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Functional Analysis — *Erratum on "Spectral analysis and long-time behaviour of a Fokker–Planck equation with a non-local perturbation"*, by DOMINIK STÜRZER and ANTON ARNOLD, communicated on 13 November 2015.[1]

KEY WORDS: Fokker-Planck, non-local perturbation, spectral analysis, exponential stability

MATHEMATICS SUBJECT CLASSIFICATION: 35B20, 35P99, 35Q84, 47D06

In Section 4 of the article [2] the choice of the weight function $\omega(\mathbf{x})$ is incorrect. Therefore the norms $\|\cdot\|_\omega$ and $|||\cdot|||_\omega$ are not equivalent. This can be resolved by replacing $\omega(\mathbf{x})$ by the new weight function

$$\omega(\mathbf{x}) := \sum_{\ell=1}^{d} \cosh \beta x_\ell,$$

with $\beta > 0$. Now $\|\cdot\|_\omega$ and $|||\cdot|||_\omega$ are equivalent norms in $\mathscr{E} = L^2(\omega)$.

Except for the Poincaré inequality (Lemma 4.2) all results of Section 4 remain true. However, Lemma 4.2 is not correct for the new weight. A counterexample on $\mathbb{R}^2$ is the sequence $\{f_n\}_{n\in\mathbb{N}} := \left\{\exp\left(-\left(\frac{x_1}{n}\right)^2 - (x_2 - n^2)^2\right)\right\}_{n\in\mathbb{N}} \subset \mathscr{E}$, for $\mathbf{k} = [1, 0]^\top$. Hence, Lemma 4.2 needs to be replaced by:

LEMMA 4.2. *There exists a constant $C > 0$ such that for all $f \in \mathscr{E}$ with $|\nabla f| \in \mathscr{E}$ there holds:*

$$\|f\|_\omega \le C\|\nabla f\|_\omega.$$

PROOF. We use the norm $|||\cdot|||_\omega$, and compute

$$|||\nabla f|||_\omega^2 = \sum_{j,\ell=1}^{d} \left(\left\| \left(\xi_j + \mathrm{i}\frac{\beta}{2}\delta_{j\ell}\right) \hat{f}\left(\boldsymbol{\xi} + \mathrm{i}\frac{\beta}{2}\mathbf{e}_\ell\right) \right\|_{L^2(\mathbb{R}^d_\xi)}^2 + \left\| \left(\xi_j - \mathrm{i}\frac{\beta}{2}\delta_{j\ell}\right) \hat{f}\left(\boldsymbol{\xi} - \mathrm{i}\frac{\beta}{2}\mathbf{e}_\ell\right) \right\|_{L^2(\mathbb{R}^d_\xi)}^2 \right)$$

[1] This note is the erratum of the paper [2] in the references.

$$\geq \sum_{j=1}^{d} \left(\left\| \left(\xi_j + \mathrm{i}\frac{\beta}{2}\right) \hat{f}\left(\xi + \mathrm{i}\frac{\beta}{2}\mathbf{e}_j\right) \right\|^2_{L^2(\mathbb{R}^d_\xi)} + \left\| \left(\xi_j - \mathrm{i}\frac{\beta}{2}\right) \hat{f}\left(\xi - \mathrm{i}\frac{\beta}{2}\mathbf{e}_j\right) \right\|^2_{L^2(\mathbb{R}^d_\xi)} \right)$$

$$\geq \left(\frac{\beta}{2}\right)^2 |||f|||^2_\omega,$$

where we used $\left|\xi_j + \mathrm{i}\frac{\beta}{2}\right| \geq \frac{\beta}{2}$ in the last step. □

Most proofs of §4 are not affected by this change of $\omega(\mathbf{x})$, only the proof of the decay estimates of $(\mathrm{e}^{t\mathscr{L}})_{t\geq 0}$ on the spaces $\mathscr{E}_k$ needs to be done differently. Actually, the Poincaré inequality is not needed for this. We still estimate $|||\mathrm{e}^{t\mathscr{L}} f|||_\omega$, using the following representation from [2]:

$$\mathscr{F}_{\mathbf{x}\to\xi}[\mathrm{e}^{t\mathscr{L}}] = \exp\Big(-\frac{\xi\cdot\xi}{2}(1-\mathrm{e}^{-2t})\Big)\hat{f}(\xi\mathrm{e}^{-t}), \quad t \geq 0.$$

But instead of considering the terms

$$\frac{\hat{f}(\xi)}{\xi^{\mathbf{k}}}\xi^{\mathbf{k}}, \quad \text{with } |\mathbf{k}| = k$$

in the norm $|||\cdot|||_\omega$ (similar to the proof of Proposition 2.17), we use that $\hat{f}(\xi) = \mathcal{O}(|\xi|^k)$ locally around the origin for $f \in \mathscr{E}_k$. We then use the Taylor expansion of $\hat{f}$ around the origin with remainder in Lagrange form: For every $\xi \in \Omega_{\beta/2}$ there holds

$$\hat{f}(\xi) = \sum_{|\mathbf{k}|=k} \frac{1}{\mathbf{k}!}\xi^{\mathbf{k}}(\nabla^{\mathbf{k}}_\xi \hat{f})(\kappa\xi), \quad \text{for some } \kappa \in [0,1].$$

For every $0 < \beta' < \beta$ we can show that, for every $\mathbf{k} \in \mathbb{N}^d$, the derivative $\nabla^{\mathbf{k}}\hat{f}(\xi)$ can be uniformly bounded in $\Omega_{\beta'/2}$ by $|||f|||_\omega$. Hence, from the Taylor expansion we obtain

$$|\hat{f}(\mathbf{z})| \leq C|\mathbf{z}|^k |||f|||_\omega, \quad \forall \mathbf{z} \in \Omega_{\beta'/2}. \tag{4.1}$$

We now use this to estimate the integrals in $|||\mathrm{e}^{t\mathscr{L}} f|||^2_\omega$, for all $t \geq 1$ and $\ell \in \{1, \ldots, d\}$:

$$\left\| \mathscr{F}[\mathrm{e}^{t\mathscr{L}}]\left(\xi \pm \mathrm{i}\frac{\beta}{2}\mathbf{e}_\ell\right) \right\|^2_{L^2(\mathbb{R}^d_\xi)} \leq \int_{\mathbb{R}^d} \mathrm{e}^{-\gamma\xi\cdot\xi} \left| \hat{f}\left(\mathrm{e}^{-t}\left(\xi \pm \mathrm{i}\frac{\beta}{2}\mathbf{e}_\ell\right)\right) \right|^2 \mathrm{d}\xi$$

$$\leq C|||f|||^2_\omega \int_{\mathbb{R}^d} \mathrm{e}^{-\gamma\xi\cdot\xi} \left| \mathrm{e}^{-t}\left(\xi \pm \mathrm{i}\frac{\beta}{2}\mathbf{e}_\ell\right) \right|^{2k} \mathrm{d}\xi$$

$$\leq C'\mathrm{e}^{-2kt}|||f|||^2_\omega,$$

where $\gamma = 1 - e^{-2}$. Extensions to a wider class of Fokker–Planck equations and a more detailed proof are given in the follow-up paper [1].

References

[1] F. Achleitner - A. Arnold - D. Stürzer, *Large-time behavior in non-symmetric Fokker–Planck equations*, To appear in: Riv. Math. Univ. Parma (N.S.), (2015).

[2] D. Stürzer - A. Arnold, *Spectral analysis and long-time behaviour of a Fokker–Planck equation with a non-local perturbation*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 25, 1 (2014), 53–89.

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