Calculus of Variations — *A notion of nonlocal Gaussian curvature*, by Paolo Podio-Guidugli, communicated on 13 November 2015.

Dedicated to the memory of Professor Enrico Magenes

Abstract. — A local notion of directional-curvature vector is proposed, which admits a nonlocal version that turns out to be the key ingredient to define both nonlocal mean curvature in the manner of [1] and [4] and nonlocal Gaussian curvature, a new notion.

1. Introduction

There is a triple point of geometry, mechanics and mathematical analysis, where three notions are met, namely, the geometrical notion of *mean curvature* of a surface, the mechanical notion of *diffusion* of a substance, and the analytical notion of *Laplacian* of a scalar-valued function. The many connections among these notions are well known; all three are *local*, in that they specify a property at a chosen position in space (and at a chosen time, but I here concentrate on spatial locality) in terms of information items gathered at that position. In recent years, Luis Caffarelli and others [5, 6, 7, 8, 9, 10] have found many good reasons to develop *nonlocal* versions of those notions. It seems to me that the interconnections of such nonlocal notions are far from being completely elucidated, especially with regard to the mechanical *coté*. This paper deals with the geometrical notion of nonlocal curvature.

Recently, a notion of nonlocal *mean* curvature (NMC) has been proposed and carefully analyzed in [1] and [4]. Moreover, in [1], a notion of nonlocal *directional* curvature (NDC) has also been proposed and it has been shown that such NMC is nothing but a circular average of NDCs. Both NMC and NDC are scalar objects, both tend to their local counterparts in an appropriate limit. I move from the latter to introduce a nonlocal directional-curvature *vector* (NDCV) and make it the essential ingredient not only, as expected, to define both NMC and NDC but also to define a new notion of nonlocal *Gaussian* curvature (NGC). I believe that both concepts, NMC and NGC, will prove relevant for certain applications in mechanics different from those which motivated Caffarelli and coworkers, e.g., to capture the nonlinear behavior of thin structures in general

and, in particular, the behavior of those 'soft' structures made of polymeric gels that may swell.

Sections 2, 3 and 4 are meant to motivate why there are good reasons to say that mean curvature plays a pivotal role at a triple point of geometry, mechanics and analysis. Firstly, in Section 2, I quickly recap the reasons why this is the case for local mean curvature; that the role of nonlocal mean curvature is the same is argued in Section 4; Section 3, where local and nonlocal diffusion is considered, bridges between the two adjacent sections. In Section 5, the bulk of this paper, the geometrical notion of directional-curvature vector is introduced and it is shown how, in both its versions, local and not, it allows to define both curvatures, mean and Gaussian; the notion of nonlocal Gaussian curvature I here propose is, to my knowledge, new.

2. MINIMAL SURFACES, MOTION BY MEAN CURVATURE, AND DIFFUSION

A classic problem in calculus of variations is to find a minimal-area surface supported on an assigned boundary curve. *Minimal surfaces*, the critical points of the minimum-area functional, are surfaces with constant mean curvature (spheres, if closed).

Motion by mean curvature is the motion of a material but massless surface (possibly identified with the complete boundary ∂E of a bounded set $E \subset \mathbb{R}^n$), driven by surface tension, assumed proportional to mean curvature H, and opposed by a dissipative force, assumed proportional to normal velocity V:

$$(1) \qquad\qquad V = H.$$

As is well-known, this nonlinear parabolic equation is likely to develop singularities. It was proposed in [12] to approximate motion by mean curvature of ∂E, in a short time interval, by evolving the function

$$(2) \qquad\qquad \tilde{\chi}_E(x) := \begin{cases} +1 & \text{if } x \in E, \\ -1 & \text{if } x \in \mathscr{C}E, \end{cases}$$

according to the linear heat equation:

$$(3) \qquad\qquad \partial_t u(x, t) = \Delta u(x, t), \quad u(x, 0) = \tilde{\chi}_E(x)$$

(in (2), $\mathscr{C}E := \mathbb{R}^n \backslash E$); to approximate mean-curvature evolution by a process of Fickian *diffusion* brings in the prototypical elliptic operator, the *Laplacian*.

We now see why the three standard notions of minimal surfaces, motion by mean curvature, and diffusion, sit at the triple point of mathematical analysis, geometry, and mechanics we mentioned in the Introduction. When consistently generalized, those three notions continue to be interlinkable in the same manner: mean curvature is the link.

3. Diffusion

Modelling diffusion is a central issue in mechanics, both continuum and statistical. While all standard models are local, hereafter I take freely from an inspiring lecture by Luis Caffarelli [6] both to offer an interesting interpretation of the Laplacian, the differential operator associated with standard local diffusion, and to introduce the integral operators associated with nonlocal diffusion; needless to say, possible mistakes and misconceptions are all mine.

3.1. Local diffusion

Let $B_\varepsilon(x)$ be a ball of radius ε centered at a point x of a region where a smooth scalar-valued field u is defined. In view of the Mean Value Theorem, of the fact that $\Delta(\cdot) = \mathrm{Div}(\nabla(\cdot))$, and of the divergence lemma, there is a point $\bar{x} \in B_\varepsilon(x)$ for which

$$\Delta u(\bar{x}) = \fint_{B_\varepsilon(x)} \Delta u = \frac{1}{vol(B_\varepsilon(x))} \int_{\partial B_\varepsilon(x)} \partial_n u;$$

here, $\partial_n u(y) = \nabla u(y) \cdot \boldsymbol{n}(y)$ is the derivative of u at a point $y = x + \varepsilon\boldsymbol{n}(y)$ of $\partial B_\varepsilon(x)$, in the direction of the outer normal $\boldsymbol{n}(y)$. Now, given that

$$vol(B_\varepsilon(x)) = \frac{1}{3}\varepsilon\, area(\partial B_\varepsilon(x)),$$

$$\lim_{\varepsilon \to 0} \Delta u(\bar{x}) = \Delta u(x) \quad \text{and} \quad \lim_{\varepsilon \to 0}\left(\frac{u(y) - u(x)}{\varepsilon} - \partial_n u(y)\right) = 0,$$

we have that

(4)
$$\Delta u(x) = 3 \lim_{\varepsilon \to 0} \varepsilon^{-2} \frac{1}{area(\partial B_\varepsilon(x))} \int_{\partial B_\varepsilon(x)} ((u(y) - u(x))$$

$$= 3 \lim_{\varepsilon \to 0} \varepsilon^{-2}\left(\fint_{\partial B_\varepsilon(x)} u - u(x)\right);$$

this shows that *the Laplacian is the limit of a scaled mean*. Here is how this analytical finding is interpreted in [6]: "The density at the point x compares itself with its values in a tiny surrounding ball. The difference between the surrounding average and the value at x, properly scaled, is the Laplacian."

Moreover, according to a stripped-to-the-bone evolution equation of type (3), we see that, within a local descriptive framework, the diffusion phenomenon is such that asymptotically (that is, provided $\partial_t u \to 0$ for $t \to \infty$) the field of interest takes at any given point the value of its average over the boundary of any of its neighbourhoods.

3.2. *Nonlocal diffusion*

In [6], Caffarelli terms "integral-diffusion" processes the solutions of evolution equations of the following type:

$$(5) \qquad \partial_t u(x, t) = L[u](x, t), \quad L[u](x, t) = \int [(u(y, t) - (u(x, t)]\kappa(x, y) \, dy,$$

where κ is a positive and symmetric kernel; along such processes, $u(x)$ compares itself with $u(y)$ over a fixed region, weighting discrepancies with $\kappa(x, y)$; an asymptotic behavior similar to that of the solutions of the heat equation is expected.

Caffarelli gives two examples of an equation of type (5): the *geostrophic equation*, a model for the evolution of ocean temperature due to the ocean-atmosphere interaction; and the *Lévy-Khintchine* equation:

$$(6) \qquad \partial_t u(x, t) = \int [(u(x + y, t) + (u(x - y, t) - 2u(x, t)] \, dm(y),$$

whose solutions are processes during which particles jump randomly in space, in a manner independent of their past path (note that $[(u(x + y, t) + (u(x - y, t) - 2u(x, t)]$ may be regarded as an approximation of the second gradient of u at x, sort of a Laplacian in disguise).

The operator

$$(7) \qquad ((-\Delta)^s u)(x) := c_{n,s} \, PV \int_{\mathbb{R}^n} (u(x) - u(y))|x - y|^{-(n+2s)} \, dy, \quad s \in (0, 1),$$

is called the *s-Laplacian* (aka *fractional Laplacian*); the *principal value PV* of the improper integral

$$\int_{\mathbb{R}^n} f(x) \, dm(x), \quad dm(x) = \kappa(x) \, dx, \quad \text{with } \kappa \text{ a singular integral kernel,}$$

is defined as follows:

$$PV \int_{\mathbb{R}^n} f(x) \, dm(x) := \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} f(x) \, dm(x).$$

This definition has the integral-diffusion format given in (5), provided one sets:

$$\kappa(x, y) = c_{n,s}|x - y|^{-(n+2s)}.$$

Another essentially equivalent definition, which has the format of equation (6), is:

$$((-\Delta)^s u)(x) = c_{n,s} \, PV \int_{\mathbb{R}^n} (u(x + y) + u(x - y) - 2u(x))|x - y|^{-(n+2s)} | \, dy,$$

where again $s \in (0, 1)$; both definitions are taken from the website www.ma.utexas.edu/mediawiki/index.php/Fractional_Laplacian.

No matter what definition one takes, the limit for $s \to 1$ restitutes the standard local Laplacian. Indeed, with reference to the fractional evolution equation

$$(8) \qquad \partial_t u + (-\Delta)^s u = 0,$$

the Fourier-Transform symbol of the fractional Laplacian is $|\xi|^{2s}$; for $s \to 1$, the standard Laplacian $(-\Delta)$, whose F-T symbol is $|\xi|^2$, is recovered (but no integral representation of the form (7) is possible for it); moreover, for $s \to 0$, the limit operator is minus the identity, whence (8) reduces to

$$\partial u + u = 0 \quad \Rightarrow \quad u(x, t) = u_0(x) \exp(-t).$$

We see that the s-dependence induces a sort of *uncertainty-principle situation*: a substance can achieve a continuum of states in between maximal diffusion accompanied by ever changing position of all its particles (the Fickian model, for $s = 1$) and no diffusion and hence no positional changes (for $s = 0$).

4. Sets of minimal s-perimeter, motions by s-nonlocal mean curvature, and s-diffusion

We learn in Section 2.2 of [1] about a recent generalization of the classic minimal surface problem, that is, the minimization of the so-called *s-perimeter functional* $\mathrm{Per}_s(E, U)$:

for a given set $U \subset \mathbb{R}^n$, to find a measurable set $E \subset \mathbb{R}^n$ such that (i) $E \backslash U$ is not modifiable (so that U plays a role of boundary datum, analogous to the role of the support curve in the classical case); (ii) the s-perimeter of E is minimal, in the sense that E minimizes the functional

$$\mathrm{Per}_s(E, U) := \mathscr{F}(E \cap U, U \backslash E) + \mathscr{F}(E \cap U, \mathscr{C}U \cap \mathscr{C}E) + \mathscr{F}(E \backslash U, U \backslash E),$$

where

$$\mathscr{F}(A, B) := \frac{1}{\omega_{n-1}} \int_A dx \int_B dy \frac{1}{|y - x|^{n+2s}}, \quad s \in (0, 1/2),$$

mimics a *mutual distance interaction* between sets A and B.

The reasons why the term "perimeter" is used in place of the classical term "area" are neatly explained in [2]; apparently, what prompts the change in nomenclature is that surfaces are regarded as (part of) the (boundary $\equiv$)perimeter of a set.

The critical points of the s-perimeter functional satisfy in a suitable weak sense the Euler-Lagrange equation:

$$(9) \qquad \int_{\mathbb{R}^n} \tilde{\chi}_E(y) |x - y|^{-(n+2s)} \, dy = 0;$$

moreover, it can be shown [3, 10] that the classical setting of this minimization problem is recovered in the following limit:

$$(10) \qquad \lim_{s \to 1/2} (1 - 2s) \operatorname{Per}_s(E, B_r) = \operatorname{Per}(E, B_r) \quad \text{for a.e. } r > 0.$$

On recalling that the critical points of the minimum-area functional are surfaces with constant mean curvature, the above results justify the idea of making use of the left side of equation (9) to introduce a *s*-dependent notion of *nonlocal mean curvature*.

   When this is done, it is only natural to study *motion by s-nonlocal mean curvature*, both as is and in its short-time approximation obtained by evolving the function $\tilde{\chi}$ defined in (2) according to the *s*-Laplacian (see [6] and [1]); that is, to approximate motions by *s*-nonlocal mean curvature by means of related *motions by s-diffusion*.

5. Curvatures, mean and Gaussian

The typical surface S we consider is a part of the complete boundary ∂E of a simply-connected bounded subset E of the three-dimensional Euclidean point space, that we identify for convenience with $\mathbb{R}^3$: $S \subset \partial E$; we denote by $\mathscr{T}_E(x)$ the tangent plane of S at its point x. Were we working in a continuum mechanical context and would it be important to distinguish between the referential and current placements of a body of interest, then we would think of E as the space region occupied by a typical body part at the current time.

5.1. Local notions

5.1.1. Algebraic definitions. In classical differential geometry of C^2-surfaces embedded in $\mathbb{R}^3$ [11], for

$$(11) \qquad \mathbb{R}^2 \ni (\zeta^1, \zeta^2) \leftrightarrow x = \hat{x}(\zeta^1, \zeta^2)$$

the equation of an orientable and oriented surface S in terms of the curvilinear coordinates ζ^α,

- the vector

$$(12) \qquad \boldsymbol{n}(x) := \frac{\boldsymbol{e}_1(x) \times \boldsymbol{e}_2(x)}{|\boldsymbol{e}_1 \times \boldsymbol{e}_2|}, \quad \boldsymbol{e}_\alpha := \partial_{\zeta^\alpha} x \quad (\alpha = 1, 2),$$

   is the unit normal at a typical point x;
- the *curvature tensor* $\boldsymbol{K}$ is the surface gradient of the normal field:

$$(13) \qquad \boldsymbol{K}(x) := {}^{s}\nabla\boldsymbol{n}(x) := -\boldsymbol{n}_{,\alpha}(x) \otimes \boldsymbol{e}^\alpha(x), \quad \boldsymbol{e}^\alpha := \partial_x \zeta^\alpha \quad (\alpha = 1, 2)$$

   (here the symbol $\otimes$ denotes dyadic product of vectors); note that $\boldsymbol{K}(x)$ is a linear map of $\mathscr{T}_E(x)$ into itself;

- the *mean curvature H* and the *Gaussian curvature G* are defined as follows in terms of the orthogonal invariants of $\boldsymbol{K}$:

(14)
$$H(x) := \frac{1}{2} \operatorname{tr} \boldsymbol{K}(x), \quad G(x) := \det \boldsymbol{K}(x),$$

and are therefore independent of any representation of type (11) of surface S. On using normal coordinates centered at anyone of its regular points, a C^2-surface can be specified locally as the graph of a smooth mapping whose Hessian can be identified (to within its sign) with the curvature tensor, so that in particular, as observed in [1], the mean curvature is measured by the Laplacian.

Definitions (14) are algebraic in nature; they cannot be carried over to nonlocal situations. There are however alternative 'transportable' definitions, which can be imitated to define nonlocal mean and Gaussian curvatures when a direct imitation of the above algebraic definitions turns out to be impracticable.

5.1.2. Geometric definitions. For $e \in \mathscr{T}_E(x)$,

(15)
$$K_e(x) := \boldsymbol{K}(x) \cdot \boldsymbol{e} \otimes \boldsymbol{e}$$

is the curvature of the so-called *e-normal section* of S, that is, of the curve obtained by intersecting S with the plane through x containing both $\boldsymbol{n}(x)$ and $\boldsymbol{e}$ (Figure 1); in [1], K_e is called the *directional curvature* of S in the direction $\boldsymbol{e}$, a scalar field over S.

It is the matter of an elementary calculation to compute the circular average of the directional curvature, and find it equal to the mean curvature:

(16)
$$\fint K_e(x) = \fint \boldsymbol{K}(x) \cdot \boldsymbol{e} \otimes \boldsymbol{e} = H(x),$$
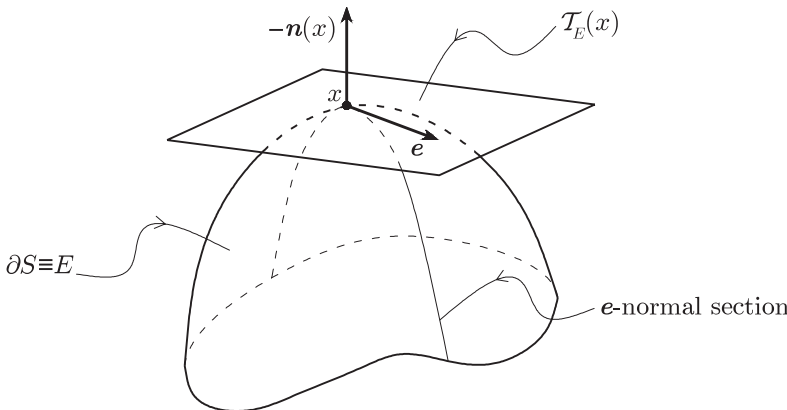

Figure 1

because

$$\oint e \otimes e := \frac{1}{2\pi} \int_0^{2\pi} e(\varphi) \otimes e(\varphi)\, d\varphi = \frac{1}{2}\left(I - n \otimes n\right)$$

(cf. Theorem 4 of [1]; here I denotes the identity mapping of V^3, the translation vector space associated with $\mathbb{R}^3$, onto itself). Furthermore, for

$$\tilde{K} := K + n \otimes n,$$

we have that

(17) $$\begin{aligned} G(x) &= \det K(x) = \tilde{K}(x)e_\alpha \times \tilde{K}(x)e_\beta \cdot \tilde{K}(x)n(x) \\ &= n_{,\alpha}(x) \times n_{,\beta}(x) \cdot n(x). \end{aligned}$$

We are now in a position to introduce a geometric notion that not only allows for expressions of the local mean and Gaussian curvatures alternative to (16) and (17), and hence to the standard algebraic expressions (14), but also allows to define nonlocal mean and Gaussian curvatures in integral forms that reduce to the corresponding local differential forms in an appropriate limit. Such notion is that of *directional-curvature vector* in the direction e at point $x \in S$:

(18) $$K(x)e =: k_e(x),$$

with which, as we shall show shortly, the following alternatives to definitions (14) can be formulated:

(19) $$H := \oint (k_e \cdot e), \quad G := k_{e_\alpha} \times k_{e_\beta} \cdot n.$$

Note that, given (11), the two directional-curvature vectors

$$k_{e_\alpha}(x) := -n_{,\alpha}(x) \quad (\alpha = 1,2)$$

carry an information content about the surface S at its point x which is completely equivalent to that of the curvature tensor $K(x)$; moreover,

$$K(x) = 2\oint k_e(x) \otimes e.$$

As we shall see next, the local notion of directional-curvature vector can be given a nonlocal version that turns out to be the key ingredient to define nonlocal mean and Gaussian curvatures.

5.2. *Nonlocal notions*

5.2.1. NMC and NDC. We begin by reproducing the definitions of nonlocal mean and directional curvatures given in [1].

– (NMC) For E an open set of $\mathbb{R}^n$ ($n \geq 3$) with a C^2 boundary, the *nonlocal mean curvature* at $x \in \partial E$ is:

$$(20) \qquad H_s(x) := \frac{1}{\omega_{n-2}} \int_{\mathbb{R}^n} \tilde{\chi}_E(y) |x - y|^{-(n+2s)} \, dy, \quad s \in (0, 1/2),$$

where taking the principal value of the integral is implied and ω_{n-2} is the Hausdorff measure of the S^{n-2} unit sphere. The definition given in [4] is the same, to within a multiplicative constant.

– (NDC) Let

$$\pi(x, e) := \{y \in \mathbb{R}^n \mid y = \rho e + h\boldsymbol{n}(x), \rho > 0, h \in \mathbb{R}\}$$

denote the half-plane through a point $x \in \partial E$ defined by the unit vector $\boldsymbol{e} \in \mathscr{T}_E(x)$ and the normal $\boldsymbol{n}(x)$ (Figure 2); moreover, let

$$y' = x + \rho e, \quad x' = x.$$

The *nonlocal directional curvature* at x in the direction e is:

$$(21) \qquad K_{s,e}(x) := \int_{\pi(x,e)} |y' - x'|^{n-2} \tilde{\chi}_E(y) |x - y|^{-(n+2s)} \, dy, \quad s \in (0, 1/2),$$

where again taking the principal value of the integral is implied: precisely, here

$$PV \int_{\pi(x,e)} \equiv \lim_{\varepsilon \to 0} \int_{\pi(x,e) \setminus B_\varepsilon(x)} .$$
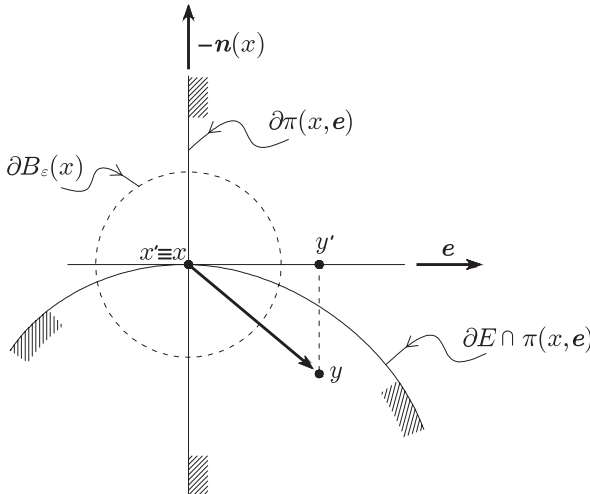
It is proved in [1] that

Figure 2

• the circular average of the NDC is equal to the NMC:

$$(22) \qquad H_s = \frac{1}{\omega^{n-2}} \int_{S^{n-2}} K_{s,e} \, d\mathcal{H}^{n-2}$$

(recall the corresponding local relation in (16));
• nonlocal directional and mean curvatures tend to their local counterparts in the limit when $s \to 1/2$; precisely,

$$(23) \qquad \lim_{s\to 1/2}(1-2s)K_{s,e} = K_e, \qquad \lim_{s\to 1/2}(1-2s)H_s = H$$

(cf. (10)).

5.2.2. NDCV. To arrive at my definition of nonlocal directional-curvature vector, I find it expedient to reconstruct in part the proof of (22) given in [1]; for simplicity, I concentrate on the case of interest in applications, and take $n = 3$.

With reference to Figure 3, a point $y \in \mathbb{R}^3$ is assigned cylindrical (not polar!) coordinates (ρ, h, φ) with respect to the origin x, a point of the tangent plane $\mathcal{T}_E(x)$, which serves as the coordinate plane $h = 0$; φ is the angle of the unit vector $e(\varphi)$ to an arbitrarily fixed vector in $\mathcal{T}_E(x)$; the volume measure is $\rho \, d\rho \, dh \, d\varphi$, and

$$\int_{\mathbb{R}^3} g(y) \, dy = \int_{S^1} d\varphi \int_0^{+\infty} d\rho \int_{-\infty}^{+\infty} dh \, \rho g(\rho, h, \varphi).$$
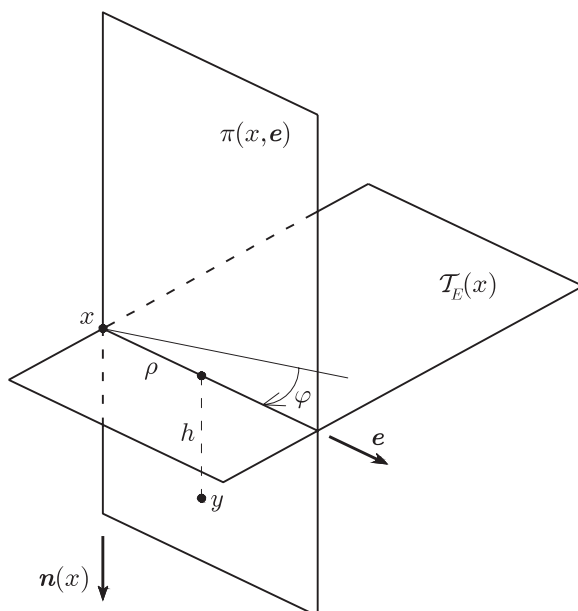

Figure 3

Now, having fixed x, pick $g(y) = |y - x|^{-(3+2s)} \tilde{\chi}_E(y)$, so that, in view of definition (20),

$$PV \int_{\mathbb{R}^3} g(y) \, dy = \omega_1 H_s.$$

Whatever $s \in (0, 1/2)$, (21) can be written as

$$(24) \quad K_{s,e}(x) = PV \int_{\pi(x,e)} ((y - x) \cdot e) g(y) \, dy = \left(PV \int_{\pi(x,e)} g(y)(y - x) \, dy \right) \cdot e.$$

Then, on setting

$$\boldsymbol{k}_{s,e}(x) := (\boldsymbol{I} - \boldsymbol{n}(x) \otimes \boldsymbol{n}(x)) PV \int_{\pi(x,e)} g(y)(y - x) \, dy,$$

we have that, whatever $x \in \partial E$,

$$(25) \qquad\qquad K_{s,e}(x) = \boldsymbol{k}_{s,e}(x) \cdot \boldsymbol{e}.$$

I propose to call $\boldsymbol{k}_{s,e}(x)$ the *nonlocal directional-curvature vector* (NVDC) at a point $x \in \partial E$. Importantly, with the use of (25) and (18), the first limit result in (23) implies that

$$\lim_{s \to 1/2} (1 - 2s) \boldsymbol{k}_{s,e}(x) = \boldsymbol{k}_e(x),$$

whatever $x \in \partial E$ and $e \in \mathcal{T}_E(x)$. Note, moreover, that

$$H_s(x) = \fint (\boldsymbol{k}_{s,e}(x) \cdot \boldsymbol{e}),$$

a relation to be compared with the first of (19).

5.2.3. NGC. I propose the following definition of *nonlocal Gaussian curvature* at $x \in \partial E$:

$$(26) \qquad\qquad G_s(x) := \boldsymbol{k}_{s,e_m}(x) \times \boldsymbol{k}_{s,e_M}(x) \cdot \boldsymbol{n}(x), \quad s \in (0, 1/2),$$

where e_m and e_M denote the unit vectors in $\mathcal{T}_E(x)$ for which the NDC is, respectively, minimal and maximal (recall the second of the local definitions (19)).

Needless to say, this definition works fine provided that, for a chosen pair $(x, s) \in (\partial E \times (0, 1/2)$, the extremals of the mapping

$$S^1 \in \boldsymbol{e} \mapsto K_{s,e} \in \mathbb{R}$$

are uniquely determined. This might not to be the case even for sets E with a smooth boundary (recall that here, as is done in the quoted literature, the surfaces

under exam are of class C^2), whence the need for an 'unconditionally robust' definition of NGC, possibly different from (26). In this connection, it is worth recalling Theorem 11 of [1], which states that for whatever disjoint subsets of S^1 a smooth surface can be found whose NDC takes its minimal value in one of the two subsets and its maximal value in the other.

REFERENCES

[1] N. ABATANGELO - E. VALDINOCI, *A notion of nonlocal curvature*, Numer. Funct. Anal. Optim. 35 (2014), 793–815.

[2] G. ALBERTI, *Distributional Jacobian and singularities of Sobolev maps*, Ric. Mat. LIV(2) (2006), 375–394.

[3] L. AMBROSIO - G. DE PHILIPPIS - L. MARTINAZZI, Γ-*convergence of nonlocal perimeter functionals*, Manuscripta Math. 134(3–4) (2011), 377–403.

[4] X. CABRÉ - M. M. FALL - J. SOLÀ-MORALES - T. WETH, *Curves and surface with constant nonlocal mean curvature: meeting Alexandrov and Delauney*, arXiv:1503.00469v1 [math.AP] 2 Mar 2015.

[5] L. CAFFARELLI, *Surfaces minimizing nonlocal energies*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 20 (2009), 281–299.

[6] L. CAFFARELLI, *The mathematical idea of diffusion*, Enrico Magenes Lecture, Pavia, March 2013.

[7] L. CAFFARELLI - J.-M. ROQUEJOFFRE - O. SAVIN, *Nonlocal minimal surfaces*, Comm. Pure Appl. Math., 63(9) (2010), 1111–1144.

[8] L. CAFFARELLI - P. SOUGANIDIS, *Convergence of nonlocal threshold dynamics approximations to front propagation*, Arch. Rational Mech. Anal. 195(1) (2010), 1–23.

[9] L. CAFFARELLI - E. VALDINOCI, *Uniform estimates and limiting arguments for nonlocal minimal surfaces*, Calc. Var. Partial Dif, 41(1–2) (2011), 203–240.

[10] L. CAFFARELLI - E. VALDINOCI, *Regularity properties of nonlocal minimal surfaces via limiting arguments*, Adv. Math. 248 (2013), 843–871.

[11] M. P. DO CARMO, *Differential Geometry of Curves and Surfaces*, Prentice Hall, New Jersey, 1976.

[12] B. MERRIMAN - J. K. BENCE - S. OSHER, *Diffusion Generated Motion by Mean Curvature*, CAM report. Dept. Math., University of California, Los Angeles, 1992.

Paolo Podio-Guidugli
Accademia Nazionale dei Lincei
Palazzo Corsini
Via della Lungara 10
00165 Rome, Italy
and
Department of Mathematics
University of Rome Tor Vergata
Via della Ricerca Scientifica 1
00133 Rome, Italy
ppg@uniroma2.it