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Measure and Integration — *Duality for A_∞ weights on the real line*, by Luigi D'Onofrio, Arturo Popoli and Roberta Schiattarella, communicated on 12 February 2016.

Abstract. — Under the same bounds on G_q-constants and A_p-constants, the optimal exponents for sharp inclusions between Gehring G_q-class of weights and Muckenhoupt A_p-class $(1 < p, q < \infty)$ are Hölder conjugate, if p and q are conjugate. This is a consequence of a representation theorem of A_∞ weights in terms of $W^{1,r}$-biSobolev maps and a duality result between G_q and A_p classes in dimension one. We prove also that sharp a priori bounds on constants correspond under the Hölder conjugate mapping $\phi(t) = \frac{t}{t-1}$.

Key words: Muckenhoupt weights, Gehring classes

Mathematics Subject Classification: 46E30, 26A48

1. Introduction

For a weight w on $\mathbb{R}^n$, i.e. for a locally integrable function $w : \mathbb{R}^n \to [0, +\infty[$ positive on a set of positive measure, we define the A_p*-constant* of w, $p > 1$, as

$$A_p(w) = \sup_Q ⨍_Q w \Big(⨍_Q w^{-\frac{1}{p-1}} \Big)^{p-1} \tag{1.1}$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the axes. Similarly, for a weight v on $\mathbb{R}^n$ we define the G_q*-constant* of v, $q > 1$, as

$$G_q(v) = \sup_Q \left[\frac{\big(⨍_Q v^q \big)^{\frac{1}{q}}}{⨍_Q v} \right]^{\frac{q}{q-1}}. \tag{1.2}$$

The class A_p of weights w such that $A_p(w) < \infty$ was introduced in 1972 by B. Muckenhoupt [30] for a characterization of weighted L^p-maximal inequalities (see also [9]). He proved that, for $w \in A_p$ such that $A_p(w) \leq A$, there exists $\rho < p$ such that $w \in A_\rho$ with $\rho = \rho(n, p, A)$ and $A_\rho(w) \leq C(n, \rho, p, A)$.

The class G_q of weights v such that $G_q(v) < \infty$ was introduced in 1973 by F. W. Gehring in the study of L^q-integrability of gradient of quasiconformal mappings. He proved that, for $v \in G_q$ with $G_q(v) \leq G$, there exists $r > q$ such that $v \in G_r$ with $r = r(n, q, G)$ and $G_r(v) \leq C(n, q, r, G)$.

We call ρ and r *self-improvement exponents*. In [37], [8], [22], [34] sharp self improvement exponents where precisely determined for $n = 1$.

Notice that the condition $G_q(v) < \infty$ corresponds to a reverse Hölder inequality

$$\left(\fint_Q v^q\right)^{\frac{1}{q}} \leq H \fint_Q v$$

with the same support Q, for any cube $Q \subset \mathbb{R}^n$ and for a certain $H > 1$. Interesting applications of such inequality to the solvability of the L^p-Dirichlet problem in the plane, in the sense of nontangential convergence and a priori L^p estimates, are obtained in [11], [12], [21] (see [40] for sharp results). Applications to regularity of quasiminima of the q-Dirichlet integral in one-dimension are described in [15] (see [1], [28] for sharp results).

Note that, by Hölder inequality

$$\text{if } 1 < p \leq \rho < \infty, \text{ then } A_p \subset A_\rho \text{ and } 1 \leq A_\rho(w) \leq A_p(w) \tag{1.3}$$

and

$$\text{if } 1 < q \leq r < \infty, \text{ then } G_r \subset G_q \text{ and } 1 \leq G_q(v) \leq G_r(v). \tag{1.4}$$

We are interested in the A_∞-class of weights given by

$$A_\infty = \bigcup_{p>1} A_p \tag{1.5}$$

and in the G_1-class of weights given by

$$G_1 = \bigcup_{q>1} G_q. \tag{1.6}$$

A relation between Muckenhoupt and Gehring classes was established by R. Coifman and C. Fefferman in 1974 (see [5]). Namely, they proved that

$$A_\infty = G_1. \tag{1.7}$$

In [19] the A_∞-constant of weights was introduced

$$A_\infty(w) = \sup_Q \fint_Q w \exp\left(\fint_Q \log\frac{1}{w}\right). \tag{1.8}$$

and the following result

$$w \in A_\infty \quad \text{iff} \quad A_\infty(w) < \infty$$

was proved.

In [11], [12] R. Fefferman proved that a weight v belongs to $G_1 = A_\infty$ iff there exists $C > 1$ such that, for any cube $Q \subset \mathbb{R}^n$

$$\fint_Q v \log \frac{v}{v_Q} \le C \fint_Q v$$

where $v_Q = \fint_Q v$. This result suggested the introduction in [29] of the G_1-constant of v

$$G_1(v) = \sup_Q \exp\Big(\fint_Q \frac{v}{v_Q} \log \frac{v}{v_Q}\Big). \tag{1.9}$$

As a consequence, we have

$$v \in G_1 \quad \text{iff} \quad G_1(v) < \infty.$$

An important issue is to establish limit relations for A_p-constants as $p \to \infty$ and for G_q-constants as $q \to 1$.

In [39] the formula

$$A_\infty(w) = \lim_{p \to \infty} A_p(w) \tag{1.10}$$

was established, while in [29] the formula

$$G_1(v) = \lim_{q \to 1} G_q(v) \tag{1.11}$$

was proved. We will give a simple proof of (1.11) relative to one dimensional case (Section 2). The facts that $G_1(v)$ is well related to G_q-constants of v and $A_\infty(w)$ is well related to A_p-constants of w, will be further analyzed also in connection with recent work of [3], [24].

There are natural questions to find the sharp A_p-class for a weight $w \in A_\infty$, and the sharp G_q-class for a weight $v \in G_1$, under suitable constraints. Let us quote [10], [26], [27], [41] for recent works on sharp exponents and constants in case $n = 1$. The question is: is anything missing in the exact relationships obtained in the literature?

One of our results is that:

For $n = 1$, given $G > 1$, one can find the sharp exponent $\sigma_1 = \sigma_1(G)$ such that for all $v \in G_1$ satisfying $G_1(v) \le G$ we have

$$v \in A_\sigma \quad \forall \sigma < \sigma_1.$$

This was a missing result from literature [3], [24].

Before stating Theorem 1.1, we give some definitions.

For $q > 1$, $G > 1$, let us introduce *the sharp transition exponents*

$$\begin{aligned} \sigma_q &= \sigma_q(G) \\ &= \inf\{\sigma \in]1, \infty[: A_\sigma(v) < \infty, \text{ for } v \in G_q \text{ such that } G_q(v) \le G\} \end{aligned} \tag{1.12}$$

and for $p > 1$, $A > 1$, set

$$(1.13) \qquad \begin{aligned} s_p &= s_p(A) \\ &= \sup\{s \in]1,\infty[\,:\, G_s(w) < \infty, \text{ for } w \in A_p \text{ such that } A_p(w) \le A\}. \end{aligned}$$

Theorem 1.1. *For $G > 1$, the sharp exponent $\sigma_1 = \sigma_1(G)$ such that for all $v \in G_1$ with $G_1(v) \le G$ we have $v \in A_\sigma$, $\forall \sigma < \sigma_1$, is given by*

$$\sigma_1(G) = \lim_{q \to 1} \sigma_q(G)$$

and coincides with the positive solution $x = \sigma_1(G)$ to the equation

$$(1.14) \qquad x e^{\frac{1}{x} - 1} = G.$$

Moreover, if we set $s_\infty = \inf_{p>1} s_p(G)$, then

$$(1.15) \qquad \frac{1}{s_\infty} + \frac{1}{\sigma_1} = 1.$$

Let us now define the *sharp improvement exponents*

$$(1.16) \qquad \begin{aligned} r_q &= r_q(G) \\ &= \sup\{r \in]1,\infty[\,:\, G_r(v) < \infty, \text{ for } v \in G_q \text{ such that } G_q(v) \le G\} \end{aligned}$$

and

$$(1.17) \qquad \begin{aligned} \rho_p &= \rho_p(A) \\ &= \inf\{\rho \in]1,\infty[\,:\, A_\rho(w) < \infty, \text{ for } w \in A_p \text{ such that } A_p(w) \le A\} \end{aligned}$$

where $A > 1$ and $G > 1$ are fixed.

In the general case $p \neq 2$, $q \neq 2$ no explicit formula is available for the exponents in (1.12), (1.13), (1.16), (1.17) which are only characterized as positive solutions of certain algebraic equations.

We have the following result which states that the exponents correspond under the Hölder conjugate mapping $\phi(x) = \frac{x}{x-1}$.

Theorem 1.2. *Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $G = A$. Let σ_q, s_p, r_q and ρ_p be defined as in* (1.12), (1.13), (1.16), (1.17). *Then*

$$(1.18) \qquad \frac{1}{r_q} + \frac{1}{\rho_p} = 1$$

and

$$(1.19) \qquad \frac{1}{\sigma_q} + \frac{1}{s_p} = 1.$$

REMARK 1.3. By Theorem 1.2 it follows that it is enough to know two of four exponents r_q, s_p, ρ_p, σ_q to obtain all the others.

Let us now consider the following *borderline sharp improvement exponents* $(A > 1, G > 1)$

$$(1.20)\qquad \begin{aligned} r_1 &= r_1(G) \\ &= \sup\{r \in]1,\infty[: G_r(v) < \infty, \text{ for } v \in G_1, \text{ such that } G_1(v) \le G\} \end{aligned}$$

and

$$(1.21)\qquad \begin{aligned} \rho_\infty &= \rho_\infty(A) \\ &= \inf\{\rho \in]1,\infty[: A_\rho(w) < \infty, \text{ for } w \in A_\infty, \text{ such that } A_\infty(w) \le A\}. \end{aligned}$$

and the *borderline sharp transition exponents* $(A > 1, G > 1)$

$$(1.22)\qquad \begin{aligned} \sigma_1 &= \sigma_1(G) \\ &= \inf\{\sigma \in]1,\infty[: A_\sigma(v) < \infty, \text{ for } v \in G_1, \text{ such that } G_1(v) \le G\} \end{aligned}$$

and

$$(1.23)\qquad \begin{aligned} s_\infty &= s_\infty(A) \\ &= \sup\{s \in]1,\infty[: G_s(w) < \infty, \text{ for } w \in A_\infty, \text{ such that } A_\infty(w) \le A\}. \end{aligned}$$

We have the following extension of Theorem 1.1 (see Section 2 for complements about equations satisfied by the exponents, described in [25], [24], [3]).

THEOREM 1.4. *For $G = A > 1$, let r_1, σ_1 and ρ_∞, s_∞ be defined by* (1.20)–(1.23). *Then we have*

$$(1.24)\qquad r_1 = \lim_{q\to 1} r_q, \quad \rho_\infty = \lim_{p\to\infty} \rho_p,$$

$$(1.25)\qquad \sigma_1 = \lim_{q\to 1} \sigma_q, \quad s_\infty = \lim_{p\to\infty} s_p.$$

Whence

$$(1.26)\qquad \frac{1}{r_1} + \frac{1}{\rho_\infty} = 1,$$

$$(1.27)\qquad \frac{1}{\sigma_1} + \frac{1}{s_\infty} = 1.$$

Let us emphasize the new conjugate relations (1.18), (1.19), (1.26), (1.27) which were missing in the quoted papers.

An important case, for which the assumptions of Theorem 1.2 are fulfilled, arises in connection with increasing homeomorphisms $h : \mathbb{R} \overset{\text{onto}}{\longrightarrow} \mathbb{R}$ which belongs to Sobolev class $W^{1,1}_{\mathrm{loc}}(\mathbb{R})$ together with their inverse h^{-1}. We say that h is a bi-Sobolev map. For $r > 1$, we say that h is a $W^{1,r}$-biSobolev map if h' and $(h^{-1})' \in L^r_{\mathrm{loc}}(\mathbb{R})$.

For a bi-Sobolev map h a duality formula holds for the weights

$$v = h', \quad w = (h^{-1})' \tag{1.28}$$

which guarantees that the regularity of one of the two relies on the other one's, namely

$$A_p(w) = G_q(v) \tag{1.29}$$

if $\frac{1}{p} + \frac{1}{q} = 1$ (see [20] and Section 3 for a simple proof).

In [35] it is proved that actually the representation of weights in terms of bi-Sobolev maps $h : \mathbb{R} \to \mathbb{R}$ expressed by (1.28) is a general fact for A_∞ weights. In this paper (Section 4) we prove the following

Theorem 1.5. *For any weight $w \in A_\infty(\mathbb{R})$ there exist a weight $v \in A_\infty(\mathbb{R})$ and a bi-Sobolev map $h : \mathbb{R} \to \mathbb{R}$ such that* (1.28) *holds true. Moreover, h is $W^{1,t}_{\mathrm{loc}}$-biSobolev for a $t > 1$. Namely,*

$$v = h' \in G_r \quad w = (h^{-1})' \in G_s \quad \forall r < r_1,\ \forall s < s_\infty$$

where $r_1 = r_1(A)$, $s_\infty = s_\infty(A)$ are defined as in (1.20), (1.23) *with $A = G = A_\infty(w)$.*

Finally, in Section 5 we will present the sharp improvement and transition *constants* appearing in the previous Theorems.

2. Notations and Preliminaries

A weight w, i.e. a nonnegative measurable function on $\mathbb{R}^n$ belongs to the A_p-class of Muckenhoupt with $p > 1$, if the $A_p(w)$-constant of w, satisfies the condition

$$A_p(w) = \sup_Q \fint_Q w \left(\fint_Q w^{\frac{-1}{p-1}} \right)^{p-1} < \infty$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$, with sides parallel to the axes.

The classes A_p are decreasing as p increases; actually

$$\rho < p \quad \Rightarrow \quad A_p(w) \le A_\rho(w) \tag{2.1}$$

hence $A_\rho \subset A_p$.

A weight v belongs to the G_q-class of Gehring with $q > 1$, if the $G_q(v)$-constant of v satisfies the condition

$$G_q(v) = \sup_Q \left[\frac{\left(\fint_Q v^q \right)^{\frac{1}{q}}}{\fint_Q v} \right]^{\frac{q}{q-1}} < \infty$$

The class G_q are increasing as q increases; actually

$$r < q \quad \Rightarrow \quad G_r(v) \leq G_q(v) \tag{2.2}$$

hence $G_q \subset G_r$.

The A_∞ class is defined as

$$A_\infty = \bigcup_{p>1} A_p$$

and the G_1 class as

$$G_1 = \bigcup_{q>1} G_q.$$

It is well known (see [5], [16], [7]) that

$$A_\infty = G_1 \tag{2.3}$$

and that a weight $w \in A_\infty$ if and only if

$$\sup_Q \fint_Q w \exp \fint_Q \log \frac{1}{w} = A_\infty(w) < \infty.$$

Moreover,

$$A_\infty(w) = \lim_{p \to \infty} A_p(w) \tag{2.4}$$

(see [39]).

It is also well known that for $v \in G_1$

$$\sup_Q \exp \left(\fint_Q \frac{v}{v_Q} \log \frac{v}{v_Q} dx \right) = G_1(v) < \infty. \tag{2.5}$$

Moreover in [29], it was proved that

$$G_1(v) = \lim_{q \to 1^+} G_q(v) \tag{2.6}$$

Let us now give a simple one-dimensional proof of (2.6).

PROOF OF (2.6). For any weight $w \in A_\infty$, we know by Theorem 1.5 that there exist a weight $v \in A_\infty$ and a bi-Sobolev map $h : \mathbb{R} \xrightarrow{\text{onto}} \mathbb{R}$, such that

$$h' = v, \quad (h^{-1})' = w.$$

By (2.4) it follows

$$A_\infty(w) = \lim_{p\to\infty} A_p(w).$$

Moreover (1.29) implies

$$A_\infty(w) = \lim_{q\to 1} G_q(v)$$

with $\frac{1}{p} + \frac{1}{q} = 1$.

The equality (2.6) follows using the other duality identity (see [6], [20])

$$G_1(v) = A_\infty(w). \qquad \square$$

A useful characterization of A_∞-condition, which represents a scale invariant version of mutual absolute continuity, is the following ([5], [36]).

PROPOSITION 2.1. *The weight* $w : \mathbb{R} \to [0, +\infty[$ *belongs to* A_∞ *if and only if there exist constants* $0 < \alpha \le 1 \le K$ *so that*

$$\frac{|E|}{|I|} \le K\left(\frac{\int_E w(x)\,dx}{\int_I w(x)\,dx}\right)^\alpha$$

for each interval I and for each measurable set $E \subset I$.

Let us now give the following.

DEFINITION 2.2. We say that a homeomorphism $h : \mathbb{R} \to \mathbb{R}$ is a bi-Sobolev map if h and h^{-1} belong to $W^{1,1}_{\text{loc}}(\mathbb{R})$. More specifically, if $h \in W^{1,r}_{\text{loc}}(\mathbb{R})$ and $h^{-1} \in W^{1,r}_{\text{loc}}(\mathbb{R})$, $1 \le r \le \infty$ then we say that h is $W^{1,r}$-biSobolev.

REMARK 2.3. It is well known ([17]) that h is bi-Sobolev iff $h' > 0$ and $(h^{-1})' > 0$ a.e.

In the papers [30] and [14] the following *self-improving* property of a weight $w \in A_p$ and of a weight $v \in G_q$ were indipendently discovered:

(2.7) For $A > 1$ there exists $\rho_p = \rho_p(A) < p$ such that

$$A_p(w) \le A \Rightarrow A_\rho(w) < \infty,\ \forall \rho \ge \rho_p$$

(2.8) For $G > 1$ there exists $r_q = r_q(G) > r$ such that

$$G_q(v) \le G \Rightarrow G_r(v) < \infty,\ \forall r \le r_q.$$

In the paper [5] the following *transition* properties of A_p-weights into G_s and of G_q-weights into A_σ were discovered:

$$\text{for } A > 1 \text{ there exists } s_p = s_p(A) > s \text{ such that} \quad A_p(w) \le A \Rightarrow G_s(w) < \infty,\ \forall s \le s_p \tag{2.9}$$

$$\text{for } G > 1 \text{ there exists } \sigma_q = \sigma_q(G) < \sigma \text{ such that} \quad G_q(v) \le G \Rightarrow A_\sigma(v) < \infty,\ \forall \sigma \ge \sigma_q \tag{2.10}$$

(see (1.16), (1.17), (1.12), (1.13)).

In the case $n = 1$, [37] and [8] in the particular case that v is a non-increasing weight, obtained exact value of self-improving exponent $r_q > q$ in (2.8) as a solution to the equation

$$\left(\frac{r_q}{r_q - q}\right)^{\frac{1}{q-1}} \left(\frac{r_q - 1}{r_q}\right)^{\frac{q}{q-1}} = G \tag{2.11}$$

(see [22] for the case of an arbitrary weight $v : \mathbb{R} \to [0, \infty[$.) In [22] also exact value for *self-improving* exponent $\rho_p < p$ in (2.7) was found as a solution to the equation

$$\frac{1}{\rho_p}\left(\frac{p-1}{p-\rho_p}\right)^{p-1} = A. \tag{2.12}$$

Later [26], [27] found exact values for *transition* exponent σ_q in (2.10) as a solution to

$$\frac{\sigma_q^{\frac{q}{q-1}}}{[1 + q(\sigma_q - 1)]^{\frac{1}{q-1}}} = G \tag{2.13}$$

and for the transition exponent s_p in (2.9) as a solution to

$$\frac{s_p}{s_p - 1}\left(\frac{s_p(p-1)}{1 + s_p(p-1)}\right)^{p-1} = A. \tag{2.14}$$

As a matter of calculations one can verify that if $\frac{1}{p} + \frac{1}{q} = 1$ and $A = G$, the following couples of exponents correspond under the Hölder conjugate mapping $\phi(x) = \frac{x}{x-1}$:

$$\frac{1}{r_q} + \frac{1}{\rho_p} = 1 \tag{2.15}$$

and

$$\frac{1}{\sigma_q} + \frac{1}{s_p} = 1. \tag{2.16}$$

Recently exact A_ρ-exponents of weights from A_∞-condition (1.8) ([25], [24]) and exact G_r exponents of weights from G_1-condition (1.9) ([3]) were found (see (1.20), (1.21)).

Namely if $A_\infty(w) = A$ then

$$A_\rho(w) < \infty \quad \text{for } \rho > \rho_\infty \tag{2.17}$$

where $\rho_\infty > 1$ solves the equation

$$\frac{e^{\rho_\infty - 1}}{\rho_\infty} = A \tag{2.18}$$

(see Theorem 3 in [24] proved by rearrangements techniques).

If $G_1(v) = G$ then

$$G_r(v) < \infty \quad \text{for } 1 < r < r_1 \tag{2.19}$$

where $r_1 > 1$ solves the equation

$$\frac{(r_1 - 1)e^{\frac{1}{r_1 - 1}}}{r_1} = G \tag{2.20}$$

(see Theorem 1.5 in [3] proved with the Bellman function tecnique).

Both previous results are sharp in the sense that for any constant $C > 1$, there exists a weight $w \in A_\infty$ with $A_\infty(w) = C$ such that w does not belong to A_{ρ_∞} where (2.18) holds for $A = C$ and there exists a weight $v \in G_1$ with $G_1(v) = C$ such that v does not belong to G_{r_1} when (2.20) holds for $G = C$.

In this paper we justify the fact that the exponents correspond under the Hölder conjugate mapping $\phi(x) = \frac{x}{x-1}$

$$\frac{1}{r_1} + \frac{1}{\rho_\infty} = 1 \tag{2.21}$$

and

$$\frac{1}{\sigma_1} + \frac{1}{s_\infty} = 1. \tag{2.22}$$

where σ_1 and s_∞ are defined by (1.22) and (1.23). To this aim, first of all, we invoke the duality relation

$$A_\infty((h^{-1})') = G_1(h') \tag{2.23}$$

which holds true whenever $h : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism such that $h \in W^{1,1}_{\text{loc}}(\mathbb{R})$ and $h^{-1} \in W^{1,1}_{\text{loc}}(\mathbb{R})$ (i.e. h is a bi-Sobolev map (see [18] and the book [17]).

Further, we notice that any weight, $w \in A_s$, is of the form $w = (h^{-1})'$ for a bi-Sobolev map h which moreover is a $W^{1,r}$-biSobolev map for a $r > 1$.

This result was suggested to us by paper [28] on quasiminima and their inverses in dimension one.

The topic of reverse Hölder inequalities plays a central role for the solvability of the Dirichlet problem for planar elliptic equations $Lu = 0$ with non-smooth coefficients, when the boundary data f belongs to some L^p-class or to BMO ([12], [40], [22], [42]). It turns out that the L^p a priori estimate on the nontangential maximal function Nu of the solution $u \in W^{1,2}_{\mathrm{loc}}$

$$\|Nu\|_{L^p(\partial\Omega)} \le c\|f\|_{L^p(\partial\Omega)}$$

is equivalent to the G_q-condition of Gehring on the harmonic measure ω_L $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$.

3. Weighted A_p and G_q classes

In this Section we present a new proof of a useful theorem due to Johnson and Neugebauer [20].

For $1 < q < \infty$ we define $G_{q,d\mu}$ the weighted Gehring class respect to the measure $d\mu$, i.e.

$$G_{q,d\mu}(v) = \sup_{J \subset \mathbb{R}} \left[\frac{\left(⨍_J v^q \, d\mu\right)^{\frac{1}{q}}}{\left(⨍_J v \, d\mu\right)} \right]^{\frac{q}{q-1}}$$

where the supremum is taken over all intervals $J \subset \mathbb{R}$.

Our proof is based on the following Lemma (see [37], [16], [33], [23]).

Lemma 3.1. *Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ and h be a bi-Sobolev map. Then*

$$h' \in A_p \Leftrightarrow (h^{-1})' \in G_{q,h'\,\mathrm{d}x} \tag{3.1}$$

and

$$A_p(h') = G_{q,h'\,\mathrm{d}x}((h^{-1})'). \tag{3.2}$$

Theorem 3.2. *Let h be a bi-Sobolev map, then*

$$A_p(h') = G_q((h^{-1})') \tag{3.3}$$

if $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By Lemma 3.1, if $h' \in A_p$, then

$$A_p(h') = G_{q,h'\,\mathrm{d}x}((h^{-1})'). \tag{3.4}$$

Now it remains to prove that

$$G_{q,h'\,\mathrm{d}x}((h^{-1})') = G_q((h^{-1})'). \tag{3.5}$$

Using the change of variable $y = h(x)$ we have:

$$\frac{\left(⨍_J((h^{-1}(y)')^q\,\mathrm{d}y\right)^{\frac{p}{q}}}{\left(⨍_J(h^{-1}(y))')\,\mathrm{d}y\right)^p} = \frac{\left(\frac{1}{|J|}\int_I((h^{-1}(h(x))')^q h'(x)\,\mathrm{d}x\right)^{\frac{p}{q}}}{\left(\frac{1}{|J|}\int_I((h^{-1}(h(x))')h'(x)\,\mathrm{d}x\right)^p} \tag{3.6}$$

$$= \frac{\left(\frac{1}{\int_I h'(x)\,\mathrm{d}x}\int_I((h^{-1})(h(x))')^q h'(x)\,\mathrm{d}x\right)^{\frac{p}{q}}}{\left(\frac{1}{\int_I h'(x)\,\mathrm{d}x}\int_I((h^{-1})(h(x))')h'(x)\,\mathrm{d}x\right)^p}$$

where $h(I) = J$.

Considering the supremum of all intervals $J \subset \mathbb{R}$ we obtain (3.5) and this completes the proof. □

4. Proof of Theorems

Proof of Theorem 1.2. Let us prove (1.18).

For $p, q \geq 1$ conjugate exponents, i.e. $\frac{1}{p} + \frac{1}{q} = 1$ and $C > 1$ let us define

$$X_q(C) = \{r \in [q, \infty[\, : G_q(v) \leq C \Rightarrow G_r(v) < \infty\} \tag{4.1}$$

and

$$Y_p(C) = \{\rho \in \,]1, p] : A_p(w) \leq C \Rightarrow A_\rho(w) < \infty\} \tag{4.2}$$

Let us show that

$$r \in X_q(C) \Leftrightarrow \rho = \frac{r}{r-1} \in Y_p(C) \tag{4.3}$$

Let us first prove that implications $\Rightarrow$ holds true.

Let $r \in X_q(C)$, then we know that for any weight v

$$G_q(v) \leq C \Rightarrow G_r(v) < \infty. \tag{4.4}$$

Define $\rho = \frac{r}{r-1}$ and let us show that

$$A_p(w) \leq C \Rightarrow A_\rho(w) < \infty. \tag{4.5}$$

Fix a weight w such that

$$A_p(w) \leq C \tag{4.6}$$

then by Theorem 1.5 there exists a biSobolev map $h : \mathbb{R} \to \mathbb{R}$ such that $w = (h^{-1})'$.

Define $v = h'$ so we have by Theorem 3.2 and (4.6)

$$G_q(v) = A_p(w) \le C.$$

Hence, by (4.4), $G_r(v) < \infty$. Using again Theorem 3.2 we have

$$A_{\frac{r}{r-1}}(w) = A_\rho(w) = G_r(v) < \infty$$

and (4.5) is proved. This implies that $\rho \in Y_p(C)$.

To prove the other implication in (4.3) we follow similar steps.

Since $\rho \in]1, p]$ iff $r = \frac{\rho}{\rho-1} \in [q, \infty[$ we deduce by (4.3) that for $C = A = G$

$$r_q(G) = \sup X_q(C) = \frac{\rho_p(A)}{\rho_p(A) - 1}$$

where $\rho_p(A) = \inf Y_p(A)$ that is (1.18) holds true.

Let us prove (1.19).

Define for $C > 1$

$$Z_q(C) = \{\sigma \in]1, \infty[: G_q(v) \le C \Rightarrow A_\sigma(v) < \infty\} \tag{4.7}$$

and

$$T_p(C) = \{s \in]1, \infty[: A_p(w) \le C \Rightarrow G_s(w) < \infty\}. \tag{4.8}$$

We prove that:

$$\sigma \in Z_q(C) \Leftrightarrow s = \frac{\sigma}{\sigma - 1} \in T_p(C). \tag{4.9}$$

Let us first prove that left hand side implies right hand side of (4.9).

Let $\sigma \in Z_q(c)$, then we know that for any weight v

$$G_q(v) \le C \Rightarrow A_\sigma(v) < \infty \tag{4.10}$$

Define $s = \frac{\sigma}{\sigma-1}$ and let us show that

$$A_p(w) \le C \Rightarrow G_s(w) < \infty. \tag{4.11}$$

Fix a weight w such that

$$A_p(w) \le C. \tag{4.12}$$

By Theorem 1.5 we know that $w = (h^{-1})'$ for a bi-Sobolev map $h : \mathbb{R} \to \mathbb{R}$. Define $v = h'$, so, by [20] and (4.12)

$$G_q(v) = A_p(w) \le C.$$

Hence, by (4.10), $A_\sigma(v) < \infty$ and also $s \in T_p(C)$.

The inverse implication is proved similarly.
By (4.9), with $C = A = G$

$$s_p(G) = \sup T_p(C) = \frac{\sigma_q(A)}{\sigma_q(A) - 1}$$

where $\sigma_q(A) = \inf Z_q(A)$ and this completes the proof. □

We have the following

LEMMA 4.1. *Let r_q, ρ_p, s_p and σ_q be as in* (1.16), (1.17), (1.12), (1.13) *with* $A = G = C > 1$. *Then*

$$q_1 < q_2 \Rightarrow r_{q_1} < r_{q_2} \tag{4.13}$$

$$p_1 < p_2 \Rightarrow \rho_{p_1} < \rho_{p_2} \tag{4.14}$$

$$p_1 < p_2 \Rightarrow s_{p_1} > s_{p_2} \tag{4.15}$$

$$q_1 < q_2 \Rightarrow \sigma_{q_1} > \sigma_{q_2} \tag{4.16}$$

PROOF. Set

$$X_q(C) = \{r \in]1, \infty[: G_q(v) \le C \Rightarrow G_r(v) < \infty\}$$

and assume $q_1 < q_2$. We fix $r \in X_{q_1}$ and show that $r \in X_{q_2}$. So, assume

$$G_{q_1}(v) \le C \Rightarrow G_r(v) < \infty \tag{4.17}$$

and suppose $G_{q_2}(v) \le C$. Since $G_{q_1}(v) \le G_{q_2}(v)$ we have also $G_{q_1}(v) \le C$.
Hence, by (4.17) $G_r(v) < \infty$.
The others following in the same way. □

PROOF OF THEOREM 1.4. More generally, let us notice that, according to [26], the exponent s_p defined by (1.13), is the unique solution to the algebraic equation

$$\frac{s_p}{s_p - 1}\left(1 + \frac{1}{s_p(p-1)}\right)^{1-p} = A_p(w) \tag{4.18}$$

and s_∞ is , according to [24], the unique solution to the algebraic equation

$$\frac{s_\infty}{s_\infty - 1} e^{-\frac{1}{s_\infty}} = A_\infty(w). \tag{4.19}$$

We will use that $A_p(w) \to A_\infty(w)$ to prove

$$\lim_{p \to \infty} s_p = s_\infty. \tag{4.20}$$

To this aim, let $p_k > 1$ be a sequence such that $p_k \to \infty$ and show that $s_{p_k} \to s_\infty$. We may assume $s_{p_k} \to s_0 \geq 1$. Define for $x > 1$ the sequence of functions

$$g_k(x) = \left[1 + \frac{1}{x(n_k - 1)}\right]^{1-n_k} \frac{x}{x-1}$$

which of course satisfies for every $x > 1$

$$\lim_{k\to\infty} g_k(x) = e^{-\frac{1}{x}} \frac{x}{x-1} = g(x).$$

Then, in every compact interval $[a, b] \subset]1, \infty[$, g_k converges uniformly to the continuous function g, since $g_k(x) \leq g_{k+1}(x)$. Hence

$$g_k(s_{p_k}) \to g(s_0) \quad \text{as } k \to \infty$$

and so

$$A_{p_k} \to e^{-\frac{1}{s_0}} \frac{s_0}{s_0 - 1} = A_\infty(w)$$

by

$$A_p(w) \to A_\infty(w) \quad \text{as } p \to \infty. \tag{4.21}$$

By (4.19) we have $s_0 = s_\infty$ and this completes the proof.

Similarly one proves the other limit relations in (1.24), (1.25). □

Proof of Theorem 1.1. The proof of Theorem 1.1 relies on the statement of Theorem 1.2 and Theorem 1.4, except for (1.14) which derives from the expression of σ_q given by (1.12), passing to the limit as $q \to 1$ which gives the equation

$$\sigma_1 e^{\frac{1}{\sigma_1} - 1} = G. \tag{4.22}$$

Notice that (1.15) is equivalent to

$$\frac{s_\infty}{s_\infty - 1} e^{-\frac{1}{s_\infty}} = G. \tag{4.23}$$

□

Proof of Theorem 1.5. Let $w : \mathbb{R} \to [0, \infty[$ be a weight in A_∞ and let us define

$$h(x) = \int_0^x w(t)\, dt. \tag{4.24}$$

Then h is a continuous non constant increasing function which is one-to-one. In fact, since $A_\infty = \bigcup_{q>1} G_q$, there exists $q > 1$ such that $h' = w \in G_q$, hence h is strictly increasing (see [15]). Actually, we have the inequality (see [15])

$$\fint_a^b (h')^q\,dx \le G_q(w)^{q-1}\Big(\fint_a^b h'\Big)^q \tag{4.25}$$

for every interval $[a,b]$.

Let us suppose by contradiction that $h = 0$ for $t \le 0$ and $h > 0$ for $t > 0$ and let $a < 0 < b$. By (4.25) we have

$$\begin{aligned}\fint_a^b (h')^q\,dx &\le G_q(w)^{q-1}\Big(\frac{b}{b-a}\fint_a^b h'\Big)^q \\ &\le G_q(w)^{q-1}\frac{b^{q-1}}{(b-a)^q}\int_0^b (h')^q\end{aligned} \tag{4.26}$$

and then a contradiction, letting $b \to 0$.

Let us show now that h is onto $\mathbb{R}$, i.e.

$$\lim_{x\to\pm\infty}\int_0^x w(t)\,dt = \pm\infty. \tag{4.27}$$

Since $w \in A_\infty$, the inequality

$$\frac{|E|}{|I|} \le K\Big(\frac{\int_E w}{\int_I w}\Big)^\alpha \tag{4.28}$$

holds true for any measurable set $E \subset I$ and suitable $0 < \alpha \le 1 \le K < \infty$.

Applying this to (see [20])

$$I_n = [0, 2^n] \quad \text{and} \quad E = [2^{n-1}, 2^n]$$

we have

$$w(I_{n-1}) \le w(I_n) \le K^{\frac{1}{\alpha}}\Big(\frac{|I_n|}{|E|}\Big)^{\frac{1}{\alpha}} w(E) = (2K)^{\frac{1}{\alpha}} w(E).$$

Hence,

$$w(I_n) = w(I_{n-1}) + w(E) \ge w(I_{n-1})\left[1 + \frac{1}{(2K)^{\frac{1}{\alpha}}}\right]$$

for any n.

Iterating we finally arrive at

$$w(I_n) \geq \left[1 + \frac{1}{(2K)^{\frac{1}{\alpha}}}\right]^n w(I_0)$$

with $I_0 = [0,1]$. This implies (4.27).

Let us now prove that

$$h' > 0 \quad \text{and} \quad (h^{-1})' > 0 \text{ a.e.}.$$

Notice that

$$h^{-1}(y) = \int_0^y \frac{ds}{w(h^{-1}(s))} = \int_0^y v(s)\, ds$$

with $v \in A_\infty$.

By (2.3) we know that

$$A_\infty(w) = G_1(v) = A = G.$$

Hence, by (1.20) $v \in G_r$ $\forall r < r_1$ and by (1.23) $w \in G_s$ $\forall s < s_\infty$. □

5. Sharp constants

In this Section we present another application of duality formula $A_p((h^{-1})') = G_q(h')$, where $\frac{1}{p} + \frac{1}{q} = 1$, in the comparison of constants which appear in the sharp inequalities. From our results it follows that for improvement and transition inequalities it is sufficient to know two of the sharp constants to obtain the others.

We are especially interested in a-priori bounds for sets of weights of the form $\{w \in A_p : A_p(w) \leq A\}$ and $\{v \in G_q : G_q(v) \leq G\}$ for given $A > 1$, $G > 1$.

Given $A > 1$, $G > 1$, let us now define the *sharp improvement constants* for $\rho > \rho_p$

$$\alpha_{p,\rho}(A) = \sup\{A_\rho(w) : w \in A_p \text{ and } A_p(w) \leq A\} \tag{5.1}$$

and for $r < r_q$

$$\gamma_{q,r}(G) = \sup\{G_r(v) : v \in G_q \text{ and } G_q(v) \leq G\} \tag{5.2}$$

where ρ_p and r_q are as in (1.17) and (1.16).

In case $p = q = 2$ the exact values of $\alpha_{2,\rho}(A)$ and $\gamma_{2,r}(G)$ were found (see [41], [10], [38]):

$$\alpha_{2,\rho}(A) = (\sqrt{A})^\rho \frac{\sqrt{A} - \sqrt{A-1}}{\left(\sqrt{A} - \frac{1}{\rho-1}\sqrt{A-1}\right)^{\rho-1}} \tag{5.3}$$

for $\rho_2 = 1 + \sqrt{\frac{A-1}{A}} < \rho \le 2$; and

$$\gamma_{2,r}(G) = (\sqrt{G})^{\frac{r}{r-1}} \frac{\sqrt{G} - \sqrt{G-1}}{(\sqrt{G} - (r-1)\sqrt{G-1})^{\frac{1}{r-1}}} \tag{5.4}$$

for $1 < r < r_2 = 1 + \sqrt{\frac{G}{G-1}}$.

We have the following

PROPOSITION 5.1. *If* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $A = G$ *then for any* $r < r_q$ *and* $\rho > \rho_p$ *we have*

$$\gamma_{q,r}(G) = \alpha_{p,\rho}(A) \tag{5.5}$$

provided $\frac{1}{\rho} + \frac{1}{r} = 1$.

PROOF. We know that $\gamma_{q,r}(G) \ge G_r(v)$ $\forall r < r_q$ and $\forall v$ such that $G_q(v) \le G$.

Now consider the constant $A_\rho(w)$ with $\rho > \rho_p$ and w such that $A_p(w) \le C$. Set $v_0 = h'$ where h is given by $h^{-1}(y) = \int_0^y w$ so that, by [20], we have

$$G_q(v_0) = A_p(w) \le C.$$

Since, $\rho > \rho_p$, by (1.29) and (1.18), we get

$$A_\rho(w) = G_{\frac{\rho}{\rho-1}}(v_0) \in \{G_r(v) : r < r_q, G_q(v) \le C\}$$

and so

$$\gamma_{q,r}(G) \ge \alpha_{p,\rho}(A). \tag{5.6}$$

Similarly, we can prove the reverse inequality to (5.6). □

Let us define the *sharp transition constants* for $\sigma > \sigma_q$

$$\tilde{\alpha}_{q,\sigma}(G) = \sup\{A_\sigma(v) : v \in G_q \text{ and } G_q(v) \le G\} \tag{5.7}$$

and for $s < s_p$

$$\tilde{\gamma}_{p,s}(A) = \sup\{G_s(w) : w \in A_p \text{ and } A_p(w) \le A\} \tag{5.8}$$

where σ_q and s_p are as in (1.12) and (1.13).

In case $p = q = 2$ the exact values of $\tilde{\alpha}_{2,\sigma}(G)$ and $\tilde{\gamma}_{2,s}(A)$ were found (see [41], [10]):

$$\tilde{\alpha}_{2,\sigma}(G) = \frac{1}{\sqrt{G}}(\sqrt{G} - \sqrt{G-1}) \left[\frac{1}{\frac{\sigma}{\sigma-1} - \frac{\sqrt{G}}{\sigma-1}(\sqrt{G} + \sqrt{G-1}} \right]^{\sigma-1} \tag{5.9}$$

for $\sigma > \sigma_2 = \sqrt{G}[\sqrt{G} + \sqrt{G-1}]$; and

$$\tilde{\gamma}_{2,s}(A) = \frac{1}{\sqrt{A}}\left[\frac{(\sqrt{A} - \sqrt{A-1})^s}{\sqrt{A} - s\sqrt{A-1}}\right]^{\frac{1}{s-1}} \tag{5.10}$$

for $s < s_2 = \sqrt{\frac{A}{A-1}}$.

We have the following result whose proof is similar to the proof of Proposition 5.1.

PROPOSITION 5.2. *If* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $A = G$ *then for* $1 < s_p$ *and* $\sigma > \sigma_q$ *we have*

$$\tilde{\alpha}_{q,\sigma}(G) = \tilde{\gamma}_{p,s}(A) \tag{5.11}$$

provided $\frac{1}{\sigma} + \frac{1}{s} = 1$.

In [41] the following Theorem was proved. Here the exponents s_p, ρ_p are defined as in (1.13), (1.17).

THEOREM 5.3. *Let* $w \in A_p$ $(p > 1)$. *Then*

i) *for* $p \geq \rho > \rho_p$ *we have the sharp inequality*

$$A_\rho(w) \leq \frac{1}{\rho_p}\left(\frac{\rho - 1}{\rho - \rho_p}\right)^{\rho-1} \tag{5.12}$$

ii) *for* $s < s_p$ *we have the sharp inequality*

$$G_s(w) \leq \frac{s_p - 1}{s_p}\left(\frac{s_p - 1}{s_p - s}\right)^{\frac{1}{s-1}}. \tag{5.13}$$

Notice that for $p = \rho$ the left hand side in (5.12) equals $A_p(w)$ (see (2.12)).

In [10] the following Theorem was proved in which the exponents r_q, σ_q are defined as in (1.16), (1.12).

THEOREM 5.4. *Let* $v \in G_q$ $(q > 1)$. *Then*

j) *for* $q \leq r < r_q$ *we have the sharp inequality*

$$G_r(v) \leq \frac{r_q - 1}{r_q}\left(\frac{r_q - 1}{r_q - r}\right)^{\frac{1}{r-1}} \tag{5.14}$$

jj) *for* $\sigma > \sigma_q$ *we have the sharp inequality*

$$A_\sigma(v) \leq \frac{1}{\sigma_q}\left(\frac{\sigma - 1}{\sigma - \sigma_q}\right)^{\sigma-1} \tag{5.15}$$

Notice that for $r = q$ the left hand side in (5.14) equals $G_q(v)$ (see (2.11)).

REMARK 5.5. We notice that, provided $A = G$, $\frac{1}{p} + \frac{1}{q} = 1$, using (2.15) the bound (5.12) for $w = (h^{-1})'$ reduces to the bound (5.14) for $v = h'$ and conversely. Moreover, using (2.16), the bound (5.13) for $w = k'$ reduces to the bound (5.15) for $v = (k^{-1})'$.

In conclusion, Theorem 5.3 can be easily deduced by Theorem 5.4 and conversely.

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