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Partial Differential Equations — *On Fitzpatrick's theory and stability of flows*, by Augusto Visintin, communicated on 13 November 2015.

Dedicated to the memory of Professor Enrico Magenes

Abstract. — The initial-value problem associated to a maximal monotone operator may be formulated as a minimization principle, on the basis of a theory that was pioneered by Brezis, Ekeland, Nayroles and Fitzpatrick. This note defines the notions of structural compactness and structural stability, and reviews results concerning the stability of maximal monotone flows under perturbations not only of data but also of the operator. This rests upon De Giorgi' theory of Γ-convergence, and on the use of an exotic nonlinear topology of weak type.

Key words: Maximal monotone operators, evolutionary Γ-convergence, Fitzpatrick theory, doubly-nonlinear parabolic equations

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Foreword. Professor Enrico Magenes was a distinguished mathematician, who founded and directed a renown school of mathematical and numerical analysis. He was also a remarkable example of human and civil engagement. I had the privilege of having him as a guide and a mentor. I then wish to express my appreciation for the initiative of the Accademia Nazionale dei Lincei to devote some issues of these Rendiconti to his memory, and thank Carlo Sbordone for inviting me to contribute.

1. Introduction

These pages illustrate a research about *structural compactness* and *structural stability* of flows, on which this author has been working in the last years. At the focus there is the behaviour of the solution of quasilinear evolutionary PDEs under perturbations not only of the data but also of the operator, as it is illustrated in Section 5.

For stationary models represented by a minimization principle, structural compactness and stability are provided by E. De Giorgi's notion of Γ-convergence— a well-known theory that was introduced in [21] and then studied in a large number of works, see e.g. the monographs [1], [8], [9], [10], [19]. Indirectly, this also applies to the corresponding Euler-Lagrange equations.

For equations that are governed by maximal monotone operators, a variational characterization was pointed out by S. Fitzpatrick in the seminal paper

[24]; see Sections 2 and 3 ahead. By combining that approach with two pioneering works of Brezis and Ekeland [12] and Nayroles [36] (which actually predated that of Fitzpatrick), this method led to the *extended BEN principle* of [43]. This provides a variational formulation for first-order flows of the form

$$D_t u + \alpha(u) \ni h \quad \text{a.e. in time } (D_t := \partial/\partial t), \tag{1.1}$$

with α maximal monotone and possibly multi-valued, and with h prescribed; see Section 4.

This theory is based on the definition of functionals that act on the Cartesian product $V \times V'$ of a Banach space V and its dual V', that are convex and lower semicontinuous and dominate the duality pairing. Along with [47], it seems convenient to introduce a somehow exotic nonlinear topology of weak type, that we label by $\tilde{\pi}$, see Section 6. This topology is aimed to trade between two opposite exigences: to provide compactness of the class of functionals, and also to yield convergence of the perturbed equation. This issue is illustrated in [47] and more systematically here in Section 7.

In Section 8 we introduce a notion of *evolutionary* Γ*-convergence of weak type.* Afterwards we state the structural compactness and stability of the initial-value problem of a large class of first-order equations for maximal monotone operators; see [51].

2. The Fitzpatrick theory

In this section we show how any maximal monotone operator can be formulated as a minimization principle, on the basis of Fitzpatrick's pioneering work [24]. We then extend this formulation to nonmonotone operators.

The Fitzpatrick Theorem. Let V be a real Banach space, and let us denote by $\langle \cdot, \cdot \rangle$ the duality pairing between V' and V. Let $\alpha : V \to \mathscr{P}(V')$ (the set of the parts of V') be a (possibly multi-valued) *measurable* operator, i.e., such that

$$g^{-1}(A) := \{v \in V : g(v) \cap A \neq \emptyset\} \tag{2.1}$$

is measurable, for any open subset A of V'. For instance, this condition is fulfilled if α is maximal monotone. (We shall always assume that α is proper, i.e., $\alpha(V) \neq \emptyset$.)

In [24] Fitzpatrick defined a function, which nowadays is called the *Fitzpatrick function*:

$$\begin{aligned} f_\alpha(v, v^*) &:= \langle v^*, v\rangle + \sup\{\langle v^* - v_0^*, v_0 - v\rangle : v_0^* \in \alpha(v_0)\} \\ &= \sup\{\langle v^*, v_0\rangle - \langle v_0^*, v_0 - v\rangle : v_0^* \in \alpha(v_0)\} \quad \forall (v, v^*) \in V \times V'. \end{aligned} \tag{2.2}$$

He noticed that f_α is convex and lower semicontinuous, being the supremum of a family of affine and continuous functions, and proved the following assertion.

Theorem 2.1 ([24]). *If α is maximal monotone then*

$$f_\alpha(v, v^*) \geq \langle v^*, v \rangle \quad \forall (v, v^*) \in V \times V', \tag{2.3}$$

$$f_\alpha(v, v^*) = \langle v^*, v \rangle \quad \Leftrightarrow \quad v^* \in \alpha(v). \tag{2.4}$$

The definition of the function f_α and Theorem 2.1 were unnoticed for several years, and were then rediscovered by Martinez-Legaz and Théra [30] and (independently) by Burachik and Svaiter [16]. This started an intense research that bridges monotone operators and convex functions; see the end of the next section for references.

Representative functions. Nowadays one says that a function $f : V \times V' \to \mathbf{R} \cup \{+\infty\}$ (variationally) *represents* a measurable operator $\alpha : V \to \mathscr{P}(V')$ whenever $f : V \times V' \to \mathbf{R} \cup \{+\infty\}$ is convex and lower semicontinuous,

$$\begin{aligned} &f(v, v^*) \geq \langle v^*, v \rangle \quad \forall (v, v^*) \in V \times V', \\ &f(v, v^*) = \langle v^*, v \rangle \quad \Leftrightarrow \quad v^* \in \alpha(v). \end{aligned} \tag{2.5}$$

One accordingly says that α is *representable,* and that f is a *representative function.* We shall denote by $\mathscr{F}(V)$ the class of the functions that fulfill the first two of these properties. Representable operators are monotone, see [24], but need not be either cyclically or maximal monotone [17].

Let us next assume that the real Banach space V is reflexive. Besides the duality between V and V', let us consider the duality between the spaces $V \times V'$ and its dual $V' \times V$, and the corresponding convex conjugation. More specifically, for any function $g : V \times V' \to \mathbf{R} \cup \{+\infty\}$, let us set

$$g^*(w^*, w) := \sup\{\langle w^*, v \rangle + \langle v^*, w \rangle - g(v, v^*) : (v, v^*) \in V \times V'\} \quad \forall (w^*, w) \in V' \times V. \tag{2.6}$$

Some examples of representative functions. Here we outline some simple examples of representative functions of maximal monotone operators related to PDEs. More may be found e.g. in [46], [47].

Example 2.1. The Fitzpatrick Theorem 2.1 generalizes the following classical result of Fenchel [23] of convex analysis. Let $\varphi : V \to \mathbf{R} \cup \{+\infty\}$ be a convex and lower semicontinuous proper function (i.e., $\varphi \not\equiv +\infty$), and denote by $\varphi^* : V' \to \mathbf{R} \cup \{+\infty\}$ and $\partial\varphi : V \to \mathscr{P}(V')$ respectively the convex conjugate function and the subdifferential of φ. Then

$$\begin{aligned} &\varphi(v) + \varphi^*(v^*) \geq \langle v^*, v \rangle \quad \forall (v, v^*) \in V \times V', \\ &\varphi(v) + \varphi^*(v^*) = \langle v^*, v \rangle \quad \Leftrightarrow \quad v^* \in \partial\varphi(v). \end{aligned} \tag{2.7}$$

In other terms, the subdifferential operator $\partial\varphi : V \to \mathscr{P}(V')$ is represented by the *Fenchel function* $(v, v^*) \mapsto \varphi(v) + \varphi^*(v^*)$.

We shall refer to (2.3), (2.4) ((2.7), resp.) as the *Fitzpatrick system* (the *Fenchel system*, resp.), and to the mapping $V \times V' \to \mathbf{R} \cup \{+\infty\} : (v, v^*) \mapsto \varphi(v) + \varphi^*(v^*)$ as a *Fenchel function.* Incidentally note that the Fenchel function need not coincide with the Fitzpatrick function $f_{\partial\varphi}$.

EXAMPLE 2.2. Let us denote by I_K the indicator function of any set K (i.e., $I_K = 0$ in K and $I_K = +\infty$ outside K). If $L : V \to V'$ is a linear, monotone and hemicontinuous operator (i.e., for any $u, v \in V$, $L(u + \lambda v) \to L(u)$ weakly in V' as $\lambda \to 0$), then it is represented by the function

$$f_L(v, v^*) = I_L(v, v^*) + \langle Lv, v\rangle \quad \forall (v, v^*) \in V \times V'. \tag{2.8}$$

EXAMPLE 2.3. Let Ω be a bounded domain of $\mathbf{R}^N$ ($N > 1$), $p \in]1, +\infty[$, and set $V := W_0^{1,p}(\Omega)$. Let a maximal monotone mapping $\vec{\gamma} : \mathbf{R}^N \to \mathscr{P}(\mathbf{R}^N)$ be represented by a function $f \in \mathscr{F}(\mathbf{R}^N)$. If

$$\exists a_1, a_2 \in \mathbf{R}^+ : \forall \vec{w} \in \mathbf{R}^N, \quad \forall \vec{z} \in \vec{\gamma}(\vec{w}), \quad |\vec{z}| \le a_1 |\vec{w}|^p + a_2, \tag{2.9}$$

it is well known that then the operator

$$\hat{\beta} : V \to \mathscr{P}(V') : v \mapsto -\nabla \cdot \vec{\gamma}(\nabla v) \tag{2.10}$$

is maximal monotone. This includes e.g. the case of the p-Laplacian: $\hat{\beta}(v) = -\nabla \cdot (|\nabla v|^{p-2} \nabla v)$, with $1 < p < +\infty$.

We claim that $\hat{\beta}$ may be represented by the function $\varphi \in \mathscr{F}(V)$ that is constructed as follows. For any $(v, v^*) \in V \times V'$, first let $\theta \in H_0^1(\Omega)^N$ be such that $-\Delta\theta = v^*$ in $\mathscr{D}'(\Omega)$, and set $\vec{\xi}_{v^*} = \nabla\theta$. Hence

$$\vec{\xi}_{v^*} \in \nabla H_0^1(\Omega)^N, \quad -\nabla \cdot \vec{\xi}_{v^*} = v^* \quad \text{in } \mathscr{D}'(\Omega). \tag{2.11}$$

Then set

$$\varphi(v, v^*) = \int_\Omega f(\nabla v, \vec{\xi}_{v^*})\, dx. \tag{2.12}$$

The function φ is convex and lower semicontinuous, and

$$\varphi(v, v^*) = \int_\Omega f(\nabla v, \vec{\xi}_{v^*})\, dx \overset{f \in \mathscr{F}(\mathbf{R}^N)}{\ge} \int_\Omega \nabla v \cdot \vec{\xi}_{v^*}\, dx = -\langle v, \nabla \cdot \vec{\xi}_{v^*}\rangle \overset{(2.11)}{=} \langle v, v^*\rangle. \tag{2.13}$$

Thus $\varphi \in \mathscr{F}(V)$. Moreover, as $f(\nabla v, \vec{\xi}_{v^*}) \ge \nabla v \cdot \vec{\xi}_{v^*}$ pointwise, in (2.13) equality holds if and only if $f(\nabla v, \vec{\xi}_{v^*}) = \nabla v \cdot \vec{\xi}_{v^*}$ a.e. in Ω. As f represents $\vec{\gamma}$, this equality is equivalent to $\vec{\xi}_{v^*} \in \vec{\gamma}(\nabla v)$ a.e. in Ω, namely by (2.11)

$$v^* \in -\nabla \cdot \vec{\gamma}(\nabla v) \quad \text{in } \mathscr{D}'(\Omega). \tag{2.14}$$

Incidentally notice that the curl-free field $\vec{\xi}_{v^*}$ is just one of the many selections $\vec{\eta} \in \vec{\gamma}(\nabla v)$ such that $\vec{\eta} \in L^2(\Omega)^N$ and $-\nabla \cdot \vec{\eta} = v^*$ in $\mathscr{D}'(\Omega)$.

Example 2.4. Let an operator $\alpha : V \to \mathscr{P}(V')$ be represented by a function $f_\alpha \in \mathscr{F}(V)$, and $L : V \to V'$ be bounded, linear and monotone. The operator $\alpha + L$ is then representable, and is represented for instance by the function

$$f(v, v^*) = f_\alpha(v, v^* - Lv) + \langle Lv, v\rangle \quad \forall (v, v^*) \in V \times V'. \tag{2.15}$$

This further generalizes what in [43] is named *extended BEN principle*, which corresponds to $L = D_t$ (in a space V of time-dependent functions). This example is at the basis of the construction that we shall illustrate in the next section.

Example 2.5. Bounded skew-adjoint operators are maximal monotone. For instance, if Ω is as above and $N \geq 1$, let us set $V = H_0^1(\Omega) \times L^2(\Omega)^N$, whence $V' = H^{-1}(\Omega) \times L^2(\Omega)^N$ (identifying $L^2(\Omega) \times L^2(\Omega)^N$ with its dual). By the Example 2.2, the bounded skew-adjoint operator

$$\Lambda : V \to V' : \begin{pmatrix} u \\ \vec{v} \end{pmatrix} \mapsto \begin{pmatrix} \nabla \cdot \vec{v} \\ \nabla u \end{pmatrix} \tag{2.16}$$

is represented by the following function (here by I_Λ we denote the indicator function of the operator Λ)

$$f(U, U^*) = I_\Lambda(U, U^*) \quad \forall (U, U^*) \in V \times V'. \tag{2.17}$$

Example 2.6. Let us define V and Λ as in the previous example, and set $W = L^2(0, T; V) \cap H^1(0, T; V')$. Let $\vec{\gamma}, \vec{\eta} : \mathbf{R}^N \to \mathscr{P}(\mathbf{R}^N)$ be maximal monotone mappings, and the operator

$$\alpha : V \to \mathscr{P}(V') : \begin{pmatrix} u \\ \vec{v} \end{pmatrix} \mapsto \begin{pmatrix} -\nabla \cdot \vec{\gamma}(\nabla u) \\ \vec{\eta}(\vec{v}) \end{pmatrix} \tag{2.18}$$

be represented by a function $f_\alpha \in \mathscr{F}(V)$. The operator $D_t + \Lambda + \alpha : W \to W'$ is then representable, and is represented for instance by the function

$$F(U, U^*) = \int_0^T f_\alpha(U, U^* - D_t U - \Lambda U)\, dt + \int_0^T \langle D_t U, U\rangle\, dt \qquad \forall (U, U^*) \in W \times W'. \tag{2.19}$$

(Here by $\langle \cdot , \cdot \rangle$ we denote the duality pairing between W' and W.)

If the operator $\vec{\eta}$ in (2.18) vanishes identically and (say)

$$F = (f, \vec{g}) \in H^1(0, T; H^{-1}(\Omega)) \times L^2(0, T; L^2(\Omega)^N),$$

then by eliminating $\vec{v}$ it is promptly checked that the vector equation

$$D_t U + \Lambda U + \alpha(U) = F \quad \text{in } V', \text{ a.e. in }]0,T[\tag{2.20}$$

is equivalent to the semilinear third-order scalar equation

$$D_t^2 u - \Delta u - D_t \nabla \cdot \vec{\gamma}(\nabla u) = D_t f - \nabla \cdot \vec{g} \quad \text{in } H^{-1}(\Omega), \text{ a.e. in }]0,T[. \tag{2.21}$$

Representation of nonmonotone operators. Next we extend the notion of representative function to nonmonotone operators. Let us denote by σ an intermediate topology between the strong and the weak topology of $V \times V'$. We shall say that a (possibly nonconvex) function f *σ-represents* a (possibly nonmonotone) measurable operator $\alpha : V \to \mathscr{P}(V')$ if

$$\begin{aligned} & f : V \times V' \to \mathbf{R} \cup \{+\infty\} \quad \text{is } \sigma\text{-lower semicontinuous}, \\ & f(v, v^*) \geq \langle v^*, v \rangle \quad \forall (v, v^*) \in V \times V', \\ & f(v, v^*) = \langle v^*, v \rangle \quad \Leftrightarrow \quad v^* \in \alpha(v). \end{aligned} \tag{2.22}$$

We shall denote by $\mathscr{E}_\sigma(V)$ the class of the functions that fulfill the first two of these conditions, and in particular by $\mathscr{E}_s(V)$ ($\mathscr{E}_w(V)$, resp.) the class corresponding to the strong (weak, resp.) topology of $V \times V'$. Clearly

$$\mathscr{F}(V) \subset \mathscr{E}_w(V) \subset \mathscr{E}_\sigma(V) \subset \mathscr{E}_s(V);$$

in order to distinguish $\mathscr{F}(V)$ from these other classes, when needed we shall refer to the respective elements as *convex representatives* and *possibly-nonconvex σ-representatives.*

Nonconvex Fitzpatrick functions. Next we modify the definition of the Fitzpatrick function (2.2) to represent a relevant class of nonmonotone operators. Let us first denote by X_w (X_s, resp.) any Banach space X equipped with the weak (strong, resp.) topology.

Proposition 2.2. *For any $z \in V$, let $\gamma_z : V \to V'$ be a* (*single-valued*) *maximal monotone operator, and define the corresponding Fitzpatrick function f_{γ_z} as in* (2.2). *Then:*

(i) *If the mapping*

$$V_w \to (V')_s : z \mapsto \gamma_z(v) \quad \forall v \in V, \tag{2.23}$$

is continuous, then the function $\varphi : V \times V' \to \mathbf{R} \cup \{+\infty\} : (v, v^*) \mapsto f_{\gamma_v}(v, v^*)$ is an element of $\mathscr{E}_w(V)$.

(ii) *If the mapping*

$$V_s \to (V')_w : z \mapsto \gamma_z(v) \quad \forall v \in V, \tag{2.24}$$

is continuous, then $\varphi \in \mathscr{E}_s(V)$.

PROOF. Let us prove part (i). By (2.24),

$$V \times V' \to \mathbf{R} \cup \{+\infty\} : (v, v^*) \mapsto \langle v^*, w \rangle - \langle \gamma_v(w), w - v \rangle \text{ is weakly continuous}, \quad \forall w \in V. \tag{2.25}$$

The supremum φ of this family w.r.t. $w \in V$, namely the Fitzpatrick function f_{γ_z}, is then weakly lower semicontinuous.

By the Fitzpatrick Theorem 2.1, for any $z \in V$

$$\begin{aligned} &f_{\gamma_z}(v, v^*) \geq \langle v^*, v \rangle \quad \forall (v, v^*) \in V \times V', \\ &f_{\gamma_z}(v, v^*) = \langle v^*, v \rangle \quad \Leftrightarrow \quad v^* \in \gamma_z(v). \end{aligned} \tag{2.26}$$

The function φ then fulfills $(2.22)_2$ and $(2.22)_3$. Thus $\varphi \in \mathscr{E}_w(V)$.

The proof of part (ii) is analogous, and is left to the reader. Loosely speaking, here the strong and the weak topologies are exchanged. □

REMARKS 2.3. (i) Part (i) of Proposition 2.2 may be applied to pseudo-monotone differential operators, e.g.

$$\begin{aligned} &\alpha(u) = -\nabla \cdot g(u, \nabla u) \qquad (\nabla\cdot := \text{div}), \\ &\alpha(u) = \nabla \times g(u, \nabla \times u) \quad (\nabla\times := \text{curl, in } \mathbf{R}^3). \end{aligned} \tag{2.27}$$

In either case it is assumed that g is single-valued, continuous w.r.t. the first argument, and nondecreasing w.r.t. the second one.

Further nonmonotone differential operators are constructed by adding a (typically lower order) continuous perturbation.

(ii) A surjectivity result for possibly nonmonotone operators α of the form of Proposition 2.2 was proved in [49]. This may be applied e.g. to several PDEs expressed in terms of pseudo-monotone operators.

3. Null-minimization

In this section we discuss two different minimization principles that are associated to representative functions.

Null-minimization principle. Let a function f represent a (possibly nonmonotone) operator $\alpha : V \to \mathscr{P}(V')$, and define the function

$$J(v, v^*) := f(v, v^*) - \langle v^*, v \rangle \quad \forall (v, v^*) \in V \times V'. \tag{3.1}$$

By $(2.22)_2$ $J \geq 0$; by $(2.22)_3$, $J(v, v^*) = 0$ if and only if $v^* \in \alpha(v)$, or equivalently

$$J(v, v^*) = 0 \ (= \inf J) \quad \Leftrightarrow \quad v^* \in \alpha(v). \tag{3.2}$$

We shall label "$J(v, v^*) = 0 \ (= \inf J)$" as a problem of *null-minimization.*

Let us distinguish between two different minimization principles:

(i) *Ordinary minimization.* $J(v, v^*)$ may be minimized by varying both v and v^*, as it is often the case when one deals with the Fitzpatrick theory. In this case it is not needed to prescribe the minimum value to vanish, since the construction of J entails this property.

(ii) *Null-minimization.* In several cases $J(v, v^*)$ is instead minimized varying just v (so for a fixed v^*); this is often the case when one studies a specific problem (e.g., a boundary-value problem). For instance this occurs for equations including a maximal monotone operator, and in particular for the (extended) BEN principle that we illustrate below.

In this latter case one must prescribe the vanishing of the minimum value. Indeed, although obviously any null-minimizer is also an ordinary minimizer, a priori it is not clear whether the former exists (if this were not the case, the infimum would be equal to $+\infty$, by convention).

This discussion also applies to nonmonotone representable operators.

An initial-value problem for an elementary ODE. We illustrate the issue that we just outlined via a simple example, along the lines of Section 7 of [49]; see also the introduction of [27]. For any $h \in L^2(0, T)$, let us consider the trivial initial-value problem

$$\begin{cases} D_t u + u = h & \text{a.e. in }]0, T[, \\ u(0) = 0. \end{cases} \tag{3.3}$$

We already know that

$$u(t) = \int_0^t e^{\tau - t} h(\tau)\, d\tau \quad \text{a.e. in }]0, T[\tag{3.4}$$

is the unique solution of this problem. However, here we are concerned with the variational formulation. Defining the functional

$$\begin{gathered} \Lambda_h : \mathscr{X}_0 := \{v \in H^1(0, T) : v(0) = 0\} \to \mathbf{R}, \\ \Lambda_h(v) := \int_0^T \left\{ \frac{1}{2}|v|^2 + \frac{1}{2}|h - D_t v|^2 - hv \right\} dt + \frac{1}{2}|v(T)|^2, \end{gathered} \tag{3.5}$$

the BEN principle reads

$$u \in \mathscr{X}_0, \quad \Lambda_h(u) = \inf \Lambda_h = 0 \quad \Leftrightarrow \quad (3.3). \tag{3.6}$$

In alternative, let us now consider the ordinary minimization problem

$$u \in \mathscr{X}_0 \quad \Lambda_h(u) = \inf \Lambda_h, \tag{3.7}$$

and wonder whether this entails (3.3).

As the functional Λ_h is convex and lower semicontinuous, by varying u with smooth functions supported in $]0, T[$, one gets the Euler-Lagrange equation

$$u - D_t^2 u + D_t h - h = 0 \quad \text{a.e. in }]0, T[. \tag{3.8}$$

Because of the convexity of the functional, this equation is equivalent to the equation (3.7). (3.8) also reads $(I - D_t)[(I + D_t)u - h] = 0$ a.e. in $]0, T[$, which by inverting the operator $I - D_t$ is equivalent to

$$u + D_t u - h = Ce^t \quad \text{a.e. in }]0, T[, \text{ for some } C \in \mathbf{R}. \tag{3.9}$$

Because of the initial condition, we thus get (displaying the dependence on C)

$$u_C(t) = \int_0^t e^{\tau - t}[h(\tau) + Ce^\tau]\, d\tau = \int_0^t e^{\tau - t} h(\tau)\, d\tau + C \sinh t \qquad \text{a.e. in }]0, T[. \tag{3.10}$$

The presence of the undetermined constant C is natural, since we prescribed just an initial condition for a second-order ODE. A second condition is needed to determine the constant C. This further condition may be the prescription that the minimum value of the functional J is zero. In the present case we know that a null-minimizer exists, and $C = 0$ corresponds to the actual solution (3.4).

Although this was just illustrated on a simple linear example for $V = \mathbf{R}$, analogous conclusions hold for nonlinear initial-value problems in infinite dimensional spaces: ordinary minimization is not sufficient to determine the solution of the initial-value problem, and prescribing the minimum value of the functional balances the number of side-conditions. Anyway, one should show that this null-minimizer actually exists.

The present discussion may be extended to the same ODE $(3.3)_1$ coupled with periodicity conditions: $u(0) = u(T)$ and $u'(0) = u'(T)$.

4. Monotone flows and extended BEN principle

In this section we illustrate how a family of first-order flows governed by a maximal monotone operator α may be formulated as a null-minimization principle, on the basis of the Example 2.4 above with $\Lambda = D_t$. For fixed u^* and u^0, we thus deal with the initial-value problem

$$\begin{cases} D_t u + \alpha(u) \ni u^* & \text{in } V', \text{ a.e. in }]0, T[\ (D_t := \partial/\partial t) \\ u(0) = u^0. \end{cases} \tag{4.1}$$

However the present analysis may be extended to the same equation coupled with conditions of time periodicity.

Extended BEN principle. Let us assume that we are given a Gelfand triplet of (real) Banach spaces

$$V \subset H = H' \subset V' \tag{4.2}$$

with continuous and dense injections, and fix any $u^* \in L^{p'}(0,T;V')$ $(2 \le p < +\infty,\ p' = p/(p-1))$, and $u^0 \in H$.

By (3.2), whenever an operator $\alpha : V \to \mathscr{P}(V')$ is represented by a function $f_\alpha \in \mathscr{F}(V)$, let us define the convex and lower semicontinuous functional

$$\begin{aligned} J(v,u^*) &:= \int_0^T [f_\alpha(v, u^* - D_t v) - \langle u^* - D_t v, v\rangle]\,dt \\ &= \int_0^T [f_\alpha(v, u^* - D_t v) - \langle u^*, v\rangle]\,dt \\ &\quad + \frac{1}{2}\|v(T)\|_H^2 - \frac{1}{2}\|u(0)\|_H^2 \end{aligned} \tag{4.3}$$

for any $v \in L^p(0,T;V) \cap W^{1,p'}(0,T;V')$ $(\subset C^0([0,T];H))$ and any $u^* \in V'$.

Consistently with the discussion of the previous section, the initial-value problem (4.1) is equivalent to the null-minimization of $J(\cdot,u^*)$, with $u(0)$ replaced by u^0. We shall refer to this equivalence as the *extended BEN principle.* This result allows to apply the extended corpus of variational techniques to investigate the structural properties of compactness and stability of (4.1).

Further spaces. Let V, H be as in (4.2), let us define the measure

$$\mu(A) = \int_A (T-t)\,dt \quad \forall A \in \mathscr{L}(0,T), \quad \text{i.e.,} \quad d\mu(t) = (T-t)\,dt, \tag{4.4}$$

and introduce the weighted Hilbert spaces

$$\begin{aligned} \tilde{\mathscr{H}} &:= \left\{ v :]0,T[\to H \text{ measurable: } \int_0^T \|v(t)\|_H^2\,d\mu(t) < +\infty \right\}, \\ \tilde{\mathscr{V}} &:= \left\{ v \in \tilde{\mathscr{H}} : \int_0^T \|v(t)\|_V^2\,d\mu(t) < +\infty \right\}, \\ \tilde{\mathscr{X}} &:= \{ v \in \tilde{\mathscr{V}} : D_t v \in \tilde{\mathscr{H}} \}\ (\subset C^0([0,T[;H)). \end{aligned} \tag{4.5}$$

Identifying $\tilde{\mathscr{H}}$ with its dual space, we get the Hilbert triplet

$$\tilde{\mathscr{V}} \subset \tilde{\mathscr{H}} = \tilde{\mathscr{H}}' \subset \tilde{\mathscr{V}}', \tag{4.6}$$

with continuous and dense injections.

Any maximal monotone operator $\alpha : V \to \mathscr{P}(V')$ canonically determines a corresponding evolutionary operator $\hat{\alpha} : \tilde{\mathscr{V}} \to \mathscr{P}(\tilde{\mathscr{V}}')$. It is easy to check that, if

α is represented by a function $f \in \mathcal{F}(V)$, then $\hat{\alpha}$ is represented by the function

$$\int_0^T f(\cdot\,,\cdot)\,d\mu(t) = \int_0^T d\tau \int_0^\tau f(\cdot\,,\cdot)\,dt \in \mathcal{F}(\tilde{\mathcal{V}}). \tag{4.7}$$

The reader will notice that the further integration does not modify the set of null-minimizers.

The initial-value problem (4.1) may then also be reformulated as the null-minimization of the functional

$$\begin{aligned} M(v,v^*) &:= \int_0^T \left\{ \int_0^\tau f_\alpha(v, v^* - D_t v)\,dt - \int_0^\tau \langle v^* - D_t v, v\rangle\,dt \right\} d\tau \\ &= \int_0^T f_\alpha(v, v^* - D_t v)\,d\mu(t) - \int_0^T \langle v^* - D_t v, v\rangle\,d\mu(t) \\ &\qquad \forall (v, v^*) \in \tilde{\mathcal{V}} \times \tilde{\mathcal{V}}' \text{ such that } D_t v \in \tilde{\mathcal{V}}'. \end{aligned} \tag{4.8}$$

The same applies if $\mathcal{F}(V)$ and $\mathcal{F}(\tilde{\mathcal{V}})$ are respectively replaced by $\mathcal{E}_\sigma(V)$ and $\mathcal{E}_\sigma(\tilde{\mathcal{V}})$ (σ being an intermediate topology between the strong and the weak topology of $V \times V'$).

Original BEN principle. The equivalence between the equation $(4.1)_1$ and the null minimization of the functional (4.3) generalizes an approach that was pioneered by Brezis and Ekeland [12] and by Nayroles [36], prior to the Fitzpatrick theory. These authors assumed α to be cyclically maximal monotone (namely, the subdifferential of a convex lower semicontinuous function), did not use the weight function $T - t$, and already pointed out the need of prescribing the minimum value (null in our case) in the minimization w.r.t. u. More specifically, they noticed that, for any convex lower semicontinuous function $\psi : V \to \mathbf{R} \cup \{+\infty\}$ and any $u^* \in L^2(0,T;V')$, the gradient flow

$$D_t u + \partial\psi(u) = u^* \quad \text{in } V', \text{ a.e. in }]0,T[\tag{4.9}$$

is tantamount to the null-minimization of the functional

$$\begin{aligned} \Phi(v) &:= \int_0^T [\psi(v) + \psi(u^* - D_t v)]\,dt - \langle u^* - D_t v, v\rangle \\ &= \int_0^T [\psi(v) + \psi(u^* - D_t v)]\,dt - \langle u^*, v\rangle \\ &\quad + \frac{1}{2}\|v(T)\|_H^2 - \frac{1}{2}\|u(0)\|_H^2 \end{aligned} \tag{4.10}$$

as v ranges in $H^1(0,T;V') \cap L^2(0,T;V)$ $(\subset C^0([0,T];H))$. This equivalence directly stems from the Fenchel system (2.7).

REMARKS. (i) The operator $D_t u + \alpha$ is maximal monotone in the affine space

$$\{v \in L^p(0,T;V) \cap W^{1,p'}(0,T;V') : v(0) = u_0\},$$

and the functional J of (4.3) may be interpreted as a representative function of this operator. A similar remark applies to the functional Φ of (4.10). In the framework of the Fitzpatrick theory the significance of the BEN principle is thus well understood.

(ii) Setting

$$g(v,t) := f_\alpha(v, u^*(t) - D_t v) - \langle u^*(t) - D_t v, v\rangle \geq 0 \quad \text{a.e. in }]0,T[, \tag{4.11}$$

the null-minimization of the time-integrated functional J of (4.3) is equivalent to the pointwise null-minimization:

$$g(u(t),t) = 0 \ (= \inf g(\cdot,t)) \quad \text{a.e. in }]0,T[, \tag{4.12}$$

in $\{v \in L^p(0,T;V) \cap W^{1,p'}(0,T;V') : v(0) = u^0\}$.

(iii) In the initial-value problem (4.1) we may assume $u^0 = 0$. If this is not the case, it suffices to replace u by $\tilde{u} = u - u^0$ and α by $\tilde{\alpha} = \alpha(\cdot + u^0)$, so that (4.1) is equivalent to

$$\begin{cases} D_t\tilde{u} + \tilde{\alpha}(\tilde{u}) \ni u^* & \text{in } V', \text{ a.e. in }]0,T[\\ \tilde{u}(0) = 0. \end{cases} \tag{4.13}$$

Thus $\tilde{u}$ is an element of the real Banach space

$$X_0^p := \{v \in L^p(0,T;V) \cap W^{1,p'}(0,T;V') : v(0) = 0\}, \tag{4.14}$$

whereas u belongs to the affine function space associated to the initial condition $u(0) = u^0$. Henceforth we shall drop the tilde, and write u instead of $\tilde{u}$.

(iv) Whenever $\psi : V \to \mathbf{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuous function(al) and $\alpha = \partial\psi$, the problem (4.1) is a *gradient flow*, and may also be set in the following form.

Find $u \in L^p(0,T;V) \cap W^{1,p'}(0,T;V')$ such that $u(0) = u^0$, and

$$\int_0^T [\psi(u) + \langle D_t u - u^*, u\rangle]\, dt \leq \int_0^T [\psi(v) + \langle D_t u - u^*, v\rangle]\, dt \qquad \forall v \in L^p(0,T;V). \tag{4.15}$$

Defining the right-hand side of this inequality as $M_u(v)$, $(4.15)_2$ also reads as a *quasi-variational inequality*:

$$M_u(u) \leq M_u(v) \quad \forall v \in L^p(0,T;V). \tag{4.16}$$

This variational structure is not in the direction of the Fitzpatrick theory. This set-up was investigated in [29] in 1978, and more recently e.g. in [6].

Methods to prove existence of a solutions. For a representable operator α, the extended BEN principle may be used to prove existence of a solution for the initial-value problem (4.1).

(i) The problem (4.1) may be approximated by a sequence of initial-value problems for which existence of a solution is already known by the classical theory, and this approximated problem may be represented as an equivalent null-minimization principle. One may then prove existence of a solution of the limit problem (4.1) by passing to the limit in this formulation, provided that appropriate uniform estimates are available. Whenever the operator α is also approximated, the use of De Giorgi's Γ-convergence is in order.

(ii) The inclusion $(4.1)_1$ may be reformulated via a *self-dual* representative function, that is, a function $V \times V' \to \mathbf{R} \cup \{+\infty\}$ that coincides with its conjugate in the duality between $V \times V'$ and its dual $V' \times V$; see e.g. [5]. This may be compared with the approach that was extensively investigated by Ghoussoub and coworkers; see e.g. [25], [26] and references therein.

(iii) Existence of a null-minimizer for the original BEN principle was directly proved by Auchmuty in [2], and for the extended BEN principle in [49], as we next illustrate.

Existence of a null-minimizer. Via representative functions, next we retrieve a variant of a classical result, namely, the surjectivity of coercive maximal monotone operators acting on a reflexive Banach space; see e.g. [3], [11], [13], [52]. The assumptions of the classical theory are weaker than those of the following result, but this does not seem to be due to any intrinsic obstruction.

THEOREM 4.1 ([49]). *Let us assume that*

$$V \text{ is a real reflexive Banach space}, \quad u^* \in V', \qquad \alpha : V \to \mathscr{P}(V') \quad \text{is maximal monotone.} \tag{4.17}$$

Let a mapping $f_\alpha \in \mathscr{F}(V)$ represent α, and be such that

$$\inf_{v^* \in V'} \frac{f_\alpha(v, v^*)}{\|v\|_V} \to +\infty \quad \text{as } \|v\|_V \to +\infty. \tag{4.18}$$

Then there exists $u \in V$ such that

$$f_\alpha(u, u^*) = \langle u^*, u \rangle. \tag{4.19}$$

This equation is equivalent to the inclusion $\alpha(u) \ni u^*$ in V'.

(The condition (4.18) is related to the coerciveness of the operator α.)

This theorem may be proved by reformulating the equation (4.19) as a minimax problem, and then applying the classical Ky Fan inequality, see [49].

The next statement extends this theorem to a class of nonmonotone operators.

THEOREM 4.2 ([49]). *Assume that*

$$\text{(4.20)}\qquad \begin{aligned}&V \text{ is a real reflexive Banach space}, \quad u^* \in V',\\ &\alpha_z : V \to \mathcal{P}(V') \text{ is maximal monotone } \forall z \in V,\end{aligned}$$

and let α_z *be represented by the restriction* $\psi_z : V \times V' \to \mathbf{R} \cup \{+\infty\}$ *of a measurable function* $V \times V \times V' \to \mathbf{R} \cup \{+\infty\} : (z, v, v^*) \mapsto \psi_z(v, v^*)$ *such that*

$$\text{(4.21)}\qquad \begin{aligned}&\inf_{v^* \in V'} \frac{\psi_v(v, v^*)}{\|v\|_V} \to +\infty \quad \text{as } \|v\|_V \to +\infty,\\ &\inf_{v \in V} \frac{\psi_v(v, v^*)}{\|v^*\|_{V'}} \to +\infty \quad \text{as } \|v^*\|_{V'} \to +\infty,\end{aligned}$$

$V \times V' \to \mathbf{R} \cup \{+\infty\} : (v, v^*) \mapsto \psi_v(v, v^*)$ *is weakly lower semicontinuous.*

For any $u^* \in V'$, *then there exists* $u \in V$ *such that*

$$\text{(4.22)}\qquad \psi_u(u, u^*) = \langle u^*, u\rangle.$$

This equation is equivalent to the inclusion $\alpha_u(u) \ni u^*$ *in* V'.

We refer the reader to [49] for the argument and for applications of this result.

REMARKS. (i) By combining the extended BEN principle with Theorem (4.1), one gets existence of a solution of the initial-value problem associated with a maximal monotone operator, see [49]. Similarly Theorem 4.2 provides existence of a solution for the initial-value problem associated with a pseudomonotone operator.

(ii) These methods may be extended to integro-differential equations of the form

$$\text{(4.23)}\qquad D_t u + \Lambda \int_0^t u(\tau)\, d\tau + \alpha(u) \ni u^* \quad \text{in } V', \text{ a.e. in }]0, T[,$$

with $\Lambda : V \to V'$ linear and monotone and $\alpha : V \to \mathcal{P}(V')$ maximal monotone. The same holds for the analogous equation with a pseudo-monotone term $\alpha_u(u)$ in place of $\alpha(u)$; for instance,

$$\text{(4.24)}\qquad \begin{aligned}&D_t u - \Delta \int_0^t u(\tau)\, d\tau - \nabla \cdot (k(u)|\nabla u|^{p-2}\nabla u) = h\\ &\qquad \text{in } H^{-1}(\Omega), \text{ a.e. in }]0, T[\ (1 < p < +\infty).\end{aligned}$$

for a positive and continuous prescribed function k.

A look at the literature. After the pioneering work of Fitzpatrick [24] and the rediscovery of those results by Martinez-Legaz and Théra [30] and by Burachik and Svaiter [16],[1] a recent but rapidly expanding literature has been devoted to this theory in the last fifteen years; see e.g. [4], [5], [7], [17], [15], [25], [26], [31], [32], [37], [38], and the related notion of *bipotential* [14].

We already sketched the origin of the extended BEN principle. The original formulation was applied in several works. For the study of doubly-nonlinear evolutionary PDEs, it was used e.g. in [39] and [41], and the extended formulation was applied in [46]. In [47] the second of the above existence methods (the one based on approximation) was used; in particular the dependence on data and operators for the solution of quasilinear maximal monotone equations was studied, by applying Γ-convergence to the null-minimization problem. This method was also used for the homogenization of evolutionary quasilinear PDEs in [44], [45] and [47].

5. Structural properties: compactness and stability

In this section we introduce the concepts of structural compactness and of structural stability, distinguishing equations from minimization principles.

5.1. Well-posedness and stability of equations

Let us consider a rather general set-up. Let X, Y be topological spaces,[2] $\mathscr{A} : X \to Y$ be a (possibly nonlinear and multi-valued) operator, $f \in Y$ be prescribed, and consider the problem

$$\mathscr{P}_{\mathscr{A},f}: \quad \text{Find } u \in X \text{ such that } \mathscr{A}u \ni f. \tag{5.1}$$

Following Hadamard, one says that this problem is *well-posed* if, for any $f \in Y$, $\mathscr{P}_{\mathscr{A},f}$ has a solution u, this is unique, and it depends continuously on the datum f. In other terms, the inverse operator $\mathscr{A}^{-1} : f \mapsto u$ is single-valued and continuous.

If the solution is not assumed to be unique, this condition may be replaced by the following one, which also includes a form of compactness. If $\{f_n\}$ is any sequence in the range $\mathscr{A}(X)$ and $f_n \to f$ in Y, then it is required that any corresponding sequence of solutions $\{u_n\}$ (i.e., such that $\mathscr{A}u_n \ni f_n$ for any n) has an accumulation point u, and that $\mathscr{A}u \ni f$. In other terms,

(i) the (possibly multi-valued) inverse operator $\mathscr{A}^{-1}$ is sequentially compact,

in the sense that it maps any bounded set to a sequentially relatively compact set, and

(ii) the graph of $\mathscr{A}^{-1}$ is sequentially closed.

[1] [Note added in proofs.] Ulisse Stefanelli just brought to the attention of this author the following note, that introduces what would then be called the Fitzpatrick function, and derives some of its properties:

N. V. Krylov: Some properties of monotone mappings. Litovsk. Mat. Sb. 22 (1982), no. 2, 80–87.

[2] Topological spaces will always be assumed Hausdorff spaces.

Whenever this closure property holds, we shall say that problem $\mathscr{P}_{\mathscr{A},f}$ is (sequentially) *stable.*[3]

5.2. Structural well-posedness and structural stability of equations

So far the operator $\mathscr{A}$ was kept fixed; next we let it vary, too. As the operator defines the structure of the system that is modelled by Problem (5.1), we shall speak of *structural* properties.

Let X, Y be topological spaces as above, and $\Phi(X, Y)$ be a family of (possibly multi-valued) operators $X \to Y$ equipped with a suitable topology, or at least a notion of convergence. We shall say that Problem (5.1) is *structurally well-posed* whenever, for any $\mathscr{A} \in \Phi(X, Y)$ and any $f \in Y$, problem $\mathscr{P}_{\mathscr{A},f}$ has a solution u, this is unique, and this depends (sequentially) continuously on both the operator $\mathscr{A}$ and the datum f; that is,

$$\begin{array}{l} f_n \to f \quad \text{in } Y \\ \mathscr{A}_n \to \mathscr{A} \quad \text{in } \Phi(X, Y) \end{array} \quad \Rightarrow \quad \mathscr{A}_n^{-1} f_n \to \mathscr{A}^{-1} f \quad \text{in } \mathscr{P}(X). \tag{5.2}$$

This presumes the convergence of data and of operators. For data it suffices to assume that the topological space Y is sequentially compact. For operators a suitable notion of sequential compactness is also needed: this must be sufficiently strong, in order to guarantee the implication above.

Whenever the solution of the nth problem is not unique, i.e. the operators $\mathscr{A}_n^{-1}$ are not single-valued, this picture must be amended. We shall say that Problem (5.1) is *structurally compact* if, whenever $\mathscr{A}$ and f respectively range through any subfamily of $\Phi(X, Y)$ and any subset Y that are (sequentially) compact in the respective topologies, $\mathscr{A}^{-1}(f)$ is confined to a (sequentially) relatively compact subset of X. (For formal reasons we do not exclude $\mathscr{A}^{-1}(f) = \emptyset$.)

We shall say that Problem (5.1) is *structurally stable* if

$$\begin{array}{l} \mathscr{A}_n u_n \ni f_n \quad \forall n \\ u_n \to u \quad \text{in } X \\ f_n \to f \quad \text{in } Y \\ \mathscr{A}_n \to \mathscr{A} \quad \text{in } \Phi(X, Y) \end{array} \quad \Rightarrow \quad \mathscr{A} u \ni f. \tag{5.3}$$

Incidentally, notice that the assumption "$\mathscr{A}_n u_n \ni f_n$ for all n" may be replaced by the weaker condition $\lim_{n\to\infty}(\mathscr{A}_n u_n - f_n) \ni 0$ in Y.

The selection of the operator space $\Phi(X, Y)$ and of its topology are the key points. This topology must strike a balance between two conflicting exigences: to be sufficiently weak in order to allow for compactness, and at the same time to be so strong to provide stability (i.e., passage to the limit in the perturbed problems).

[3] As we shall be concerned with applications involving weak-type topologies, we shall always refer to *sequential* compactness and continuity. For the sake of brevity, sometimes we shall omit this qualification.

Although stability might appear as the key requirement, compactness is also relevant, since it provides the existence of sequences that fulfill the hypotheses of (5.3). (Stability without compactness risks of being empty ...)

Remark. The above definitions depend on the functional set-up of the problem. We shall search for the weakest spaces that guarantee existence of a solution, in order to allow for the extension of our results to equations as general as possible. This stipulation will induce us to deal with weak topologies; this turns out to be one of the main sources of difficulty of this analysis.

5.3. Well-posedness and stability of minimization problems

Next we formulate analogous issues for variational problems. In this case the structure of the problem is determined by a functional, rather than an operator, and first we keep it fixed.

Let X be a topological space, J be a functional $X \to \mathbf{R} \cup \{+\infty\}$, and consider the minimization problem

$$(5.4) \qquad \mathscr{M}_J: \quad \text{Find } u \in X \text{ such that } J(u) = \inf J.$$

One says that

(5.5) $\mathscr{M}_J$ is *well-posed in the sense of Tychonov* if
any minimizing sequence converges to a minimum point,

or equivalently,
(i) a minimum point exists, and
(ii) any minimizing sequence is convergent.

This definition is rather restrictive, and in particular requires the uniqueness of the minimizer. One then defines a weaker notion:

(5.6) $\mathscr{M}_J$ is *well-posed in the generalized sense of Tychonov* if
any minimizing sequence has a subsequence
that converges to a minimum point,

or equivalently,
(i) any minimizing sequence has a convergent subsequence, and
(ii) the limit of any convergent minimizing sequence is a minimum point.

These concepts are illustrated e.g. in [22]. The two notions of well-posedness and generalized well-posedness for minimization may respectively be regarded as the pendant of the well-posedness and stability for equations that we defined above. In this case the functional J is kept fixed, as we did above for the operator $\mathscr{A}$. Next we extend these notions letting J vary.

5.4. Structural compactness and structural stability of minimization problems

Let X be a topological space as above, and $\mathscr{G}$ be a family of functionals $X \to \mathbf{R} \cup \{+\infty\}$, that we shall equip with a suitable topology. We shall deal with the corresponding family of minimization problems $\{\mathcal{M}_J : J \in \mathcal{G}\}$. We shall say that this family is *structurally compact* if

(i) the family $\mathcal{G}$ is (sequentially) compact, that is, any sequence $\{J_n\}$ in $\mathcal{G}$ has a convergent subsequence,

(ii) the sequence $\{u_n\}$ of minimizers is confined to a (sequentially) relatively compact subset of X.

We shall say that Problem (5.1) is *structurally stable* if

$$\begin{cases} J_n(u_n) = \inf J_n & \forall n \\ u_n \to u & \text{in } X \\ J_n \to J & \text{in } \mathcal{G} \end{cases} \quad \Rightarrow \quad J(u) = \inf J. \tag{5.7}$$

If this holds one often says that $\mathcal{G}$ is equipped with a *variational convergence.*

Incidentally, notice that the assumption $J_n(u_n) = \inf J_n$ for all n may be replaced by the weaker condition $J_n(u_n) - \inf J_n \to 0$.

In the next section we shall see that De Giorgi's notion of Γ-convergence provides natural results of structural compactness and structural stability.

For a large class of functionals, there is a natural relation between compactness and stability of a minimization problem and compactness and stability of the corresponding Euler-Lagrange equation. In this sense the minimized functional accounts for both the operator and the data of the associated equation.

Analogously to what we saw for equations, the selection of the topology of the space of functionals $\mathcal{G}$ is crucial, and may not be an obvious choice. Here also the requirements of structural compactness and structural stability are in competition, and a trade is needed. In the next section we shall see that Γ-convergence with respect to a nonstandard weak-type topology is especially appropriate, apparently more than other variational convergences, like Mosco-convergence.

6. A nonlinear weak topology

In this section we introduce the concepts of structural compactness and of structural stability, distinguishing equations from minimization principles.

In this section we deal with Γ-compactness and Γ-stability of the class $\mathcal{F}(V)$ of representative functions, that we defined in Section 2. This analysis is based on introducing what we shall refer to as a *nonlinear weak topology* of the space $V \times V'$. In this section we revisit Sections 4 and 5 of [47] and Section 5 of [51], wherein the reader may find the proofs of the results that are here stated.

Motivation. Let V be a real reflexive and separable Banach space, $\alpha : V \to \mathcal{P}(V')$ be a maximal monotone operator, and f_α be a representative function of α. We are concerned with the structural properties of the null-minimization of functionals of the form (3.1), viz.,

$$J(v, v^*) := f_\alpha(v, v^*) - \langle v^*, v \rangle \quad \forall (v, v^*) \in V \times V'. \tag{6.1}$$

On the other hand, dealing with flows, we first define the following spaces of time-dependent functions:

$$\mathcal{V} := L^2(0,T;V), \quad \mathcal{X} := L^2(0,T;V) \cap H^1(0,T;V').$$

On the basis of the extended BEN principle of Section 4, we consider the null-minimization of functionals of the form

$$\tilde{J}(v,v^*) := \int_0^T f_\alpha(v, v^* - D_t v)\,dt - \int_0^T \langle v^* - D_t v, v\rangle\,dt \tag{6.2}$$
$$\forall (v,v^*) \in \mathcal{V} \times \mathcal{V}' \text{ such that } D_t v \in \mathcal{V},$$

that is, for any $v \in \mathcal{X}$ and any $v^* \in L^2(0,T;V')$.

In particular we are concerned with the Γ-convergence of sequences of functionals $\{J_n\}$ and $\{\tilde{J}_n\}$, associated to sequences $\{\alpha_n\}$. Dealing with the sequence $\{J_n\}$, let us consider the weak topology of $V \times V'$, and concentrate our attention on the duality pairing $\langle v^*, v\rangle$. Because of the definition of (sequential) Γ-convergence in the weak topology of $V \times V'$, the exigence arises of passing to the limit in $\langle v_n^*, v_n\rangle$ whenever $v_n \rightharpoonup v$ in V and $v_n^* \rightharpoonup v^*$ in V'.[4] This difficulty may simply be removed by confining v_n^* to a compact subset of V': for instance, a bounded subset of H, if

$$V \subset H = H' \subset V' \quad \text{is a Hilbert triple with compact embeddings.} \tag{6.3}$$

For the functional (6.2) the problem of passing to the limit in the integral

$$-\int_0^T \langle v_n^* - D_t v_n, v_n\rangle\,dt = -\int_0^T \langle v_n^*, v_n\rangle\,dt + \int_0^T \langle D_t v_n, v_n\rangle\,dt \tag{6.4}$$

looks less easy. For the first integral on the right it suffices to assume that v_n^* varies in a compact subset of $\mathcal{X}'$: for instance, a bounded subset of $L^2(0,T;H)$ if (6.3) holds. To pass to the limit in the second integral is more challenging. Incidentally notice that, even assuming that $v_n(0) \to v(0)$ in H, the lower semicontinuity property

$$\liminf_n \int_0^T \langle D_t v_n, v_n\rangle\,dt = \liminf_n \frac{1}{2}\|v_n(T)\|_H^2 - \lim_n \frac{1}{2}\|v_n(0)\|_H^2 \tag{6.5}$$
$$\geq \frac{1}{2}\|v(T)\|_H^2 - \frac{1}{2}\|v(0)\|_H^2 = \int_0^T \langle D_t v, v\rangle\,dt$$

is not sufficient: for the purpose of Γ-convergence, the convergence of the integrals is needed.

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[4] We denote the strong, and weak convergence respectively by $\to$ and $\rightharpoonup$.

REMARK. If we prescribe the solution v to be T-periodic in time, then $\int_0^T \langle D_t v_n, v_n\rangle\, dt = 0$ for any n, and there is no difficulty in passing to the limit in this integral.

A possible answer. In order to overcome the difficulty that we just pointed out, one might assume a stronger form of convergence for the sequence $\{v_n\}$. For the Γ-stability it is then necessary that the null-minimizers, that is the solutions of the initial-value problems, converge in this stronger sense; to this purpose further a priori estimates are needed. This is feasible, and was performed in Sect. 8 of [47].

However this is not fully satisfactory, since it puts a severe restriction on the extension of this method to more general equations, like for instance: equations including maximal monotone operators that explicitly depend on time, pseudo-monotone operators, doubly nonlinear parabolic equations, and so on.

In alternative, one may modify the variational formulation by applying a further time-integration, as here we briefly outline. Let us first define the measure μ as in (4.4), and the functional

$$M(v,v^*) := \int_0^T \left\{ \int_0^\tau f_\alpha(v, v^* - D_t v)\, dt - \int_0^\tau \langle v^* - D_t v, v\rangle\, dt \right\} d\tau \tag{6.6}$$

$$= \int_0^T f_\alpha(v, v^* - D_t v)\, d\mu(t) - \int_0^T \langle v^* - D_t v, v\rangle\, d\mu(t)$$

$$\forall (v,v^*) \in \mathscr{V} \times \mathscr{V}' \text{ such that } D_t v \in \mathscr{V}.$$

As $\tilde{J} \ge 0$ (see (6.2)), it is clear that

$$\begin{aligned}&\text{a pair } (v,v^*) \in \mathscr{V} \times \mathscr{V}' \text{ is a null-minimizer of } \tilde{J}\\ &\text{if and only it is a null-minimizer of } M.\end{aligned} \tag{6.7}$$

Dealing with the null-minimization of M, the necessity arises of passing to the limit in the integral

$$-\int_0^T d\tau \int_0^\tau \langle v_n^* - D_t v_n, v_n\rangle\, dt = -\int_0^T \langle v_n^*, v_n\rangle\, d\mu(t) + \int_0^T \langle D_t v_n, v_n\rangle\, d\mu(t).$$

Under the assumption (6.3), one may pass to the limit in the first integral whenever $v_n^* \rightharpoonup v^*$ in $L^2(0,T;H)$. On the other hand, assuming that $v_n(0) \to v(0)$ in H,

$$\int_0^T \langle D_t v_n, v_n\rangle\, d\mu(t) = \frac{1}{2}\int_0^T D_t(\|v_n\|_H^2)\, d\mu(t) \tag{6.8}$$

$$= \int_0^T \|v_n\|_H^2\, dt - \frac{T}{2}\|v_n(0)\|_H^2$$

$$\to \frac{1}{2}\int_0^T \|v\|_H^2\, dt - \frac{T}{2}\|v(0)\|_H^2 = \int_0^T \langle D_t v, v\rangle\, d\mu(t).$$

(This explains why we applied a further time-integration to our functional.) This approach was pursued in Section 8 of [47].

A nonlinear topology of weak type. We have seen the role of the convergence of

$$\langle v_n^*, v_n \rangle \quad \text{and} \quad \int_0^T \langle v_n^* - D_t v_n, v_n \rangle \, dt$$

when dealing with the functionals (6.1) and (6.2), respectively. This prompts us to complement the weak topology of $V \times V'$ with the convergence $\langle v_n^*, v_n \rangle \to \langle v^*, v \rangle$ when dealing with the functional (6.1), and similarly to complement the weak topology of $\mathscr{V} \times \mathscr{V}'$ with the convergence $\int_0^T \langle v_n^*, v_n \rangle \, dt \to \int_0^T \langle v^*, v \rangle \, dt$ when dealing with the functional (6.2).

More specifically, with reference to the space $V \times V'$, let us first define the mapping associated to the duality pairing:

$$\pi : V \times V' \to \mathbf{R} : (v, v^*) \mapsto \langle v^*, v \rangle. \tag{6.9}$$

Let us name *nonlinear weak topology* of $V \times V'$, and denote by $\tilde{\pi}$, the coarsest among the topologies of this space that are finer than the product of the weak topology of V by the weak topology of V', and for which the mapping π is continuous. For any sequence $\{(v_n, v_n^*)\}$ in $V \times V'$, thus

$$\begin{aligned} &(v_n, v_n^*) \underset{\tilde{\pi}}{\to} (v, v^*) \quad \text{in } V \times V' \quad \Leftrightarrow \\ &v_n \rightharpoonup v \quad \text{in } V, \quad v_n^* \rightharpoonup v^* \quad \text{in } V', \quad \langle v_n^*, v_n \rangle \to \langle v^*, v \rangle, \end{aligned} \tag{6.10}$$

and similarly for nets. (The nonlinearity is obvious: a linear combination of two converging sequences need not converge.)

This construction is extended to the space $\mathscr{V} \times \mathscr{V}'$ in an obvious way, by defining the mapping $\pi : \mathscr{V} \times \mathscr{V}' \to \mathbf{R} : (v, v^*) \mapsto \int_0^T \langle v^*, v \rangle \, dt$:

$$\begin{aligned} &(v_n, v_n^*) \underset{\tilde{\pi}}{\to} (v, v^*) \quad \text{in } \mathscr{V} \times \mathscr{V}' \quad \Leftrightarrow \\ &v_n \rightharpoonup v \quad \text{in } \mathscr{V}, \quad v_n^* \rightharpoonup v^* \quad \text{in } \mathscr{V}', \quad \int_0^T \langle v_n^*, v_n \rangle \, dt \to \int_0^T \langle v^*, v \rangle \, dt, \end{aligned} \tag{6.11}$$

and similarly for nets.

7. Γ-COMPACTNESS AND Γ-STABILITY OF REPRESENTATIVE FUNCTIONS

In this section we review and discuss some results of Γ-compactness and Γ-stability with respect to the nonlinear topology of weak type that we introduced in the previous section.

Γ-*compactness and* Γ-*stability.* As the weak topology and the nonlinear weak topology $\tilde{\pi}$ are nonmetrizable, one must be cautious in dealing with *sequential* Γ-convergence of the functionals J_ns with respect to either topology.[5] However, it is known that bounded subsets of spaces equipped with the weak topology are metrizable; the same applies to the weak star topology if the space has a pre-dual. This also holds for the nonlinear weak topology $\tilde{\pi}$ of $V \times V'$; see Section 4 of [47], where it is proved that, if V is a separable real Banach space, then bounded subsets of $V \times V'$ equipped with the nonlinear weak topology $\tilde{\pi}$ are metrizable.[6]

We remind the reader that we denote by $\mathscr{F}(V)$ the class of the convex representative functions $f : V \times V' \to \mathbf{R} \cup \{+\infty\}$, see (2.5).

Theorem 7.1 ($\Gamma\tilde{\pi}$-compactness and $\Gamma\tilde{\pi}$-stability) ([47], [51]). *Let V be a real reflexive and separable Banach space, and $\{\psi_n\}$ be an equi-coercive sequence in $\mathscr{F}(V)$ in the sense that, for any $C \in \mathbf{R}$,*

$$\sup_{n\in\mathbf{N}} \{\|v\|_V + \|v^*\|_{V'} : (v, v^*) \in V \times V', \psi_n(v, v^*) \le C\} < +\infty. \tag{7.1}$$

Then: (i) *there exists $\psi : V \times V' \to \mathbf{R} \cup \{+\infty\}$ such that, up to extracting a subsequence, ψ_n $\Gamma\tilde{\pi}$-converges to ψ both topologically and sequentially;*[7]

(ii) *this entails that $\psi \in \mathscr{F}(V)$;*

(iii) *if α_n (α, resp.) is the operator that is represented by ψ_n (ψ, resp.), then for any sequence $\{(v_n, v_n^*)\}$ in $V \times V'$*

$$v_n^* \in \alpha_n(v_n) \quad \forall n, \quad (v_n, v_n^*) \underset{\tilde{\pi}}{\to} (v, v^*) \quad \Rightarrow \quad v^* \in \alpha(v). \tag{7.2}$$

The first part of this theorem rests upon two main issues:

(i) the equivalence between topological and sequential $\Gamma\tilde{\pi}$-convergence, which stems from the metrizability of bounded subsets of $V \times V'$ equipped with the topology $\tilde{\pi}$;

(ii) the following classical result of compactness:

Lemma 7.2 ([21]). *If a topological space Z has a countable basis, then every sequence $\{f_n\}$ of functions $Z \to \mathbf{R} \cup \{\pm\infty\}$ has a Γ-convergent subsequence.*

---

[5] We remind the reader that, for functions defined on a topological space, the definition of Γ-convergence involves the filter of the neighborhoods of each point; see e.g. [1], [19]. If the space is metrizable, that notion may equivalently be formulated in terms of the family of converging sequences, but this does not apply in general. We shall refer to these two notions as *topological* and *sequential* Γ-convergence, respectively. If not otherwise specified, reference to the topological notion should be understood.

A different approach consists in the following: (i) to embed compactly the domain (here e.g. $V \times V'$) into a larger space Z; (ii) to extend the functionals by the value $+\infty$ in $Z \setminus (V \times V')$; (iii) to deal with Γ-convergence w.r.t. the strong topology of Z.

[6] We shall often refer to the space $V \times V'$, but for time-dependent functions almost all of our discussion will take over to $\mathscr{V} \times \mathscr{V}'$.

[7] By $\Gamma\sigma$-convergence we mean the Γ-convergence with respect to a topology σ.

The Γ-closedness of $\mathscr{F}(V)$ plays a key role in the analysis of the structural stability of initial-value problems, see [47]. We stress that this rests upon the definition of $\tilde{\pi}$-convergence, and for instance fails for weak convergence.

The case of time-dependent functions. The above results that were stated in V take over verbatim to spaces of time-dependent functions, by replacing the space V with $\mathscr{V} = L^2(0, T; V)$. The next assertion bridges the two set-ups.

PROPOSITION 7.3 ([47]). *Let a function $\psi \in \mathscr{F}(V)$ be such that*

$$\forall C \in \mathbf{R}, \quad \sup\{\|v\|_V + \|v^*\|_{V'} : (v, v^*) \in V \times V', \psi(v, v^*) \le C\} < +\infty. \tag{7.3}$$

Then the functional

$$\Psi(v, v^*) := \int_0^T \psi(v(t), v^*(t))\, dt \quad \forall (v, v^*) \in \mathscr{V} \times \mathscr{V}' \tag{7.4}$$

is an element of $\mathscr{F}(\mathscr{V})$. Moreover, ψ represents an operator $\alpha : V \to \mathscr{P}(V')$ if and only if Ψ represents the corresponding operator

$$\hat{\alpha} : \mathscr{V} \to \mathscr{P}(\mathscr{V}') : v \mapsto \alpha(v(\cdot)). \tag{7.5}$$

REMARK. Although $\mathscr{F}(\mathscr{V})$ is stable by $\Gamma\tilde{\pi}$-converge, a major question arises: if a sequence of integral functionals of the memoryless form $\int_0^T f_{\alpha,n}(v, v^* - D_t v)\, dt$ $\Gamma\tilde{\pi}$-converges, is the limit a memoryless integral functional, too?

This question found a positive answer in [51].

Alternative topologies. Besides the nonlinear weak topology $\tilde{\pi}$, let us define the following linear topologies:

— ω is the product of the weak topology of V by the weak star topology of V',
— *ws* is the product of the weak topology of V by the strong topology of V',
— *sw*∗ is the product of the strong topology of V by the weak star topology of V',
— *s* is the strong topology of $V \times V'$.

Let us briefly discuss Γ-compactness and Γ-stability of representative functions with respect to these topologies. The Γ-compactness (i.e., part (i) of Theorem 7.1) is extended verbatim to the four topologies above; on the other hand the Γ-stability (i.e., parts (ii) and (iii) of Theorem 7.1) is only extended to the topologies *ws*, *sw*∗ and *s* and fails for ω, because of the occurrence of the duality-product term in the definition of $\mathscr{F}(V)$. Among the weak-type topologies above, the nonlinear weak topology $\tilde{\pi}$ is thus the only one for which Corollary 7.1 holds. (This is the main reason why we introduced this topology.)

The next statement concerns the structural stability of the class of Fenchel functions.

Proposition 7.4 ([47]). *Let us denote by τ any of the topologies ω, $\tilde{\pi}$, ws, sw$*$. Any equi-coercive sequence $\{\psi_n\}$ of Fenchel functions $\Gamma\tau$-converges to some function ψ, up to extracting a subsequence. If $\tau = ws$ or $\tau = sw*$ then ψ is a Fenchel function, whereas if $\tau = \omega$ or $\tau = s$ then ψ need not belong to that class.* [*There are counterexamples; the question instead is open for $\tau = \tilde{\pi}$.*]

In any case ψ is a representative function in the sense of (2.5).

Proofs, examples and counterexamples are provided in Section 5 of [47].

About Mosco-convergence. After [35], a sequence of functionals is called *Mosco-convergent* whenever it simultaneously $\Gamma\omega$- and Γs-converges to the same function. This notion is rich of properties, and is often assumed in the study of stability. We do not consider it in this paper, nor we did in [47] and [51], because in this research we are concerned not only with structural stability but also with structural compactness, and results of Mosco-compactness are rather rare, as far as this author can see.

For instance, it is known that monotone sequences of lower semicontinuous convex functionals Mosco-converge, see e.g. [1] p. 298. But monotonicity is a rather restrictive assumption, especially in the present context. For instance, it does not apply to Fenchel functions $V \times V' \to \mathbf{R} \cup \{+\infty\} : (v, v^*) \mapsto \varphi(v) + \varphi^*(v^*)$, since these functions are self-dual (in the sense that we already specified), and conjugation inverts inequalities, that is, $\varphi_1 \le \varphi_2$ entails $\varphi_2^* \le \varphi_1^*$. The application of this result to the larger class of representative functions a priori is not excluded, although it may look unlikely.

8. Evolutionary Γ-convergence of weak-type, and structural stability of maximal monotone flows

In preparation for Theorem 8.3, first we announce an extension of De Giorgi's notion of Γ-convergence to operators (rather than functionals) that act on time-dependent functions ranging in a Banach space X, and state a related result of compactness, see [51]. Successively we claim a result of structural compactness and stability for the initial-value problem for equations governed by maximal monotone operators, see [47] and [51]. This may be regarded as a first step towards more general problems, like doubly-nonlinear flows (see Sections 7, 8 of [51]) and pseudo-monotone flows (see e.g. [49]).

Let X be a real separable and reflexive Banach space, and $p \in [1, +\infty[$. For any operator $f : L^p(0, T; X) \to L^1(0, T) : w \mapsto f_w$, let us set

$$[f, \xi](w) = \int_0^T f_w(t)\xi(t)\, d\mu(t) \quad \forall w \in L^p(0, T; X),\ \forall \xi \in L^\infty(0, T). \tag{8.1}$$

It is of particular interest to deal with memoryless operators of the form

$$\begin{aligned} &f_w(t) = \varphi(t, w(t)) && \forall w \in L^p(0, T; X), \text{ for a.e. } t \in]0, T[, \\ &\varphi :]0, T[\times X \to \mathbf{R}^+ && \text{being a } \textit{normal function}, \end{aligned} \tag{8.2}$$

i.e., φ is globally measurable and $\varphi(t,\cdot)$ is lower semicontinuous for a.e. $t \in]0,T[$.

THEOREM 8.1 ([51]). *Let X, p be as above, and $\{\varphi_n\}$ be a sequence of normal functions $]0,T[\times X \to \mathbf{R}^+$. Assume that*

$$\exists C_1, C_2, C_3 > 0 : \forall n, \forall w \in X, \\ C_1\|w\|_X^p \le \varphi_n(t,w) \le C_2\|w\|_X^p + C_3 \quad \text{for a.e. } t \in]0,T[, \forall n, \tag{8.3}$$

$$\varphi_n(t,0) = 0 \quad \text{for a.e. } t \in]0,T[, \forall n. \tag{8.4}$$

Then: (i) *There exists a normal function $\varphi :]0,T[\times X \to \mathbf{R}^+$ such that, defining the operators $f, f_n : L^2_\mu(0,T;X) \to L^1_\mu(0,T)$ for any n as in* (8.2), *possibly extracting a subsequence,* (*denoting by $L^\infty_+(0,T)$ the cone of the nonnegative functions of $L^\infty(0,T)$*))

$$[f_n,\xi] \text{ sequentially weakly } \Gamma\text{-converges to } [f,\xi] \\ \text{in } L^p(0,T;X), \forall \xi \in L^\infty_+(0,T). \tag{8.5}$$

(ii) *If φ_n does not explicitly depend on t for any n, then the same holds for φ.*

The argument of [51] rests upon the classical Lemma 7.2 of Γ-compactness, on other properties of the ordinary Γ-convergence, and on Theorem 5.1 of [28].

REMARKS 8.2. (i) Whenever (8.5) holds, we shall say that

$$f_n \text{ sequentially } \Gamma\text{-converges to } f \\ \text{in the weak topology of } L^p(0,T;X) \text{ and} \\ \text{in the weak topology of } L^1(0,T). \tag{8.6}$$

This definition of evolutionary Γ-convergence is not equivalent to that of [40], nor to that of [20], [33], [34]. In those works Γ-convergence is assumed for almost any $t \in]0,T[$, whereas here it is set just weakly in $L^1(0,T)$.

(ii) The present definition of parameter-dependent Γ-convergence is based on testing on functions of time, but it might equivalently be reformulated in terms of set-valued functions. The present set-up fits that of Chap. 16 of [19], see also references therein. However here we deal with functions of time that are integrable w.r.t. the ordinary Lebesgue measure, rather than more general measures as in [19]. Anyway, [19] does not encompass the above theorem, which is based on [28].

A quasilinear parabolic equation in abstract form. Let V and H be real separable Hilbert spaces, with

$$V \subset H = H' \subset V' \tag{8.7}$$

with continuous, compact and dense embeddings.

Let a sequence $\{\alpha_n\}$ of operators and one $\{h_n\}$ of functions be such that

(8.8) $$\forall n, \alpha_n : V \to \mathscr{P}(V') \text{ is maximal monotone,}$$

(8.9) $$\exists a, b > 0 : \forall n, \forall (v, v^*) \in \operatorname{graph}(\alpha_n), \quad \langle v^*, v\rangle \geq a\|v\|_V^2 - b,$$

(8.10) $$\exists C_1, C_2 > 0 : \forall n, \forall (v, v^*) \in \operatorname{graph}(\alpha_n), \quad \|v^*\|_{V'} \leq C_1\|v\|_V + C_2,$$

(8.11) $$\alpha_n(0) \ni 0 \quad \forall n,$$

(8.12) $$u_n^0 \to u^0 \quad \text{in } H,$$

(8.13) $$h_n \to h \quad \text{in } L^2(0, T; V').$$

Notice that the condition (8.11) is not restrictive: if it is not satisfied, it may be recovered by selecting any $v_n^* \in \alpha_n(0)$ for any n, and then replacing α_n by $\bar{\alpha}_n(\cdot) = \alpha_n(\cdot) - v_n^*$.

For instance, if Ω is a bounded Lipschitz domain of $\mathbf{R}^N$ $(N \geq 1)$, and $\{\vec{\gamma}_n\}$ is a sequence of maximal monotone mappings $\mathbf{R}^N \to \mathscr{P}(\mathbf{R}^N)$, one may take

(8.14) $$H = L^2(\Omega), \quad V = H_0^1(\Omega), \quad \alpha_n(v) = -\nabla \cdot \vec{\gamma}_n(\nabla v) \quad \text{in } \mathscr{D}'(\Omega).$$

If $N = 3$, denoting the outward-oriented unit normal vector-field on $\partial\Omega$ by $\vec{\nu}$, one may also deal with

(8.15) $$\begin{aligned} &H = \{\vec{v} \in L^2(\Omega)^3 : \nabla \cdot \vec{v} = 0 \text{ a.e. in } \Omega)\},\\ &V = \{\vec{v} \in H : \nabla \times \vec{v} \in L^2(\Omega)^3, \vec{\nu} \times \vec{v} = \vec{0} \text{ in } H^{-1/2}(\partial\Omega)^3\},\\ &\vec{\alpha}_n(\vec{v}) = \nabla \times \vec{\gamma}_n(\nabla \times \vec{v}) \quad \text{in } \mathscr{D}'(\Omega)^3, \ \forall \vec{v} \in V. \end{aligned}$$

We are concerned with the structural properties of the following sequence of flows in weak form:

(8.16) $$\begin{cases} u_n \in L^2(0, T; V) \cap H^1(0, T; V'),\\ D_t u_n + \alpha_n(u_n) \ni h_n \quad \text{in } V', \text{ a.e. in }]0, T[,\\ u(0) = u^0. \end{cases}$$

It is well known, see e.g. [3], [11], [52], that, under the above assumptions, for any n this initial-value problem has one and only one solution u_n, and that the sequence $\{u_n\}$ is bounded in $\mathscr{X}$.

Several regularity results are also known to hold for this problem. At variance with [47], here we do not use them since their assumptions are somehow restrictive.

Notice that any operator $\alpha_n : V \to \mathscr{P}(V')$ canonically induces a global-in-time operator $\hat{\alpha}_n : \mathscr{V} \to \mathscr{P}(\mathscr{V}')$. We may thus reformulate the equation $(8.16)_1$ globally-in-time. The structural compactness and stability of the corresponding initial-value problem were proved in Theorem 8.3 of [47]. However a priori this allows for the onset of long-memory effects. The following more satisfactory result is proved in Theorem 6.3 of [51] for the pointwise-in-time formulation.

THEOREM 8.3 ([51]) (Structural compactness and stability). *Let* (8.7)–(8.13) *be fulfilled, and for any n let* u_n *be the solution of problem* (8.16). *Then:*

(i) *There exists* $u \in \mathcal{V}$ *such that, possibly extracting a subsequence,*

$$u_n \rightharpoonup u \quad \text{in } \mathcal{V}. \tag{8.17}$$

(ii) *There exists a function* $\varphi \in \mathcal{F}(V)$ *such that, setting*

$$\varphi_n = (\pi + I_{\alpha_n})^{**} \ (\in \mathcal{F}(V)), \tag{8.18}$$

$$\psi_{n,w}(t) = \varphi_n(w(t)), \quad \psi_w(t) = \varphi(w(t)), \tag{8.19}$$
$$\text{for a.e. } t \in]0, T[, \ \forall w \in L^2_\mu(0, T; V \times V'), \ \forall n,$$

then $\psi_n, \psi : L^2_\mu(0, T; V \times V') \to L^1_\mu(0, T)$ *and, possibly extracting a subsequence,*

$$\begin{aligned} &\psi_n \text{ sequentially } \Gamma\text{-converges to } \psi \\ &\text{in the topology } \tilde{\pi} \text{ of } L^2_\mu(0, T; V \times V') \text{ and} \\ &\text{in the weak topology of } L^1_\mu(0, T) \ (cf. \ (8.5)). \end{aligned} \tag{8.20}$$

(iii) *Denoting by* $\alpha : V \to \mathcal{P}(V')$ *the monotone operator that is represented by* φ, u *solves the corresponding initial-value problem*

$$\begin{cases} u \in \mathcal{V}, \\ D_t u + \alpha(u) \ni h \quad \text{in } V', \text{ a.e. in }]0, T[, \\ u(0) = u^0. \end{cases} \tag{8.21}$$

This result may also be extended to more general flows, including doubly nonlinear flows of the form

$$D_t \partial\gamma(u) + \alpha(u) \ni u^*, \tag{8.22}$$

$$\alpha(D_t u) + \partial\gamma(u) \ni u^*; \tag{8.23}$$

for a maximal monotone operator α and a convex and lower semicontinuous function γ; see [51].

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