w)^2\varphi(v)\varphi(w)\,dv\,dw\Big)^{\frac{3}{2}} \\
&\quad = \frac{1}{2^{\frac{3}{2}}}\Big(\iint (v-w)^2\varphi^{\frac{2}{3}}(v)\varphi^{\frac{2}{3}}(w)\varphi^{\frac{1}{3}}(v)\varphi^{\frac{1}{3}}(w)\,dv\,dw\Big)^{\frac{3}{2}} \\
&\quad \le \frac{1}{2^{\frac{3}{2}}}\iint |v-w|^3\varphi(v)\varphi(w)\,dv\,dw.
\end{aligned} \tag{35}$$

Now, by injecting (35) in (33) and (34) instead of (29), we get weaker inequalities.

The sharp inequalities (27) and (28) provide a sharp control of the evolution in time of $T(t)$ and $I(t)$. We collect this time decay into the following.

PROPOSITION 8. *Let f be a solution of the nonlinear friction equation* (5), *with initial data in Ω_{2p} with p large enough. Then, the temperature $T(t)$ and the entropy $I(t)$ defined by* (14, 15) *decay in time, and*

$$\begin{aligned}
T(t) &\le \frac{T(0)}{(1+2tT(0)^{\frac{1}{2}})^2}, \\
I(t) &\le \frac{I(0)}{(1+\sqrt[3]{2}tI(0)^{\frac{1}{3}})^3}.
\end{aligned} \tag{36}$$

PROOF. On $T(t)$ we have

$$\frac{d}{dt}T(t)^{-\frac{1}{2}} = -\frac{1}{2}T(t)^{-\frac{3}{2}}\frac{d}{dt}T(t) = \frac{1}{2}T(t)^{-\frac{3}{2}}I(t) \geq 2 \Leftrightarrow \frac{d}{dt}T(t) \leq -4T(t)^{\frac{3}{2}}$$

which implies

$$\begin{aligned} -2\Big(\frac{1}{T(t)^{\frac{1}{2}}} - \frac{1}{T(0)^{\frac{1}{2}}}\Big) &\leq -4t \\ \Leftrightarrow \frac{1}{T(t)^{\frac{1}{2}}} &\geq \frac{1}{T(0)^{\frac{1}{2}}} + 2t = \frac{1+2tT(0)^{\frac{1}{2}}}{T(0)^{\frac{1}{2}}} \\ \Leftrightarrow T(t)^{\frac{1}{2}} &\leq \frac{T(0)^{\frac{1}{2}}}{1+2tT(0)^{\frac{1}{2}}} \\ \Leftrightarrow T(t) &\leq \frac{T(0)}{(1+2tT(0)^{\frac{1}{2}})^2}. \end{aligned}$$

A weaker estimate had been already obtained by [2] in Theorem 1 through the classical Cauchy–Schwarz inequality. On $I(t)$ in an analogous way,

$$\begin{aligned} \frac{d}{dt}I(t)^{-\frac{1}{3}} = -\frac{1}{3}I(t)^{-\frac{4}{3}}\frac{d}{dt}I(t) \geq \sqrt[3]{2} \Leftrightarrow -\frac{d}{dt}I(t) \geq 3\sqrt[3]{2}I(t)^{\frac{4}{3}} \\ \Leftrightarrow \frac{d}{dt}I(t) \leq -3\sqrt[3]{2}I(t)^{\frac{4}{3}} \end{aligned}$$

which implies

$$\begin{aligned} -\Big(\frac{1}{I(t)^{\frac{1}{3}}} - \frac{1}{I(0)^{\frac{1}{3}}}\Big) &\leq -\sqrt[3]{2}t \\ \Leftrightarrow \frac{1}{I(t)^{\frac{1}{3}}} &\geq \frac{1}{I(0)^{\frac{1}{3}}} + \sqrt[3]{2}t = \frac{1+\sqrt[3]{2}tI(0)^{\frac{1}{3}}}{I(0)^{\frac{1}{3}}} \\ \Leftrightarrow I(t)^{\frac{1}{3}} &\leq \frac{I(0)^{\frac{1}{3}}}{1+\sqrt[3]{2}tI(0)^{\frac{1}{3}}} \\ \Leftrightarrow I(t) &\leq \frac{I(0)}{(1+\sqrt[3]{2}tI(0)^{\frac{1}{3}})^3}. \end{aligned}$$

□

4. SOURCE-TYPE SOLUTIONS COOL SLOWLY

Let us consider the solution of the nonlinear friction equation (5) with initial data $f_0 \in \Omega_2$ satisfying $\int v^2 \log(1+|v|) f_0(v)\,dv < \infty$ and let us denote $T(0) =$

$\int v^2 f_0(v)\,dv$ its initial temperature. If $f_\infty(v,t) = \frac{1}{2}\delta\big(v + \frac{1}{2t}\big) + \frac{1}{2}\delta\big(v - \frac{1}{2t}\big)$ is the self similar solution defined in (13), we can denote by A a positive time such that

$$\int v^2 f_\infty(v,A)\,dv = T(0) \tag{37}$$

which means

$$A = \frac{1}{2T(0)^{\frac{1}{2}}}. \tag{38}$$

If we consider now the solution $f_A(v,t)$ with the self similar profile $f_\infty(v,A)$ as initial data, by uniqueness it is

$$f_A(v,t) = \frac{1}{2}\delta\Big(v - \frac{1}{2(t+A)}\Big) + \frac{1}{2}\delta\Big(v + \frac{1}{2(t+A)}\Big) = f_\infty(v, t+A) = f_{\infty,A}(v,t)$$

and therefore

$$T(f_A(t)) = \frac{T(g_\infty)}{(A+t)^2} = \frac{1}{4(t+A)^2} \tag{39}$$

for any positive t. Of course the two solutions $f(t)$ and $f_A(t)$ share the same initial temperature

$$T(0) = T(f_A(0)).$$

Let us consider

$$F(t) = T(t)^{-\frac{1}{2}} - T(f_A(t))^{-\frac{1}{2}}.$$

We have

$$T(f_A(t))^{-\frac{1}{2}} = 2t + T(0)^{-\frac{1}{2}}.$$

Since $\frac{d}{dt}(T(t)^{-\frac{1}{2}}) \geq 2$ (see (27)) and $F(0) = 0$, then $F(t) \geq 0$ for all $t \geq 0$. This implies

$$T(t) \leq T(f_A(t)), \quad t \geq 0.$$

Therefore, the temperature at each instant t of the generic solution $f(t)$ starting with initial temperature $T(0)$ does not match the temperature of the self similar solution starting with the same temperature at the same instant t, but at an instant $t+d$, where the delay d can be found by letting

$$T(f_A(t+d)) = T(t)$$

and so by (39)

$$\frac{1}{4(t+A+d)^2} = T(t) \Leftrightarrow t+A+d = \frac{1}{2}T(t)^{-\frac{1}{2}} \Leftrightarrow d = \frac{1}{2}T(t)^{-\frac{1}{2}} - (t+A).$$

So, the cooling time of the generic solution is bounded from above by the cooling time of the self similar solution starting with the same temperature. In other words, for a fixed initial temperature, the self similar solution has the slowest cooling time.

5. Conclusions

The finding of the exact rate of cooling of any solution to the dissipative Boltzmann equation for a granular gas is one of the relevant unsolved physical problems in dissipative gas dynamics. Among simplified models, the nonlinear friction equation introduced by McNamara and Young in [9, 10] allows for precise results in this direction. In this paper, we showed that the temperature of the self-similar solution (the homogeneous cooling state) has an explicit time decay which is slower than the time decay of any other solution which starts with the same temperature at time $t = 0$. This result, whenever true for the full Boltzmann equation, would be physically relevant, in that it indicates that the homogeneous cooling state exhibits a maximal cooling time with respect to any other solution starting with the same energy.

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6. Appendix

We are going to prove that $T(g(v,t))$, $I(g(v,t))$ and $J(g(v,t))$ are weakly continuous.

Proposition 9. *Let $g(v,t) = \frac{1}{t} f\left(\frac{v}{t}, t\right)$ where $f(v,t)$ is the solution of the Cauchy problem* (5)*. Then, $g(v,t)$ satisfies*

$$\begin{aligned} &\lim_{t\to+\infty} \int y^{2p} g(y,t)\,dy = \int y^{2p} g_\infty(y)\,dy, \quad p \geq 0 \\ &\lim_{t\to+\infty} I(g(v,t)) = I(g_\infty) \\ &\lim_{t\to+\infty} J(g(v,t)) = J(g_\infty). \end{aligned} \tag{40}$$

PROOF. Let us begin by proving the convergence of the moments T_{2p}. Since for (12) $g(t) \rightharpoonup g_\infty$, for all test function $\varphi \in C_c(\mathbb{R})$ we have

$$\lim_{t\to+\infty} \int \varphi(y) g(t, y)\, dy = \int \varphi(y) g_\infty(y)\, dy = \frac{1}{2}\varphi\Big(\frac{1}{2}\Big) + \frac{1}{2}\varphi\Big(-\frac{1}{2}\Big).$$

Since $g(v, t)$ has compact support K contained in $[-1, 1]$, for all $t > 0$ we can write for $\varphi \in C_c(\mathbb{R})$ satisfying $\varphi(y) = 1$ for $y \in K$

$$\begin{aligned}\lim_{t\to+\infty} \int y^{2p} g(t, y)\, dy &= \lim_{t\to+\infty} \int y^{2p} \varphi(y) g(t, y)\, dy \\ &= \int y^{2p}\varphi(y) g_\infty(y)\, dy = \int y^{2p} g_\infty(y)\, dy.\end{aligned}$$

In order to prove the convergence of the entropy $I(g(v, t))$ to $I(g_\infty)$, we have to control

$$\begin{aligned}&\iint |x - y|^3 (g(x, t) g(y, t) - g_\infty(x) g_\infty(y))\, dx\, dy \\ &= \iint |x - y|^3 (g(x, t) - g_\infty(x)) g(y, t)\, dx\, dy \\ &\quad + \iint |x - y|^3 (g(y, t) - g_\infty(y)) g_\infty(x)\, dx\, dy \\ &= A(t) + B(t).\end{aligned}$$

Let us call

$$h(y, t) = \int |x - y|^3 (g(x, t) - g_\infty(x))\, dx$$

and so

$$\begin{aligned}A(t) &= \int g(y, t) h(y, t)\, dy, \\ B(t) &= \int g_\infty(y) h(y, t)\, dy = \frac{1}{2} h\Big(\frac{1}{2}, t\Big) + \frac{1}{2} h\Big(-\frac{1}{2}, t\Big).\end{aligned}$$

(in the second integral we have exploited the symmetry between x and y). Since $g(\cdot, t)$ has compact support $K \subset [-1, 1]$, uniformly in t, and $g(t) \rightharpoonup g_\infty$, it is easy to show that for all $y \in \mathbb{R}$ we have $h(y, t) \to 0$ for $t \to +\infty$ and so $B(t) \to 0$. As for $A(t)$, since $g(t)$ is a density for all t and has compact support K we can bound

$$|A(t)| \le \|h(t)\|_{L^\infty(K)} \int g(y, t)\, dy = \|h(t)\|_{L^\infty(K)}.$$

Now, for all $t > 0$ one can consider

$$h(t, \cdot) : K \subset \mathbb{R} \to \mathbb{R}.$$

Due to the uniform bound on the moments of $g(v, t)$ (see Theorem 1), the set $\{h(t, \cdot)\}_{t>0}$ is a bounded set of equicontinuous functions on K. So for Ascoli Arzelà theorem it is relatively compact in $L^\infty(K)$ and since $\lim_{t\to+\infty} h(t, y) = 0$ for all y it follows that there exists a sequence $t_n \to +\infty$ such that

$$\lim_{t_n\to+\infty} \|h(t_n)\|_{L^\infty(K)} = 0.$$

Therefore $I(g(v, t_n)) \to I(g_\infty)$ for a sequence $t_n \to +\infty$. By a standard argument this implies that $I(g(v, t)) \to I(g_\infty)$ for $t \to +\infty$. The proof of the convergence of $J(g(v, t))$ is completely analogous. □

References

[1] G. I. Barenblatt, *Scaling, Self-similarity, and Intermediate Asymptotics*, Cambridge Univ. Press, NewYork, 1996.

[2] D. Benedetto - E. Caglioti - M. Pulvirenti, *A kinetic equation for granular media*, Mat. Mod. Numer. Anal. 31 (1997), 615–641.

[3] D. Benedetto - E. Caglioti - J. A. Carrillo - M. Pulvirenti, *A non Maxwellian distribution for one-dimensional granular media*, J. Statist. Phys. 91 (1998), 979–990.

[4] E. Caglioti - C. Villani, *Homogeneous cooling states are not always good approximation to granular flows*, Arch. Rational Mech. Anal. 163 (2002), 329–343.

[5] J. A. Carrillo - R. J. McCann - C. Villani, *Kinetic equilibration rates for granular media and related equations: entropy dissipation and mass transportation estimates*, Revista Mat. Iberoamericana 19 (2003), 1–48.

[6] J. Dolbeault - G. Toscani, *Best matching Barenblatt profiles are delayed*, Journal of Physics A: Math. Theor. 48 (2015).

[7] J. Dolbeault - G. Toscani, *Nonlinear diffusions: extremal properties of Barenblatt profiles, best matching and delays*, arXiv:1501.03646 (2015).

[8] H. Li - G. Toscani, *Long-time asymptotics of kinetic models of granular flows*, Arch. Rational Mech. Anal. 172 (2004), 407–428.

[9] S. McNamara - W. R. Young, *Inelastic collapse and clumpining a one-dimensional granular medium*, Phys. Fluids A 4 (1992), 496–504.

[10] S. McNamara - W. R. Young, *Kinetics of a one-dimensional granular medium in the quasi-elastic limit*, Phys. Fluids A 5 (1993), 34–45.

[11] G. Toscani, *One-dimensional kinetic models of granular flows*, RAIRO Modél. Math. Anal. Numér. 34 (2000), 1277–1292.

[12] J. L. Vázquez, *The porous medium equation*, Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford (2007).

[13] C. Villani, *Mathematics of granular materials*, J. Statist. Phys. no. 2–4 (2006), 781–822.

[14] Y. B. Zel'dovich - A. Kompaneets, *Towards a theory of heat conduction with thermal conductivity depending on the temperature*, Collection of papers dedicated to 70th birthday of Academician AF Ioffe, Izd. Akad. Nauk SSSR, Moscow pp. 61–71 (1950).

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