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Measure and Integration — *A sharp quantitative estimate for the surface areas of convex sets in* $\mathbb{R}^3$, by Menita Carozza, Flavia Giannetti, Francesco Leonetti and Antonia Passarelli di Napoli, communicated on 12 February 2016.

Abstract. — Let $E \subset B \subset \mathbb{R}^3$ be closed, bounded, convex sets. It is known that the monotonicity of the surface areas holds, i.e. $\mathscr{H}^2(\partial E) \le \mathscr{H}^2(\partial B)$. Here we give a quantitative estimate of the difference of the surface areas from below depending on the Hausdorff distance between E and B. Moreover, we construct an example which shows the sharpness of our result.

Key words: Convex sets, surface areas, Hausdorff distance

Mathematics Subject Classification: 52A15

1. Introduction

For two bounded convex sets E and B, with $E \subset B \subset \mathbb{R}^n$, it is well known that

$$\mathscr{H}^{n-1}(\partial E) \le \mathscr{H}^{n-1}(\partial B), \tag{1.1}$$

i.e. the monotonicity of the surface area (see, for instance, page 52 of [1] and Lemma 2.4 in [2]). In (1.1), ∂E is the boundary of E, $\mathscr{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure, thus $\mathscr{H}^{n-1}(\partial E)$ is the measure of the boundary of E, the surface area of E for short. A first quantitative version of (1.1) has been proven in the planar case $n = 2$ in [5], where the authors express the deficit $\delta(E, B)$ between the perimeters of E and B in terms of the diameter of the set B and the Hausdorff distance $h(E, B)$ between E and B. An improved version of the estimate in [5] has been established in [3], where $\delta(E, B)$ is expressed by the measure of a section of B orthogonal to a line segment which realizes the Hausdorff distance between E and B.

More precisely, if E and B are two closed bounded convex sets, with $\emptyset \ne E \subsetneq B \subset \mathbb{R}^2$, it has been proven the following sharp estimate

$$\mathscr{H}^1(\partial E) + \frac{4|h(E,B)|^2}{2\sqrt{\left(\frac{\mathscr{H}^1(B\cap\mathscr{L})}{2}\right)^2 + |h(E,B)|^2} + \mathscr{H}^1(B \cap \mathscr{L})} \le \mathscr{H}^1(\partial B), \tag{1.2}$$

where $b \in B$ and $P(b) \in E$ are such that $|b - P(b)| = h(E, B)$; moreover, $\mathscr{L}$ is the orthogonal line to the vector $b - P(b)$ through $P(b)$, see Figure 1.

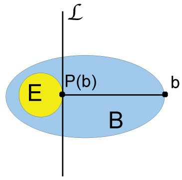

Figure 1. The line $\mathscr{L}$ through $P(b)$ orthogonal to the vector $b - P(b)$

The aim of this paper is to extend the estimate (1.2) to convex sets in $\mathbb{R}^3$. Namely we prove the following

THEOREM 1.1. *Let us consider two closed, bounded and convex sets* $\emptyset \neq E \subsetneq B \subset \mathbb{R}^3$. *Then*

$$\mathscr{H}^2(\partial E) + \frac{\pi d |h(E,B)|^2}{\sqrt{d^2 + |h(E,B)|^2} + d} \leq \mathscr{H}^2(\partial B), \tag{1.3}$$

where $h(E,B)$ *is the Hausdorff distance between* E *and* B, $d = \mathrm{dist}(P(b), \partial B \cap \partial S^+)$, *where* $b \in B$ *and* $P(b) \in E$ *are such that* $|b - P(b)| = h(E,B)$ *and*

$$S^+ = \{x \in \mathbb{R}^3 : \langle b - P(b), P(b) - x\rangle \geq 0\},$$

(*see Figure* 2).

The main idea is to estimate from below the difference of the surfaces of E and B with the surface of a cone contained in $B \backslash E$ whose base belongs to the plane supporting the convex E, as the monotonicity of the surface areas holds. This argument reveals to be successful exactly as it was to estimate from below the difference between the perimeters of convex sets in the plane with a triangle in our previous paper [3]; see also [4].

In the last section we will give an example of two convex sets in $\mathbb{R}^3$, showing that the result of previous Theorem is sharp. Indeed, we will show that there exist $E \subsetneq B \subset \mathbb{R}^3$ for which in estimate (1.3) equality occurs.

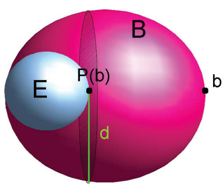

Figure 2. Sets E and B, points b and $P(b)$, distance d

2. Proof of theorem 1.1

If $d = 0$ there is nothing to prove since the inequality (1.3) reduces to the monotonicity property in (1.1), so we assume that $d > 0$. For E and B closed, bounded sets with $E \subset B$, recall that the distance $h(E, B)$ can be written as follows

$$h(E,B) = \max_{x\in B} \min_{y\in E} |x-y|. \tag{2.1}$$

Let $b \in B$ and $P(b) \in E$ such that

$$h(E,B) = |b - P(b)|. \tag{2.2}$$

It turns out that $b \in B \backslash E$ and $P(b)$ is the projection of b onto the closed convex set E. Please, note that $P(b) \in \partial E$ and the plane through $P(b)$ orthogonal to $\tilde{h} = b - P(b) \neq 0$ is a supporting one for the convex set E. Set

$$S^+ = \{x \in \mathbb{R}^3 : \langle \tilde{h}, P(b) - x\rangle \geq 0\}$$

then

$$\partial S^+ = \{x \in \mathbb{R}^3 : \langle \tilde{h}, P(b) - x\rangle = 0\}$$

and $E \subset S^+$. By virtue of the monotonicity in (1.1) and since $E \subset B \cap S^+$, we have

$$\begin{aligned}\mathscr{H}^2(\partial E) &\leq \mathscr{H}^2(\partial(B\cap S^+)) \leq \mathscr{H}^2((\partial B \cap S^+) \cup (B \cap \partial S^+)) \\ &\leq \mathscr{H}^2(\partial B\cap S^+) + \mathscr{H}^2(B\cap \partial S^+).\end{aligned} \tag{2.3}$$

On the other hand

$$\mathscr{H}^2(\partial B) = \mathscr{H}^2(\partial B\cap S^+) + \mathscr{H}^2(\partial B \backslash S^+) \tag{2.4}$$

Combining (2.3) and (2.4), we get

$$\mathscr{H}^2(\partial E) \leq \mathscr{H}^2(\partial B) - \mathscr{H}^2(\partial B\backslash S^+) + \mathscr{H}^2(B\cap\partial S^+). \tag{2.5}$$

Let us represent the closed simple curve $\partial B \cap \partial S^+$ in the polar coordinate (ρ, ϑ) on ∂S^+ with

$$\rho = g(\vartheta), \quad \vartheta \in [0, 2\pi] \tag{2.6}$$

where the polar system has the origin coinciding with $P(b)$. Note that $P(b)$ is in the interior of $B \cap \partial S^+$ since we are assuming $d = \operatorname{dist}(P(b), \partial B \cap \partial S^+) > 0$ and that $\partial B \cap \partial S^+$ is the boundary of a convex, open, bounded set in ∂S^+. Therefore $g(\vartheta)$ is Lipschitz continuous (see [6]).

Let us consider the closed cone τ having $B \cap \partial S^+$ as base and vertex b and note that

$$\tau \subset B \cap (\overline{\mathbb{R}^3 \backslash S^+}),$$

so we can use again (1.1) obtaining

$$\mathscr{H}^2(\partial\tau) \le \mathscr{H}^2(\partial(B \cap \overline{(\mathbb{R}^3 \setminus S^+)})). \tag{2.7}$$

The surface area of τ is given by

$$\mathscr{H}^2(\partial\tau) = \mathscr{H}^2(B \cap \partial S^+) + \mathscr{H}^2(\Sigma),$$

where Σ denotes the lateral surface of the cone τ. Let us parametrize the surface Σ as follows

$$\begin{cases} x(s,\vartheta) = sg(\vartheta)\cos\vartheta \\ y(s,\vartheta) = sg(\vartheta)\sin\vartheta \quad (s,\vartheta) \in [0,1) \times [0,2\pi), \\ z(s,\vartheta) = h - sh \end{cases} \tag{2.8}$$

where $h = |\tilde{h}| = h(E,B)$ and the axis z is in the direction of $\tilde{h}$. The Jacobi matrix of this parametrization of Σ is given by

$$\begin{pmatrix} g(\vartheta)\cos\vartheta & sg'(\vartheta)\cos\vartheta - s \cdot g(\vartheta)\sin\vartheta \\ g(\vartheta)\sin\vartheta & sg'(\vartheta)\sin\vartheta + s \cdot g(\vartheta)\cos\vartheta \\ -h & 0 \end{pmatrix} \tag{2.9}$$

and therefore

$$\begin{aligned} \mathscr{H}^2(\Sigma) &= \int_0^1 s\,ds \int_0^{2\pi} \sqrt{h^2[(g'(\vartheta))^2 + (g(\vartheta))^2] + (g(\vartheta))^4}\,d\vartheta \\ &= \frac{1}{2}\int_0^{2\pi} \sqrt{h^2[(g'(\vartheta))^2 + (g(\vartheta))^2] + (g(\vartheta))^4}\,d\vartheta \\ &\ge \frac{1}{2}\int_0^{2\pi} \sqrt{h^2 g^2(\vartheta) + g^4(\vartheta)}\,d\vartheta \\ &= \frac{1}{2}\int_0^{2\pi} g(\vartheta)\sqrt{h^2 + g^2(\vartheta)}\,d\vartheta. \end{aligned} \tag{2.10}$$

On the other hand, we have that

$$\mathscr{H}^2(B \cap \partial S^+) = \int_0^{2\pi}\int_0^{g(\vartheta)} \rho\,d\rho\,d\vartheta = \frac{1}{2}\int_0^{2\pi} g^2(\vartheta)\,d\vartheta \tag{2.11}$$

and then

$$\mathscr{H}^2(\partial\tau) \ge \frac{1}{2}\int_0^{2\pi} g^2(\vartheta)\,d\vartheta + \frac{1}{2}\int_0^{2\pi} g(\vartheta)\sqrt{h^2 + g^2(\vartheta)}\,d\vartheta. \tag{2.12}$$

Inequalities (2.7) and (2.12) merge into the following

$$\begin{aligned}\mathscr{H}^2(\partial B\backslash S^+) + \mathscr{H}^2(B\cap\partial S^+) &= \mathscr{H}^2(\partial(B\cap(\overline{\mathbb{R}^3\backslash S^+}))) \geq \mathscr{H}^2(\partial\tau)\\ &\geq \frac{1}{2}\int_0^{2\pi} g^2(\vartheta)\,d\vartheta + \frac{1}{2}\int_0^{2\pi} g(\vartheta)\sqrt{h^2+g^2(\vartheta)}\,d\vartheta,\end{aligned}$$

and so by (2.11)

$$\mathscr{H}^2(\partial B\backslash S^+) \geq \frac{1}{2}\int_0^{2\pi} g(\vartheta)\sqrt{h^2+g^2(\vartheta)}\,d\vartheta. \tag{2.13}$$

Plugging this estimate into (2.5), we get

$$\mathscr{H}^2(\partial E) \leq \mathscr{H}^2(\partial B) - \frac{1}{2}\int_0^{2\pi} g(\vartheta)\sqrt{h^2+g^2(\vartheta)}\,d\vartheta + \frac{1}{2}\int_0^{2\pi} g^2(\vartheta)\,d\vartheta \tag{2.14}$$

i.e.,

$$\mathscr{H}^2(\partial E) + \frac{1}{2}\int_0^{2\pi} g(\vartheta)[\sqrt{h^2+g^2(\vartheta)} - g(\vartheta)]\,d\vartheta \leq \mathscr{H}^2(\partial B). \tag{2.15}$$

Observe now, as one can easily check, that the integrand in (2.15) is an increasing function with respect to $g(\vartheta)$. Therefore, since $g(\vartheta) \geq d$ where $d = \operatorname{dist}(P(b), \partial B\cap\partial S^+)$, estimate (2.15) gives the following

$$\mathscr{H}^2(\partial E) + \frac{1}{2}\int_0^{2\pi} d(\sqrt{h^2+d^2} - d)\,d\vartheta \leq \mathscr{H}^2(\partial B), \tag{2.16}$$

i.e.

$$\mathscr{H}^2(\partial E) + \pi d(\sqrt{h^2+d^2} - d) \leq \mathscr{H}^2(\partial B). \tag{2.17}$$

Hence

$$\mathscr{H}^2(\partial E) + \pi\frac{dh^2}{\sqrt{h^2+d^2}+d} \leq \mathscr{H}^2(\partial B), \tag{2.18}$$

Recalling that $h = h(E, B)$, we get the conclusion. □

REMARK 2.1. Actually, the estimate (2.15) is stronger than (1.3) when we are not in a symmetric case; nevertheless the formulation given in (1.3) has the advantage to make more comprehensible the statement of the main result.

3. Example

In this section we will give an example of two convex sets in $\mathbb{R}^3$, showing that the result of Theorem 1.1 is sharp.

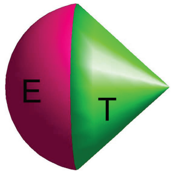

Figure 3. The half ball E and the cone T

Let E be the half ball of radius d, C the maximum circle contained in ∂E, T the cone with base C of height h and $B = E \cup T$ as in Figure 3.

Obviously we have

$$\mathscr{H}^2(\partial E) = 3\pi d^2, \quad \mathscr{H}^2(\partial B) = 2\pi d^2 + \pi d\sqrt{d^2+h^2}$$
$$h(E,B) = h.$$

Moreover, if we note that in this case b coincides with the vertex of the cone T, the point $P(b)$ that realizes the Hausdorff distance between E and B is exactly the center of C and the supporting plane $\mathscr{L}$ of E at $P(b)$ is the one containing C, we have that

$$d = \operatorname{dist}(P(b), \partial B \cap \partial S^+).$$

We obviously have

$$\mathscr{H}^2(\partial B) - \mathscr{H}^2(\partial E) = \frac{\pi d h^2}{\sqrt{d^2+h^2}+d}.$$

Therefore, in this case, in estimate (1.3) equality occurs.

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