Rend. Lincei Mat. Appl. 27 (2016), 117–125
DOI 10.4171/RLM/726

Calculus of Variations — *A new nonautonomous chain rule in BV*, by Virginia De Cicco, communicated on 13 November 2015.[1]

Abstract. — The aim of this note is to present a new nonautonomous chain rule formula for the distributional derivative of the composite function $v(x) = B(x, u(x))$, where $u : \mathbb{R}^N \to \mathbb{R}$ is a scalar function of bounded variation and B admits a special integral form in terms of a locally bounded function $b(x, t)$, with $b(\cdot, t)$ of bounded variation. It is an useful tool especially in view to applications to semicontinuity results for integral functional (see [1, 8, 9, 10]) and to conservation laws (see [5, 6]).

Key words: Chain rule, BV functions, lower semicontinuity

Mathematics Subject Classification: 49J45, 49Q20

1. Introduction

In this note we present a new nonautonomous chain rule formula in the scalar case for the distributional derivative of the composite function $v(x) = B(x, u(x))$, with $u : \mathbb{R}^N \to \mathbb{R}$ a scalar function of bounded variation and $B(x, t) = \int_0^t b(x, s)\, ds$, where $b(x, t)$ is locally bounded (which implies that $B(x, \cdot)$ is Lipschitz continuous) and $b(\cdot, t)$ has bounded variation.

In 1967, A. I. Vol'pert in [13] considers a general B in the autonomous case and by requiring the Lipschitz continuity of B, proved that the following identity holds in the sense of measures:

$$Dv = \nabla B(u)\nabla u \mathscr{L}^N + \nabla B(\tilde{u}) D^c u + [B(u^+) - B(u^-)]\nu_u \mathscr{H}^{N-1} \llcorner J_u, \tag{1}$$

where

$$Du = \nabla u \mathscr{L}^N + D^c u + \nu_u \mathscr{H}^{N-1} \llcorner J_u \tag{2}$$

is the usual decomposition of Du in its absolutely continuous part ∇u with respect to the Lebesgue measure $\mathscr{L}^N$, its Cantor part $D^c u$ and its jumping part, which is represented by the restriction of the $(N-1)$-dimensional Hausdorff measure to the jump set J_u. Moreover, ν_u denotes the measure theoretical unit normal to J_u, $\tilde{u}$ is the approximate limit and u^+, u^- are the approximate limits from both sides of J_u.

[1] This paper is related to a talk given by the author at "XXV Convegno Nazionale di Calcolo delle Variazioni"—Levico, 2–6 February 2015.

The validity of (1) is stated also in the vectorial case (see Theorem 3.96 in [4] for $B \in C^1$). The situation is significantly more complicated if B is only a Lipschitz continuous function. In this case, the general chain rule is false, while a weaker form of the formula was proved by Ambrosio and Dal Maso in [3] (see also [12]).

On the other hand, in some recent papers a remarkable effort is devoted to establish chain rule formulas with an explicit dependence on the space variable x (see [1, 5, 8, 9, 10]). Notice that the new term of derivation with respect to x needs a particular attention. The proofs are achieved by regularizing $B(\cdot, t)$ with fixed t, by applying the Ambrosio–Dal Maso formula to the regularized functions and finally by passing to the limit in each term.

More recently, a very general nonautonomous formula is proven in [2] for vector functions $u \in BV$. Here, the first assumption is a C^1 dependence of $B(x, \cdot)$ with an uniform bound on $\partial_t B(x, t)$. Concerning the x-derivative, it is required the existence of a Radon measure σ bounding from above all measures $|D_x B(\cdot, t)|$, uniformly with respect to $t \in \mathbb{R}$.

The aim of this note is to consider the special case of

$$B(x, t) = \int_0^t b(x, s)\, ds.$$

In the spirit of Theorem 3.1 below proved in [9] we find a chain rule in this situation. We assume that b is BV in x and it is locally bounded (then $B(\cdot, t)$ is BV and $B(x, \cdot)$ is Lipschitz continuous) and we find an explicit form for the term involving the x-derivation, which is described in [9] by a Fubini's type inversion of integration order.

In the spirit of [2] we require the existence of a Radon measure $\bar{\sigma}$ bounding from above all measures $|D_x b(\cdot, t)|$, uniformly with respect to $t \in \mathbb{R}$. We prove that for any $u \in BV_{\mathrm{loc}}$ the composite function $v(x) = B(x, u(x))$ belongs to BV_{loc} and it is shown the existence of a countably $\mathscr{H}^{N-1}$-rectifiable set $\mathscr{N}$, independent of u and containing the jump set of $B(\cdot, t)$ for every $t \in \mathbb{R}$, such that the jump set of v is contained in $\mathscr{N} \cup J_u$. A chain rule is obtained (see Theorem 4.2) by requiring further uniformity conditions, but without assuming any continuity assumptions. The result here presented will be proven in a forthcoming paper.

2. Definitions and preliminaries

In this section we recall some preliminary results and basic definitions (see [4] and [11]).

Let E be a measurable subset of $\mathbb{R}^N$. The *density* $D(E; x)$ of E at a point $x \in \mathbb{R}^N$ is defined by

$$D(E; x) = \lim_{\varrho \to 0} \frac{\mathscr{L}^N(E \cap B_\rho(x))}{\omega_N \rho^N},$$

whenever this limit exists, where ω_N is the measure of the unit ball and $B_\rho(x)$ denotes the ball centered at x with radius ρ.

Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $u : \Omega \to \mathbb{R}$ be a measurable function. The *upper and lower approximate limits* of u at a point $x \in \Omega$ are defined as

$$u^+(x) = \inf\{t \in \mathbb{R} : D(\{u > t\}; x) = 0\},\\ u^-(x) = \sup\{t \in \mathbb{R} : D(\{u < t\}; x) = 0\}, \tag{3}$$

respectively. The quantities $u^+(x)$, $u^-(x)$ are well defined (possibly equal to $\pm\infty$) at every $x \in \Omega$, and $u^-(x) \le u^+(x)$. The functions $u^+, u^- : \Omega \to [-\infty, \infty]$ are Borel measurable.

We say that u is *approximately continuous* at a point $x \in \Omega$ if $u^+(x) = u^-(x) \in \mathbb{R}$. In this case, we set $\tilde{u}(x) = u^+(x) = u^-(x)$ and call $\tilde{u}(x)$ the *approximate limit* of u at x. The set of all points in Ω where u is approximately continuous is a Borel set which will be denoted by C_u and called the set of *approximate continuity* of u. The set $S_u = \Omega \backslash C_u$ will be referred to as the set of *approximate discontinuity* of u.

Finally, by u^* we denote the *precise representative* of u which is defined by

$$u^*(x) = \frac{u^+(x) + u^-(x)}{2}$$

if $u^+(x), u^-(x) \in \mathbb{R}$, $u^*(x) = 0$ otherwise.

A locally integrable function u is said to be *approximately differentiable* at a point $x \in C_u$ if there exists $\nabla u(x) \in \mathbb{R}^N$ such that

$$\lim_{\rho \to 0} \frac{1}{\rho^{N+1}} \int_{B_\rho(x)} |u(y) - \tilde{u}(x) - \langle \nabla u(x), y - x \rangle| \, dy = 0. \tag{4}$$

Here, $\langle \cdot, \cdot \rangle$ stands for scalar product in $\mathbb{R}^N$. The vector $\nabla u(x)$ is called the *approximate differential* of u at x.

A function $u \in L^1(\Omega)$ is said to be of *bounded variation* if its distributional gradient Du is an $\mathbb{R}^N$-valued Radon measure in Ω and the total variation $|Du|$ of Du is finite in Ω. The space of all functions of bounded variation in Ω is denoted by $BV(\Omega)$, while the notation $BV_{\mathrm{loc}}(\Omega)$ will be reserved for the space of those functions $u \in L^1_{\mathrm{loc}}(\Omega)$ such that $u \in BV(\Omega')$ for every open set $\Omega' \subset\subset \Omega$.

Let $u \in BV(\Omega)$. Then it can be proved that

$$\lim_{\rho \to 0} \fint_{B_\rho(x)} |u(y) - \tilde{u}(x)| \, dy = 0 \quad \text{for } \mathscr{H}^{N-1}\text{-a.e. } x \in C_u$$

and that u is approximately differentiable for $\mathscr{L}^N$-a.e. x. Moreover, the functions u^- and u^+ are finite $\mathscr{H}^{N-1}$-a.e. and for $\mathscr{H}^{N-1}$-a.e. $x \in S_u$ there exists a unit vector $\nu_u(x)$ such that

$$\lim_{\rho \to 0} \fint_{B_\rho^\pm(x; \nu_u(x))} |u(y) - u^\pm(x)| \, dy = 0, \tag{5}$$

where $B_\rho^+(x;\nu_u(x)) = \{y \in B_\rho(x) : \langle y - x, \nu_u(x)\rangle > 0\}$, and $B_\varrho^-(x;\nu_u(x))$ is defined analogously. The set of all points in S_u where the equalities in (5) are satisfied is called the *jump set* of u and is denoted by J_u.

If u is a BV function, we denote by $D^a u$ the absolutely continuous part of Du with respect to Lebesgue measure. The singular part, denoted by $D^s u$, is split into two more parts, the *jump part* $D^j u$ and the *Cantor part* $D^c u$, defined by

$$D^j u = D^s u \llcorner J_u, \quad D^c u = D^s u - D^j u.$$

Finally, we denote by $\tilde{D}u$ the *diffuse part* of Du, defined by

$$\tilde{D}u = D^a u + D^c u.$$

3. The chain rule in $BV(\mathbb{R}^N)$ proven in [9]

In the paper [9] the authors deal with a general chain rule formula in $BV(\mathbb{R}^N)$ for functions whose dependence in x is BV. More precisely, the following theorem is proved for particular functions of the type $B(x,t) = \int_0^t b(x,s)\,ds$.

Theorem 3.1. *Let $b : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a Borel function. Assume that*

(α) *the function $b(x,t)$ is locally bounded;*
(β) *for every $t \in \mathbb{R}$ the function $b(\cdot,t) \in BV(\mathbb{R}^N)$;*
(γ) *for any compact set $H \subset \mathbb{R}$,*

$$\int_H |D_x b(\cdot,t)|(\mathbb{R}^N)\,dt < +\infty,$$

where $D_x b(\cdot,t)$ is the distributional gradient of the map $x \mapsto b(x,t)$.

Then for every $u \in BV(\mathbb{R}^N) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$, the function $v : \mathbb{R}^N \to \mathbb{R}$, defined by

$$v(x) := \int_0^{u(x)} b(x,t)\,dt,$$

belongs to $BV_{\mathrm{loc}}(\mathbb{R}^N)$ and for any $\phi \in C_0^1(\mathbb{R}^N)$ we have

$$\begin{aligned}(6)\quad \int_{\mathbb{R}^N} \nabla\phi(x) v(x)\,dx = &-\int_{-\infty}^{+\infty} dt \int_{\mathbb{R}^N} \operatorname{sgn}(t)\chi^*_{\Omega_{u,t}}(x)\phi(x)\,dD_x b(x,t)\\ &- \int_{\mathbb{R}^N} \phi(x) b(x,u(x))\nabla u(x)\,dx - \int_{\mathbb{R}^N} \phi(x)\tilde{b}(x,\tilde{u}(x))\,dD^c u\\ &- \int_{J_u} \phi(x)\left[\int_{u^-(x)}^{u^+(x)} b^*(x,t)\,dt\right]\nu_u(x)\,d\mathcal{H}^{N-1},\end{aligned}$$

where J_u is the jump set of u, $\Omega_{u,t} = \{x \in \mathbb{R}^N : t$ belongs to the segment of end-points 0 and $u(x)\}$ and $\chi^*_{\Omega_{u,t}}$ and $b^*(\cdot, t)$ are, respectively, the precise representatives of $\chi_{\Omega_{u,t}}$ and $b(\cdot, t)$.

Remark 3.2. Notice that $b^*(x,t) = (b^+(x,t) + b^-(x,t))/2$, where $b^+(x,t)$ and $b^-(x,t)$ are the upper and lower approximate limits of $b(\cdot,t)$ at a point x. The function $b(\cdot,t)$ is approximately continuous at a point x if $b^+(\cdot,t) = b^-(\cdot,t) \in \mathbb{R}$. In this case, we set $\tilde{b}(\cdot,t) = b^+(\cdot,t) = b^-(\cdot,t)$. By Lemma 3.1 in [9] the functions $\tilde{b}(x,t)$, $b^+(x,t)$, $b^-(x,t)$ and $b^*(x,t)$ are locally bounded Borel functions. Moreover, if $b(x,t) \equiv b(t)$, then (6) reduces to the well known chain rule formula for the composition of BV functions with a Lipschitz function, while, in the special case that $b(x,t) \equiv b(x)$, (6) gives the formula for the derivative of the product of two BV functions.

4. An explicit chain rule

In this section we will present the result and we will write more explicitly the first term appearing in the right hand side of formula (6).

Let $b : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a Borel function. Assume that

(i) the function $b(x,t)$ is locally bounded;
(ii) for every $t \in \mathbb{R}$ the function $b(\cdot,t) \in BV(\mathbb{R}^N)$;
(iii) the measure

$$\bar{\sigma} := \bigvee_{t\in\mathbb{R}} |D_x b(\cdot,t)|$$

is a Radon measure, where $\bigvee$ denotes the least upper bound in the space of nonnegative Borel measures.

Remark 4.1. As in Remark 3.5 in [2], since we will consider $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$, condition (iii) can be replaced by the following local version

(iii)$_{\mathrm{loc}}$ for every compact set $H \subset \mathbb{R}$ the measure

$$\bar{\sigma}_H := \bigvee_{t\in H} |D_x b(\cdot,t)|$$

is a Radon measure.

For simplicity we will omit the explicit dependence of $\bar{\sigma}$ on H. By (iii), we have that $\bar{\sigma} \ll \mathscr{H}^{N-1}$ and, if we define

$$\mathcal{N} = \left\{x \in \mathbb{R}^N : \liminf_{r\to 0} \frac{\bar{\sigma}(B_r(x))}{r^{N-1}} > 0\right\},$$

then $\bar{\mathcal{N}}$ is a $\mathcal{H}^{N-1}$-rectifiable set. We omit the dependence of $\bar{\mathcal{N}}$ of H in the local version (see Remark 3.5 in [2]).

Moreover we consider the following assumptions:

(iv) there exists a Borel set $\mathcal{N}_0 \subset \mathbb{R}^N$ with $\mathcal{L}^N(\mathcal{N}_0) = 0$ such that the approximate differential $\nabla_x b(x,t)$ of the function $y \mapsto b(y,t)$ at x exists for every $x \in \mathbb{R}^N \backslash \mathcal{N}_0$ and for every $t \in \mathbb{R}$ and

$$\frac{dD_x b(\cdot,t)}{d\mathcal{L}^N}(x) = \nabla_x b(x,t)$$

for every $x \in \mathbb{R}^N \backslash \mathcal{N}_0$ and for every $t \in \mathbb{R}$;

(v) there exists a Borel set $\mathcal{N}_1 \subseteq \mathbb{R}^N$ with $\bar{\sigma}(\mathcal{N}_1) = 0$ such that the following limit

$$\lim_{r\downarrow 0} \frac{D_x^c b(\cdot,t)(B_r(x))}{\bar{\sigma}(B_r(x))} = \frac{dD_x^c b(\cdot,t)}{d\bar{\sigma}}(x)$$

exists for every $x \in \mathbb{R}^N \backslash \mathcal{N}_1$ and for every $t \in \mathbb{R}$ and this equality holds, where $\frac{dD_x^c b(\cdot,t)}{d\bar{\sigma}}(x)$ is Radon–Nikodým derivative at x of the Cantor part of the measure $D_x b(\cdot,t)$ w.r.t. $\bar{\sigma}$;

(vi) there exists a Borel set $\mathcal{N}_2 \subset \mathbb{R}^N$ with $\mathcal{H}^{N-1}(\mathcal{N}_2) = 0$ such that the one-sided limits $b^+(x,t)$ and $b^-(x,t)$ defined by

$$\lim_{r\downarrow 0} \fint_{B_r^\pm(x)} |b(y,t) - b^\pm(x,t)|\, dy = 0$$

exist for every $x \in \mathbb{R}^N \backslash \mathcal{N}_2$ and for every $t \in \mathbb{R}$, where $B_r^\pm(x)$ are the two half balls determined by the normal $\nu_{\bar{\mathcal{N}}}$, and

$$\frac{dD_x^j b(\cdot,t)}{d\mathcal{H}^{N-1}}(x) = [b^+(x,t) - b^-(x,t)]\nu_{\bar{\mathcal{N}}}(x)$$

for every $x \in \mathbb{R}^N \backslash \mathcal{N}_2$ and for every $t \in \mathbb{R}$.

By (vi) the functions $b^\pm : (\mathbb{R}^N \backslash \mathcal{N}_2) \times \mathbb{R} \to \mathbb{R}$ are locally bounded Borel functions.

Moreover for all $x \in \mathbb{R}^N \backslash (\bar{\mathcal{N}} \cup \mathcal{N}_2)$ and $t \in \mathbb{R}$ there exists the limit

$$\tilde{b}(x,t) = \lim_{r\to 0} \fint_{B_r(x)} b(y,t)\, dy.$$

For all $x \in \mathbb{R}^N \backslash (\bar{\mathcal{N}} \cup \mathcal{N}_2)$ the function $t \mapsto \tilde{b}(x,t)$ is a locally bounded Borel functions. If assumptions (i)–(vi) hold, then for every $t \in \mathbb{R}$ the following decomposition formula holds

$$(D_x b)(\cdot,t) = (\nabla_x b)(x,t)\mathscr{L}^N + \frac{dD_x^c b(\cdot,t)}{d\bar\sigma}(x)\bar\sigma + [b^+(x,t) - b^-(x,t)]\nu_{\bar{\mathcal{N}}}(x)\mathscr{H}^{N-1} \llcorner \bar{\mathcal{N}}, \tag{7}$$

in the sense of measures.

THEOREM 4.2. *Let $b : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a Borel function satisfying* (i)–(vi). *Then, for every $u \in BV(\mathbb{R}^N) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$, the function $v : \mathbb{R}^N \to \mathbb{R}$, defined by*

$$v(x) := \int_0^{u(x)} b(x,t)\,dt,$$

belongs to $BV_{\mathrm{loc}}(\mathbb{R}^N)$ and for any $\phi \in C^1_0(\mathbb{R}^N)$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \nabla\phi(x)v(x)\,dx \\
&\quad = -\int_{\mathbb{R}^N} \phi(x)\left[\int_0^{u(x)} \nabla_x b(x,t)\,dt\right]dx - \int_{\mathbb{R}^N} \phi(x)b(x,u(x))\nabla u(x)\,dx \\
&\qquad - \int_{\mathbb{R}^N} \phi(x)\left[\int_0^{\tilde u(x)} \frac{dD_x^c b}{d\bar\sigma}(x,t)\,dt\right]d\bar\sigma - \int_{\mathbb{R}^N} \phi(x)\tilde b(x,\tilde u(x))\,dD^c u \\
&\qquad - \int_{\bar{\mathcal{N}}\cup J_u} \phi(x)\left[\int_0^{u^+(x)} b^+(x,t)\,dt - \int_0^{u^-(x)} b^-(x,t)\,dt\right]\nu_{\bar{\mathcal{N}}\cup J_u}(x)\,d\mathscr{H}^{N-1},
\end{aligned} \tag{8}$$

where it is understood that for $\mathscr{H}^{N-1}$-a.e. $x \in \bar{\mathcal{N}} \cap J_u$ the normal $\nu_{\bar{\mathcal{N}}\cup J_u}$ is choosen equal to $\nu_{\bar{\mathcal{N}}}$.

COROLLARY 4.3. *Let $b : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a Borel function satisfying*

(i) *the function $b(x,t)$ is locally bounded;*
(ii) *for every $t \in \mathbb{R}$ the function $b(\cdot,t) \in W^{1,1}(\mathbb{R}^N)$ and there exists a Borel set $\mathcal{N}_1 \subseteq \mathbb{R}^N$ such that $\mathscr{H}^{N-1}(\mathcal{N}_1) = 0$ such that*

$$b(x,t) = \tilde b(x,t)$$

for every $x \in \mathbb{R}^N \setminus \mathcal{N}_1$ and every $t \in \mathbb{R}$;
(iii) *for every compact set $H \subseteq \mathbb{R}$ the function*

$$g_H(x) := \sup_{t\in H} |\nabla_x b(x,t)|$$

belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^N)$;

(iv) *there exists a Borel set* $\mathcal{N}_2 \subseteq \mathbb{R}^N$ *such that* $\mathcal{L}^N(\mathcal{N}_2) = 0$ *such that the approximate gradient* $\nabla_x b(x,t)$ *of the function* $y \mapsto b(y,t)$ *at* x *exists for every* $x \in \mathbb{R}^N \setminus \mathcal{N}_2$ *and every* $t \in \mathbb{R}$.

Then, for every $u \in BV(\mathbb{R}^N) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$, *the function* $v : \mathbb{R}^N \to \mathbb{R}$, *defined by*

$$v(x) := \int_0^{u(x)} b(x,t)\, dt,$$

belongs to $BV_{\mathrm{loc}}(\mathbb{R}^N)$ *and for any* $\phi \in C^1_0(\mathbb{R}^N)$ *we have*

$$\begin{aligned}\int_{\mathbb{R}^N} \nabla\phi(x) v(x)\, dx = &\int_{\mathbb{R}^N} \phi(x) \left[\int_0^{u(x)} \nabla_x b(x,t)\, dt \right] dx \\ &- \int_{\mathbb{R}^N} \phi(x) b(x,u(x)) \nabla u(x)\, dx - \int_{\mathbb{R}^N} \phi(x) \tilde{b}(x,\tilde{u}(x))\, dD^c u \\ &- \int_{J_u} \phi(x) \left[\int_{u^-(x)}^{u^+(x)} \tilde{b}(x,t)\, dt \right] \nu_u(x)\, d\mathcal{H}^{N-1}.\end{aligned}$$

Remark 4.4. This corollary improves Proposition 1.2 in [8] where $\mathcal{N}_2 = \emptyset$ and $b(x,\cdot)$ is continuous for a.e. $x \in \mathbb{R}^N$.

References

[1] M. Amar - V. De Cicco - P. Marcellini - E. Mascolo, *Weak lower semicontinuity for non coercive polyconvex integrals*, Adv. Calc. Var., I (2008), no. 2 171–191.

[2] L. Ambrosio - G. Crasta - V. De Cicco - G. De Philippis, *A nonautonomous chain rule formula in* $W^{1,p}$ *and in BV*, Manuscripta Math. 140 (2013) no. 3, 461–480.

[3] L. Ambrosio - G. Dal Maso, *A general chain rule for distributional derivatives*, Proc. Amer. Math. Soc., 108, (1990), 691–702.

[4] L. Ambrosio - N. Fusco - D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford University Press, (2000).

[5] G. Crasta - V. De Cicco, *A chain rule formula in BV and applications to conservation laws*, SIAM J. Math. Anal. 43 (2011), no. 1, 430–456.

[6] G. Crasta - V. De Cicco - G. De Philippis, *Kinetic formulation and uniqueness for scalar conservation laws with discontinuous flux*, Comm. Partial Differential Equations, 40 (2015), no. 4, 694–726.

[7] V. De Cicco, *Lower semicontinuity for nonautonomous surface integrals*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl., 26 (2015), 1–21.

[8] V. De Cicco - N. Fusco - A. Verde, *On* L^1*-lower semicontinuity in BV*, J. of Convex Anal., 12 (2005), 173–185.

[9] V. De Cicco - N. Fusco - A. Verde, *A chain rule formula in BV and application to lower semicontinuity*, Calc. Var. Partial Differential Equations, 28 (2007), no. 4, 427–447.

[10] V. De Cicco - G. Leoni, *A chain rule in* $L^1(\mathrm{div};\mathbb{R}^N)$ *and its applications to lower semicontinuity*, Calc. Var. Partial Differential Equations, 19, (2004), no. 1, 23–51.

[11] H. Federer, *Geometric measure theory*, Springer, Berlin, (1969).
[12] G. Leoni - M. Morini, *Necessary and sufficient conditions for the chain rule in* $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^d)$ *and* $BV_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^d)$, Comm. Pure Appl. Math., 58 (2005), no. 8, 1051–1076.
[13] A. I. Vol'pert, *Spaces BV and quasilinear equations*, Mat. Sb. (N.S.) 73 (115) (1967), 255–302.

Received 17 June 2015,
and in revised form 24 July 2015.

Virginia De Cicco
Dipartimento SBAI Scienze di Base e Applicate all'Ingegneria
Sapienza Università di Roma
Via A. Scarpa 16
I-00161 Rome, Italy
virginia.decicco@sbai.uniroma1.it