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History of Mathematics — *Peano on definition of surface area*, by Gabriele H. Greco, Sonia Mazzucchi and Enrico M. Pagani, communicated on 15 January 2016.[1]

On the occasion of the 150th anniversary of the birth of Giuseppe Peano.

Abstract. — In this paper we investigate the evolution of the concept of surface area in Peano's mathematical research, taking into account the main role played by Grassmann's geometric-vector calculus and Peano's theory on derivative of measures. Geometric (in *Applicazioni geometriche*, 1887) and bi-vector (in *Calcolo geometrico*, 1888) Peano's approaches to surface area, culminating into the celebrated Peano's paper *Sulla definizione dell'area d'una superficie*, presented by Casorati for publication on *Rendiconti dell'Accademia dei Lincei* in 1890 and re-proposed in Peano's textbook *Lezioni di analisi infinitesimale* (1893), mark the development of this topic during the first half of the last century. Moreover we will present some remarkable contributions on surface area that are inspired and/or closely related to Peano's definition.

Key words: Grassmann–Peano geometric-vector calculus, area of polygons, bi-vectorial definition of area, Peano's surface area, Lebesgue's surface area

Mathematics Subject Classification: 01A65, 28-03, 28A75, 15-03, 15A75

1. Introduction

In 1882 Peano at the age of 24 discovers that the definition of area of a surface presented by Serret in his *Course d'Analyse* [73, (1868) vol. 2, p. 296] was not correct. According to Serret's proposal, the area of a surface should be given by the limit of the area of the inscribed polyhedral surfaces, but this definition cannot be applied even to a cylindrical surface. In fact Peano observes that in this case it is possible to choose a suitable sequence of inscribed polyhedral surfaces whose areas converge to infinity (see [62, (1890)], [64, (1902) pp. 300–301]).

Genocchi, Peano's teacher, dampens the enthusiasm of the young mathematician, by communicating him that a similar counterexample was already been discovered by Schwarz. In fact, in a letter of May 26, 1882, Genocchi writes Schwarz [71, (1890) vol. 2, p. 369]:

[1] Presented by G. Letta.

> C'est précisément Mr. Peano, qui m'amène à vous parler d'un autre sujet. Devant aborder la quadrature des surfaces courbes, il s'est aperçu que la définition d'une aire courbe donnée par Serret n'était pas bonne, et m'a expliqué les raisons qui ne lui permettaient pas de l'adopter. Alors je l'ai informé du jugement que vous en aviez porté dans plusieurs de vos lettres (20 et 26 décembre 1880, 8 janvier 1881), ce qui l'a beaucoup intéressé.

Genocchi and Schwarz[2] were conscious of the problem and of the lack of a "correct" definition of surface area, suitable to handle at least the area of elementary figures. In 1882 Genocchi [20, p. 323] writes another letter to Schwarz and invites him to propose an alternative definition, but Schwarz declines and stresses the difficulties:

> Vous avez voulu que je donne la rectification de la définition incomplète; mais ce n'est pas facile. On peut rectifier cette définition *de plusieures manières* et il me semble qu'il suffit de donner expressément une seule possibilité qui convient avec la définition donnée par Sturm.

Sturm's definition [79, (1877) vol. 1, p. 427] is the following:

> On appelle aire d'une surface courbe, terminée à un contour quelconque, la limite vers laquelle tend l'aire d'une surface polyédrique composée de faces planes, qui en diminuant toutes indéfiniment, tendent à devenir tangentes à la surface considérée. On suppose d'ailleurs que le contour qui termine la surface polyédrique se rapproche indéfiniment de celui qui termine la surface courbe.

We can think that the drawbacks communicated by Schwarz to Genocchi were also due to the lack of a choice criterion between the several possible definitions of surface area. In any case a "good" definition of surface area should at least be compatible with the Lagrange formula of area of a Cartesian surface[3]:

$$\iint_D \sqrt{1 + |\nabla f(x, y)|^2}\, dx\, dy \tag{1.1}$$

for C^1 functions $f : D \to \mathbb{R}$ (D being any rectangular subset of $\mathbb{R}^2$).

Peano's geometrical definition of surface area, given in *Applicazioni geometriche del calcolo infinitesimale* [60, (1887) p. 164], overcomes the drawbacks of Serret's approach, yielding the Lagrange formula (1.1). Peano's bi-vectorial proposal, introduced in *Calcolo geometrico secondo l'Ausdehnugslehre di H. Grassmann* [61, (1888)], is deeply influenced by Grassmann's geometric-vector calculus in affine spaces, that gives a mathematical formalization of geometrical

[2] Schwarz communicates his counterexample also to Hermite (see [43, (1883) p. 35]), to Casorati and to Beltrami (1880); see the correspondence between Casorati and Peano in Gabba [32, (1957)].

[3] Nowadays we know that the Lagrange formula is sufficient to define the area of a C^1-submanifold, but the extension of the formula (1.1) from a rectangle D to a more general 2-dimensional set is not trivial and hides some pitfalls.

and physical concepts (vectors, mechanical couple of vectors and their related moments, and so on) and allows also to take into account properties related to orientation, without using drawings or tricky and intuitive constructions. A final definition of surface area is given by Peano in the celebrated paper *Sulla definizione dell'area d'una superficie* [62, (1890)], presented by Casorati for publication on *Rendiconti dell'Accademia dei Lincei*. It is not surprising that Peano's definition via Grassmann's calculus is suitable to handle oriented integrals and, consequently, to prove main results (such as Stokes theorem and Green formula), and to develop formulae leading Cartan to the theory of integration of 2-forms[4]. Peano's bi-vectorial approach to surface area is re-proposed in his textbook *Lezioni di analisi infinitesimale* [63, (1893)].

Besides Peano's proposal, in the literature several definitions of surface area have been given[5]: nowadays the most famous and commonly accepted as definitive are grounded on Hausdorff measures.

The aim of the present paper is the investigation of the evolution and use of the concept of surface area in Peano's works, taking into account the main role played by Grassmann's geometric-vector calculus and Peano's theory on derivative of measures. Peano's approach to surface area marks the development of this topic during the first half of the last century. In the sequel we will also present contributions concerning surface area that are inspired and/or closely related to Peano's definition.

Peano's definition of measure of surfaces is grounded on elementary formulae of area of planar polygons (see Theorems 3.1, 3.3 and their proofs). The surprising absence of results concerning area of planar polygons in several modern encyclopedic books (see for example Alexandrov [1, (2005)] and Berger [9, (1977)]) motivates us to try to trace the history of such formulae that, as we shall see, are deeply connected with mechanics (statics, in particular) and can be found in their final form in the works by Möbius and Bellavitis. The generalization of the formula of area from planar to non-planar polygons and, finally, to non-planar closed curves allowed Peano to specify and to evaluate area of surface at an infinitesimal level (see Section 5).

The paper is organized as follows. In Section 2 the main definitions and results on Grassmann's geometric-vector calculus are presented in a modern fashion, according to Greco, Pagani [40, (2010)]. In Section 3 we recall Peano's bi-vectorial integral formula and other ways of associating a *number* to a given oriented

[4] In 1899 Cartan [17, p. 242] introduces the calculus of differential forms:

> Ce calcul présente aussi de nombreuses analogies avec le calcul de Grassmann; il est d'ailleures identique au calcul géométrique dont se sert M. Burali-Forti dans un Livre récent (*Introduction à la Géométrie différentielle, suivant la méthode de Grassmann*, Gauthier-Villars, 1898).

Burali-Forti was one of the prominent scholars of Peano. Together with Marcolongo he developed and applied Grassmann's vector calculus to geometry, mechanics and physics [12, (1909)].

[5] For a detailed presentation of the several possible definitions of surface area, see Cesari [28, (1954)] and Federer [29, (1969)].

closed curve. In addition we outline some theorems and their proofs in order to clarify the role played by graded exterior algebra in the construction of a definition of surface area. In Section 4 the historical development of the formulae of area of polygons and volume of polyhedra is investigated. Section 5 is devoted to the various descriptions of Peano's definition of surface area (i.e., *geometrical* and *bi-vectorial*). In Section 6 we will list main propositions and theorems about area given by Peano in his works. In Section 7 we recall some significant mathematical contributions inspired and/or closely related to Peano's definition. In particular we present the re-formulations of Peano's and Geöcze's surface area (due to several mathematicians) in order to make them coincident with Lebesgue's surface area.

This article concerns some historical aspects. From a methodological point of view, we are focussed on primary sources, that is on mathematical facts and not on historical accounts or interpretations of these facts by other scholars of history of mathematics.

2. Grassmann–Peano geometric-vector calculus on three dimensional spaces

We present here the Grassmann–Peano geometric-vector calculus[6]. The aim of this section is to understand the mathematical basis used by Peano in the construction of his notion of surface area. Such a formalism will be useful not only to clarify the genesis of the vectorial formulae for area of polygons and volume of polyhedra, but also to understand the deep connection between some concepts of statics (points, applied forces, momenta, Poinsot couple and so on) and of geometry (geometric forms of first, second, third and fourth-degree, namely, points, vectors, bi-points, bi-vectors, tri-points, tri-vectors, quadri-points).

Peano is one of the first mathematicians who presents Grassmann's work [37, (1844)], [38, (1862)], [36] to the mathematical community. Actually he rebuilds Grassmann's calculus using an original "functional" approach that relies only on the assignment of a volume form on a given affine space (see Greco, Pagani [40] for a detailed presentation of this subject).

For convenience of the reader we choose to rebuild here *Grassmann graded exterior algebra on the ordinary* 3*-dimensional space* (Grassmann algebra, for short), using an approach based on the usual notion of graded exterior algebra on a 4-dimensional *Möbius vector space*. A 4-dimensional Möbius vector space is a couple $(\mathbb{W}, \omega)$, where $\mathbb{W}$ is 4-dimensional vector space and $\omega : \mathbb{W} \to \mathbb{R}$ is a non-vanishing linear form, called *mass*. Given a Möbius vector space $(\mathbb{W}, \omega)$, let us consider the subspace of $\mathbb{W}$:

$$\mathbb{V}_3 := \{w \in \mathbb{W} : \omega(w) = 0\} \tag{2.1}$$

[6] Concerning the aspects of the theory based on the affine structure of the space, we follow Greco, Pagani [40, (2010)].

and the affine subspace of $\mathbb{W}$:

$$\mathbb{P}_3 := \{w \in \mathbb{W} : \omega(w) = 1\}. \tag{2.2}$$

Elements of $\mathbb{V}_3$ and $\mathbb{P}_3$ will be called *ω-vectors* and *ω-points* of $(\mathbb{W}, \omega)$, respectively. The elements of $\mathbb{W}$ with $\omega(x) \neq 0$ are called *weighted ω-points*. Therefore, the linear form ω allows a non ambiguous selection of the *ω-vectors* and the *ω-points* from the elements of the Möbius vector space $\mathbb{W}$. A linear function $f : \mathbb{W} \to \mathbb{W}$ will be called *barycentric*, if $\omega \circ f = \omega$. Clearly, a barycentric linear function maps ω-points (resp. ω-vectors) into ω-points (resp. ω-vectors).

The affine space

$$(\mathbb{P}_3, \mathbb{V}_3, -) \tag{2.3}$$

where "$-$" stands for the difference between elements of $\mathbb{W}$, is a 3-dimensional affine space. Affine maps from $\mathbb{P}_3$ to $\mathbb{P}_3$ are the restrictions to $\mathbb{P}_3$ of the barycentric linear maps from Möbius vector space $\mathbb{W}$ into itself. By endowing $\mathbb{V}_3$ with a positive definite symmetric bilinear form $\langle \cdot, \cdot \rangle$, the affine space (2.3) becomes a 3-dimensional Euclidean space; in the sequel with "$\| \cdot \|$" we denote its associated norm.

Let us consider the graded exterior algebras $\mathbb{G}(\mathbb{W})$ and $\mathbb{G}(\mathbb{V}_3)$ on the vector spaces $\mathbb{W}$ and $\mathbb{V}_3$ respectively. The product in $\mathbb{G}(\mathbb{W})$ and in $\mathbb{G}(\mathbb{V}_3)$ will be denoted as juxtaposition of symbols[7]. We have explicitly

$$\mathbb{G}(\mathbb{W}) = \Lambda^0(\mathbb{W}) \oplus \Lambda^1(\mathbb{W}) \oplus \Lambda^2(\mathbb{W}) \oplus \Lambda^3(\mathbb{W}) \oplus \Lambda^4(\mathbb{W})$$

where $\Lambda^0(\mathbb{W}) := \mathbb{R}$ and $\Lambda^k(\mathbb{W})$, $k = 1, \ldots, 4$, is the vector space generated by the products of k elements of $\mathbb{W}$. The elements of $\Lambda^k(\mathbb{W})$ are called *geometric forms of degree k*. Since $\mathbb{W}$ is generated by ω-points, it is worth observing that $\Lambda^k(\mathbb{W})$ is generated by the products of k ω-points. In a similar way we have

$$\mathbb{G}(\mathbb{V}_3) = \Lambda^0(\mathbb{V}_3) \oplus \Lambda^1(\mathbb{V}_3) \oplus \Lambda^2(\mathbb{V}_3) \oplus \Lambda^3(\mathbb{V}_3)$$

where $\Lambda^0(\mathbb{V}_3) := \mathbb{R}$ and $\Lambda^k(\mathbb{V}_3)$, $k = 1, 2, 3$, are linear combinations of products of k elements of $\mathbb{V}_3$. The elements of $\Lambda^k(\mathbb{V}_3)$ are called *geometric vector forms of degree k*. Exterior powers $\Lambda^k f : \Lambda^k(\mathbb{W}) \to \Lambda^k(\mathbb{W})$ of a barycentric linear function f from $\mathbb{W}$ to itself, maps $\Lambda^k(\mathbb{V}_3)$ into $\Lambda^k(\mathbb{V}_3)$.

The inner product $\langle \cdot, \cdot \rangle$ is extended to $\mathbb{G}(\mathbb{V}_3)$ by the following positions:

$$\langle u_1 u_2, v_1 v_2 \rangle := \det \begin{pmatrix} \langle u_1, v_1 \rangle & \langle u_1, v_2 \rangle \\ \langle u_2, v_1 \rangle & \langle u_2, v_2 \rangle \end{pmatrix} \tag{2.4}$$

[7] Recall that the algebras $\mathbb{G}(\mathbb{W})$ and $\mathbb{G}(\mathbb{V}_3)$ are anticommutative, i.e. $xy = (-1)^{rs} yx$, for any $x \in \Lambda^r(\mathbb{W})$, $y \in \Lambda^s(\mathbb{W})$. Clearly, $\mathbb{G}(\mathbb{V}_3)$ is a sub-algebra of $\mathbb{G}(\mathbb{W})$; moreover, due to anti-commutativity of the product and to the dimension of the spaces $\mathbb{W}$ and $\mathbb{V}_3$ we have that $\Lambda^k(\mathbb{W}) = 0$ (for $k > 4$) and $\Lambda^k(\mathbb{V}_3) = 0$ (for $k > 3$). Recall that $\dim \Lambda^k(\mathbb{W}) = \binom{4}{k}$ (for $0 \leq k \leq 4$), and that $\dim \Lambda^k(\mathbb{V}_3) = \binom{3}{k}$ (for $0 \leq k \leq 3$).

$$\langle u_1u_2u_3, v_1v_2v_3\rangle := \det\begin{pmatrix} \langle u_1, v_1\rangle & \langle u_1, v_2\rangle & \langle u_1, v_3\rangle \\ \langle u_2, v_1\rangle & \langle u_2, v_2\rangle & \langle u_2, v_3\rangle \\ \langle u_3, v_1\rangle & \langle u_3, v_2\rangle & \langle u_3, v_3\rangle \end{pmatrix} \tag{2.5}$$

for every $u_1, u_2, u_3, v_1, v_2, v_3 \in \mathbb{V}_3$. Hence, with respect to the associated norm we have

$$\|u_1u_2\| = \text{usual area of the parallelogram with edges } u_1 \text{ and } u_2, \tag{2.6}$$

$$\|u_1u_2u_3\| = \text{usual volume of the parallelepiped with edges } u_1, u_2, u_3. \tag{2.7}$$

Consequently, for every $A, B, C, D \in \mathbb{P}_3$ we have

$$\|(B-A)(C-A)\| = 2(\text{area of the triangle } ABC), \tag{2.8}$$

$$\|(B-A)(C-A)(D-A)\| = 6(\text{volume of the tetrahedron } ABCD). \tag{2.9}$$

Thus, if the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ form an orthonormal basis of $\mathbb{V}_3$, then the bi-vectors $\mathbf{e}_1\mathbf{e}_2$, $\mathbf{e}_2\mathbf{e}_3$ and $\mathbf{e}_1\mathbf{e}_3$ form an orthonormal basis for $\Lambda^2(\mathbb{V}_3)$. On the other hand, for every vector $x \in \mathbb{V}_3$ and for distinct $i, j \in \{1, 2, 3\}$ let x_{ij} denote the orthogonal projection of x on the coordinate-plane generated by the vectors $\mathbf{e}_i, \mathbf{e}_j$. Then for every vectors $x, y \in \mathbb{V}_3$

$$xy = x_{12}y_{12} + x_{23}y_{23} + x_{13}y_{13} \tag{2.10}$$

and, with respect to the norm associated with the inner product (2.4), from Pythagorean theorem it follows that

$$\|xy\|^2 = \|x_{12}y_{12}\|^2 + \|x_{23}y_{23}\|^2 + \|x_{13}y_{13}\|^2, \tag{2.11}$$

because the bi-vectors $x_{12}y_{12}$, $x_{23}y_{23}$ and $x_{13}y_{13}$ are the orthogonal projections of the bi-vector xy on $\mathbf{e}_1\mathbf{e}_2$, $\mathbf{e}_2\mathbf{e}_3$ and $\mathbf{e}_1\mathbf{e}_3$, respectively. The norm $\|xy\|^2$ can also be expressed in terms of coordinates of x and y; in fact, if $x = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ and $y = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$ for $a_i, b_i \in \mathbb{R}$, then

$$\|xy\|^2 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2, \tag{2.12}$$

since $xy = (a_1b_2 - a_2b_1)\mathbf{e}_1\mathbf{e}_2 + (a_1b_3 - a_3b_1)\mathbf{e}_1\mathbf{e}_3 + (a_2b_3 - a_3b_2)\mathbf{e}_2\mathbf{e}_3$.[8]

The linear form $\omega : \mathbb{W} \to \mathbb{R}$ can be extended to a linear map from the whole $\mathbb{G}(\mathbb{W})$ to $\mathbb{G}(\mathbb{V}_3)$ by means of the following relations

$$\omega(1) = 0,$$
$$\omega(P_0) = 1,$$

[8] It is worth noticing that, due to fact that $\dim(\mathbb{V}_3) = 3$, there is an orthogonal isomorphism from $\Lambda^2(\mathbb{V}_3)$ onto $\mathbb{V}_3$ which maps the bi-products xy into the usual cross products $x \wedge y$, for every $x, y \in \mathbb{V}_3$. In particular, $\|xy\| = \|x \wedge y\|$.

$$\omega(P_0P_1) = P_1 - P_0,$$
$$\omega(P_0P_1P_2) = (P_1 - P_0)(P_2 - P_0),$$
$$\omega(P_0P_1P_2P_3) = (P_1 - P_0)(P_2 - P_0)(P_3 - P_0)$$

for every $P_0, P_1, P_2, P_3 \in \mathbb{P}_3$[9]. Actually $\Lambda^k(\mathbb{V}_3)$ is a vector subspace of $\Lambda^k(\mathbb{W})$ and the linear map ω connects the graded algebras $\mathbb{G}(\mathbb{W})$ and $\mathbb{G}(\mathbb{V}_3)$ in the following way:

(2.13) $$\omega(\Lambda^k(\mathbb{W})) = \Lambda^{k-1}(\mathbb{V}_3) = \mathrm{Ker}(\omega_{|\Lambda^{k-1}(\mathbb{W})}) \quad \text{for } k = 1, \dots, 4.$$

Restrictions of ω to $\Lambda^k(\mathbb{W})$, denoted by ω_k, are called $(k-1)$-*vector-masses* because, by the first equality of formula (2.13), ω transforms a k-degree geometric form into geometric vector forms of $(k-1)$-degree. The second equality says that

(2.14) ω is null only on the geometric vector forms.

In particular, $\omega \circ \omega = 0$ and the following *reduction formula* holds:

(2.15) $$x = P\omega(x) + \omega(Px), \quad \text{for every } P \in \mathbb{P}_3,\ x \in \Lambda^k(\mathbb{W}),\ k = 0, 1, 2, 3, 4.$$

In the sequel, concerning the possibility of associating a bi-vector to a closed curve, a remarkable fact, due to $\dim \mathbb{V}_3 = 3$, is the property that

(2.16) $$\Lambda^2(\mathbb{V}_3) = \{vw : v, w \in \mathbb{V}_3\},$$

namely that any linear combination of bi-vectors is still a bi-vector.

A quadri-point $ABCD$ (with $A, B, C, D \in \mathbb{P}_3$ regarded as the four vertices of a tetrahedron) suggests the construction of particular bases of the spaces $\Lambda^k(\mathbb{W})$ and $\Lambda^k(\mathbb{V}_3)$, whenever they are not co-planar (i.e. the vectors $B - A$, $C - A$, $D - A$ are linearly independent). In fact, with respect to this tetrahedron $ABCD$

- the four vertices A, B, C, D are a basis of $\Lambda^1(\mathbb{W})$,
- the six edges AB, AC, AD, BC, BD, CD are a basis of $\Lambda^2(\mathbb{W})$,
- the four faces ABC, ACD, ABD, BCD are a basis of $\Lambda^3(\mathbb{W})$, and
- the tetrahedron $ABCD$ is a basis of $\Lambda^4(\mathbb{W})$;
- the vectors $B - A, C - A, D - A$ form a basis for $\Lambda^1(\mathbb{V}_3)$,
- the bi-vectors $(B - A)(C - A)$, $(B - A)(D - A)$, $(C - A)(D - A)$ form a basis for $\Lambda^2(\mathbb{V}_3)$, and
- the tri-vector $(B - A)(C - A)(D - A)$ forms a basis for $\Lambda^3(\mathbb{V}_3)$.

[9] The extension of the linear form ω on the whole graded algebra $\mathbb{G}(\mathbb{W})$ can be uniquely determined by the following two conditions:

- $\omega : \mathbb{G}(\mathbb{W}) \to \mathbb{G}(\mathbb{V}_3)$ is linear and $\omega(P) = 1$ for every $P \in \mathbb{P}_3$,
- $\omega(xy) = \omega(x)y + (-1)^{\deg(x)} x\omega(y)$ (*graded Leibnitz rule*).

If A', B', C', D' are the four vertices of another tetrahedron, then

$$A'B'C'D' = \frac{\det(B' - A', C' - A', D' - A')}{\det(B - A, C - A, D - A)} ABCD, \tag{2.17}$$

where $(B' - A', C' - A', D' - A')$ is the 3×3 matrix whose columns are the coordinates of the ω-vectors $B' - A', C' - A', D' - A'$ along the basis $B - A$, $C - A, D - A$. Equality (2.17) enlightens the geometrical interpretation of a quadri-point in terms of an oriented volume. The equality between two elements $x, y \in \Lambda^k(\mathbb{W})$ can be expressed by means the following condition:

$$x = y \quad \Leftrightarrow \quad xz = yz \quad \text{for every } z \in \Lambda^{4-k}(\mathbb{W}). \tag{2.18}$$

Concerning the volume of a tetrahedron or, in general, of an oriented polyhedron (i.e., a region delimited by a closed oriented polyhedral surface), Peano[10] proves the following property, referred later in Section 4.

(2.19) Formula for the volume of oriented polyhedra. *Let us consider an oriented closed polyhedral surface $\mathscr{S}$ made of triangular faces $A_iB_iC_i$ with $A_i, B_i, C_i \in \mathbb{P}_3$ and $i = 1, \dots, n$. Then the sum of the oriented volumes*

$$\sum_{i=1}^{n} PA_iB_iC_i \tag{$*$}$$

of the tetrahedra $PA_iB_iC_i$ does not depend on the choice of the vertex $P \in \mathbb{P}_3$[11].

Let $ABCD$ be a tetrahedron of unitary volume. By (2.17) there are real numbers a_i depending on P such that $PA_iB_iC_i = a_iABCD$. If in addition the polyhedral surface $\mathscr{S}$ of (2.19) is the boundary of a *convex* polyhedron $\mathfrak{S}$, then the sum

[10] See *Applicazioni geometriche del calcolo infinitesimale* [60, (1887) pp. 26–27], *Calcolo geometrico* [61, (1888) p. 66], *Lezioni di analisi infinitesimale* [63, (1893) vol. 2, p. 35].

[11] *Proof of* 2.19. The closedness of the oriented polyhedral surface made of triangular faces $A_iB_iC_i$ amount to the condition $\omega(\sum_{i=1}^n A_iB_iC_i) = 0$; hence, by (2.14) the element $\sum_{i=1}^n A_iB_iC_i$ is a tri-vector. This implies that there exist four points O, X, Y, Z, such that $\sum_{i=1}^n A_iB_iC_i = (X - O)(Y - O)(Z - O)$. From the reduction formula (2.15) the required independence of $\sum_{i=1}^n PA_iB_iC_i$ from the point $P \in \mathbb{P}_3$ follows; indeed $OXYZ = P\omega(OXYZ) + \omega(POXYZ) = P\omega(OXYZ) = P(X - O)(Y - O)(Z - O) = \sum_{i=1}^n PA_iB_iC_i$ for every $P \in \mathbb{P}_3$. (*Remark*: the independence of $\sum_{i=1}^n PA_iB_iC_i$ from $P \in \mathbb{P}_3$ has been proved starting from the equality $\omega(\sum_{i=1}^n A_iB_iC_i) = 0$. It is worth noting that the converse is still valid; hence, the equality $\omega(\sum_{i=1}^n A_iB_iC_i) = 0$ holds if and only if "$v \sum_{i=1}^n A_iB_iC_i = 0$ for every $v \in \mathbb{V}_3$". In other words, a system of triangular surfaces $\{A_iB_iC_i\}_{i=1}^n$ forms an oriented closed polyhedral surface if and only if the sum $\sum_{i=1}^n PA_iB_iC_i$ does not depend on the point $P \in \mathbb{P}_3$).

$a := \sum_{i=1}^{n} a_i$ is the signed volume of $\mathfrak{S}$. In fact, each a_i is the signed volume of the tetrahedron $PA_iB_iC_i$ with respect to the unit $ABCD$; therefore, chosen a point $P \in \mathfrak{S}$, the tetrahedra $\{PA_iB_iC_i\}_{i=1}^{n}$ are equi-oriented and form a decomposition of $\mathfrak{S}$; consequently, the sum $a = \sum_{i=1}^{n} a_i$ (which is independent from P) is the signed volume of $\mathfrak{S}$.

Several elements of the graded exterior algebras $\mathbb{G}(\mathbb{W})$ and $\mathbb{G}(\mathbb{V}_3)$ admit interesting geometrical and mechanical interpretation. Let $A, B, C, D \in \mathbb{P}_3$. Then

- a bi-point AB can be seen as the "vector" $B - A$ "applied in the point" A[12];
- a tri-point ABC can be seen as the "bi-vector" $(B - A)(C - A)$ "applied in the point" A[13];
- a quadri-point $ABCD$ can be seen as the "tri-vector" $(B - A)(C - A)(D - A)$ "applied in the point" A[14];
- a bi-vector can be seen as a Poinsot couple[15] or as the oriented boundary of a triangle[16];
- a tri-vector can be seen as the oriented surface of a tetrahedron[17].

Besides mechanical interpretations, a force applied in a point can be represented by a bi-point and, hence, a system of applied forces can be represented by an element of $\Lambda^2(\mathbb{W})$, more precisely as a sum of bi-points. The equivalence (by the point of view of mechanical equilibrium) between two systems of applied forces $\{A_iB_i\}_{i=1,\ldots,n}$ and $\{C_jD_j\}_{j=1,\ldots,m}$ becomes the equality between the corresponding elements $\sum_i A_iB_i$ and $\sum_j C_jD_j$ of $\Lambda^2(\mathbb{W})$. Indeed by means of equality (2.17), given a bi-point PQ, the product A_iB_iPQ can be recognized as the axial moment of the force A_iB_i with respect to the axis passing through P and Q. As a consequence the equality (2.18) between elements of $\Lambda^2(\mathbb{W})$ reduces the equivalence between two systems of applied forces to the equality of their axial moments with respect to every axis.

Pursuing the analogy with statics, the image of the operator ω acting on $\Lambda^2(\mathbb{W})$ represents the resultant of a system of forces (a special case of the 1-vector-mass introduced above). The reduction formula (2.15) can be directly translated into the reduction formula for a system of forces: given an arbitrary point P, a system of forces x is (statically) equivalent to a system formed by the resultant $\omega(x)$ applied in P, and by the Poinsot couple $\omega(Px)$. As a particular case, a system of forces with vanishing resultant, can be represented by an element of $\Lambda^2(\mathbb{V}_3)$. It is interesting to note that Poinsot's theorem "the sum of Poinsot couples is a Poinsot couple" emerges naturally from the structure of vector space of $\Lambda^2(\mathbb{V}_3)$, as expressed by (2.16).

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[12] Since $AB = A(B - A)$.

[13] Since $ABC = A(B - A)(C - A)$.

[14] Since $ABCD = A(B - A)(C - A)(D - A)$.

[15] Since $(B - A)(C - A) = B(C - A) - A(C - A)$.

[16] Since $(B - A)(C - A) = AB + BC + CA$.

[17] Since $(B - A)(C - A)(D - A) = BCD - ACD + ABD - ABC$.

3. Bi-vectors associated with oriented closed curves

Peano's program toward a definition of area of a surface is determining, for a given oriented closed (not necessarily planar) curve $\mathscr{P}$, the maximum value of the area delimited by its parallel projections on arbitrary planes.

For this purpose Peano introduces the definition that two oriented closed curves are said to be *equipollent* whenever the area delimited by their parallel projections on any plane are equal.

Peano realizes his program by considering the following decisive step: "given an oriented closed (not necessarily planar) curve $\mathscr{P}$, determining in magnitude and in position a triangle $\mathscr{T}$ equipollent to $\mathscr{P}$". Clearly, in this case, the maximum value of the area is obtained by orthogonally projecting $\mathscr{P}$ on a plane parallel to the triangle $\mathscr{T}$: it is equal to the area of $\mathscr{T}$.

3.1. Closed poligonal lines. Let $\mathscr{P}$ be an oriented closed polygonal line with consecutive vertices $A_1, \dots, A_n$ belonging to $\mathbb{P}_3$. With the polygonal line $\mathscr{P}$ Peano associates a bi-vector $\beta(\mathscr{P})$ defined by

$$\beta(\mathscr{P}) := A_1A_2 + A_2A_3 + \cdots + A_{n-1}A_n + A_nA_1. \tag{3.1}$$

In view of (2.14) and (2.16), $\beta(\mathscr{P})$ is a bi-vector since $\sum_{i=1}^{n} A_iA_{i+1}$ is a geometric form of degree two such that $\omega(\sum_{i=1}^{n} A_iA_{i+1}) = \sum_{i=1}^{n}(A_i - A_{i+1}) = 0$[18].

Oriented triangular contours (that is, oriented closed polygonal lines with $n = 3$) are the simplest oriented closed curves. If $\mathscr{T}$ is the oriented contour of a triangle with consecutive vertices X, Y and Z, then its associated bi-vector is

$$\beta(\mathscr{T}) = XY + YZ + ZX. \tag{3.2}$$

It is worth observing that any bi-vector is the associated bi-vector with an oriented triangular contour. Indeed, for every $u, v \in \mathbb{V}_3$, choose an arbitrary point $X \in \mathbb{P}_3$ and define the points $Y := X + u$ and $Z := X + v$; then the bi-vector uv is the associated bi-vector with the oriented contour $\mathscr{T}$ of the triangle with consecutive vertices X, Y and Z, since $u = Y - X$, $v = Z - X$ and

$$\beta(\mathscr{T}) = XY + YZ + ZX = (Y - X)(Z - X). \tag{3.3}$$

Hence, for any oriented closed polygonal line $\mathscr{P}$, there exists an oriented triangular contour $\mathscr{T}$ such that

$$\beta(\mathscr{T}) = \beta(\mathscr{P}). \tag{3.4}$$

This $\mathscr{T}$ is the triangle required by the Peano's program introduced above (see Theorem 3.3).

With an oriented closed *planar* polygonal line $\mathscr{P}$ (i.e., its vertices $A_1, \dots, A_n$ belong to a given affine oriented plane π) Peano associates a real number,

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[18] $A_{n+1} := A_1$

denoted here by 'area($\mathscr{P}$)' and defined in the following way. Let the orientation of π be given by a tri-point RST of unitary area. The vector space of linear combinations of tri-points of π that is 1-dimensional, is generated by the element RST. Then, fixed an arbitrary point $P \in \pi$, we define

$$\text{(3.5)} \qquad \text{area}(\mathscr{P}) := \text{“the real number } a \text{ such that } \sum_{i=1}^{n} PA_iA_{i+1} = aRST\text{”}.$$

The reals numbers a_i such that

$$\text{(3.6)} \qquad PA_iA_{i+1} = a_iRST,$$

are said to be the *oriented areas* of the tri-points (or, triangles) PA_iA_{i+1} with respect to unit RST[19]. Therefore, by definition (3.5), one has area($\mathscr{P}$) $= \sum_{i=1}^{n} a_i$.

The quantity 'area($\mathscr{P}$)' is termed by Peano [signed] *area bounded by the oriented closed planar polygonal line* $\mathscr{P}$. As observed by Peano, this area coincides with the usual area if the polygonal line (with consecutive distinct vertices) is convex or, more generally, is not self-intersecting[20].

The definition (3.5) of 'area($\mathscr{P}$)' is correlated to the bi-vector $\beta(\mathscr{P})$ and, moreover, is independent of the choice of the point $P \in \pi$. Indeed the following theorem holds.

Theorem 3.1 (Planar polygons, bi-vectors and area[21]). *Let* $\mathscr{P}$ *be an oriented closed polygonal line with consecutive vertices* $A_1, \dots, A_n$ *belonging to an affine plane* π *of* $\mathbb{P}_3$. *Then the bivector* $\beta(\mathscr{P})$ *is parallel to* π *and the following equalities hold:*

$$\text{(3.7)} \qquad X\beta(\mathscr{P}) = \sum_{i=1}^{n} PA_iA_{i+1} \quad \textit{for every point } X, P \in \pi$$

and, equivalently,

$$\text{(3.8)} \qquad \beta(\mathscr{P}) = \sum_{i=1}^{n} (A_i - P)(A_{i+1} - P) \quad \textit{for every point } P \in \pi.$$

[19] In other words, $|a_i|$ are the areas of the triangles PA_iA_{i+1}; the sign of a_i are positive (negative) if such triangles have the same (opposite) orientation with respect to the unit RST.

[20] We wish to call attention of the reader that a given set may be bounded by several closed polygonal lines having different (signed) areas. For instance, a triangle of unitary area, with vertices X, Y, Z, may be bounded by the closed polygonal lines $\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3$, where $\mathscr{P}_1$ has ordered vertices X, Y, Z, $\mathscr{P}_2$ has ordered vertices X, Y, Z, X, Y, Z and $\mathscr{P}_3$ has ordered vertices X, Z, Y. Their corresponding (signed) areas are equal to $1, 2, -1$, respectively.

[21] See Peano's *Applicazioni geometriche del calcolo infinitesimale* [60, (1887) p. 21], *Calcolo geometrico* [61, (1888) p. 59], *Lezioni di analisi infinitesimale* [63, (1893) vol. 2, p. 32].

Let RST be a tri-point of π of unitary area as above. By the definition (3.5) and the equality (3.7) it is clear that

$$X\beta(\mathscr{P}) = \operatorname{area}(\mathscr{P})RST. \tag{3.9}$$

Consequently, by (2.8) one has

$$\beta(\mathscr{P}) = 2\operatorname{area}(\mathscr{P})uv \tag{3.10}$$

for every vectors $u, v \in \mathbb{V}_3$ parallels to π such that $\|uv\| = 1$ and the bi-vector uv induces on π the same orientation of RST, and

$$|\operatorname{area}(\mathscr{P})| = \frac{1}{2}\|\beta(\mathscr{P})\|, \tag{3.11}$$

that is, the absolute value of the area bounded by an oriented closed planar polygonal line $\mathscr{P}$ is one half of the norm of the associated bi-vector $\beta(\mathscr{P})$.

Formula (3.5) allows the evaluation of the signed area of a planar polygon $\mathscr{P}$ in terms of the coordinates of its vertices $A_1, \ldots, A_n$. Let $A_i := O + x_i\mathbf{e}_1 + y_i\mathbf{e}_2$ where $O \in \mathbb{P}_3$, $x_i, y_i \in \mathbb{R}$ for $i = 1, \ldots, n$. Since the points A_i belong to an affine plane π, generated by the point O and by the orthonormal vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, by formula (3.5) with $P := O$ the signed area area($\mathscr{P}$) of the polygon $\mathscr{P}$ with respect to the orientation of π given by the unitary bi-vector $\mathbf{e}_1\mathbf{e}_2$, one has the well known expression (see Gauss in [16, (1810) pp. 362–363] and Jacobi [45, (1866)]).

$$\operatorname{area}(\mathscr{P}) = \frac{1}{2}\sum_{i=1}^{n}\begin{vmatrix} x_i & x_{i+1} \\ y_i & y_{i+1} \end{vmatrix}. \tag{3.12}$$

Corollary 3.2. *The areas bounded by two oriented closed coplanar polygonal lines are equal if and only if their associated bi-vectors are equal.*

Theorem 3.3 (Non-planar polygons, bi-vectors and equipollence[22]). *For every oriented closed polygonal line $\mathscr{P}$ (not necessarily planar) there exists an oriented triangular contour $\mathscr{T}$, given by the bi-vector associated with $\mathscr{P}$, equipollent to $\mathscr{P}$.*

Corollary 3.4. *Two oriented closed polygonal lines (not necessarily planar) are equipollent if and only if their associated bi-vectors are equal.*

For convenience of the reader, we give the proofs of the previous theorems accordingly to Grassmann–Peano geometric-vector calculus.

Proof of theorem 3.1. Since $\beta(\mathscr{P}) = \sum_{i=1}^{n} A_iA_{i+1}$ is a linear combination of bi-points of the plane π, the bi-vector $\beta(\mathscr{P})$ is parallel to π. Therefore, there exist three points $X, Y, Z \in \mathbb{P}_3$ in the plane π such that $\beta(\mathscr{P}) = (Y - X)(Z - X)$. Given a generic point $P \in \mathbb{P}_3$ in the plane π, we have $\omega_4(PXYZ) =$

[22] See Peano's *Calcolo geometrico* [61, (1888) p. 137].

$(X-P)(Y-P)(Z-P)=0$ (these three vectors $(X-P),(Y-P),(Z-P)$ are linearly dependent!); therefore, by the reduction formula (2.15), we have

$$XYZ = P\omega_3(XYZ) + \omega_4(PXYZ) = P\omega_3(XYZ) = P\beta(\mathscr{P}) = \sum_{i=1}^{n} PA_iA_{i+1}$$

and the conclusion follows. □

Proof of theorem 3.3. Let $A_1, \dots, A_n$ be the consecutive vertices of the closed polygonal line $\mathscr{P}$ and let $A_{n+1} := A_1$. By (3.3) and (3.4) there exist three points X, Y, Z such that $\beta(\mathscr{P}) = (Y-X)(Z-X)$. Now, let $\mathscr{T}$ be the oriented contour of the triangle with consecutive vertices X, Y and Z. Given a generic plane π, let us consider a vector $u \in \mathbb{V}_3$ non-parallel to the plane π. The relation (3.1) implies that

$$(Y-X)(Z-X)u = \sum_{i=1}^{n} A_iA_{i+1}u. \tag{3.13}$$

Now, let us consider the projections $\tilde{A}_i$, $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$ on π along the vector u of the points A_i, X, Y, Z, respectively. Then

$$(\tilde{Y}-\tilde{X})(\tilde{Z}-\tilde{X})u = \sum_{i=1}^{n} \tilde{A}_i\tilde{A}_{i+1}u.\text{[23]} \tag{3.14}$$

Observe that $F := (\tilde{Y}-\tilde{X})(\tilde{Z}-\tilde{X}) - \sum_{i=1}^{n} \tilde{A}_i\tilde{A}_{i+1}$ is a bi-vector parallel to the plane π, and, on the other hand, by (3.14) we have that $Fu = 0$; therefore $F = 0$, because u is not parallel to plane π. Hence

$$(\tilde{Y}-\tilde{X})(\tilde{Z}-\tilde{X}) = \sum_{i=1}^{n} \tilde{A}_i\tilde{A}_{i+1}; \tag{3.15}$$

that is, the bi-vectors associated with the projection on π of $\mathscr{P}$ and $\mathscr{T}$ are equal. In other words, by previous Theorem 3.1 the area bounded by the projection on π of the polygonal line $\mathscr{P}$ is equal to the area of the projection on π of the oriented triangular contour $\mathscr{T}$. □

3.2. Closed C^1 curves. After considering polygonal lines, in *Calcolo geometrico* [61, (1888) pp. 136–137] Peano (see also [63, (1893) pp. 233–234]) defines the bi-vector associated with a C^1 oriented closed curve in the following way. Let

[23] The equality (3.14) follows from (3.13), because for every points $B_1, B_2 \in \mathbb{P}_3$ and for every $\lambda_1, \lambda_2 \in \mathbb{R}$ we have $B_1B_2u = (B_1+\lambda_1 u)(B_2+\lambda_2 u)u$.

$A : [t_0, t_1] \to \mathbb{P}_3$ be a piecewise C^1 closed curve. The bi-vector $\beta(A)$ associated with A is given by[24]

$$\beta(A) := \int_{t_0}^{t_1} A(t)A'(t)\,dt. \tag{3.16}$$

As in the case of closed planar polygonal lines, Peano defines the [signed] *area delimited by a C^1 closed planar curve A*, here denoted by 'area(A)'. Let π be the affine plane of A, oriented by a tri-point RST of π of unitary area. Given an arbitrary point $P \in \pi$, he defines

$$\text{area}(A) := \text{“the real number } a \text{ such that } \int_{t_0}^{t_1} PA(t)A'(t)\,dt = aRST\text{”}. \tag{3.17}$$

As before, the vector space of linear combinations of tri-points of π is 1-dimensional and is generated by the element RST. Hence the notion (3.17) of area is well defined since the integral $\int_{t_0}^{t_1} PA(t)A'(t)\,dt$ is a tri-point of the plane π, and is independent of $P \in \pi$. In fact, as in Theorem 3.1, we have

$$X\beta(\mathscr{P}) = \int_{t_0}^{t_1} PA(t)A'(t)\,dt \quad \text{for every point } X, P \in \pi. \tag{3.18}$$

Being $PA(t)A'(t)$ a tri-point of π, there exists $a(t) \in \mathbb{R}$ such that $PA(t)A'(t) = a(t)RST$. Consequently, by (3.17),

$$\text{area}(A) = \int_{t_0}^{t_1} a(t)\,dt \tag{3.19}$$

regarded by Peano as the area that has swept by the segment $PA(t)$ when t varies. Moreover, by the definition (3.17) and the equality (3.18) it is clear that

$$X\beta(A) = \text{area}(A)RST \quad \text{and} \quad |\text{area}(A)| = \frac{1}{2}\|\beta(A)\|; \tag{3.20}$$

[24] $A'(t)$ denotes the derivative of A at t. The 2-degree geometric form (3.16) is a bi-vector because its vector-mass is null:

$$\omega\Big(\int_{t_0}^{t_1} A(t)A'(t)\,dt\Big) = \int_{t_0}^{t_1} \omega(A(t)A'(t))\,dt = \int_{t_0}^{t_1} A'(t)\,dt = A(t_1) - A(t_0) = 0.$$

Alternatively, due to the fact that A is a closed curve, the following formula holds:

$$\int_{t_0}^{t_1} A(t)A'(t)\,dt = \int_{t_0}^{t_1} (A(t) - O)A'(t)\,dt \tag{$*$}$$

for every point $O \in \mathbb{P}_3$. Consequently it is evident that the integral $\int_{t_0}^{t_1} A(t)A'(t)\,dt$ is a bi-vector because the integrand $(A(t) - O)A'(t)$ of the right hand side is a bi-vector.

that is, the absolute value of the area bounded by a C^1 oriented closed planar curve A is one half of the norm of the associated bi-vector $\beta(A)$. Finally, as shown for closed planar polygonal lines, by (2.8) one has

$$\beta(A) = 2\,\mathrm{area}(A)uv \tag{3.21}$$

for every vectors $u, v \in \mathbb{V}_3$ parallels to π such that $\|uv\| = 1$ and the bi-vector uv induces on π the same orientation of RST.

Analogously to Theorem 3.3 and its Corollary 3.4, for spatial curves we have:

- *for every C^1 oriented closed curve A (not necessarily planar) there exists an oriented triangular contour $\mathscr{T}$, given by the bi-vector $\beta(A)$, equipollent to A;*
- *two C^1 oriented closed curves (not necessarily planar) are equipollent if and only if their associated bi-vectors are equal.*

It is evident that definitions (3.16) and (3.17) extend definitions (3.1) and (3.5) respectively, whenever closed polygonal lines are regarded as piecewise linearly parameterized curves.[25]

Peano observes that for arbitrary C^1 closed (non necessarily planar) curve A the bi-vector $\beta(A)$ associated with A is the limit of the bi-vectors associated with the polygonal lines inscribed in the curve A; more precisely we have

$$\beta(A) = \lim_{\{t_i\}} \sum_{i=0}^{m-1} A(t_i)A(t_{i+1}) \tag{3.22}$$

where the limit is evaluated on the subdivisions $0 = t_0 < \cdots < t_i < t_{i+1} < \cdots < t_m = 1$ of the interval $[0, 1]$ for $\max\{t_{i+1} - t_i : i = 0, \ldots, m-1\} \to 0$.

The evaluation of the bi-vector $\beta(A)$ in terms of an orthogonal coordinate representation of the curve A yields the following well known formulae (3.23) and (3.24) for planar and spatial curves, respectively.

(3.23) When A is a C^1 closed planar curve in a plane π, without loss of generality, we may assume $A(t) = O + x(t)\mathbf{e}_1 + y(t)\mathbf{e}_2$, where O is a fixed point of π, and $x, y : [t_0, t_1] \to \mathbb{R}$ are C^1 functions. A straightforward calculation shows that

$$\beta(A) = \int_{t_0}^{t_1} A(t)A'(t)\,dt = \Big(\int_{t_0}^{t_1} (x(t)y'(t) - x'(t)y(t))\,dt\Big)\mathbf{e}_1\mathbf{e}_2.$$

Therefore, by (3.21), $\mathrm{area}(A) = \frac{1}{2}\displaystyle\int_{t_0}^{t_1} (x(t)y'(t) - x'(t)y(t))\,dt$, accordingly to Green formula.

[25] In fact, given a closed polygonal line of vertices $A_1, \ldots, A_n$, for any linear parameterization of a segment $[A_i, A_{i+1}]$ of the form $A : [t_i, t_{i+1}] \to \mathbb{P}_3$, with $A(t) := A_i + \frac{t-t_i}{t_{i+1}-t_i}(A_{i+1} - A_i)$, we have $\displaystyle\int_{t_i}^{t_{i+1}} A(t)A'(t)\,dt = A_iA_{i+1}$.

(3.24) In the case of spatial closed curves, we have that, for every plane π, *the projection along a direction u (non-parallel to π) of the bi-vector* (3.16) *on π is equal to the bi-vector associated with the projected curve*. Hence, assuming $A(t) = O + x(t)\mathbf{e}_1 + y(t)\mathbf{e}_2 + z(t)\mathbf{e}_3$, where O is a fixed point of $\mathbb{P}_3$, and $x, y, z : [t_0, t_1] \to \mathbb{R}$ are C^1 functions, from the equality (3.23) it follows that

$$\int_{t_0}^{t_1} A(t)A'(t)\,dt = \left(\int_{t_0}^{t_1} \begin{vmatrix} x(t) & y(t) \\ x'(t) & y'(t) \end{vmatrix} dt\right)\mathbf{e}_1\mathbf{e}_2 + \left(\int_{t_0}^{t_1} \begin{vmatrix} y(t) & z(t) \\ y'(t) & z'(t) \end{vmatrix} dt\right)\mathbf{e}_2\mathbf{e}_3 + \left(\int_{t_0}^{t_1} \begin{vmatrix} z(t) & x(t) \\ z'(t) & x'(t) \end{vmatrix} dt\right)\mathbf{e}_3\mathbf{e}_1; \tag{3.25}$$

Finally, from (2.11) we have

$$\left\|\int_{t_0}^{t_1} A(t)A'(t)\,dt\right\|^2 = \left(\int_{t_0}^{t_1} \begin{vmatrix} x(t) & y(t) \\ x'(t) & y'(t) \end{vmatrix} dt\right)^2 + \left(\int_{t_0}^{t_1} \begin{vmatrix} y(t) & z(t) \\ y'(t) & z'(t) \end{vmatrix} dt\right)^2 + \left(\int_{t_0}^{t_1} \begin{vmatrix} z(t) & x(t) \\ z'(t) & x'(t) \end{vmatrix} dt\right)^2. \tag{3.26}$$

3.3. Closed continuous curves. Finally, concerning an oriented closed continuous curve A, not necessarily C^1, Peano [62, (1890)] suggests, for the bi-vector associated with A, the following definition

$$\beta(A) := \lim_{\{t_i\}} \sum_{i=0}^{m-1} A(t_i)A(t_{i+1}) \tag{3.27}$$

where the limit is evaluated on the subdivisions $0 = t_0 < \cdots < t_i < t_{i+1} < \cdots < t_m = 1$ of the interval $[0, 1]$ for $\max\{t_{i+1} - t_i : i = 0, \ldots, m-1\} \to 0$. Peano does not specify general properties of A in order for the limit exist. An example of curve for which the bi-vector (3.27) does not exist will be given in Subsection 3.4.

In view of (3.22), it is evident that definition (3.27) is consistent with (3.16) in the C^1 case. The existence of the limit (3.27) for a given closed curve A guarantees the existence of the analogous limits for any parallel projection of A on a plane; more clearly, the bi-vector $\beta(A^*)$ associated with the curve A^*, projection of the curve A on the plane π along a vector u (non-parallel to π), is equal to the projection of the bi-vector $\beta(A)$ on π.

3.4. Some examples. In this Subsection we present three examples, the first is due to Peano.

Ex. (i) Peano formulation on equipollence of closed spatial is expressed by equality of their associated bi-vectors also in the case of oriented closed piecewise C^1 curves (see Corollary 3.4). An interesting example of equipollence given by Peano (see [61, (1888) p. 138], [62, (1890)] and [63, (1893) p. 234]) concerns a non-planar closed curve A equipollent to a circle. This curve is a piecewise C^1 map A with value in $\mathbb{P}_3$ formed by a cylindrical helix of radius r and pitch $2\pi h$ and three rectilinear pieces, two horizontal and one vertical.

Let O be a point of $\mathbb{P}_3$; let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthonormal base of $\mathbb{V}_3$ and let $A_1 := O + r\mathbf{e}_1$, $A_2 := O + r\mathbf{e}_1 + 2\pi h\mathbf{e}_3$ and $A_3 := O + 2\pi h\mathbf{e}_3$, $A_4 := O$. The cylindrical helix, joining A_1 to A_2 is parameterized by

$$A(t) := O + r\cos t\mathbf{e}_1 + r\sin t\mathbf{e}_2 + ht\mathbf{e}_3 \quad \text{for } 0 \le t \le 2\pi;$$

while the three rectilinear pieces are the segments A_2A_3, A_3A_4, A_4A_1, linearly parameterized. By a straightforward calculations of the integral (3.16) we have that $2\pi r^2\mathbf{e}_1\mathbf{e}_2$ is the associated bi-vector with the curve A, that is equal to the bi-vector associated with (hence, equipollent to) the orthogonal projection (that is a circle) of the curve A on the plane $\mathbf{e}_1, \mathbf{e}_2$. This is consistent with the observation following Theorem 3.1, the elementary area of the circle being one half of the norm of bi-vector above.

Ex. (ii) *Uniform convergence of closed polygonal lines does not imply convergence of their associated bi-vectors.* Let $\varepsilon \in (0,1)$ and $n \in \mathbb{N}$. Let $P_0 := (0,0)$, $P_1 := (1,0)$, $P_2 := (1,1)$, $P_3 := (0,1)$, $R_\varepsilon := (0, 1-\varepsilon)$, $Q_\varepsilon := (1, 1-\varepsilon)$ be points of $\mathbb{R}^2$. Now we define the oriented closed polygonal lines $\mathscr{P}$ and $\mathscr{P}_{\varepsilon,n}$.

The first polygonal line $\mathscr{P}$ is given by the ordered sequence of points

$$P_0, P_1, P_2, P_3, P_0.$$

The second polygonal line $\mathscr{P}_{\varepsilon,n}$ is given by the ordered sequence of points

$$P_0, P_1, P_2, P_3, \overbrace{R_\varepsilon, Q_\varepsilon, P_2, P_3}^{n \text{ times}}, P_0.$$

where the group of points $R_\varepsilon, Q_\varepsilon, P_2, P_3$ is repeated n-times. Using suitable parameterizations of $\mathscr{P}$ and $\mathscr{P}_{\varepsilon,n}$, the distance of the two polygonal lines is equal to ε. Moreover area$(\mathscr{P}) = 1$ while area$(\mathscr{P}_{\varepsilon,n}) = 1 + n\varepsilon$. Therefore the sequence $\{\mathscr{P}_{\frac{1}{m}, m^2}\}_m$ uniformly converges to $\mathscr{P}$ when $m \to \infty$ and $\lim_{m\to\infty} \|\beta(\mathscr{P}_{\frac{1}{m}, m^2})\| = 2\lim_{m\to\infty} \text{area}(\mathscr{P}_{\frac{1}{m}, m^2}) = +\infty$. In conclusion, the sequence $\{\mathscr{P}_{\frac{1}{m}, m^2}\}_m$ uniformly converges to $\mathscr{P}$, but the associated bi-vectors $\beta(\mathscr{P}_{\frac{1}{m}, m^2})$ does not converge.

Ex. (iii) *An example of a closed continuous curve that has not an associated bi-vector*. Let $P_0 := (0,0)$, $Q_n := \left(\frac{1}{2^n}, 0\right)$, $R_n := \left(\frac{1}{2^n}, \frac{1}{2^n}\right)$ for every natural number $n \geq 0$. Now consider the continuous curve $\mathscr{P}$, given by joining with segments the pairs of consecutive points of the following infinite sequence

$$P_0, Q_0, \overbrace{R_0, Q_1, Q_0}^{2^2 \text{ times}}, R_0, Q_1, \ldots, \overbrace{R_n, Q_{n+1}, Q_n}^{2^{2n+2} \text{ times}}, R_n, Q_{n+1}, \ldots, P_0.$$

Moreover, for every n, define the polygonal lines $\mathscr{P}_n$, inscribed in $\mathscr{P}$, given by the following finite sequence of points

$$P_0, Q_0, \overbrace{R_0, Q_1, Q_0}^{2^2 \text{ times}}, R_0, Q_1, \ldots, \overbrace{R_n, Q_{n+1}, Q_n}^{2^{2n+2} \text{ times}}, R_n, Q_{n+1}, P_0.$$

An elementary calculation yields for the area corresponding to the polygonal line $\mathscr{P}_n$ the value $\text{area}(\mathscr{P}_n) = 1 + n + \sum_{i=0}^{n} \frac{1}{2^{2i+2}}$. Therefore, since $\lim_{n\to\infty} \|\beta(\mathscr{P}_n)\| = 2 \lim_{n\to\infty} \text{area}(\mathscr{P}_n) = +\infty$, the bi-vector associated with the closed continuous curve $\mathscr{P}$ does not exist.

4. Möbius and Bellavitis on the area of polygons and volume of polyhedra

At the beginning of the 19th century an increasing interest is devoted to the study of polygons and polyhedra. This interest is paved by the researches by Legendre and Poinsot, who follow the way traced by Euclid, Kepler, Descartes and Euler. Legendre in 1794 gives a proof of the famous Euler's formula (1750) for polyhedra:

$$V - E + F = 2, \tag{4.1}$$

where V, E and F denote the number of vertices, edges and faces, respectively. On the other hand, Poinsot [66], according to the "Géométrie de situation" of Leibnitz [51], in 1810 started the classification of polygons and polyhedra, discovering some new "star polyhedra". In 1813 Cauchy [24] gave a new proof of Euler's formula (4.1) showing that there are no star polyhedra different from those described by Poinsot. Moreover, urged by Legendre, Cauchy gave the famous rigidity theorem for convex polyhedra, as he said in *Sur les polygones et les polyèdres* [25, p. 87]:

> [...] chercher la démonstration du théorème renfermé dans la définition 9, placée à la tète du onzième Livres *Elements d'Euclide*, savoir que deux polyèdres convexes sont égaux lorsqu'ils sont compris sous un même nombre de faces égales chacune à chacune.

One of the first book devoted to polyhedra was written by Descartes [30], but many other authors devote their efforts to the study of this topic.

An evidence of the importance which was given to polygons and polyhedra in the 19th century is the *Gran Prix* “Perfectionner dans quelque point important la théorie géométrique des polyèdres” organized in 1858 by the Accademy of Sciences of Paris. Indeed the Accademy decided to assign a prize only in presence of a significative and revolutionary contribution to the theory of polyhedra. Several important scientists participate, including Möbius. As other participants, Möbius’s goal was to provide a complete classification of polyhedra, but very soon he discovered that this is really an arduous task and decided to change his aim, proposing an innovative work concerning the concept of orientation. Despite of this, the Accademy does not judge any contribution sufficiently important and does not assign the prize to any participant.

Among several results present in the mathematical literature, we restrict ourselves to analyze in details the works of Möbius and Bellavitis, due to their influence on Peano. Formula (3.5) for evaluating area of polygons can be found for the first time in Möbius’s *Barycentrische Calcul* [57, (1827) p. 201], where it appears in a remark, as an application of the analogous formula for triangles and as a direct geometrical consequence of the notion of barycentric coordinates. Bellavitis presents the formula (3.5) for evaluating area of polygons in *Teoremi generali per determinare le aree dei poligoni e i volumi di poliedri* [6, (1834)] as a “trivial consequence” of a well known property due to Poinsot [67, (1811) pp. 53–54]. Bellavitis says indeed:

> [In the formula of area] one can see the property satisfied by a system of applied forces with vanishing resultant to be equivalent to a couple, independently of the common point in which the forces are translated.

Later Möbius himself deduced formula (3.5) for evaluating area of polygons, as an application of statics, in his book *Der Statik* [55, (1837) pp. 61–64]. In our opinion this correlation with statics, where the couples of consecutive vertices (= bi-points) of a closed polygonal line are interpreted as forces with vanishing resultant, is important from a historical point of view and may be emphasized into the following:

Theorem 4.1 (Metatheorem). *The following two propositions are equivalent:*

(4.2) *Formula of area* (3.5) *for planar polygons holds.*
(4.3) *Any system of planar forces with vanishing resultant is equivalent to a couple.*

According to Bellavitis also the formula of volume of polyhedra (2.19) can be seen as a consequence of the static theorem of Poinsot: “the sum of couples is a couple” [67, (1811) p. 56]. Later, references to formula (3.5) for evaluating area of polygons and formula (2.19) can be found in Bellavitis’s *Metodo delle equipollenze* [8, (1838) pp. 95–97] and *Sposizione del metodo delle equipollenze* [7, (1854)]. Concerning Möbius, both formulae (3.5) and (2.19) can be found in his article appeared in 1865 *Über die Bestimmung des Inhaltes eines Polyhedres* [56, pp. 486, 494].

The methods of proof of Bellavitis and Möbius are quite different. Bellavitis is one of the first mathematicians developing vector calculus, and he uses it in most of his proofs; moreover the deep connection between statics and geometry is strongly emphasized. It is also worthwhile to note that Bellavitis applies the duality relation between polygons and polyhedra, then it is not surprising that formula (3.5) for evaluating area of polygons and formula (2.19) appear in the same article. In the work by Möbius emerges the revolutionary concept of orientation. Möbius is conscious that orientability of polyhedra is an important condition for the validity of the formula of volume, and it cannot be ignored, as well as he was aware of the existence of non oriented polyhedra. In Bellavitis's work the necessity of orientability is not transparent, and he handles only with polyhedra which are dual of polygons that are oriented by construction.

We did not find any trace (but we cannot exclude it) concerning "area of non-planar polygons" either in Möbius or in Bellavitis even if a statement similar to Metatheorem 4.1 is still valid for non-planar polygons and non-planar forces:

Theorem 4.2 (Metatheorem). *The following two propositions are equivalent:*

(4.4) *For any closed polygonal line* (*not necessarily planar*) *there exists a triangle such that the area of any projection of the polygonal line on an arbitrary plane is equal to the area of the projection of the triangle.*

(4.5) *Any system of forces with vanishing resultant is equivalent to a couple.*

5. Peano's definitions of area

In Peano's works we recognize two definitions of area of a non-planar surface, the first one, referred as *geometrical*, is based on the notion of Jordan-Peano area of planar sets; the second one, referred in the following as *bi-vectorial* (and further split into *planar* and *spatial*), is grounded in the notion of bi-vector associated with a closed curve bounding a piece of surface.

In *Applicazioni Geometriche del Calcolo Infinitesimale* [60, p. 164] Peano introduces his geometrical definition of surface area in the following terms:

> Area of non-planar surfaces. Let us consider an arbitrary surface. Performing an orthogonal projection on a plane, we get a plane figure; we assume that this figure has an "area propria" [i.e., it is Peano-Jordan measurable] and that the given surface can be decomposed into parts having the same property.
>
> Let us decompose the given surface into pieces and, after carrying these pieces arbitrarily in the space, let us project all these pieces on the same plane. The sum of the areas of these projections is an area of a planar set, depending on the decomposition of the surface and on the way its pieces are located. The supremum of the values of these planar areas will be defined as the *area* of the surface.
>
> It follows immediately from this definition that the area of an arbitrary surface is greater than its orthogonal projection on an arbitrary plane.

Paraphrasing the content of the Peano's paper *Sulla definizione dell'area d'una superficie* [62, (1890), p. 56], we may have the following *planar bi-vectorial definition* of area[26], that may help the reader in comparing the two definitions of surface area given by Peano:

> [Let us consider an arbitrary surface delimited by a closed oriented curve. Performing an orthogonal projection on a plane, we get a plane figure delimited by a closed oriented curve; we assume that to the latter there corresponds a bi-vector which magnitude[27] gives the planar area of the figure, and that the given surface can be decomposed into pieces having the same property. Let us decompose the given surface into pieces and, after carrying these pieces arbitrarily in the space, let us project all these pieces on a same plane. The sum of areas delimited by the closed oriented curves of these projections depends on the decomposition of the surface and on the way its pieces are located. The supremum of these sums will be defined as the (*planar*) *bi-vectorial area* of the surface.]

In the same paper [62, (1890)] Peano examines, also by a historical viewpoint, various definitions of surface area and restates his definition. He starts by presenting the definitions of length of a convex planar arc and the area of a convex surface, given by Archimedes, as the limit of inscribed and circumscribed polygons and, respectively, as the limit of inscribed and circumscribed convex polyhedral surfaces. Peano, aware of the fact that Archimede's proposal is suitable enough to define the area of a cylindrical surface, tries to propose a definition of surface area preserving the analogy between length of arcs and area of surfaces present in Archimede's work. In the case of non-planar curves a good definition of length can be obtained by considering only the inscribed polygons, but in the case of surfaces, Peano observes that Archimede's definition cannot be applied to the non convex ones. Peano's aim is to extend Archimede's definition in order to handle more general surfaces, such as the concave ones.

Later, Peano [62, (1890)] criticizes the definitions of surface area present in the literature, including Serret's definition, and explaining that

> The main mistake of Serret is his belief that the plane passing through three points of a surface tends to its tangent plane.

He criticizes also Lagrange's definition [49, (1813) Chap. 14, p. 300] saying that "the result has been obtained by Lagrange by means of a not exact statement". He also criticizes Harnach's [41, (1885) Vol. II, p. 195] modification of Serret's definition, saying that, even if the faces of the polyhedron considered by Harnach tend to the tangent planes, Harnach's definition fails even in the case of a

[26] The word *planar* is related to the fact that this definition relies on the evaluation of bi-vectors associated with *planar* closed curves.

[27] In Peano [62, (1890)] the term "magnitude of a bi-vector uv" (*grandezza* in italian) means area delimited by the triangle X, Y, Z such that $uv = (Y - X)(Z - X) = XY + YZ + ZX$. Therefore this magnitude is one half of the norm of the bi-vector, the norm of bi-vectors being defined in Section 2.

cartesian surface of equation $z = f(x, y)$. Peano also recalls that the non correctness of Serret's definition has already been noted by Schwarz. The definition proposed by Hermite [44, (1887) p. 36] as a consequence of Schwarz's remark, even if considered sufficiently "rigorous" by Peano, is not completely satisfactory, because depends on the choice of the coordinate system.

Finally, Peano [62, (1890)] observes that any difficulty can be overcome by using the concept of *oriented* area, attributed by him to Chelini, Möbius, Bellavitis, Grassmann and Hamilton[28]. The bi-vectorial definition of non-planar surface area of Peano is based on the concept of Grassmann's *bi-vector*: Peano extends the equipollence between closed polygonal lines and triangles (see Theorem 3.3) to arbitrary oriented closed curves. Thus oriented closed lines can be represented by bi-vectors:

> Given a closed (not planar) line l, it is always possible to determine a closed planar line or bi-vector l' in such a way that by projecting both lines on an arbitrary plane, with parallel rays along an arbitrary direction, the [signed] areas defined by their projections coincide.

The logical evidence of this proposition is not trivial for a modern reader[29]. We observe that Peano [62, (1890)], by presenting the mathematical instruments for the proof, emphasizes the role of bi-vectors:

> This proposition is a direct consequence of the sum, or composition, of bi-vectors [since such a sum is a bi-vector,] when the line l is polygonal. The usual limiting procedure allows one to prove this fact when l is described by a point having finite derivative, and also in other cases.

The trivialness of the first part of this quotation, in the case of polygonal curves, is a consequence of Theorem 3.3 of Section 3. Concerning the second part, the approximation of a line by means of polygons provides the direct way to transfer properties of oriented closed polygonal lines to oriented closed continuous curves (see (3.27)). It is worthwhile to note that limits of polygons and triangles are included in the topological concepts introduced by Peano concerning

[28] It is interesting to note that Chelini, Möbius, Bellavitis and Grassmann in their work refer directly to Poinsot.

[29] In a letter [32, (1957) p. 867] written by Peano to Casorati (October 26, 1889) to submit his paper [62, (1890)] to Rendiconti dell'Accademia dei Lincei, we find:

> These closed contours are analogous, by duality, to *segments* or *vectors*; they can be identified with the *couples* of mechanics. According to Grassmann they are products of two vectors and can be called *bi-vectors*. Let us call the magnitude of a bi-vector C the area, in absolute value, of the triangle T described in the previous Theorem. If, by projecting C on a tern of orthogonal planes, one obtains the areas a, b, c, then the size of C is given by $\sqrt{a^2 + b^2 + c^2}$. The bi-vectors can be added, or composed, in an analogous way as the vectors, and more precisely as the couples of forces. If a part of a surface is decomposed into pieces, the bi-vector (or contour) of that surface is the sum of the bi-vectors of its pieces, as in the case of an arc of a line is decomposed into pieces, the vector (cord) of the arc is the sum (resultant) of the vectors of its pieces.

geometric forms (see Section 2). The condition of finite derivative, besides guaranteeing the continuity of the curve, assures that any projection of the closed line is the boundary of a set which is measurable in the sense of Jordan–Peano.

Moreover, Peano [62, (1890)] underlines that area must be thought as "oriented":

> Areas must be considered by taking their signs into account.

This part underlines the fact that the orientation of closed lines has always to be taken into account and this aspect becomes fundamental in the case of self-intersecting lines.

Thanks to the notion of equipollence between closed lines, Peano observes that:

> If one projects orthogonally a (not planar) closed line l on a variable plane, the maximum of the area delimited by the projection of l is equal to the magnitude of the bi-vector l. This maximum is achieved by projecting on a plane on which l lies.

In 1890 Peano [62] presents a new and more clear formulation of its definition of surface area, in the following referred as *spatial bi-vector definition* of area[30]:

> The area of a portion of surface is the upper limit of the sum of the magnitudes of bi-vectors corresponding to its parts.

More pragmatically, we may re-state this quotation in the following way:

> [Given an arbitrary non-planar surface, we consider a decomposition into pieces. For each of these pieces we consider its oriented boundary and the magnitude of the corresponding bi-vector. The supremum, with respect to all decompositions of the surface, of the sums of the magnitudes of the bi-vectors of the pieces of the decomposition, will be defined as the (*spatial*) *bi-vectorial area* of the surface.]

With this formulation Peano provides the fundamental property leading to the formula of area (1.1):

> The bi-vector corresponding to an infinitesimal part of the surface lies on the tangent plane; the ratio between its size and the area of that part is equal to 1.

In this way Peano shows the complete analogy between length of arcs and area of surfaces: in fact Peano observes that the direction of the vector with endpoints on an infinitesimal arc coincides with the tangent, and the rate between their lengths is equal to one[31]. Commenting on this fact, we may say

[30] The word *spatial* is related to the fact that this definition relies on a direct evaluation of bi-vectors associated with *spatial*, non necessarily planar, closed curves.

[31] Besides these properties of areas, Peano gives an estimate of the difference between the lengths of an arc and its cord and between the area of a surface and its bi-vector.

that Peano's definition grasps the essence of the measure of area at infinitesimal level.[32]

This fact may be summarized into the following theorem, relevant in obtaining the integral formula (6.2) for area of a C^1 non planar surface. Let $P : [0,1] \times [0,1] \to \mathbb{P}_3$ be a C^1 surface. For any $x, y \in (0,1)$ let's consider the infinitesimal square $Q_{a,b,\varepsilon} := [a, a+\varepsilon] \times [b, b+\varepsilon]$ with $a, b \in (0,1)$, its counterclockwise oriented boundary $\partial^+ Q_{a,b,\varepsilon}$ and the infinitesimal element of surface $P(Q_{a,b,\varepsilon})$.

Theorem 5.1. *The ratio between the magnitude of the bi-vector $\beta(P(\partial^+ Q_{a,b,\varepsilon}))$, associated with the closed curve $P(\partial^+ Q_{a,b,\varepsilon})$ (image of $\partial^+ Q_{a,b,\varepsilon}$ under P), and the Peano's area of the piece of surface $P(Q_{a,b,\varepsilon})$ tends to* 1 *when (a,b) tends (x,y) and ε tends to* 0^+.

The idea of projection on planes and the selection of the projection which maximizes the area is present also in Carathèodory's work of 1914 [15], and further developed by Hausdorff, who extends Carathèodory's results in the case of Hausdorff measures with integer exponent [42]. Nowadays the most famous measure is the Hausdorff measure, which allows one to define the measure of rather general sets by including also the concept of dimension. One of the first results proved by Hausdorff is the Lagrange formula of area (1.1).

6. Use of the concept of area by Peano

By analyzing the complete Peano's production, we found the following works containing applications of the notion of surface area: *Applicazioni geometriche* [60, (1887)], *Calcolo geometrico* [61, (1888)], *Lezioni di analisi infinitesimale* [63, (1893)] and *Formulario mathematico* (1895–1908).

Peano, by means of the notions of inner and outer measures on Euclidean spaces of dimension $1, 2, 3$, that have been introduced by him in *Sull'integrabilità delle funzioni* [59, (1883)], refounds in *Applicazioni geometriche* [60, (1887)] the notion of Riemann integral and extends it to abstract measures. The development of the theory of measure is based on a solid topological and logical ground and on a deep knowledge of set theory.

Peano in *Applicazioni geometriche* and later Jordan in *Cours d'Analyse* [46, (1893)] develop the well known concepts of classical measure theory: measurability, change of variables, fundamental theorems of calculus. These classical

[32] An elementary example is the following. Let S be a spherical surface of radius R and C an oriented circle obtained by intersecting S with a plane having distance $h < R$ from the center of the spherical surface. Denoting by $r = \sqrt{R^2 - h^2}$ the radius of C, we have that

$$\lim_{h \to R^-} \frac{\text{Area of the spherical cup delimited by } C}{\frac{1}{2}\|\beta(C)\|} = \lim_{h \to R^-} \frac{\frac{2\pi r^2}{1+h/R}}{\pi r^2} = 1$$

concepts of measure theory are developed by Peano also in *Lezioni di analisi infinitesimale* [63, (1893)] and in *Formulario mathematico* (1895–1908), where areas of planar sets delimited by several classical curves are evaluated [65, (1908) pp. 390–407] and, chiefly, alternative definitions of integral are given [65, (1908) p. 442].

The mathematical tools employed by Peano were really innovative both on geometrical and topological level. Peano used extensively the geometric vector calculus introduced by Grassmann (see Sections 2 and 3). A revolutionary tool is also the notion of differentiation of distributive set functions, that suggests to regard area of a non-planar surface as a distributive (or, finitely additive) set function and to compare it, at infinitesimal level, with the area of a planar set. In this context the evaluation of the area of a non-planar surface is reduced to the integration of a numerical function obtained by differentiation of the area of a non-planar surface with respect to the area of planar sets[33].

In this rich mathematical context Peano gives his first definition of area of non-planar surfaces (see first quotation of Section 5) and derives general formulae for planar and non-planar surfaces.

(6.1) *Formula for planar area* (see [60, (1887, Th. 47, p. 237)], [63, (1893, Vol. 2, §394 pp. 224–226)]). Let $A, B : [t_0, t_1] \to \mathbb{R}^2$ be two C^1 functions, such that the segments $A(t)B(t)$ and $A(t')B(t')$ have empty intersection for any $t, t' \in [t_0, t_1]$, $t \neq t'$. The set spanned by the segment $A(t)B(t)$, with $t \in [t_0, t_1]$, namely the set $\bigcup_{t\in[t_0,t_1]} A(t)B(t)$ has an area s given by the formula

$$s = \frac{1}{2}\int_{t_0}^{t_1} (B(t) - A(t)) \cdot \Big(\frac{dA(t)}{dt} + \frac{dB(t)}{dt}\Big)\, dt$$

where, following Peano's terminology, $(B(t) - A(t)) \cdot \big(\frac{dA(t)}{dt} + \frac{dB(t)}{dt}\big)$ denotes the norm of the bi-vector given by the product of the vectors $(B(t) - A(t))$ and $\big(\frac{dA(t)}{dt} + \frac{dB(t)}{dt}\big)$.[34]

(6.2) *Formula for non-planar area* (see [60, (1887, Th. 49, p. 243)], [63, (1893, Vol. 2, §396 pp. 229–232)]). Let $P : D \to \mathbb{R}^3$ be a C^1 function over $D := \{(u, v) \in \mathbb{R}^2 : a < u < b,\ \theta_0(u) < v < \theta_1(u)\}$ where θ_0 and θ_1 are contin-

[33] As observed in our paper *Peano on derivative of measures: strict derivative of distributive set functions* [39, (2010)], differentiation of distributive set functions gives a mathematical implementation of the *massa-density paradigm* (mass and volume are distributive set functions and the density is obtained by differentiating mass with respect to volume).

[34] In modern language, the norm of this bi-vector is the norm of the cross product $(B(t) - A(t)) \wedge \big(\frac{dA(t)}{dt} + \frac{dB(t)}{dt}\big)$. Therefore the formula (6.1) becomes

$$s = \frac{1}{2}\int_{t_0}^{t_1} \Big\| (B(t) - A(t)) \wedge \Big(\frac{dA(t)}{dt} + \frac{dB(t)}{dt}\Big) \Big\|\, dt.$$

uous functions defined on the interval $[a, b]$. The surface formed by points $P(u, v)$, with $(u, v) \in D$, has an area s given by the formula

$$s = \int_a^b du \int_{\theta_0(u)}^{\theta_1(u)} v(u, v)\, dv$$

where $v(u, v)$ is the norm of the bi-vector, product of the vectors $\frac{\partial P}{\partial u}$ and $\frac{\partial P}{\partial v}$.[35]

Formula (6.2), already obtained by Peano from his geometrical definition of area of surfaces, is proved by him also using his bi-vectorial definition. This coincidence is valid in the case of C^1 surfaces, but it does not hold for arbitrary surfaces.

Peano uses formulae (6.1) and (6.2) to obtain classical formulae for elementary surfaces (planar and non-planar). Moreover from (6.1) he derives in ([60, (1887), p. 242]) and in ([63, (1893, Vol. 2, §394 pp. 225–226)]) formulae that have been recovered one century later by Mamikon A. Mnatsakanian in his paper *On the area of the region on a developable surface* [54, (1981)].

Particular instances of formula (6.1), considered by Peano, are the following:

(6.3) The point A moves along a straight line and the angle of the segment AB with that line is constant;
(6.4) The point A is fixed;
(6.5) The segment AB is tangent at the point A to the curve described by A;
(6.6) The segment AB is of constant length and normal to the curve described by its midpoint.

In the case (6.5), formula (6.1) becomes

$$s = \frac{1}{2} \left| \int_{t_0}^{t_1} \det \begin{pmatrix} v_1(t) & v_2(t) \\ v_1'(t) & v_2'(t) \end{pmatrix} dt \right|,$$

where $v_1(t)$, $v_2(t)$ are the components of the vector $B(t) - A(t)$ and $t \in [t_0, t_1]$. It is clear from this formula, that the area depends only on the differences of the points $B(t) - A(t)$ and not on the particular positions of the points $A(t)$, $B(t)$. As a consequence of this fact, Peano derives the content of what is nowadays stated as Mamikon's Theorem: *the area of a tangent sweep of a curve is equal to the area of its corresponding tangent cluster*. For instance, the three following figures (see Figure 1) have the same area, because the first two figures are swept by equal tangent vectors to the inner ellipsis, while the third one is their corresponding tangent cluster. The areas marked by the same letter have the same area as well.

[35] In modern language, the norm of this bi-vector is the norm of the cross product $\frac{\partial P}{\partial u}(u, v) \wedge \frac{\partial P}{\partial v}(u, v)$; see footnote 8. Therefore the formula (6.2) becomes

$$s = \int_a^b du \int_{\theta_0(u)}^{\theta_1(u)} \left\| \frac{\partial P}{\partial u}(u, v) \wedge \frac{\partial P}{\partial v}(u, v) \right\| dv.$$

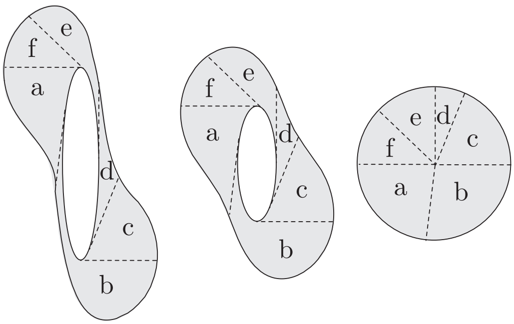

Figure 1

Mamikon's theorem has several applications, as it enables one to obtain area of complicated planar figures almost without calculation, by reducing the problem to the calculus of area of simple figures; see, for examples, Mamikon A. Mnatsakanian and Apostol in [4], [2] and [3].[36]

Surprisingly, in the five editions of *Formulario mathematico*, Peano does not present neither the geometrical nor the bi-vectorial definition of surface area, but he prefers to adopt another definition. This definition, due to Borchardt [10, (1854) p. 369][37], is stated in [64, (1902) p. 300] and in [65, (1908) p. 384] by the following limit (if any):

$$\lim_{h\to 0^+} \frac{\text{Volum}\{x \in \mathbb{R}^3 : \text{dist}(x, S) < h\}}{2h} \tag{6.7}$$

for every set S of points in $\mathbb{R}^3$ of null volume.

In *Formulario mathematico* [64, (1902) pp. 300–301] the well-known counter-example to the definition of Serret on surface area is given. This is based on the construction of a polyhedral surface $\mathscr{S}_{m,n}$, with m, n positive integers, inscribed into a cylinder of height 1 and radius 1, formed by mn triangles with the following vertices:

$$\left(\cos\left[\frac{2\pi r}{m}\right], \sin\left[\frac{2\pi r}{m}\right], \frac{s}{n}\right), \left(\cos\left[\frac{2\pi[r+1]}{m}\right], \sin\left[\frac{2\pi[r+1]}{m}\right], \frac{s}{n}\right),$$
$$\left(\cos\left[\frac{\pi[2r+1]}{m}\right], \sin\left[\frac{\pi[2r+1]}{m}\right], \frac{s+1}{n}\right)$$

[36] In [5, (2009)] Apostol and Mnatsakanian, using Mamikon's theorem, prove the property of Roberval: "The area of a cycloidal sector is three times the area described by the generating disk along its motion". This property was proved by Peano [63, (1893) Vol. 2, §395 pp. 226–228] using (6.1).

[37] Borchardt's surface area, usually called Minkowski area, was rediscovered and extended to arbitrary point set surfaces by Minkowski [53, (1901)] 47 years later.

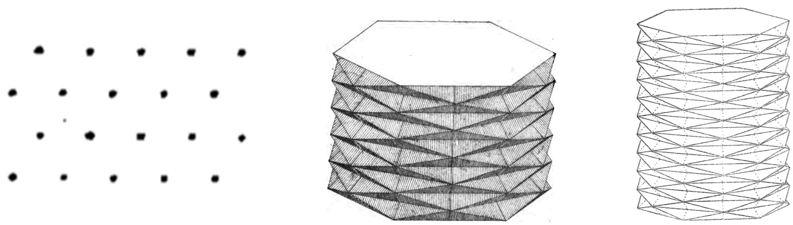

Figure 2

and by mn triangles with the following vertices:

$$\left(\cos\left[\frac{2\pi r}{m}\right], \sin\left[\frac{2\pi r}{m}\right], \frac{s}{n}\right), \left(\cos\left[\frac{\pi[2r-1]}{m}\right], \sin\left[\frac{\pi[2r-1]}{m}\right], \frac{s+1}{n}\right),$$
$$\left(\cos\left[\frac{\pi[2r+1]}{m}\right], \sin\left[\frac{\pi[2r+1]}{m}\right], \frac{s+1}{n}\right)$$

with $r = 0, 1, \ldots, m-1$ and $s = 0, 1, \ldots, n-1$.

The pictures of Figure 2 show the positions of vertices of triangles in the plane development of the cylindrical surface (as appears in Peano [64, (1902) pp. 300–301], with $m = 5$, $n = 3$) and the shapes of the polyhedral surfaces $\mathscr{S}_{m,n}$ (as appears in Hermite [43, (1883) p. 36] with $m = 6$, $n = 10$ and in Schwarz [71, (1890) vol. 2, p. 311] with $m = 6$, $n = 20$).

A straightforward calculations gives the area $a_{m,n}$ of the polyhedral surface $\mathscr{S}_{m,n}$:

$$a_{m,n} = 2m \sin\left(\frac{\pi}{m}\right)\sqrt{1 + 4n^2 \sin^4 \frac{\pi}{2m}}. \tag{6.8}$$

Clearly

$$\lim_{m\to\infty} a_{m,m} = 2\pi, \quad \lim_{m\to\infty} a_{m,m^2} = 2\pi\sqrt{1+\frac{\pi^4}{4}}, \quad \lim_{m\to\infty} a_{m,m^3} = +\infty \tag{6.9}$$

Consequently the limit of the area of the polyhedra $\mathscr{S}_{m,n}$ for $m, n \to \infty$ does not exist.

7. On the influence of Peano on definition of surface area

With Lebesgue's Thesis *Intégrale, Longueur, Aire* [50, (1902)], Peano's definition of surface area acquires notoriety. Lebesgue is acquainted with the bi-vectorial definition of surface area given by Peano in 1890, but ignores the original definition of 1887 and any other contribution of this Author (with the exception of the Peano's curve). As a consequence of this, it is not surprising that, in almost all contributions on the definition of surface area, references to the other Peano's works on surface area (in particular, to the books *Applicazioni geometriche* [60, (1887)] *Calcolo geometrico* [61, (1888)] and *Lezioni di analisi infinitesimale* [63, (1893)]) are absent.

Lebesgue's area of a parameterized surface is defined by him as the *lower limit of the area of the polyhedral surfaces that approximate uniformly the surface*.

Concerning the notion of surface area, now we recall various contributions and implementations inspired and/or closely related to Peano's definition. The various implementations correspond to the different ways of defining the "area of the orthogonal projection on a plane" of a piece of surface.

In the mathematical literature, we find definitions of surface area that implement Peano's inequality, namely the "area of a surface is greater or equal to the area of its orthogonal projection on an arbitrary plane". Other definitions of surface area implement the Peano's bi-vectorial inequality, namely that the "area of a surface bounded by a closed oriented contour is greater or equal to the magnitude of the bi-vector associated with the contour itself". In this case, the implementations correspond to the different ways to associate a number with a given oriented closed curve.

After Schwarz and Peano, as observed by Radó in [70, (1956) p. 513], "many definitions [of surface area] have been proposed, and an enormous amount of efforts have been expended in the study of [...] various concepts of surface area". For this reason we are forced to present only some contributions. Interested readers may find detailed historical account and mathematical facts in Cesari's book *Surface area* [28, (1954)] and Radó's book *Length and area* [69, (1948)].

In addition to the ones given by Peano, remarkable definitions are the Lebesgue's and Geöcze's surface area. The original definitions of Peano and Geöcze provide an evaluation of surface area that is greater than or equal to Lebesgue's surface area. Observe that Peano's and Geöcze's surface area relies on the evaluation of the area of the orthogonal projection on planes of pieces of the given surface. Therefore many authors have proposed different ways to define the area of a plane surface, in order to make Peano's and Geöcze's areas coincident with Lebesgue's area for a wider class of continuous parametric surfaces (see Radó [68, (1928)] and Cecconi [26, (1950)], [27, (1951)]).

Cesari [28, (1956)] reformulates the definitions given by Peano and Geöcze in a suitable way in order "to preserve" usual elementary area of polyhedral surfaces and, above all, lower semicontinuity. Cesari states the following theorem:

THEOREM 7.1. *For every continuous surface S we have $\mathscr{L}(S) = \mathscr{V}(S) = \mathscr{P}(S)$, where $\mathscr{L}(S)$, $\mathscr{V}(S)$ and $\mathscr{P}(S)$ denote Lebesgue area, Geöcze area and Peano area respectively.*

More precisely, Peano's[38] and Geöcze's definitions are reformulated by Cesari in terms of *topological index* of oriented closed planar curves. This index is denoted with $O(P, \gamma)$ by Cesari, where γ is an oriented closed planar curve,

[38] Here we refer to the *planar* bi-vectorial definition of surface area; see Section 5.

and $P = (x, y)$ is a point of the plane π of γ. It is worth observing that, as for the bi-vector associated with an oriented closed planar curve, the integral $\int_\pi |O(P,\gamma)|\,dx\,dy$ (denoted in the following by $v(\gamma,\pi)$) is interpreted, as "area of the planar surface delimited by γ" (see Cesari [70, (1956) p. 104]).

Now, let S be a parametric surface in $\mathbb{R}^3$, parameterized by a continuous $\varphi : A \to S$ (i.e., $S = \varphi(A)$), where A is an *admissible* set[39]. Given a plane α and a curve γ in A, let us denote with $\gamma^{*\alpha}$ the orthogonal projection on α of the image γ^* of γ under the parameterization φ.

The reformulation of Peano's area of the surface S, given by Cesari (see [28, (1956) p. 137]), is the following:

$$\mathscr{P}(S) := \sup_{\{\gamma_i\}_i} \sum_i \sup_\alpha v(\gamma_i^{*\alpha}, \alpha) \tag{7.1}$$

where $\{\gamma_i\}_i$ runs over all finite families of simple closed polygonal curves in A delimiting non-overlapping regions, and α runs over all planes in $\mathbb{R}^3$.

Concerning Geöcze's surface area, let us consider the coordinate planes α_{xy}, α_{yz} and α_{zx} in Euclidean space. The reformulation of Geöcze's area of the surface S, given by Cesari (see [28, (1956) p. 117]), is the following:

$$\mathscr{V}(S) := \sup_{\{\gamma_i\}_i} \sum_i \sqrt{[v(\gamma_i^{*\alpha_{xy}}, \alpha_{xy})]^2 + [v(\gamma_i^{*\alpha_{yz}}, \alpha_{yz})]^2 + [v(\gamma_i^{*\alpha_{zx}}, \alpha_{zx})]^2}. \tag{7.2}$$

A great deal of research has been dedicated to find an axiomatic characterization of a notion of surface area, namely to the problem of establishing properties characterizing univocally the notion of surface area. Cecconi in [27, (1951)] gives the following properties characterizing Lebesgue surface area (and, consequently, Peano area (7.1) and Geöcze area (7.2)):

Theorem 7.2. *Let Φ be a functional defined over all continuous parametric surfaces S on 2-cells. Then Φ coincides with Lebesgue surface area if the following properties are satisfied:*

(7.3) Φ *is lower semi-continuous;*
(7.4) Φ *coincides with usual elementary area for polyhedral surfaces;*
(7.5) Φ *is super-additive*[40]*;*
(7.6) Φ *satisfies Peano inequality*[41].

[39] Among the admissible sets (see Cesari [28, (1956) p. 27]), we mention: planar sets delimited by a Jordan simple curve or finite union of such sets, and open sets.

[40] Namely, for every subdivision of the surface S in pieces, the area of S is greater than the sum of the areas of the various pieces.

[41] Namely, for every surface S, and for every plane α, one has $\Phi(S) \geq \mathrm{mis}(\{P \in \alpha : O(P; C_\alpha) \neq 0\})$, where C_α is the orthogonal projection of the contour of S on α, and $O(P; C_\alpha)$ is the topological index of P with respect to the curve C_α defined above.

In the proof of this Theorem, given by Cecconi, a crucial step consists in the inequality $\mathscr{P}(S) \leq \Phi(S) \leq \mathscr{L}(S)$ that, together with the equality $\mathscr{P}(S) = \mathscr{L}(S)$ (see Theorem 7.1), leads to the expected coincidence $\Phi(S) = \mathscr{L}(S)$.[42]

In addition to Lebesgue, Radó, Cesari and Cecconi, many other mathematicians have been influenced by Peano's definition of surface area; for instance, Geöcze [34, (1910) p. 68], [35, (1913)], Young [81, (1919)], Burkill [13, (1923)], Fréchet [31, (1925)], Severi [74, (1927)], Caccioppoli [14, (1930)].

Young in [81, (1919)], [82, (1920)] and [83, (1921)] defines the "area of a closed skew curve"[43] following a procedure that is equivalent, in essence, to the spatial bi-vectorial definition of surface area given by Peano[44].

Severi in [74, (1927)] provides integral formulae for surface area, starting from the spatial bi-vectorial definition of surface area given by Peano. More precisely, in [74, (1927) p. 475] he uses the integral

$$\int (P(t) - O) \wedge P'(t)\, dt \tag{7.7}$$

($\wedge$ denoting the usual cross product in $\mathbb{R}^3$) to associate a vector with the spatial closed curve $P(t)$ in $\mathbb{R}^3$, "in agreement" with formula (3.16) given by Peano[45].

Caccioppoli, pioneer of the theory of sets of finite perimeter, develops his theory of surface area in several papers, written in the period 1927–1952. He is aware of the various definitions of surface area, due to Lebesgue, Peano, Young, Burkill, Banach, Schauder, Geöcze and Radó. Among these, in [14, (1930)], he appreciates mainly the definition given by Peano:

> [...] the most vital aspect of the idea of Peano is the presence of a vectorial definition, not merely numerical, of surface area.

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[42] Caccioppoli in [14, (1930)] proposes the problem of axiomatizing the notion of surface area. Various answers are given by Zwirner [84, (1937–38)], Scorza-Dragoni [72, (1946)], Stampacchia [78, (1946)] and Pagni [58, (1950)], followed by Cecconi [27, (1951)].

[43] Fréchet [31, (1925)], with respect to Young's paper [81, (1919)], observes: M. W. Young [...] obtient d'ailleurs des résultats très remarquables—et [...] il croit pouvoir préférer à la définition par la plus petite limite des surfaces polyédrales voisines [,] une definition basée comme celle de Peano sur la notion "d'aire minima d'une courbe fermée".

[44] In Young [82, (1920) p. 346] a mechanical interpretation of the area of a closed planar polygonal curve is present: "Join the corresponding points $A, P_1, P_2, \ldots, P_{m-1}, A$, on the curve to form an inscribed polygon. Imagine forces, represented in magnitude, line of action, and sense, by the sides of this polygon [...]. Denote by F the sum of the moments of these forces about any point in the plane, *i.e.* the moment of the resulting couple, counted positive when anti-clockwise sense. Then, if, the norm e [the maximum length of the segments of the subdivision] tends to zero, so that $m \to \infty$, the number F has a unique limit $2A$, the limit A [sic] is called *the area of the curve*". It is worth observing that this limit, in general, does not exit, as shown in Ex. (iii) in Subsection 3.4.

[45] The vector (7.7), due to the fact that $P(t)$ is closed, is independent of the point O; moreover, in view of equality (*) of the footnote 24, it is orthogonal to the oriented 2-plane associated with the bi-vector (3.16), has the same norm and orientation. Notice that formula (7.7) is meaningful only in three dimensional spaces, whereas formula (3.16) is meaningful in space of any dimension.

Moreover, comparing Peano's and Lebesgue's definitions, Caccioppoli writes:

> [A point of view], that appears more simple and rigorous, has been adopted by Lebesgue: the area of a surface is defined as the lower limit of the areas of the approximating polyhedral surfaces. The other point of view, more vague, but more fecund, is the one of Peano: with any piece of surface is associated an *oriented* planar area, playing the same role of a vector for a piece of a curve; in this way, similar to what happens in evaluating the length of a curve by inscribed polygonal curves, it is possible to obtain a set of values that approximate the area of the surface; the area is defined as the upper bound of this set.
>
> [...] The sterility of Lebesgue's definition is related to the absolute absence of a notion of *oriented element* of area.

Some members of the School of Peano [47, p. 187] worked on the notion of surface area, referring to Peano's works on surface area: Sibirani [75, (1906)], [76, (1914)], Viglezio [80, (1920–21)], Cassina [18, (1922)], [19, (1937)], [20, (1950)], [22, 21, (1951)], [23, (1961)].

In addition to the book of Cesari [28, (1956)] we mention some recent review papers, referring to Peano's notion of surface area: Borgato [11, (1993)], Gandon and Perrin [33, (2009)].

It is not rare to find review papers that forget mentioning Peano, whereas they report contributions of authors that refer to Peano's definition. For instance, in a review paper [77, (1931)], Smith does not mention the definition of Peano, attributing it to de la Vallee-Poussin [48, (1903)] but refers to several papers in which Peano's contribution is explicitly evoked (Schwarz [71, (1890)], Lebesgue [50, (1902)], Mangoldt [52, (1902)], de la Vallee–Poussin [48, (1903)], Geöcze [34, (1910)] and Young [81, (1919)]).

References

[1] A. D. Alexandrov, *Convex Polyhedra*, Springer-Verlag, New York, 2005.

[2] T. M. Apostol - M. A. Mnatsakanian, *Subtangents—an aid to visual calculus*, American Mathematical Monthly, 109:525–533, 2002.

[3] T. M. Apostol - M. A. Mnatsakanian, *Tangents and subtangents used to calculate areas*, American Mathematical Monthly, 109:900–908, 2002.

[4] T. M. Apostol - M. A. Mnatsakanian, *The method of sweeping tangents*, Mathematical Gazette, 92:97–102, 2008.

[5] T. M. Apostol - M. A. Mnatsakanian, *New insight into cycloidal areas*, American Mathematical Monthly, 116:598–611, 2009.

[6] G. Bellavitis, *Teoremi generali per determinare le aree dei poligoni e i volumi di poliedri col mezzo delle distanze dei loro vertici*, Annali delle Scienze del Regno Lombardo-Veneto, 4:256–264, 1834.

[7] G. Bellavitis, *Sposizione del metodo delle equipollenze*, Memorie di Matematica e di Fisica della Società Italiana delle Scienze, 25 (2):225–309, 1854.

[8] G. Bellavitis, *Metodo delle equipollenze*, Annali delle Scienze del Regno Lombardo-Veneto, 8:85–121, 1938.

[9] M. Berger, *Géométrie*, CEDIC, Paris, 1977.

[10] C.-W. Borchardt, *Sur la quadrature définie des surfaces courbes*, Journal de Mathématiques Pures et appliquées, 19:369–394, 1854.

[11] M. T. Borgato, *Giuseppe Peano: tra Analisi e Geometria*, In Peano e i fondamenti della matematica. Atti del convegno, Modena 22–24 ottobre 1991, pp. 139–169. Mucchi, Modena, 1993.

[12] C. Burali-Forti - R. Marcolongo, *Elementi di calcolo vettoriale con numerose applicazioni alla geometria, alla meccanica e alla fisica-matematica*, Zanichelli, Bologna, 1909 (2nd edition, 1921).

[13] J. C. Burkill, *The expression of area as an integral*, Proceedings of London Mathematical Society, 22:311–336, 1923.

[14] R. Caccioppoli, *Trasformazioni piane, superficie quadrabili, integrali di superficie*, Rend. Circ. Mat. Palermo, 54:217–262, 1930.

[15] C. Carathéodory, *Über das lineare Mass von Punktmengen—eine Verallgemeinerung des Längenbrgriffs*, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse, pp. 404–426, 1914.

[16] L. N. M. Carnot, *Geometrie der Stellung oder über die Anwendung der Analysis auf Geometrie*, (vol. II). Altona, 1810.

[17] E. Cartan, *Sur certaines expressions différentielles et le problème de Pfaff*, Annales scientifiques de l'École Normale Supérieure, 16:239–332, 1899.

[18] U. Cassina, *Volume, area, lunghezza e curvatura di una figura*, Atti della R. Accademia delle Scienze di Torino, 57:205–216, 1922.

[19] U. Cassina, *Sull'equivalenza fra le figure geometriche e le nozioni di volume, area e lunghezza*, Periodico di Matematiche, 17:1–13, 1937.

[20] U. Cassina, *L'area di una superficie curva nel carteggio inedito di Genocchi con Schwarz ed Hermite*, Rend. Istituto Lombardo di Scienze e Lettere, 83:311–328, 1950.

[21] U. Cassina, *Lunghezze aree e volumi*, In M. Villa, editor, Repertorio di Matematiche, pp. 319–335. CEDAM, Padova, 1951.

[22] U. Cassina, *Sulla teoria delle grandezze e dell'equivalenza*, In M. Villa, editor, Repertorio di Matematiche, pp. 295–318. CEDAM, Padova, 1951.

[23] U. Cassina, *Critica dei principî della matematica e questioni di logica*, Edizioni Cremonese, Roma, 1961.

[24] A. L. Cauchy, *Recherches sur les polyèdres*, Première partie, J. École Polytechnique, 9:68–98, 1813.

[25] A. L. Cauchy, *Sur les polygones et les polyèdres (Deuxieme partie)*, J. École Polytechnique, 9:87–93, 1813.

[26] J. Cecconi, *Su di una congettura di T. Radó*, Rend. Sem. Mat. Univ. Padova, 19:342–366, 1950.

[27] J. Cecconi, *Sull'area di Peano e sulla definizione assiomatica dell'area di una superficie*, Rend. Sem. Mat. Univ. Padova, 20:307–314, 1951.

[28] L. Cesari, *Surface Area*, Princeton University Press, Princeton, 1956.

[29] H. Federer, *Geometric measure theory*, Springer, New York, 1969.

[30] P. J. Federico, *Descartes on Polyhedra: A Study of the "De Solidorum Elementis"*, Springer-Verlag, New York, 1982.

[31] M. Fréchet, *Sur le prolongement des fonctionnelles semi-continues et sur l'aire des surfaces courbes*, Fundamenta Mathematicae, 7:210–224, 1925.

[32] A. Gabba, *La definizione di area di una superficie curva ed un carteggio inedito di Casorati con Schwarz e Peano*, Istituto Lombardo di Scienze e Lettere, 91:857–883, 1957.

[33] S. Gandon - Y. Perrin, *Le problème de la définition de l'aire d'une surface gauche: Peano et Lebesgue*, Arch. Hist. Exact. Sci., 63:665–704, 2009.

[34] Z. de Geöcze, *Quadratures des surfaces courbes*, Mathematische und naturwissenschaftliche Berichte aus Ungarn, 26:1–88, 1910.

[35] Z. de Geöcze, *Sur la quadrature des variétés*, C.R.A.Sc. Paris, 157:910–912, 1913.

[36] H. G. Grassmann, *Gesammelte Werke*, Teubner, Leipzig, 1894–1911.

[37] H. G. Grassmann, *A New Branch of Mathematics, The Ausdehnungslehre of 1844 and Other Works*, Open Court Publ. Comp., Chicago, 1995.

[38] H. G. Grassmann, *Extension Theory (Translation of Ausdehnungslehre, 2nd edition, 1862)*, American Mathematical Society, 2000.

[39] G. H. Greco - S. Mazzucchi - E. Pagani, *Peano on derivative of measures: strict derivative of distributive set functions*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl., 21:305–339, 2010.

[40] G. H. Greco - E. Pagani, *Reworking on affine exterior algebra of Grassmann: Peano and his School*, Istituto Lombardo, Rendiconti Classe di Scienze Matematiche e Naturali, 144:17–52, 2010.

[41] A. Harnack, *Lehrbuch der Differential- und Integralrechnung* (vol. 2), Teubner, Leipzig, 1885.

[42] F. Hausdorff, *Dimension und aüsseres Mass*, Math. Ann., 79:157–179, 1918.

[43] C. Hermite, *Cours de M. Hermite professé pendant le* 2^e *semestre 1881–82*, Hermann, Paris, 1883.

[44] C. Hermite, *Cours de M. Hermite professé pendant le* 2^e *semestre 1881–82*, Hermann, Paris, 1887.

[45] C. G. J. Jacobi, *Regel zur Bestimmung des Inhalts der Sternpolygone*, Journal für die reine und angewandte Mathematik, 65:173–174, 1866.

[46] C. Jordan, *Cours d'Analyse de l'École Polytechnique* (3 vol.), Gauthier-Villars, Paris, 1893 (2nd edtion).

[47] H. C. Kennedy, *Life and Works of Giuseppe Peano*, D. Reidel Publ. Co, Dodrecht, 1980.

[48] C. de La Vallée Poussin, *Sur la définition de l'aire des surfaces courbes*, Annales de la Societé scientifique de Bruxelles, 27 A:90–91, 1903.

[49] J.-L. Lagrange, *Théorie des fonctions analytiques*, Paris, 1813.

[50] H. Lebesgue, *Intégrale, longueur, aire*, Annali di Matematica Pura e Applicata, 7:231–359, 1902.

[51] G. W. Leibniz, *La caractéristique géométrique*, Vrin, Paris, 1995.

[52] H. von Mangoldt, *Anwendung der Differential- und Integralrechnung auf Kurven und Flächen*, In Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, vol. III, 3, pp. 64–67. Teubner, Leipzig, 1902.

[53] H. Minkowski, *Ueber die Begriffe Länge, Oberfläche und Volumen*, Jahresbericht der Deutschen Mathematiker-Vereinigung, 9:115–121, 1901.

[54] M. A. Mnatsakanian, *On the area of the region on a developable surface*, Dokl. Akad. Nauk Armenian. SSR, 2:97–102, 1981.

[55] A. F. Möbius, *Lehrbuch der Statik*, Leipzig, 1837.

[56] A. F. Möbius, *Über die Bestimmung des Inhaltes eines Polyedres*, Leipziger Sitzungsberichte math.-phys. Classe, 17:31–68, 1865.

[57] A. F. Möbius, *Der Barycentrische Calcul*, Verlag von J. A. Barth, Leipzig, 1927.

[58] M. Pagni, *Sulla definizione dell'area di una superficie per via assiomatica*, Rend. Sem. Mat. Univ. Padova, 19:303–316, 1950.

[59] G. Peano, *Sull'integrabilità delle funzioni*, Atti della R. Accademia delle Scienze di Torino, 18:439–446, 1882.

[60] G. Peano, *Applicazioni Geometriche del calcolo infinitesimale*, Fratelli Bocca Editori, 1887.

[61] G. Peano, *Calcolo geometrico secondo Ausdehnungslehre di H. Grassmann*, Fratelli Bocca, Torino, 1888.

[62] G. Peano, *Sulla definizione dell'area d'una superficie*, Atti della Reale Accademia dei Lincei, Rendiconti, 6 (1st sem.):54–57, 1890.

[63] G. Peano, *Lezioni di analisi infintesimale*, (2 vol.). Candeletti, Torino, 1893.

[64] G. Peano, *Formulaire Mathématique*, Fratelli Bocca, Torino, 4th edition, 1902–3.

[65] G. Peano, *Formulario Mathematico*, Fratelli Bocca, Torino, 5th edition, 1908.

[66] L. Poinsot, *Mémoire sur les polygones et les polyèdres*, J. École Polytechnique, 4:16–49, 1810.

[67] L. Poinsot, *Éléments de Statique*, Maillet-Bachelier, Paris, 1811.

[68] T. Radó, *Sur l'aire des surfaces continues*, In Atti del Congresso Internazionale dei Matematici, Bologna 1928. Vol. VI (t. II), pp. 353–360. Zanichelli, Bologna, 1930.

[69] T. Radó, *Length and Area*, American Mathematical Society, New York, 1948.

[70] T. Radó, *Review of "L. Cesari, Surface area"*, Bull. Amer. Math. Soc., 62:513–515, 1956.

[71] H. A. Schwarz, *Gesammelte mathematische Abhandlungen* (2 vol.), Springer, Berlin, 1890.

[72] G. Scorza-Dragoni, *Sulla definizione assiomatica dell'area di una superficie*, Rend. Sem. Mat. Univ. Padova, 15:8–24, 1946.

[73] J.-A. Serret, *Cours de calcul différentiel et intégral* (2 vol.), Gauthier-Villars, Paris, 1868^1, 1879^2, 1886^3, 1894^4, 1900^5, 1911^6.

[74] F. Severi, *Riflessioni sull'area di una sperficie curva*, Rendiconti della R. Accademia dei Lincei, 5:471–479, 1927.

[75] F. Sibirani, *Sulla definizione di area di una superficie curva*, Periodico di Matematiche, 21:32–43, 1906.

[76] F. Sibirani, *Sulla lunghezza delle linee e sull'area delle superficie*, Rend. Istituto Lombardo di Scienze e Lettere, 47:383–398, 1914.

[77] H. L. Smith, *The notion of area of surface*, Mathematics News Letter, 5:36–43, 1931.

[78] G. Stampacchia, *Sulla definizione assiomatica dell'area di una superficie rettificabile*, Rendiconti della Accademia dei Lincei. Classe di Scienze fisiche, matematiche e naturali, 2:542–546, 1947.

[79] C. Sturm, *Cours d'analyse de l'École polytechnique* (2 vol.), Gauthier-Villars, 1857–59^1, 1863–4^2, 1877^5.

[80] E. Viglezio, *Aree di curve piane*, Atti della R. Accademia delle Scienze di Torino, 56:89–96, 1920–21.

[81] W. H. Young, *On the area of surfaces*, Proceedings of the Royal Society of London, S. A, 96:71–81, 1919.

[82] W. H. Young, *On a formula for an area*, Proceedings of London Mathematical Society, 18:339–374, 1920.

[83] W. H. Young, *On the triangulation method of defining the area of a surface*, Proceedings of London Mathematical Society, 19:117–152, 1921.

[84] Giuseppe Zwirner, *Sulla definizione d'area di una superficie*, Atti Ist. Veneto Sci. Lett. Arti, 97:409–416, 1937–38.

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