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Partial Differential Equations — *Continuity estimates for p-Laplace type operators in Orlicz–Zygmund spaces*, by Fernando Farroni, communicated on 11 March 2016.

Abstract. — We study the Dirichlet problem

$$\begin{cases} \operatorname{div} \mathcal{A}(x, \nabla u) = \operatorname{div} f & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$, with $N \geq 2$. The vector field $\mathcal{A} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies the tipical growth and coercivity conditions of the p-Laplacian type operator with $p > 1$. We prove existence and uniqueness results in the case the vector field f belongs to the Orlicz–Zygmund space $\mathscr{L}^q \log^{-\alpha} \mathscr{L}(\log\log \mathscr{L})^{-\beta}(\Omega, \mathbb{R}^N)$, $q = \frac{p}{p-1}$, $\alpha > 0$ and $\beta \in \mathbb{R}$ or $\alpha = 0$ and $\beta > 0$. In particular, the gradient of the solution belongs to $\mathscr{L}^p \log^{-\alpha} \mathscr{L}(\log\log \mathscr{L})^{-\beta}(\Omega, \mathbb{R}^N)$. Further, we provide estimates implying the continuity of the operator which carries any given f into the gradient field ∇u of the solution.

Key words: Nonlinear elliptic equations, continuity estimates

Mathematics Subject Classification: 35J60

1. Introduction

Let Ω be a bounded Lipschitz domain of $\mathbb{R}^N$, $N \geq 2$. We consider the Dirichlet problem

$$\begin{cases} \operatorname{div} \mathcal{A}(x, \nabla u) = \operatorname{div} f & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\mathcal{A} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory vector field satisfying the following conditions for a.e. $x \in \Omega$ and all $\xi, \eta \in \mathbb{R}^N$

$$\mathcal{A}(x, 0) = 0 \tag{2}$$

$$\langle \mathcal{A}(x, \xi) - \mathcal{A}(x, \eta), \xi - \eta \rangle \geq a|\xi - \eta|^2(|\xi| + |\eta|)^{p-2} \tag{3}$$

$$|\mathcal{A}(x, \xi) - \mathcal{A}(x, \eta)| \leq b|\xi - \eta|(|\xi| + |\eta|)^{p-2} \tag{4}$$

where $p > 1$, $0 < a \leq b$.

Let $f = (f^1, f^2, \dots, f^N)$ be a vector field of class $\mathscr{L}^s(\Omega, \mathbb{R}^N)$, $1 \leq s \leq q$ where q is the conjugate exponent to p, i.e. $pq = p + q$.

Definition 1.1. A function $u \in \mathscr{W}_0^{1,r}(\Omega)$, $\max\{1, p-1\} \le r \le p$, is a solution of (1) if

$$\int_\Omega \langle \mathcal{A}(x, \nabla u), \nabla\varphi \rangle \, dx = \int_\Omega \langle f, \nabla\varphi \rangle \, dx, \tag{5}$$

for every $\varphi \in C_0^\infty(\Omega)$.

By a routine argument, if $s \ge r/(p-1)$, it can be seen that the identity (5) still holds for functions $\varphi \in \mathscr{W}^{1, \frac{r}{r-p+1}}(\Omega)$ with compact support. By virtue of assumptions (2)–(4), the model case we have in mind is represented by a Dirichlet problem involving the p-Laplace operator, namely

$$\begin{cases} \operatorname{div}|\nabla u|^{p-2}\nabla u = \operatorname{div} f & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{6}$$

We shall refer to a solution in the sense of Definition 1.1 as a distributional solution or (as some people say) as a very weak solution [20, 24]. Moreover, if $r < p$, a solution may have infinite energy, namely $|\nabla u| \notin \mathscr{L}^p(\Omega)$.

Existence and uniqueness for a solution to problem (1) is a well studied problem and known results available in literature depending on the degree of regularity of the right hand side of (1). Beside the standard fact that a solution exists, it is unique and belongs to $\mathscr{W}_0^{1,p}(\Omega)$ if $f \in \mathscr{L}^q(\Omega, \mathbb{R}^N)$, the existence of a solution $u \in \mathscr{W}_0^{1,1}(\Omega)$ to problem (1) is obtained in [4] when $\operatorname{div} f$ belongs to $\mathscr{L}^1(\Omega, \mathbb{R}^N)$. On the other hand, uniqueness of solutions to problem (1) generally fails, as shown by a classical counterexample due to J. Serrin [29]. The presence of such "pathological" examples gave rise to new possible definitions of a solution to problem (1). This is the case of the so-called duality solutions [30], the approximation solutions (SOLA) [4], the entropy solutions [28, 23, 5]. Regularity properties of such solutions are given in [25, 26]. Unfortunately, the tecniques developped in these papers do not provide uniqueness if we consider distributional solutions and still this question is not completely clear, unless for the special case $p = 2$ [3, 14]. In this particular case, the understanding of the problem in the scale of Sobolev spaces is available, see e.g. [22]. Until now, the case $p \neq 2$ has been treated under suitable regularity assumptions on f, that is to say f belongs to function spaces slightly larger than the natural setting $\mathscr{L}^q(\Omega, \mathbb{R}^N)$, where q denotes the conjugate exponent to p. In particular, we refer the reader to [17] (see also [11]) and to [12] where the problem is settled in the grand Sobolev spaces and in Zygmund spaces respectively). We refer also to [3] for the case $p = N$. We want to point out that the study of elliptic problems in the setting of Zygmund–Orlicz spaces has been previously considered in [27] and [6] where the higher integrability of solutions to degenerate elliptic equations of p-Laplacian type is proved.

Our aim is to study existence and uniqueness for the problem (1) in the case where the vector field f belongs to the Orlicz–Zygmund space

$$\mathscr{L}^q \log^{-\alpha} \mathscr{L} (\log\log \mathscr{L})^{-\beta}(\Omega, \mathbb{R}^N)$$

$q = \frac{p}{p-1}$, $\alpha > 0$ and $\beta \in \mathbb{R}$.

We recall that the Zygmund space $\mathscr{L}^q \log^{-\alpha} \mathscr{L}(\log\log \mathscr{L})^{-\beta}(\Omega, \mathbb{R}^N)$, for $1 < q < \infty$, $\alpha, \beta \in \mathbb{R}$ is the Orlicz space generated by the function

$$\Phi(t) \equiv \Phi_{p,\alpha,\beta}(t) = t^q \log^{-\alpha}(a+t)(\log\log(a+t))^{-\beta}, \quad t \geq 0,$$

for suitable large values of the constant $a \geq e^e$ (whose choich is immaterial) and it is usually equipped with the Luxemburg norm $\|\cdot\|_{L^\Phi(\Omega)}$ (see Section 2.3 for the definitions). In order to simplify our notation, we will often write $[\![\cdot]\!]_{p,\alpha,\beta}$ instead of $\|\cdot\|_{L^\Phi(\Omega)}$.

The expected regularity for the solution u of (1) corresponds to the fact that the gradient ∇u belongs to $\mathscr{L}^p \log^{-\alpha} \mathscr{L}(\log\log \mathscr{L})^{-\beta}(\Omega, \mathbb{R}^N)$. Indeed, for α and β positive, some explicit example could be given by means of a radial solutions; for instance the function

$$u(x) = \int_{|x|}^{r_0} \frac{d\rho}{[\rho^N |\log \rho|^{1-\alpha} |\log|\log \rho||^{\sigma}]^{\frac{1}{p}}}$$

with $\sigma > 1 - \beta$, is a solution to the model case (6) when f is given by

$$f(x) = F(|x|)\frac{x}{|x|} \quad F(r) = -\frac{1}{[r^N |\log r|^{1-\alpha} |\log|\log r||^{\sigma}]^{\frac{p-1}{p}}}$$

We state our result for positive values of the parameter α. The case $\alpha = 0$ will be treated in the last section for reader's convenience. Our first result is the following.

THEOREM 1.1. *Let $1 < p < \infty$, $p \neq 2$.*

For each $f \in \mathscr{L}^q \log^{-\alpha} \mathscr{L}(\log\log \mathscr{L})^{-\beta}(\Omega, \mathbb{R}^N)$, with $pq = p + q$ and $0 < \alpha < \frac{p}{|p-2|}$, $\beta \in \mathbb{R}$, the problem (1) admits a unique solution $u : \Omega \to \mathbb{R}$, such that $\nabla u \in \mathscr{L}^p \log^{-\alpha} \mathscr{L}(\log\log \mathscr{L})^{-\beta}(\Omega, \mathbb{R}^N)$. There exists a constant $C > 0$, $C = C(N, p, \alpha, \beta, a, b)$, such that the following estimate holds true

$$[\![\nabla u]\!]^p_{p,\alpha,\beta} \leq C[\![f]\!]^q_{q,\alpha,\beta} \tag{7}$$

Moreover the operator

$$\mathcal{H} : \mathscr{L}^q \log^{-\alpha} \mathscr{L}(\log\log \mathscr{L})^{-\beta}(\Omega, \mathbb{R}^N) \to \mathscr{L}^p \log^{-\alpha} \mathscr{L}(\log\log \mathscr{L})^{-\beta}(\Omega, \mathbb{R}^N) \tag{8}$$

which carries a given vector field f into the gradient field ∇u is continuous.

Related to previous theorem, we provide also a precise continuity estimate for the operator $\mathcal{H}$. To this aim, we state our next result.

THEOREM 1.2. *Let $1 < p < \infty$, $p \neq 2$.*

There exists a constant $C > 0$, $C = C(N, p, \alpha, \beta, a, b)$, such that, if f and g belong to $\mathscr{L}^q \log^{-\alpha} \mathscr{L}(\log\log \mathscr{L})^{-\beta}(\Omega, \mathbb{R}^N)$ with $pq = p + q$, $0 < \alpha < \frac{p}{|p-2|}$ and $\beta \in \mathbb{R}$ then

$$(9)\qquad [\![\mathcal{H}f - \mathcal{H}g]\!]^p_{p,\alpha,\beta}$$
$$\le C([\![f-g]\!]^{\gamma}_{q,\alpha,\beta}[\![|f|+|g|]\!]^{1-\gamma}_{q,\alpha,\beta})^q\Big(1-\log\frac{[\![f-g]\!]_{q,\alpha,\beta}}{[\![|f|+|g|]\!]_{q,\alpha,\beta}}\Big)^{\beta^+},$$

where $\beta^+ = \max\{\beta, 0\}$ *and*

$$\gamma = 1 - \alpha\frac{p-2}{p} \quad \textit{if } p > 2 \tag{10}$$

$$\gamma = \frac{p}{q}\Big(1 - \alpha\frac{2-p}{p}\Big) \quad \textit{if } 1 < p < 2 \tag{11}$$

Estimate (9) has been established in [12] for $\beta = 0$, therefore Theorem 1.1 allow us to extend $\mathcal{H}$ as a continuous operator to a space larger that $\mathscr{L}^q \log^{-\alpha}\mathscr{L}(\Omega, \mathbb{R}^N)$ as long as $\beta > 0$. On the other hand, if $\beta < 0$, the space $\mathscr{L}^q \log^{-\alpha}\mathscr{L}(\log\log\mathscr{L})^{-\beta}$ is continuously embedded into $\mathscr{L}^q \log^{-\alpha}\mathscr{L}(\Omega, \mathbb{R}^N)$, so our results provides higher integrability of the solutions found in [12]. We recall that our estimates are useful also in describing some quantitative properties of mappings of bounded distortion (see e.g. [11, 10]), whose coordinate functions always solve an equation as in (1) with structural assumptions of type (2)–(4). Other sharp estimates related to mappings of bounded distortion (precisely to quasiconformal mappings) can be found also in [8, 9].

The paper is organized as follows. Section 2 is devoted to the basic notation and definitions; in particular, we will introduce the grand Sobolev spaces and we give the definition of a general Orlicz space. Section 3 collects technical lemmas allowing the construction of a new norm in the space $\mathscr{L}^q \log^{-\alpha}\mathscr{L}(\log\log\mathscr{L})^{-\beta}(\Omega)$. This new norm (which turns to be equivalent to the Luxemburg one) is introduced in Section 4 and we distinguish the two cases $\alpha > 0$ and $\beta \in \mathbb{R}$ from $\alpha = 0$ and $\beta > 0$ for reader's convenience. In Section 5, we prove Theorems 1.1 and 1.2. Our starting point consists of an estimate below the natural exponent (see Corollary 5.2 below) for problem (1) as done in [17, 21]. Our proofs strongly rely on the fact that the new norm introduced in Section 4 takes into account only the behaviour of the $\mathscr{L}^r$-norms of a function in $\mathscr{L}^q \log^{-\alpha}\mathscr{L}(\log\log\mathscr{L})^{-\beta}(\Omega)$ for $r < q$ and so it better fits the estimates of Section 5. Finally, in Section 6 we state and prove two results similar to Theorem 1.1 and Theorem 1.2 in the setting of Orlicz–Zygmund space $\mathscr{L}^q(\log\log\mathscr{L})^{-\beta}(\Omega, \mathbb{R}^N)$.

2. Preliminary results

2.1. Basic notation

We indicate that quantities $\mu, \sigma \ge 0$ are equivalent by writing $\mu \sim \sigma$; namely, $\mu \sim \sigma$ will mean that there exist constants $c_1, c_2 > 0$ such that $c_1\mu \le \sigma \le c_2\mu$.

From now on, Ω will denote a bounded Lipschitz domain in $\mathbb{R}^N$. For a function $v \in \mathscr{L}^p(\Omega)$ with $1 \le p < \infty$ we set

$$\|v\|_p = \Big(\fint_\Omega |v|^p \, dx \Big)^{\frac{1}{p}}$$

Barred integrals denote averages, namely $\fint_\Omega = \frac{1}{|\Omega|} \int_\Omega$.

2.2. *Grand Lebesgue and grand Sobolev spaces*

For $1 < p < \infty$ we denote by $\mathscr{L}^{p)}(\Omega)$ *the grand–Lebesgue space* consisting of all functions $v \in \bigcap_{0<\varepsilon\le p-1} L^{p-\varepsilon}(\Omega)$ such that

$$\|v\|_{p)} = \sup_{0<\varepsilon\le p-1} \varepsilon^{\frac{1}{p}} \Big(\fint_\Omega |v|^{p-\varepsilon} \, dx \Big)^{\frac{1}{p-\varepsilon}} < \infty.$$

Moreover

$$\|v\|_{p)} \sim \sup_{0<\varepsilon\le p-1} \Big(\varepsilon \fint_\Omega |v|^{p-\varepsilon} \, dx \Big)^{\frac{1}{p-\varepsilon}}.$$

The Marcinkiewicz class weak–$\mathscr{L}^p(\Omega)$ is contained in $\mathscr{L}^{p)}(\Omega)$ (see [20, Lemma 1.1]).

More generally, if $\alpha > 0$ we denote by $\mathscr{L}^{\alpha,p)}(\Omega)$ *the grand–Lebesgue space* consisting of all functions $v \in \bigcap_{0<\varepsilon\le p-1} \mathscr{L}^{p-\varepsilon}(\Omega)$ such that

$$\|v\|_{\alpha,p)} = \sup_{0<\varepsilon\le p-1} \varepsilon^{\frac{\alpha}{p}} \Big(\fint_\Omega |v|^{p-\varepsilon} \, dx \Big)^{\frac{1}{p-\varepsilon}} < \infty.$$

2.3. *Orlicz and Zygmund spaces*

We need to recall some basic properties of Orlicz spaces; for more details on Orlicz spaces we refer to [1] and also to [16].

Let $\Phi : [0, \infty) \to [0, \infty)$ be a Young function, that is $\Phi(0) = 0$, Φ is increasing and convex. If Ω is a open subset of $\mathbb{R}^N$, we define the Orlicz space $L^\Phi(\Omega)$ generated by the Young function Φ as the set of measurable functions $f : \Omega \to \mathbb{R}$ such that

$$\int_\Omega \Phi\Big(\frac{|f(x)|}{\lambda} \Big) \, dx < \infty,$$

for some $\lambda > 0$. This space is equipped with the Luxemburg norm

$$\|f\|_{L^\Phi(\Omega)} = \inf \Big\{ \lambda > 0 : \int_\Omega \Phi\Big(\frac{|f(x)|}{\lambda} \Big) \, dx \le 1 \Big\}.$$

We shall need to consider the Zygmund space $\mathscr{L}^q \log^{-\alpha} \mathscr{L}(\Omega)$, for $1 < q < \infty$, $\alpha > 0$. This is the Orlicz space generated by the function

$$\Phi(t) = t^q \log^{-\alpha}(a + t), \quad t \geq 0,$$

where $a \geq \mathrm{e}$ is a suitably large constant, so that Φ is increasing and convex on $[0, \infty[$. The choice of a will be immaterial. More explicitly, for a measurable function f on Ω, $f \in \mathscr{L}^q \log^{-\alpha} \mathscr{L}(\Omega)$ simply means that

$$\int_\Omega |f|^q \log^{-\alpha}(a + |f|)\, dx < \infty.$$

Equipped with the Luxemburg norm $\mathscr{L}^q \log^{-\alpha} \mathscr{L}(\Omega)$ is a Banach space. An equivalent norm to the Luxemburg one, which involves the norms in $\mathscr{L}^{q-\varepsilon}(\Omega)$, for $0 < \varepsilon \leq q - 1$ has been introduced in [12] (see also [13]) by defining

$$\|f\|_{\mathscr{L}^q \log^{-\alpha} \mathscr{L}} = \left\{ \int_0^{\varepsilon_0} \varepsilon^{\alpha-1} \|f\|_{q-\varepsilon}^q \, d\varepsilon \right\}^{1/q}$$

for any f measurable function defined on Ω. Here $\varepsilon_0 \in]0, q-1]$ is fixed. A simple application of the Lebesgue dominated convergence theorem proves that

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{\alpha/q} \|f\|_{q-\varepsilon} = 0, \tag{12}$$

for all $f \in \mathscr{L}^q \log^{-\alpha} \mathscr{L}(\Omega)$, see [15]. We stress that (12) does not hold uniformly, as f varies in a bounded set of $\mathscr{L}^q \log^{-\alpha} \mathscr{L}(\Omega)$. Nevertheless, one can prove the following (see again [12]).

Lemma 2.1. *For each compact subset $M \subset \mathscr{L}^q \log^{-\alpha} \mathscr{L}(\Omega)$, condition* (12) *holds uniformly for $f \in M$, that is*

$$\lim_{\varepsilon \downarrow 0} \left(\sup_{f \in M} \varepsilon^{\alpha/q} \|f\|_{q-\varepsilon} \right) = 0.$$

3. Integral estimates

In this Section we collect a series of integral estimates. To this aim, we let $\Gamma = \Gamma(\alpha)$ be the well known Euler Gamma function defined as

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt$$

for $\alpha > 0$.

Lemma 3.1. *For any $\alpha > 0$, $\gamma \in \mathbb{R}$ and $\varepsilon_0 \in (0, 1)$*

$$\lim_{\lambda \to \infty} \frac{\lambda^\alpha}{\log^\gamma \lambda} \int_0^{\varepsilon_0} t^{\alpha-1} |\log t|^\gamma e^{-\lambda t}\, dt = \Gamma(\alpha) \tag{13}$$

PROOF. In order to simplify our notation, we define

$$I(\lambda) = \frac{\lambda^\alpha}{\log^\gamma \lambda} \int_0^{\varepsilon_0} t^{\alpha-1} |\log t|^\gamma e^{-\lambda t}\, dt \tag{14}$$

By the change of variable $s = \lambda t$ we get

$$I(\lambda) = \int_0^{\varepsilon_0 \lambda} s^{\alpha-1} \left(\frac{\log \lambda - \log s}{\log \lambda} \right)^\gamma e^{-s}\, ds \tag{15}$$

For $s \in (0, \infty)$, we define

$$h_\lambda(s) = s^{\alpha-1} \left| \frac{\log \lambda - \log s}{\log \lambda} \right|^\gamma e^{-s} \chi_{(0,\varepsilon_0 \lambda)}(s) \tag{16}$$

$$h(s) = s^{\alpha-1} e^{-s} \tag{17}$$

so that

$$I(\lambda) = \int_0^\infty h_\lambda(s)\, ds \tag{18}$$

$$\Gamma(\alpha) = \int_0^\infty h(s)\, ds \tag{19}$$

Clearly $h_\lambda \to h$ a.e. in $(0, \infty)$ as $\lambda \to \infty$, therefore our proof is a matter of using the dominated convergence theorem to get

$$\lim_{\lambda \to \infty} I(\lambda) = \Gamma(\alpha)$$

We distinguish two cases, depending on the sign of γ.

CASE 1: $\gamma \geq 0$. For $\lambda > e$ and for all $s \in (0, \infty)$ we have

$$h_\lambda(s) \leq g(s) = s^{\alpha-1}(1 + |\log s|)^\gamma e^{-s}$$

We observe that $g \in \mathscr{L}^1(0, \infty)$, so we pass to the limit as $\lambda \to \infty$ in (15) and we get (13).

CASE 2: $\gamma < 0$. For sufficiently large values of λ we have

$$\lambda > \varepsilon_0^{-2}$$

Then, we split the integral $I(\lambda)$ as the sum of two terms

$$I(\lambda) = I_1(\lambda) + I_2(\lambda)$$

where

$$I_1(\lambda) = \int_0^{\sqrt{\lambda}} s^{\alpha-1}\Big(\frac{\log\lambda - \log s}{\log\lambda}\Big)^{\gamma} e^{-s}\,ds \tag{20}$$

$$I_2(\lambda) = \int_{\sqrt{\lambda}}^{\varepsilon_0\lambda} s^{\alpha-1}\Big(\frac{\log\lambda - \log s}{\log\lambda}\Big)^{\gamma} e^{-s}\,ds \tag{21}$$

Since γ is negative we have

$$\Big(\frac{\log\lambda - \log s}{\log\lambda}\Big)^{\gamma} \le 2^{-\gamma} \quad \text{for all } s \in (0, \sqrt{\lambda})$$

so we pass to the limit as $\lambda \to \infty$ in (20) with the aid of the dominated convergece theorem and we obtain

$$\lim_{\lambda\to\infty} I_1(\lambda) = \Gamma(\alpha) \tag{22}$$

In order to complete our proof, it remains to show that

$$\lim_{\lambda\to\infty} I_2(\lambda) = 0 \tag{23}$$

Observe that

$$I_2(\lambda) \le \frac{(-\log\varepsilon_0)^{\gamma} e^{-\frac{\sqrt{\lambda}}{2}}}{(\log\lambda)^{\gamma}} \int_{\sqrt{\lambda}}^{\varepsilon_0\lambda} s^{\alpha-1} e^{-\frac{s}{2}}\,ds \tag{24}$$

since $\tilde{g}(s) = s^{\alpha-1}e^{-\frac{s}{2}}$ is a function in $\mathscr{L}^1(0,\infty)$ the right hand side of (24) approches to zero as $\lambda \to \infty$ and so (23) immediately follows. □

LEMMA 3.2. *For any $\gamma > 0$ and $\varepsilon_0 \in (0,1)$*

$$\lim_{\lambda\to\infty} \log^{\gamma}\lambda \int_0^{\varepsilon_0} |\log t|^{-\gamma-1} e^{-\lambda t}\,\frac{dt}{t} = \frac{1}{\gamma} \tag{25}$$

PROOF. Let us observe that

$$h_\lambda(t) = \frac{|\log t|^{-\gamma-1}}{t} e^{-\lambda t} \in \mathscr{L}^1(0,\varepsilon_0) \quad \text{for all } \lambda > 0$$

as long as $\gamma > 0$. Indeed, an integration by parts yelds

$$\log^{\gamma}\lambda \int_0^{\varepsilon_0} |\log t|^{-\gamma-1} e^{-\lambda t}\,\frac{dt}{t} = \log^{\gamma}\lambda\,\frac{(-\log\varepsilon_0)^{-\gamma}}{\gamma} e^{-\varepsilon_0\lambda} + \frac{\lambda}{\gamma}\log^{\gamma}\lambda \int_0^{\varepsilon_0} (-\log t)^{-\gamma} e^{-\lambda t}\,dt \tag{26}$$

The first term at the right hand side of (26) approches to zero as $\lambda \to \infty$, while the second term can be handled as in the proof of Lemma 4.2 (it is sufficient to follow the argument there for $\alpha = 1$ and when γ is exchanged with $-\gamma$) so the proof ends by passing to the limit as $\lambda \to \infty$ in (26). □

4. Properties of the space $\mathscr{L}^q \log^{-\alpha} \mathscr{L}(\log\log \mathscr{L})^{-\beta}(\Omega)$

In this section, we study the Zygmund space $\mathscr{L}^p \log^{-\alpha} \mathscr{L}(\log\log \mathscr{L})^{-\beta}(\Omega, \mathbb{R}^N)$, for $1 < q < \infty$, $\alpha > 0$ and $\beta \in \mathbb{R}$. This is the Orlicz space generated by the function

$$\Phi(t) = t^q \log^{-\alpha}(a+t)(\log\log(a+t))^{-\beta}, \quad t \geq 0,$$

where $a \geq e^e$ is a suitably large constant so that Φ is increasing and convex on $[0, \infty)$. In particular, the main goal of the section is the construction of a new norm on $\mathscr{L}^q \log^{-\alpha} \mathscr{L}(\log\log \mathscr{L})^{-\beta}(\Omega)$ which is equivalent to the Luxembourg one and only takes into account the L^r-norms of a function in $\mathscr{L}^q \log^{-\alpha} \mathscr{L}(\log\log \mathscr{L})^{-\beta}(\Omega)$, whenever $r < p$. This will be useful in order to get the proofs of Theorem 1.1 and Theorem 1.2 that we will provide later.

The first case we take into account corresponds to the choice $\alpha > 0$. To do that, we consider a measurable function f defined on Ω and then we set

$$\|f\|_{\mathscr{L}^q \log^{-\alpha} \mathscr{L}(\log\log \mathscr{L})^{-\beta}} = \left\{ \int_0^{\varepsilon_0} \varepsilon^{\alpha-1} |\log \varepsilon|^{-\beta} \|f\|_{q-\varepsilon}^q \, d\varepsilon \right\}^{1/q} \tag{27}$$

Here $\varepsilon_0 \in]0, 1[$ is fixed. Our task is to prove the following

Lemma 4.1. *Let $\beta \in \mathbb{R}$ and let $\varepsilon_0 \in (0,1)$ A measurable function f belongs to $\mathscr{L}^q \log^{-\alpha} \mathscr{L}(\log\log \mathscr{L})^{-\beta}(\Omega)$*

$$\|f\|_{\mathscr{L}^q \log^{-\alpha} \mathscr{L}(\log\log \mathscr{L})^{-\beta}} < \infty. \tag{28}$$

Moreover, $\| \, \|_{\mathscr{L}^q \log^{-\alpha}(\log\log \mathscr{L})^{-\beta} \mathscr{L}}$ is a norm equivalent to the Luxemburg one, that is, there exist constants $C_i = C_i(q, \alpha, a, \varepsilon_0)$, $i = 1, 2$, such that for all $f \in \mathscr{L}^q \log^{-\alpha} \mathscr{L}(\log\log \mathscr{L})^{-\beta}(\Omega)$

$$C_1 [\![f]\!]_{\mathscr{L}^q \log^{-\alpha} \mathscr{L}(\log\log \mathscr{L})^{-\beta}} \leq \|f\|_{\mathscr{L}^q \log^{-\alpha} \mathscr{L}(\log\log \mathscr{L})^{-\beta}} \leq C_2 [\![f]\!]_{\mathscr{L}^q \log^{-\alpha} \mathscr{L}(\log\log \mathscr{L})^{-\beta}}.$$

The case $\alpha = 0$ and $\beta > 0$ should be treated separately. We consider a measurable function f defined on Ω and then we set

$$\|f\|_{\mathscr{L}^q (\log\log \mathscr{L})^{-\beta}} = \left\{ \int_0^{\varepsilon_0} \varepsilon^{-1} |\log \varepsilon|^{-\beta-1} \|f\|_{q-\varepsilon}^q \, d\varepsilon \right\}^{1/q} \tag{29}$$

Here $\varepsilon_0 \in]0, 1[$ is fixed. Our task is to prove the following

Lemma 4.2. *Let $\beta > 0$ and let $\varepsilon_0 \in (0,1)$ A measurable function f belongs to $\mathscr{L}^q(\log\log\mathscr{L})^{-\beta}(\Omega)$ if and only if*

$$\|f\|_{\mathscr{L}^q(\log\log\mathscr{L})^{-\beta}(\Omega)} < \infty. \tag{30}$$

Moreover, $\|\ \|_{\mathscr{L}^q(\log\log\mathscr{L})^{-\beta}}$ is a norm equivalent to the Luxemburg one, that is, there exist constants $C_i = C_i(q,\alpha,a,\varepsilon_0)$, $i = 1,2$, such that for all $f \in \mathscr{L}^q(\log\log\mathscr{L})^{-\beta}(\Omega)$

$$C_1[\![f]\!]_{\mathscr{L}^q(\log\log\mathscr{L})^{-\beta}} \le \|f\|_{\mathscr{L}^q(\log\log\mathscr{L})^{-\beta}} \le C_2[\![f]\!]_{\mathscr{L}^q(\log\log\mathscr{L})^{-\beta}}.$$

We only prove Lemma 4.1 since Lemma 4.2, can be proved in the same way. Before we give the proof of Lemma 4.2, we state some preliminary results.

Proof of Lemma 4.1. It is immediately seen that $\|\ \|_{\mathscr{L}^q\log^{-\alpha}\mathscr{L}(\log\log\mathscr{L})^{-\beta}}$ defined by (27) is a norm. Therefore, in the remaing part of the proof we show that (27) defines a norm equivalent to the Luxemburg one. Observe that for $a \ge e^e$ we have

$$|f|^q(a+|f|)^{-\varepsilon} \le |f|^{q-\varepsilon} \le 2^{q-1}[a^q + |f|^q(a+|f|)^{-\varepsilon}],$$

a.e. in Ω, hence integrating

$$\fint_\Omega |f|^q(a+|f|)^{-\varepsilon}\,dx \le \|f\|_{q-\varepsilon}^{q-\varepsilon} \le 2^{q-1}a^q + 2^{q-1}\fint_\Omega |f|^q(a+|f|)^{-\varepsilon}\,dx.$$

Observe that

$$\int_0^{\varepsilon_0} \varepsilon^{\alpha-1}|\log\varepsilon|^{-\beta}\,d\varepsilon$$

is finite and it is bounded by some constant $C_2 = C_2(\alpha,\beta,\varepsilon_0)$. This in turn implies

$$\begin{aligned}
&\int_0^{\varepsilon_0} \varepsilon^{\alpha-1}|\log\varepsilon|^{-\beta}\left[\fint_\Omega |f|^q(a+|f|)^{-\varepsilon}\,dx\right]d\varepsilon \\
&\quad\le \int_0^{\varepsilon_0} \varepsilon^{\alpha-1}|\log\varepsilon|^{-\beta}\|f\|_{q-\varepsilon}^{q-\varepsilon}\,d\varepsilon \\
&\quad\le 2^{q-1}a^q C_2(\alpha,\beta,\varepsilon_0) + 2^{q-1}\int_0^{\varepsilon_0} \varepsilon^{\alpha-1}|\log\varepsilon|^{-\beta}\left[\fint_\Omega |f|^q(a+|f|)^{-\varepsilon}\,dx\right]d\varepsilon.
\end{aligned} \tag{31}$$

Fubini's Theorem implies

$$\begin{aligned}
&\int_0^{\varepsilon_0} \varepsilon^{\alpha-1}|\log\varepsilon|^{-\beta}\left[\fint_\Omega |f|^q(a+|f|)^{-\varepsilon}\,dx\right]d\varepsilon \\
&\quad= \fint_\Omega |f|^q\left(\int_0^{\varepsilon_0} \varepsilon^{\alpha-1}|\log\varepsilon|^{-\beta}(a+|f|)^{-\varepsilon}\,d\varepsilon\right)dx
\end{aligned} \tag{32}$$

From Lemma 3.1, we see that

$$\int_0^{\varepsilon_0} t^{\alpha-1}|\log t|^{-\beta} e^{-\lambda t}\,dt \sim \lambda^{-\alpha}(\log\lambda)^{-\beta} \quad \text{for all } \lambda > e^e$$

and the constants for this equivalence only depend on ε_0, α and β. So in particular, we have

$$\begin{aligned}
(33)\qquad & C_3 \fint_\Omega |f|^q \log^{-\alpha}(a+|f|)\log^{-\beta}(\log(a+|f|))\,dx \\
&\le \int_0^{\varepsilon_0} \varepsilon^{\alpha-1}|\log\varepsilon|^{-\beta}\|f\|_{q-\varepsilon}^{q-\varepsilon}\,d\varepsilon \\
&\le C_4\left[1+\fint_\Omega |f|^q \log^{-\alpha}(a+|f|)\log^{-\beta}(\log(a+|f|))\,dx\right]
\end{aligned}$$

for some positive contants.

Assume now that f satisfies (30). As

$$\|f\|_{q-\varepsilon}^{q-\varepsilon} \le \|f\|_{q-\varepsilon}^{q} + 1$$

we see that the first term of (33) is finite, so $f \in \mathscr{L}^q \log^{-\alpha}\log^{-\beta}\log\mathscr{L}(\Omega)$. Furthermore, if $\|f\|_{\mathscr{L}^q \log^{-\alpha}\log^{-\beta}\log\mathscr{L}} = 1$, then (33) implies

$$\fint_\Omega |f|^q \log^{-\alpha}(a+|f|)\log^{-\beta}(\log(a+|f|))\,dx \le C_5$$

for a constant independent of f. By homogeneity,

$$(34)\qquad [\![f]\!]_{\mathscr{L}^q \log^{-\alpha}\mathscr{L}(\log\log\mathscr{L})^{-\beta}} \le C_5\|f\|_{\mathscr{L}^q\log^{-\alpha}\log^{-\beta}\log\mathscr{L}}$$

for all f.

Let $f \in \mathscr{L}^q \log^{-\alpha}\mathscr{L}(\log\log\mathscr{L})^{-\beta}(\Omega)$; if $\beta > 0$ the space

$$\mathscr{L}^q \log^{-\alpha}\mathscr{L}(\log\log\mathscr{L})^{-\beta}(\Omega)$$

is continuosly embedded into $\mathscr{L}^q \log^{-\alpha_0}\mathscr{L}(\Omega)$ as long as $\alpha < \alpha_0$. Therefore, the fact that $\mathscr{L}^q \log^{-\alpha_0}\mathscr{L}(\Omega)$ is in turn continuously embedded into the grand Lebesgue space $\mathscr{L}^{\alpha_0, q)}(\Omega)$ (see [20]), there exists a constant $C_6 > 0$ such that

$$\|f\|_{q-\varepsilon} \le C_6\varepsilon^{-\alpha_0/q}[\![f]\!]_{\mathscr{L}^q\log^{-\alpha}\mathscr{L}(\log\log\mathscr{L})^{-\beta}},$$

Therefore,

$$\|f\|_{q-\varepsilon}^q = \|f\|_{q-\varepsilon}^{q-\varepsilon}\|f\|_{q-\varepsilon}^{\varepsilon} \le \|f\|_{q-\varepsilon}^{q-\varepsilon} C_7 [\![f]\!]^{\varepsilon}_{\mathscr{L}^q\log^{-\alpha}\mathscr{L}(\log\log\mathscr{L})^{-\beta}}$$

and by (33) we get (28).

Infact, if $[\![f]\!]_{\mathscr{L}^q \log^{-\alpha}\mathscr{L}(\log\log\mathscr{L})^{-\beta}} = 1$, then we have

$$\|f\|_{\mathscr{L}^q \log^{-\alpha}\log^{-\beta}\log\mathscr{L}} \le C_8$$

and by homogeneity we conclude with the reverse inequality to (34). We remark that all constants C_i appearing in the previous estimates are such that $C_i = C_i(q, \alpha, a, \varepsilon_0)$. □

Remark 4.3. We examine the dependence of $\|\ \|_{\mathscr{L}^q \log^{-\alpha}\log^{-\beta}\log\mathscr{L}}$ defined by (27), on the parameter ε_0. For fixed $0 < \varepsilon_0 \le \varepsilon_1 < 1$, by Hölder's inequality we have

$$\|f\|_{q-\varepsilon} \le \|f\|_{q-\varepsilon\frac{\varepsilon_0}{\varepsilon_1}},$$

and hence

$$\begin{aligned}(35)\qquad & \int_0^{\varepsilon_0} \varepsilon^{\alpha-1}|\log\varepsilon|^{-\beta}\|f\|_{q-\varepsilon}^q \, d\varepsilon \\ & \le \int_0^{\varepsilon_1} \varepsilon^{\alpha-1}|\log\varepsilon|^{-\beta}\|f\|_{q-\varepsilon}^q \, d\varepsilon \\ & \le \left(\frac{\varepsilon_1}{\varepsilon_0}\right)^{\alpha}\left(\frac{-\log\varepsilon_1}{-\log\varepsilon_0}\right)^{-\beta} \int_0^{\varepsilon_0} \varepsilon^{\alpha-1}|\log\varepsilon|^{-\beta}\|f\|_{q-\varepsilon}^q \, d\varepsilon.\end{aligned}$$

5. Existence and uniqueness

Before proving existence and uniqueness for the solution of problem (1), we recall some known facts. We assume that $\mathcal{A} = \mathcal{A}(x, \xi)$ satisfies (2)–(4) and we consider here the equations

$$\operatorname{div} \mathcal{A}(x, \nabla u) = \operatorname{div} f \quad \text{in } \Omega, \tag{36}$$

$$\operatorname{div} \mathcal{A}(x, \nabla v) = \operatorname{div} g \quad \text{in } \Omega, \tag{37}$$

It will be important, for our purposes, to consider first the case where $f, g \in \mathscr{L}^{q-\varepsilon q}(\Omega, \mathbb{R}^N)$, $0 < \varepsilon < \min\{1/p, 1/q\}$. We need a first tecnical result concerning solutions $u, v \in \mathscr{W}^{1,p-\varepsilon p}(\Omega)$ to (36) and (37), respectively, such that

$$u - v \in \mathscr{W}_0^{1,p-\varepsilon p}(\Omega)$$

It is then possible to prove the following

Lemma 5.1. *There exist* $0 < \varepsilon_p(N) < \min\{1/p, 1/q\}$ *and a constant* $C > 0$, $C = C(N, p, a, b)$, *such that the following uniform estimate holds*

$$\|\nabla u - \nabla v\|_{p-\varepsilon p}^{p} \le C\varepsilon^{\frac{p}{p-2}} \| \, |\nabla u| + |\nabla v| \, \|_{p-\varepsilon p}^{p} + C\|f - g\|_{q-\varepsilon q}^{q}, \quad \text{if } p > 2, \tag{38}$$

$$\|\nabla u - \nabla v\|_{p-\varepsilon p}^{p} \le C\varepsilon^{\frac{p}{2-p}} \| \, |\nabla u| + |\nabla v| \, \|_{p-\varepsilon p}^{p} + C\|f - g\|_{q-\varepsilon q}^{p} \| \, |\nabla u| + |\nabla v| \, \|_{q-\varepsilon q}^{p(2-p)}, \quad \text{if } 1 < p < 2, \tag{39}$$

for every $0 < \varepsilon < \varepsilon_p(N)$.

The proof relies on the use of the Hodge decomposition (see e.g. [21]) and it is achieved as in [17], then we omit it. As in [12] we immediately get the following corollary.

Corollary 5.2. *Under the assumptions of Lemma* 5.1, *there exist* $0 < \varepsilon_p(N) < \min\{1/p, 1/q\}$ *and a constant* $C > 0$, $C = C(N, p, a, b)$, *such that, for any* $0 < \varepsilon < \varepsilon_p(N)$ *the following uniform estimate holds if* $u = v = 0$ *on* $\partial\Omega$

$$\|\nabla u - \nabla v\|_{p-\varepsilon p}^{p} \le C\varepsilon^{\frac{p}{|p-2|}} \| \, |f| + |g| \, \|_{q-\varepsilon q}^{q} + C\|f - g\|_{q-\varepsilon q}^{q\lambda} \| \, |f| + |g| \, \|_{q-\varepsilon q}^{q(1-\lambda)} \tag{40}$$

where $\lambda = \min\{1, p-1\}$.

Proof of Theorem 1.1. We start by proving uniqueness. First, we write estimate (40) for $f = g$ observing that it reduces to

$$\|\nabla u - \nabla v\|_{p-\varepsilon p}^{p} \le C\varepsilon^{\frac{p}{|p-2|}} \|f\|_{q-\varepsilon q}^{q}. \tag{41}$$

Under the assumptions that we have on the parameters α and β, the space $\mathscr{L}^q \log^{-\alpha} \mathscr{L}(\log\log \mathscr{L})^{-\beta}(\Omega, \mathbb{R}^N)$ is continuously embedded into the space $\mathscr{L}^q \log^{-\frac{p}{|p-2|}} \mathscr{L}(\Omega, \mathbb{R}^N)$ and uniqueness follows by (12) letting $\varepsilon \downarrow 0$ in (41).

In order to prove the existence of a solution, as a preliminary step, we show that, if $(f_n)_n$ is a converging sequence in $\mathscr{L}^q \log^{-\alpha} \mathscr{L}(\log\log \mathscr{L})^{-\beta}(\Omega, \mathbb{R}^N)$, such that for each n

$$\begin{cases} \operatorname{div} \mathcal{A}(x, \nabla u_n) = \operatorname{div} f_n \\ u_n = 0 \quad \text{on } \partial\Omega \end{cases} \tag{42}$$

then $(\nabla u_n)_n$ is a Cauchy sequence in $\mathscr{L}^p \log^{-\alpha} \mathscr{L}(\log\log \mathscr{L})^{-\beta}(\Omega, \mathbb{R}^N)$. To prove this, by Corollary 5.2 we get

$$\|\nabla u_m - \nabla u_n\|_{p-\varepsilon p}^{p} \le C\varepsilon^{\frac{p}{|p-2|}} \| \, |f_m| + |f_n| \, \|_{q-\varepsilon q}^{q} + C\|f_m - f_n\|_{q-\varepsilon q}^{q\lambda} \| \, |f_m| + |f_n| \, \|_{q-\varepsilon q}^{q(1-\lambda)} \tag{43}$$

where $\lambda = \min\{1, p-1\}$. Arguing as in the proof of uniqueness, fixed $\sigma > 0$, we find $\vartheta \in (0, 1]$ such that, if $0 < \varepsilon < \vartheta\varepsilon_p(N)$, then

$$\varepsilon^{\frac{p}{|p-2|}} \| \, |f_m| + |f_n| \, \|_{q-\varepsilon q}^{q} < \sigma,$$

for all $m, n \in \mathbb{N}$. Hence (43) yields

$$\|\nabla u_m - \nabla u_n\|_{p-\varepsilon p}^{p} \le C(\sigma + \|f_m - f_n\|_{q-\varepsilon q}^{q\lambda} \| \, |f_m| + |f_n| \, \|_{q-\varepsilon q}^{q(1-\lambda)}). \tag{44}$$

We multiply both sides by $\varepsilon^{\alpha-1}|\log\varepsilon|^{-\beta}$ and we integrate with respect to ε over $(0, \vartheta\varepsilon_p(N))$. For $\delta = \varepsilon p/\vartheta \geq \varepsilon p$, we have

$$\|\nabla u_m - \nabla u_n\|_{p-\varepsilon p} \geq \|\nabla u_m - \nabla u_n\|_{p-\delta},$$

hence

$$(45)\qquad \int_0^{\vartheta\varepsilon_p(N)} \varepsilon^{\alpha-1}|\log\varepsilon|^{-\beta}\|\nabla u_m - \nabla u_n\|_{p-\varepsilon p}^p \, d\varepsilon \geq \left(\frac{\vartheta}{p}\right)^{\alpha} \int_0^{\varepsilon_0} \delta^{\alpha-1}\left|\log\frac{\theta\delta}{p}\right|^{-\beta} \|\nabla u_m - \nabla u_n\|_{p-\delta}^p \, d\delta,$$

where $\varepsilon_0 = p\varepsilon_p(N)$.

We consider the case $\beta > 0$. We make use of the following inequality

$$(46)\qquad \left(-\log\frac{\theta\delta}{p}\right)^{-\beta} \geq \left(\frac{-\log\varepsilon_p(N) - \log\theta}{-\log\varepsilon_0}\right)^{-\beta}(-\log\delta)^{-\beta} \quad \text{for } \delta \in (0, \varepsilon_0)$$

which can be justified by observing that the function

$$R(\delta) = \frac{-\log\frac{\theta\delta}{p}}{-\log\delta} = 1 + \frac{-\log\frac{\theta}{p}}{-\log\delta}$$

is increasing in the interval $(0, \varepsilon_0)$. Therefore, from (45) we get

$$(47)\qquad \int_0^{\vartheta\varepsilon_p(N)} \varepsilon^{\alpha-1}|\log\varepsilon|^{-\beta}\|\nabla u_m - \nabla u_n\|_{p-\varepsilon p}^p \, d\varepsilon \geq \left(\frac{\vartheta}{p}\right)^{\alpha}\left(\frac{-\log\varepsilon_p(N) - \log\theta}{-\log\varepsilon_0}\right)^{-\beta} \int_0^{\varepsilon_0} \delta^{\alpha-1}|\log\delta|^{-\beta}\|\nabla u_m - \nabla u_n\|_{p-\delta}^p \, d\delta,$$

On the other hand,

$$\int_0^{\vartheta\varepsilon_p(N)} \varepsilon^{\alpha-1}|\log\varepsilon|^{-\beta} \, d\varepsilon \leq \frac{(\vartheta\varepsilon_p(N))^{\alpha}}{\alpha}(-\log\varepsilon_p(N) - \log\theta)^{-\beta}$$

and (setting here $\delta = \varepsilon q$) by Hölder's inequality

$$(48)\qquad \int_0^{\vartheta\varepsilon_p(N)} \varepsilon^{\alpha-1}|\log\varepsilon|^{-\beta}\|f_m - f_n\|_{q-\varepsilon q}^{q\lambda}\| \, |f_m| + |f_n| \, \|_{q-\varepsilon q}^{q(1-\lambda)} \, d\varepsilon$$
$$\leq q^{-\alpha}\left[\int_0^{\varepsilon_1} \delta^{\alpha-1}\left|\log\frac{\delta}{q}\right|^{-\beta} \|f_m - f_n\|_{q-\delta}^q \, d\delta\right]^{\lambda} \times \left[\int_0^{\varepsilon_1} \delta^{\alpha-1}\left|\log\frac{\delta}{q}\right|^{-\beta} \| \, |f_m| + |f_n| \, \|_{q-\delta}^q \, d\delta\right]^{1-\lambda},$$

where $\varepsilon_1 = q\varepsilon_p(N)$. When $p > 2$ since $\varepsilon_1 < \varepsilon_0$ each integral over $(0, \varepsilon_1)$ can be obviously reduced to an integral over $(0, \varepsilon_0)$. In case $p < 2 < q$, we can use (35) to do the same. Therefore, recalling definition (27), from (44) we get

$$
\begin{aligned}
\|\nabla u_m - \nabla u_n\|^p_{\mathscr{L}^p \log^{-\alpha} \mathscr{L}(\log\log \mathscr{L})^{-\beta}} &\le C\sigma \\
&+ C\vartheta^{-\alpha}(C_0 - \log\theta)^{\beta} \|f_m - f_n\|^{q\lambda}_{\mathscr{L}^q \log^{-\alpha} \mathscr{L}(\log\log \mathscr{L})^{-\beta}} \\
&\times \| \, |f_m| + |f_n| \, \|^{q(1-\lambda)}_{\mathscr{L}^q \log^{-\alpha} \mathscr{L}(\log\log \mathscr{L})^{-\beta}}
\end{aligned}
\tag{49}
$$

with no restrictions on $m, n \in \mathbb{N}$. Now, as the sequence $(f_n)_n$ conveges in $\mathscr{L}^q \log^{-\alpha} \mathscr{L}(\log\log \mathscr{L})^{-\beta}(\Omega)$, we have

$$
\vartheta^{-\alpha}(C_0 - \log\theta)^{\beta} \|f_m - f_n\|^{q\lambda}_{\mathscr{L}^q \log^{-\alpha} \mathscr{L}(\log\log \mathscr{L})^{-\beta}} \| \, |f_m| + |f_n| \, \|^{q(1-\lambda)}_{\mathscr{L}^q \log^{-\alpha} \mathscr{L}(\log\log \mathscr{L})^{-\beta}} < \sigma,
$$

provided m and n are sufficiently large, hence

$$
\|\nabla u_m - \nabla u_n\|^p_{\mathscr{L}^q \log^{-\alpha} \mathscr{L}(\log\log \mathscr{L})^{-\beta}} \le C\sigma
$$

and so $(\nabla u_n)_n$ is a Cauchy sequence as desired.

Now we are in a position to prove existence of a solution to problem (1), for a given $f \in \mathscr{L}^q \log^{-\alpha} \mathscr{L}(\log\log \mathscr{L})^{-\beta}(\Omega, \mathbb{R}^N)$. Indeed, we approximate the vector field f by $f_n \in \mathscr{L}^q(\Omega, \mathbb{R}^N)$, $n = 1, 2, \ldots$, such that $f_n \to f$ in $\mathscr{L}^q \log^{-\alpha} \mathscr{L}(\log\log \mathscr{L})^{-\beta}(\Omega, \mathbb{R}^N)$, and for each n we consider the (unique) solution u_n to the problem

$$
\begin{cases}
\operatorname{div} \mathcal{A}(x, \nabla u_n) = \operatorname{div} f_n \\
u_n \in \mathscr{W}^{1,p}_0(\Omega)
\end{cases}
\tag{50}
$$

By what we have seen above, $(u_n)_n$ converges in $\mathscr{W}^1_0 \mathscr{L}^p \log^{-\alpha}(\log\log \mathscr{L})^{-\beta}(\Omega)$, that is, there exists $u \in \mathscr{W}^1_0 \mathscr{L}^p \log^{-\alpha}(\log\log \mathscr{L})^{-\beta}(\Omega)$ such that $u_n \to u$. To conclude that u solves (1), we only need to note that by (4) we can pass to the limit as $n \to \infty$ into the equation of (50), getting

$$
\operatorname{div} \mathcal{A}(x, \nabla u) = \operatorname{div} f,
$$

since we easily see that $\mathcal{A}(x, \nabla u_n) \to \mathcal{A}(x, \nabla u)$ in $\mathscr{L}^1(\Omega, \mathbb{R}^N)$.

Estimate (7) follows by the same argument used above, integrating with respect to ε. Also continuity of the operator $\mathcal{H}$ follows. Indeed, if $f_n \to f$ in $\mathscr{L}^q \log^{-\alpha} \mathscr{L}(\log\log \mathscr{L})^{-\beta}(\Omega)$ then

$$
\nabla u_n = \mathcal{H} f_n \to \nabla u = \mathcal{H} f \quad \text{in } \mathscr{L}^p \log^{-\alpha} \mathscr{L}(\log\log \mathscr{L})^{-\beta}(\Omega).
$$

If $\beta < 0$, a similar argument proves the result, we only have to substitute (46) by

$$
\left(-\log \frac{\theta\delta}{p}\right)^{-\beta} \ge (-\log\delta)^{-\beta} \quad \text{for } \delta \in (0, \varepsilon_0)
\tag{51}
$$

to get

$$\int_0^{\vartheta\varepsilon_p(N)} \varepsilon^{\alpha-1}|\log\varepsilon|^{-\beta}\|\nabla u_m - \nabla u_n\|_{p-\varepsilon p}^p\, d\varepsilon \geq \int_0^{\varepsilon_0} \delta^{\alpha-1}|\log\delta|^{-\beta}\|\nabla u_m - \nabla u_n\|_{p-\delta}^p\, d\delta, \tag{52}$$

instead of (47). □

Proof of Theorem 1.2. Let now $0 < \alpha < p/|p-2|$. We start by considering the case $\beta > 0$. We let $f, g \in \mathscr{L}^q \log^{-\alpha}\mathscr{L}(\log\log\mathscr{L})^{-\beta}(\Omega, \mathbb{R}^N)$. Denote by u and v the solutions of (36) and (37), of class $\mathscr{W}_0^1\mathscr{L}^p\log^{-\alpha}\mathscr{L}(\log\log\mathscr{L})^{-\beta}(\Omega)$, respectively. To prove (9), we multiply both sides of (43) by $\varepsilon^{\alpha-1}|\log\varepsilon|^{-\beta}$ and integrate with respect to ε over $(0, \vartheta\varepsilon_p(N))$, for fixed $\vartheta \in (0,1]$. By arguing as in the proof of (45) and (48), we have

$$\int_0^{\vartheta\varepsilon_p(N)} \varepsilon^{\alpha-1}|\log\varepsilon|^{-\beta}\|\nabla u - \nabla v\|_{p-\varepsilon p}^p\, d\varepsilon \geq \left(\frac{\vartheta}{p}\right)^\alpha \left(\frac{-\log\varepsilon_p(N) - \log\theta}{-\log\varepsilon_0}\right)^{-\beta} \int_0^{\varepsilon_0} \delta^{\alpha-1}|\log\delta|^{-\beta}\|\nabla u - \nabla v\|_{p-\delta}^p\, d\delta, \tag{53}$$

$$\begin{aligned}\int_0^{\vartheta\varepsilon_p(N)} &\varepsilon^{\alpha-1}|\log\varepsilon|^{-\beta}\|f-g\|_{q-\varepsilon q}^{q\lambda}\|\,|f|+|g|\,\|_{q-\varepsilon q}^{q(1-\lambda)}\, d\varepsilon \\ &\leq q^{-\alpha}\left[\int_0^{\varepsilon_1} \delta^{\alpha-1}\left|\log\frac{\delta}{q}\right|^{-\beta}\|f-g\|_{q-\delta}^q\, d\delta\right]^\lambda \\ &\quad\times \left[\int_0^{\varepsilon_1} \delta^{\alpha-1}\left|\log\frac{\delta}{q}\right|^{-\beta}\|\,|f|+|g|\,\|_{q-\delta}^q\, d\delta\right]^{1-\lambda},\end{aligned} \tag{54}$$

respectively, where as above $\varepsilon_0 = p\varepsilon_p(N)$, $\varepsilon_1 = q\varepsilon_p(N)$ and $\lambda = \min\{1, p-1\}$. On the other hand,

$$\int_0^{\vartheta\varepsilon_p(N)} \varepsilon^{\frac{p}{|p-2|}+\alpha-1}|\log\varepsilon|^{-\beta}\|\,|f|+|g|\,\|_{q-\varepsilon q}^q\, d\varepsilon \leq \frac{(\vartheta\varepsilon_p(N))^{\frac{p}{|p-2|}}}{q^\alpha}\int_0^{\varepsilon_1} \delta^{\alpha-1}\left|\log\frac{\delta}{q}\right|^{-\beta}\|\,|f|+|g|\,\|_{q-\delta}^q\, d\delta \tag{55}$$

and therefore we get

$$\begin{aligned}\|\nabla u - \nabla v\|^p_{\mathscr{L}^p\log^{-\alpha}\mathscr{L}(\log\log\mathscr{L})^{-\beta}} &\leq C\vartheta^{\frac{p}{|p-2|}-\alpha}(C_0 - \log\theta)^\beta\|\,|f|+|g|\,\|^q_{\mathscr{L}^p\log^{-\alpha}\mathscr{L}(\log\log\mathscr{L})^{-\beta}} \\ &\quad + C\vartheta^{-\alpha}(C_0-\log\theta)^\beta\|f-g\|^{q\lambda}_{\mathscr{L}^p\log^{-\alpha}\mathscr{L}(\log\log\mathscr{L})^{-\beta}} \\ &\quad \times \|\,|f|+|g|\,\|^{q(1-\lambda)}_{\mathscr{L}^p\log^{-\alpha}\mathscr{L}(\log\log\mathscr{L})^{-\beta}}\end{aligned} \tag{56}$$

If we choose ϑ such that

$$\vartheta^{\frac{p}{|p-2|}} = \left(\frac{\|f-g\|^q_{\mathscr{L}^q \log^{-\alpha} \log^{-\beta} \log \mathscr{L}}}{\| \, |f| + |g| \, \|^q_{\mathscr{L}^q \log^{-\alpha} \log^{-\beta} \log \mathscr{L}}} \right)^{\lambda}$$

we obtain estimate (9).

If $\beta < 0$, a similar argument proves the result, we only have to substitute (53) by

$$\int_0^{\vartheta \varepsilon_p(N)} \varepsilon^{\alpha-1} |\log \varepsilon|^{-\beta} \|\nabla u - \nabla u\|^p_{p-\varepsilon p} \, d\varepsilon \geq \int_0^{\varepsilon_0} \delta^{\alpha-1} |\log \delta|^{-\beta} \|\nabla u - \nabla u\|^p_{p-\delta} \, d\delta, \tag{57}$$

The proof is complete. □

6. Results in the space $\mathscr{L}^q(\log\log \mathscr{L})^{-\beta}$

In this section we state and prove two results similar to Theorem 1.1 and Theorem 1.2, with the difference that our next result are settled in the Orlicz–Zygmund space $\mathscr{L}^q(\log\log \mathscr{L})^{-\beta}(\Omega, \mathbb{R}^N)$. Whenever the parameter α equals to zero, we use the notation $[\![\cdot]\!]_{q,\beta}$ instead of $[\![f]\!]_{q,0,\beta}$.

Theorem 6.1. *Let $1 < p < \infty$, $p \neq 2$. For each $f \in \mathscr{L}^q(\log\log \mathscr{L})^{-\beta}(\Omega, \mathbb{R}^N)$, with $pq = p + q$ and $\beta > 0$, the problem* (1) *admits a unique solution $u : \Omega \to \mathbb{R}$, such that $\nabla u \in \mathscr{L}^p(\log\log \mathscr{L})^{-\beta}(\Omega, \mathbb{R}^N)$. There exists a constant $C > 0$, $C = C(N, p, \alpha, a, b)$, such that the following estimate holds true*

$$[\![\nabla u]\!]^p_{p,\beta} \leq C [\![f]\!]^q_{q,\beta} \tag{58}$$

Moreover the operator $\mathcal{H}$ is continuous.

The proof of Theorem 6.1 is achieved exactly as for Theorem 1.1, therefore we omit it. Let us turn to the following next result.

Theorem 6.2. *Let $1 < p < \infty$, $p \neq 2$ and let $\beta > 0$. There exists a constant $C > 0$, $C = C(N, p, \alpha, a, b)$, such that, if f and g belong to $\mathscr{L}^q(\log\log \mathscr{L})^{-\beta} \cdot (\Omega, \mathbb{R}^N)$, with $pq = p + q$ then*

$$[\![\mathcal{H}f - \mathcal{H}g]\!]^p_{p,\beta} \leq C([\![f-g]\!]^{\gamma}_{q,\beta} [\![|f| + |g|]\!]^{1-\gamma}_{q,\beta})^q \left(1 - \log \frac{[\![f-g]\!]_{q,\beta}}{[\![|f| + |g|]\!]_{q,\beta}}\right)^{\beta+1} \tag{59}$$

where

$$\gamma = 1 \quad \textit{if } p > 2 \tag{60}$$

$$\gamma = \frac{p}{q} \quad \textit{if } 1 < p < 2 \tag{61}$$

PROOF. Let $f, g \in \mathscr{L}^q(\log\log\mathscr{L})^{-\beta}(\Omega, \mathbb{R}^N)$. Denote by u and v the solutions of (36) and (37), of class $\mathscr{W}^1\mathscr{L}^q(\log\log\mathscr{L})^{-\beta}(\Omega)$, respectively. To prove (9), we multiply both sides of (43) by $\varepsilon^{-1}|\log\varepsilon|^{-\beta-1}$ and integrate with respect to ε over $(0, \vartheta\varepsilon_p(N))$, for fixed $\vartheta \in (0, 1]$. By arguing as in the proof of (45) and (48), we have

$$(62)\qquad \int_0^{\vartheta\varepsilon_p(N)} \varepsilon^{-1}|\log\varepsilon|^{-\beta-1}\|\nabla u - \nabla v\|_{p-\varepsilon p}^p\, d\varepsilon \geq \left(\frac{-\log\varepsilon_p(N) - \log\theta}{-\log\varepsilon_0}\right)^{-\beta-1}\int_0^{\varepsilon_0}\delta^{-1}|\log\delta|^{-\beta-1}\|\nabla u - \nabla v\|_{p-\delta}^p\, d\delta,$$

$$(63)\qquad \int_0^{\vartheta\varepsilon_p(N)} \varepsilon^{-1}|\log\varepsilon|^{-\beta-1}\|f - g\|_{q-\varepsilon q}^{q\lambda}\|\,|f| + |g|\,\|_{q-\varepsilon q}^{q(1-\lambda)}\, d\varepsilon \leq \left[\int_0^{\varepsilon_1}\delta^{-1}\left|\log\frac{\delta}{q}\right|^{-\beta-1}\|f - g\|_{q-\delta}^q\, d\delta\right]^{\lambda} \times \left[\int_0^{\varepsilon_1}\delta^{-1}\left|\log\frac{\delta}{q}\right|^{-\beta-1}\|\,|f| + |g|\,\|_{q-\delta}^q\, d\delta\right]^{1-\lambda},$$

respectively, where as above $\varepsilon_0 = p\varepsilon_p(N)$, $\varepsilon_1 = q\varepsilon_p(N)$ and $\lambda = \min\{1, p-1\}$. On the other hand,

$$(64)\qquad \int_0^{\vartheta\varepsilon_p(N)} \varepsilon^{\frac{p}{|p-2|}-1}|\log\varepsilon|^{-\beta-1}\|\,|f| + |g|\,\|_{q-\varepsilon q}^q\, d\varepsilon \leq \frac{(\vartheta\varepsilon_p(N))^{\frac{p}{|p-2|}}}{q^\alpha}\int_0^{\varepsilon_1}\delta^{-1}\left|\log\frac{\delta}{q}\right|^{-\beta-1}\|\,|f| + |g|\,\|_{q-\delta}^q\, d\delta$$

and therefore we get

$$(65)\qquad \|\nabla u - \nabla v\|_{\mathscr{L}^p(\log\log\mathscr{L})^{-\beta}}^p \leq C\vartheta^{\frac{p}{|p-2|}}(C_0 - \log\theta)^{\beta+1}\|\,|f| + |g|\,\|_{\mathscr{L}^q(\log\log\mathscr{L})^{-\beta}}^q + C(C_0 - \log\theta)^{\beta+1}\|f - g\|_{\mathscr{L}^q(\log\log\mathscr{L})^{-\beta}}^{q\lambda}\|\,|f| + |g|\,\|_{\mathscr{L}^q(\log\log\mathscr{L})^{-\beta}}^{q(1-\lambda)}$$

If we choose ϑ such that

$$\vartheta^{\frac{p}{|p-2|}} = \left(\frac{\|f - g\|_{\mathscr{L}^q(\log\log\mathscr{L})^{-\beta}}^q}{\|\,|f| + |g|\,\|_{\mathscr{L}^q(\log\log\mathscr{L})^{-\beta}}^q}\right)^{\lambda}$$

we obtain estimate (59). □

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