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Functional Analysis — *On maximal and potential operators with rough kernels in variable exponent spaces*, by Humberto Rafeiro and Stefan Samko, communicated on 12 February 2016.

Abstract. — In the framework of variable exponent Lebesgue and Morrey spaces we prove some boundedness results for operators with rough kernels, such as the maximal operator, fractional maximal operator, sharp maximal operators and fractional operators. The approach is based on some pointwise estimates.

Key words: Rough operators, variable exponent Lebesgue spaces, variable exponent Morrey spaces

Mathematics Subject Classification: 42B25, 46E30

1. Introduction

The main operators of harmonic analysis, maximal, singular and of potential type, with the so called rough kernel have been widely studied, see e.g. [5, 6, 7, 8, 9, 12, 16, 17, 23, 24, 25, 26, 29, 34, 39] and references therein. The study of operators with rough kernels was based on the usage of the rotation method, which goes back to [15].

The theory of variable exponent spaces has received a thrust in recent years, due mainly to some applications, for example in the modeling of electro-rheological fluids [4, 3, 37] as well as thermo-rheological fluids [11], in the study of image processing [1, 2, 13, 14, 18, 19, 41] and in differential equations with non-standard growth [28, 33]. For details on variable Lebesgue spaces one can refer to [21, 22, 31, 32] and the references therein.

We want to study the boundedness of some rough operators in the framework of variable exponent Lebesgue and Morrey spaces. Since the classical proof is based upon the rotation method which is not well suited for the case of variable exponents we obtained some pointwise estimates in Section 3 to circumvent this problem.

Operator with rough kernel have already been considered in the variable exponent setting in [20], where their study was based on the extrapolation theory. Our approach is based upon some pointwise estimates and do not use extrapolation theorems and allows, in particular, to consider potential operators and fractional maximal functions of variable order $\alpha(x)$.

The paper is arranged as follows: In Section 2 we give necessary preliminaries on variable exponent Lebesgue and Morrey spaces. In Section 3 we provide pointwise estimates for maximal and potential operators with rough kernels. Sections 4 and 5 contain the main results on the boundedness of such operators.

2. Variable exponent spaces

2.1. *Variable Lebesgue spaces*

We refer to the books [21, 22], but recall some basics we need on *variable exponent Lebesgue spaces*. Let $U \subseteq \mathbb{R}^n$ be an open set and $p(\cdot)$ be a real-valued measurable function on U with values in $[1, \infty)$. We suppose that

$$1 \leq p_- \leq p(x) \leq p_+ < \infty, \tag{1}$$

where $p_- := \operatorname{ess\,inf}_{x \in U} p(x)$, $p_+ := \operatorname{ess\,sup}_{x \in U} p(x)$. As usual, by $p'(x) = p(x)/(p(x)-1)$ we indicate the conjugate exponent of $p(x)$ and we have the relations $(p')_+ = (p_-)'$ and $(p')_- = (p_+)'$. By $L^{p(\cdot)}(U)$ we denote the space of real-valued measurable functions f on U such that

$$I_{p(\cdot)}(f) := \int_U |f(x)|^{p(x)} \, \mathrm{d}x < \infty.$$

Equipped with the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\eta > 0 : I_{p(\cdot)}\left(\frac{f}{\eta}\right) \leq 1\right\},$$

this is a Banach function space. The variable Lebesgue exponent norm has the following property

$$\|f^\lambda\|_{p(\cdot)} = \|f\|^\lambda_{\lambda p(\cdot)}, \tag{2}$$

for $\lambda \geq \frac{1}{p_-}$. In the subsequent sections, we will also need the relation between the modular $I_{p(\cdot)}(f)$ and the norm.

LEMMA 2.1. *For every $f \in L^{p(\cdot)}(U)$, the inequalities*

$$\|f\|^{p_+}_{L^{p(\cdot)}(U)} \leq I_{p(\cdot)}(f) \leq \|f\|^{p_-}_{L^{p(\cdot)}(U)}, \quad \text{if } \|f\|_{L^{p(\cdot)}(U)} \leq 1, \tag{3}$$

$$\|f\|^{p_-}_{L^{p(\cdot)}(U)} \leq I_{p(\cdot)}(f) \leq \|f\|^{p_+}_{L^{p(\cdot)}(U)}, \quad \text{if } \|f\|_{L^{p(\cdot)}(U)} \geq 1, \tag{4}$$

are valid.

In the sequel we use the well known log-*condition*

$$|p(x) - p(y)| \leq \frac{A}{-\ln|x-y|}, \quad |x-y| \leq \frac{1}{2}, \quad x, y \in U, \tag{5}$$

where $A = A(p) > 0$ does not depend on x, y. In the case U is unbounded, we also use *the decay condition*: there exists a number $p_\infty \in (1, \infty)$, such that

$$|p(x) - p_\infty| \leq \frac{A}{\ln(e + |x|)}. \tag{6}$$

In the sequel we use the following notation. For an open set $U \subseteq \mathbb{R}^n$, by $\mathbb{P}^{\log}(U)$ we denote the set of exponents $p(x)$ with $1 < p_- \leq p_+ < \infty$, satisfying the log and decay conditions (the latter required if U is unbounded).

By $\mathfrak{P}(U)$ we denote the set of exponents p with $1 < p_- \leq p_+ < \infty$ such that the maximal operator is bounded in the space $L^{p(\cdot)}(U)$.

The following fact is known, see [38] or [31, Theorem 2.62].

LEMMA 2.2. *Let U be a bounded open set in $\mathbb{R}^n$ and $p \in \mathbb{P}^{\log}(U)$. In the case* $\sup_{x \in U} \alpha(x)p(x) < n$, *then*

$$\left\| \frac{\chi_{U \setminus B(x,r)}}{|x - \cdot|^{n-\alpha(x)}} \right\|_{p'(\cdot)} \leq C r^{\alpha(x) - \frac{n}{p(x)}}, \tag{7}$$

where the constant C does not depend on x.

2.2. Variable exponent Morrey spaces

For more about Morrey spaces see [27, 35, 36]. Let λ be a measurable function on U with values in $[0, n]$. We define the variable Morrey space $L^{p(\cdot),\lambda(\cdot)}(U)$ as the set of all real-valued measurable functions on U such that

$$I_{p(\cdot),\lambda(\cdot)}(f) := \sup_{x \in U, r > 0} r^{-\lambda(x)} \int_{U(x,r)} |f(y)|^{p(y)}\, dy < \infty,$$

where $U(x, r) = U \cap B(x, r)$. The norm in the space $L^{p(\cdot),\lambda(\cdot)}(U)$ can be introduced in two forms, namely

$$\|f\|_1 = \inf\left\{\eta > 0 : I_{p(\cdot),\lambda(\cdot)}\left(\frac{f}{\eta}\right) \leq 1\right\}$$

and

$$\|f\|_2 := \sup_{x \in U, r > 0} \|r^{-\frac{\lambda(x)}{p(\cdot)}} f \chi_{U(x,r)}\|_{p(\cdot)}, \tag{8}$$

which are equal, see [10]. We take

$$\|f\|_{L^{p(\cdot),\lambda(\cdot)}(U)} = \|f\|_1 \quad \text{or} \quad \|f\|_{L^{p(\cdot),\lambda(\cdot)}(U)} = \|f\|_2$$

whichever is more convenient. We also have the following important property (see [10, 30]).

LEMMA 2.3. *For every $f \in L^{p(\cdot),\lambda(\cdot)}(U)$, the inequalities*

$$\|f\|^{p_+}_{L^{p(\cdot),\lambda(\cdot)}(U)} \le I_{p(\cdot),\lambda(\cdot)}(f) \le \|f\|^{p_-}_{L^{p(\cdot),\lambda(\cdot)}(U)}, \quad \text{if } \|f\|_{L^{p(\cdot),\lambda(\cdot)}(U)} \le 1, \tag{9}$$

$$\|f\|^{p_-}_{L^{p(\cdot),\lambda(\cdot)}(U)} \le I_{p(\cdot),\lambda(\cdot)}(f) \le \|f\|^{p_+}_{L^{p(\cdot),\lambda(\cdot)}(U)}, \quad \text{if } \|f\|_{L^{p(\cdot),\lambda(\cdot)}(U)} \ge 1, \tag{10}$$

are valid.

Similarly to property (2), from (8) we have

$$\|f^\sigma\|_{L^{p(\cdot),\lambda(\cdot)}(U)} = \sup_{x\in U, r>0} \|r^{\frac{-\lambda(x)}{\sigma p(\cdot)}} f\chi_{U(x,r)}\|^\sigma_{L^{\sigma p(\cdot)}(U)} \tag{11}$$

$$= \|f\|^\sigma_{L^{\sigma p(\cdot),\lambda(\cdot)}(U)}, \tag{12}$$

whenever $\sigma \ge 1/p_-$.

The following theorem is known, see [10].

THEOREM 2.4. *Let U be an open bounded set in $\mathbb{R}^n$, $1 < p_- \le p(x) \le p_+ < \infty$, $0 \le \lambda(x) \le \lambda_+ < n$ and $p \in \mathbb{P}^{\log}(U)$. Then the maximal operator*

$$Mf(x) = \sup_{r>0} \frac{1}{r^n} \int_{U(x,r)} |f(y)|\, \mathrm{d}y$$

is bounded in the space $L^{p(\cdot),\lambda(\cdot)}(U)$.

3. POINTWISE ESTIMATES OF MAXIMAL AND FRACTIONAL ROUGH OPERATORS

To deal with the boundedness problem of rough operators, we will use some pointwise estimates related to the maximal rough operator, the fractional maximal rough operator and the fractional rough operator.

3.1. *The case of the maximal operator*

Let us consider the maximal operator with rough kernel

$$M_\Omega f(x) := \sup_{r>0} \frac{1}{r^n} \int_{|y|<r} |\Omega(y) f(x-y)|\, \mathrm{d}y, \tag{13}$$

in the case $\Omega = 1$, we simply write $M_\Omega f = Mf$. It is known that the operator M_Ω is bounded in $L^p(\mathbb{R}^n)$, $1 < p < \infty$, if $\Omega \in L^1(\mathbb{S}^{n-1})$ and Ω is a homogeneous function of degree 0, i.e.

$$\Omega(\lambda x) = \Omega(x) \tag{14}$$

for all $\lambda > 0$ and $x \in \mathbb{R}^n$, see e.g. [40, p. 72].

LEMMA 3.1. *Let $f \in L^{s'}_{\mathrm{loc}}(\mathbb{R}^n)$, Ω satisfy* (14) *and $\Omega \in L^s(\mathbb{S}^{n-1})$ with $s > 1$. Then*

$$M_\Omega f(x) \leq \frac{\|\Omega\|_{L^s(\mathbb{S}^{n-1})}}{n^{\frac{1}{s}}} (M(|f|^{s'})(x))^{\frac{1}{s'}}. \tag{15}$$

PROOF. We have

$$\begin{aligned}
\frac{1}{r^n}\int_{|y|<r} |\Omega(y)f(x-y)|\,\mathrm{d}y &= \frac{1}{r^n}\int_0^r \varrho^{n-1}\,\mathrm{d}\varrho \int_{\mathbb{S}^{n-1}} |\Omega(\sigma)f(x-\varrho\sigma)|\,\mathrm{d}\sigma \\
&\leq \frac{\|\Omega\|_{L^s(\mathbb{S}^{n-1})}}{r^n}\int_0^r \varrho^{\frac{n-1}{s}}\,\mathrm{d}\varrho\Big(\varrho^{n-1}\int_{\mathbb{S}^{n-1}} |f(x-\varrho\sigma)|^{s'}\,\mathrm{d}\sigma\Big)^{\frac{1}{s'}} \\
&\leq \frac{\|\Omega\|_{L^s(\mathbb{S}^{n-1})}}{n^{\frac{1}{s}} r^{n-\frac{n}{s}}}\Big(\int_0^r \varrho^{n-1}\,\mathrm{d}\varrho \int_{\mathbb{S}^{n-1}} |f(x-\varrho\sigma)|^{s'}\,\mathrm{d}\sigma\Big)^{\frac{1}{s'}} \\
&= \frac{\|\Omega\|_{L^s(\mathbb{S}^{n-1})}}{n^{\frac{1}{s}}}\Big(\frac{1}{r^n}\int_{|y|<r} |f(x-y)|^{s'}\,\mathrm{d}y\Big)^{\frac{1}{s'}}
\end{aligned}$$

where we used (14), the Hölder inequality on $\mathbb{S}^{n-1}$ and on the interval $(0, r)$. The inequality (15) now follows. □

3.2. The case of fractional maximal operator

The fractional maximal operator $M_{\Omega,\alpha}$ is defined as

$$M_{\Omega,\alpha} f(x) := \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|y|<r} |\Omega(y)|\,|f(x-y)|\,\mathrm{d}y. \tag{16}$$

LEMMA 3.2. *Let $f \in L^{p(\cdot)}(U)$, $0 < \alpha < n$, $1 < p_- \leq p_+ < \infty$ and q be defined pointwise by $1/q(x) = 1/p(x) - \alpha/n$. Then we have the following pointwise estimate*

$$M_{\Omega,\alpha} f(x) \tag{17}$$

$$\leq [M_{|\Omega|^{\frac{n}{n-\alpha}}}(|f(\cdot)|^{\frac{p(\cdot)}{q(\cdot)}\frac{n}{n-\alpha}})(x)]^{1-\frac{\alpha}{n}}\Big(\int_U |f(y)|^{p(y)}\,\mathrm{d}y\Big)^{\frac{\alpha}{n}}. \tag{18}$$

PROOF. Since $\frac{p(y)}{q(y)} + \frac{\alpha p(y)}{n} = 1$, we have

$$\begin{aligned}
&\frac{1}{r^{n-\alpha}}\int_{|x-y|<r} |\Omega(x-y)|\,|f(y)|^{\frac{p(y)}{q(y)}}|f(y)|^{\frac{\alpha p(y)}{n}}\,\mathrm{d}y \\
&\leq \Big(\frac{1}{r^n}\int_{|x-y|<r} |\Omega(x-y)|^{\frac{n}{n-\alpha}}|f(y)|^{\frac{n}{n-\alpha}\frac{p(y)}{q(y)}}\,\mathrm{d}y\Big)^{1-\alpha/n}[I_{p(\cdot)}(f)]^{\alpha/n} \\
&\leq [M_{|\Omega|^{\frac{n}{n-\alpha}}}(|f(\cdot)|^{\frac{p(\cdot)}{q(\cdot)}\frac{n}{n-\alpha}})(x)]^{1-\alpha/n}[I_{p(\cdot)}(f)]^{\alpha/n}
\end{aligned}$$

which shows (18). □

3.3. *The case of sharp maximal operator*

The sharp maximal operator, also known as the *Fefferman–Stein operator*, is a very well-known operator in harmonic analysis. We now introduce the rough sharp maximal operator as

$$M_\Omega^\# f(x) = f_\Omega^\#(x) = \sup_{r>0} \frac{1}{r^n} \int_{|y|<r} |\Omega(y)|\, |f(x-y) - f_{B(x,r)}|\, \mathrm{d}y$$

where f_B is the integral average, namely

$$f_B := \frac{1}{|B|} \int_B f(y)\, \mathrm{d}y.$$

LEMMA 3.3. *Let Ω satisfy* (14) *and $\Omega \in L^1(\mathbb{S}^{n-1})$. Then*

$$|f_\Omega^\#(x)| \le M_\Omega f(x) + \frac{1}{n} Mf(x) \|\Omega\|_{L^1(\mathbb{S}^{n-1})}. \tag{19}$$

PROOF. We have

$$\begin{aligned} &\frac{1}{r^n} \int_{|y|<r} |\Omega(y)|\, |f(x-y) - f_{B(x,r)}|\, \mathrm{d}y \\ &\quad \le \frac{1}{r^n} \int_{|y|<r} |\Omega(y)|\, |f(x-y)|\, \mathrm{d}y + \frac{1}{r^n} \int_{|y|<r} |\Omega(y)|\, |f_{B(x,r)}|\, \mathrm{d}y \\ &\quad = I + II. \end{aligned}$$

It is immediate that

$$I \le M_\Omega f(x). \tag{20}$$

We now estimate II:

$$\begin{aligned} II &\le Mf(x) \cdot \frac{1}{r^n} \int_{|y|<r} |\Omega(y)|\, \mathrm{d}y \\ &= Mf(x) \cdot \frac{1}{r^n} \int_0^r \varrho^{n-1}\, \mathrm{d}\varrho \int_{\mathbb{S}^{n-1}} |\Omega(\sigma)|\, \mathrm{d}\sigma \\ &= \frac{1}{n} Mf(x) \|\Omega\|_{L^1(\mathbb{S}^{n-1})} \end{aligned} \tag{21}$$

where the first inequality follows from the fact that $f_{B(x,r)} \le Mf(x)$. Taking into account (20) and (21) we obtain the desired inequality (19). $\square$

3.4. The case of fractional operator

In a similar fashion to the maximal operator with rough kernel (13) we can define the fractional operator with rough kernel

$$I_\Omega^{\alpha(x)} f(x) := \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha(x)}} f(y)\, \mathrm{d}y. \tag{22}$$

Lemma 3.4. *Let $f \in L^{s'}_{\mathrm{loc}}(\mathbb{R}^n)$, $1 < s < \infty$, Ω satisfy (14), $\Omega \in L^s(\mathbb{S}^{n-1})$ and $\alpha(x) > 0$ almost everywhere. Then*

$$|I_\Omega^{\alpha(x)}(f\chi_{B(x,r)})(x)| \leq \frac{1}{\alpha(x)^{\frac{1}{s}}} \|\Omega\|_{L^s(\mathbb{S}^{n-1})} r^{\frac{\alpha(x)}{s}} (I^{\alpha(x)}(|f|^{s'}\chi_{B(x,r)})(x))^{\frac{1}{s'}}, \tag{23}$$

at all points $x \in \mathbb{R}^n$ such that $\alpha(x) > 0$.

Proof. We have

$$\begin{aligned}
&\int_{B(x,r)} \frac{\Omega(x-y)}{|x-y|^{n-\alpha(x)}} f(y)\, \mathrm{d}y \\
&\quad = \int_0^r \varrho^{\alpha(x)-1}\, \mathrm{d}\varrho \int_{\mathbb{S}^{n-1}} \Omega(\sigma) f(x-\varrho\sigma)\, \mathrm{d}\sigma \\
&\quad \leq \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \int_0^r \varrho^{\frac{\alpha(x)-1}{s}}\, \mathrm{d}\varrho \Big(\varrho^{\alpha(x)-1} \int_{\mathbb{S}^{n-1}} |f(x-\varrho\sigma)|^{s'}\, \mathrm{d}\sigma \Big)^{\frac{1}{s'}} \\
&\quad \leq \frac{\|\Omega\|_{L^s(\mathbb{S}^{n-1})}}{\alpha(x)^{\frac{1}{s}}} r^{\frac{\alpha(x)}{s}} \Big(\int_0^r \varrho^{\alpha(x)-1}\, \mathrm{d}\varrho \int_{\mathbb{S}^{n-1}} |f(x-\varrho\sigma)|^{s'}\, \mathrm{d}\sigma \Big)^{\frac{1}{s'}} \\
&\quad = \frac{\|\Omega\|_{L^s(\mathbb{S}^{n-1})}}{\alpha(x)^{\frac{1}{s}}} r^{\frac{\alpha(x)}{s}} \Big(\int_{B(x,r)} \frac{|f|^{s'}(y)}{|x-y|^{n-\alpha(x)}}\, \mathrm{d}y \Big)^{\frac{1}{s'}} \\
&\quad = C\|\Omega\|_{L^s(\mathbb{S}^{n-1})} r^{\frac{\alpha(x)}{s}} (I^{\alpha(x)}(|f|^{s'}\chi_{B(x,r)}))^{\frac{1}{s'}},
\end{aligned}$$

where we used (14), the Hölder inequality on $\mathbb{S}^{n-1}$ and on the interval $(0, r)$. The inequality (23) now follows. □

Corollary 3.5. *Let $\Omega \in L^s(\mathbb{S}^{n-1})$, $1 < s < \infty$, $\inf_{x \in U} \alpha(x) > 0$ and $f \in L^{s'}_{\mathrm{loc}}(\mathbb{R}^n)$. Then*

$$|I_\Omega^{\alpha(x)}(f\chi_{B(x,r)})(x)| \leq C\|\Omega\|_{L^s(\mathbb{S}^{n-1})} r^{\alpha(x)} (M(|f|^{s'})(x))^{\frac{1}{s'}}. \tag{24}$$

PROOF. The proof follows from the estimate (23) and the well-known pointwise estimate

$$I^{\alpha(x)}(f\chi_{B(x,r)}) \leq Cr^{\alpha(x)}Mf(x)$$

see, e.g. [31, Theorem 2.55] for the case of variable α. □

LEMMA 3.6. *Let $\Omega \in L^s(\mathbb{S}^{n-1})$, $1 < s < \infty$, $\alpha \in L^\infty(U)$. Then*

$$|I_\Omega^{\alpha(x)}(f\chi_{\mathbb{R}^n\setminus B(x,r)})(x)| \leq C\|\Omega\|_{L^s(\mathbb{S}^{n-1})} r^{\frac{\alpha(x)}{s}-\frac{\beta(x)}{s'}}(I^{\alpha(x)+\beta(x)}(|f|^{s'}\chi_{\mathbb{R}^n\setminus B(x,r)}))^{\frac{1}{s'}} \tag{25}$$

where β is an arbitrary function chosen so as

$$\inf_{x\in U}\left[\beta(x) - \frac{\alpha(x)}{s-1}\right] > 0 \tag{26}$$

and $C = C(\alpha, s, \beta)$.

PROOF. We have

$$\begin{aligned}
&\int_{|x-y|>r} \frac{\Omega(x-y)}{|x-y|^{n-\alpha(x)}} f(y)\,\mathrm{d}y \\
&\quad = \int_r^\infty \varrho^{\alpha(x)-1}\,\mathrm{d}\varrho \int_{\mathbb{S}^{n-1}} \Omega(\sigma) f(x-\varrho\sigma)\,\mathrm{d}\sigma \\
&\quad \leq \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \int_r^\infty \varrho^{\frac{\alpha(x)-1}{s}-\frac{\beta(x)}{s'}}\,\mathrm{d}\varrho \Big(\varrho^{\alpha(x)+\beta(x)-1}\int_{\mathbb{S}^{n-1}} |f(x-\varrho\sigma)|^{s'}\,\mathrm{d}\sigma\Big)^{\frac{1}{s'}},
\end{aligned}$$

where $\beta > 0$ will be chosen later. Hence

$$\begin{aligned}
&\int_{|x-y|>r} \frac{\Omega(x-y)}{|x-y|^{n-\alpha(x)}} f(y)\,\mathrm{d}y \\
&\quad \leq C\|\Omega\|_{L^s(\mathbb{S}^{n-1})} r^{\frac{\alpha(x)}{s}-\frac{\beta(x)}{s'}} \Big(\int_r^\infty \varrho^{\alpha(x)+\beta(x)-1}\,\mathrm{d}\varrho \int_{\mathbb{S}^{n-1}} |f(x-\varrho\sigma)|^{s'}\,\mathrm{d}\sigma\Big)^{\frac{1}{s'}}
\end{aligned}$$

under the choice $\beta(x) > \frac{\alpha(x)}{s-1}$. Consequently,

$$\begin{aligned}
&\int_{|x-y|>r} \frac{\Omega(x-y)}{|x-y|^{n-\alpha(x)}} f(y)\,\mathrm{d}y \\
&\quad \leq C\|\Omega\|_{L^s(\mathbb{S}^{n-1})} r^{\frac{\alpha(x)}{s}-\frac{\beta(x)}{s'}} \Big(\int_{|x-y|>r} \frac{|f|^{s'}(y)}{|x-y|^{n-\alpha(x)-\beta(x)}}\,\mathrm{d}y\Big)^{\frac{1}{s'}},
\end{aligned}$$

which completes the proof. □

LEMMA 3.7. *Let U be a bounded open set, $p \in \mathbb{P}^{\log}(U)$,*

$$\sup_{x\in U} p(x)\alpha(x) < n, \tag{27}$$

$\inf_{x\in U} \alpha(x) > 0$, $\Omega \in L^s(\mathbb{S}^{n-1})$, $f \in L^{p(\cdot)}(U)$ *and* $1 < s < \infty$. *Then*

$$|I_\Omega^{\alpha(x)}(f\chi_{U\setminus B(x,r)})(x)| \le C\|\Omega\|_{L^s(\mathbb{S}^{n-1})} r^{\alpha(x)-\frac{n}{p(x)}} \|f\|_{L^{p(\cdot)}(U)}. \tag{28}$$

PROOF. By (25) we have

$$\begin{aligned} &|I_\Omega^{\alpha(x)}(f\chi_{U\setminus B(x,r)})(x)| \\ &\quad \le C\|\Omega\|_{L^s(\mathbb{S}^{n-1})} r^{\frac{\alpha(x)}{s}-\frac{\beta(x)}{s'}} \left(\int_{|x-y|>r} \frac{|f|^{s'}(y)}{|x-y|^{n-\alpha(x)-\beta(x)}}\,\mathrm{d}y\right)^{\frac{1}{s'}}. \end{aligned} \tag{29}$$

We apply the Hölder inequality with the variable exponent $\frac{p(x)}{s'}$ and get

$$\begin{aligned} &|I_\Omega^{\alpha(x)}(f\chi_{U\setminus B(x,r)})(x)| \\ &\quad \le C\|\Omega\|_{L^s(\mathbb{S}^{n-1})} r^{\frac{\alpha(x)}{s}-\frac{\beta(x)}{s'}} \|\,|f|^{s'}\|_{\frac{p(\cdot)}{s'}}^{\frac{1}{s'}} \left\|\frac{\chi_{U\setminus B(x,r)}}{|x-\cdot|^{n-\alpha(x)-\beta(x)}}\right\|_{\left(\frac{p(\cdot)}{s'}\right)'}^{\frac{1}{s'}}. \end{aligned}$$

By the property (2) and the estimate (7), we then obtain

$$|I^\alpha(x)_\Omega(f\chi_{U\setminus B(x,r)})(x)| \le C\|\Omega\|_{L^s(\mathbb{S}^{n-1})} r^{\alpha(x)-\frac{n}{p(x)}} \|f\|_{p(\cdot)},$$

with (7) applicable provided that

$$\sup_{x\in U}[p(x)\alpha(x) + p(x)\beta(x)] < ns'. \tag{30}$$

Since we also have a restriction

$$\inf_{x\in U}[\beta(x) - \alpha(x)/(s-1)] > 0, \tag{31}$$

it is not hard to see that the choice of $\beta(x)$ satisfying both (30) and (31) is possible by the assumption (27). □

LEMMA 3.8. *Let U be a bounded open set, $\Omega \in L^s(\mathbb{S}^{n-1})$, $1 < s < \infty$, $\inf_{x\in U} \alpha(x) > 0$, $p \in \mathbb{P}^{\log}(U)$ and $\sup_{x\in U}[\lambda(x) + \alpha(x)p(x)] < n$. Then*

$$|I_\Omega^{\alpha(x)}(f\chi_{U\setminus B(x,r)})(x)| \le C\|\Omega\|_{L^s(\mathbb{S}^{n-1})} r^{\alpha(x)-\frac{n-\lambda(x)}{p(x)}} \tag{32}$$

when $I_{p(\cdot),\lambda(\cdot)}(f) \le 1$.

PROOF. From (25) we have

$$\begin{aligned}&\int_{|x-y|>r}\Omega(x-y)\frac{f(y)}{|x-y|^{n-\alpha(x)}}\,\mathrm{d}y\\&\le\|\Omega\|_{L^s(\mathbb{S}^{n-1})}r^{\frac{\alpha(x)}{s}-\frac{\beta(x)}{s'}}\underbrace{\left(\int_{|x-y|>r}\frac{|f|^{s'}(y)}{|x-y|^{n-\alpha(x)-\beta(x)}}\,\mathrm{d}y\right)^{\frac{1}{s'}}}_{J}\end{aligned}$$

when $\inf\left[\beta(x)-\frac{a(x)}{s-1}\right]>0$. We now estimate J. By a dyadic decomposition and using the log-condition on the exponent p we obtain

$$\begin{aligned}(33)\qquad J&\le C\sum_{j=1}^{\infty}\int_{2^jr<|x-y|<2^{j+1}r}(2^jr)^{-\frac{\lambda(x)s'}{p(y)}}|f|^{s'}(y)|x-y|^{\alpha(x)+\beta(x)-n+\frac{\lambda(x)s'}{p(x)}}\,\mathrm{d}y\\&\le C\sum_{j=1}^{\infty}\|(2^jr)^{-\frac{\lambda(x)s'}{p(\cdot)}}|f|^{s'}(\cdot)\|_{L^{\frac{p(\cdot)}{s'}}(B(x,2^{j+1}r))}\\&\qquad\| |x-\cdot|^{\alpha(x)+\beta(x)-n+\frac{\lambda(x)s'}{p(x)}}\|_{L^{\left(\frac{p(\cdot)}{s'}\right)'}(U\backslash B(x,2^jr))}.\end{aligned}$$

The term $\|(2^jr)^{-\frac{\lambda(x)s'}{p(\cdot)}}|f|^{s'}(\cdot)\|_{L^{\frac{p(\cdot)}{s'}}(B(x,2^{j+1}r))}$ can be estimated by the modular, namely

$$\begin{aligned}I_{\frac{p(\cdot)}{s'}}((2^jr)^{-\frac{\lambda(x)s'}{p(\cdot)}}|f|^{s'}(\cdot))&\le C(2^{j+1}r)^{-\lambda(x)}\int_{B(x,2^{j+1}r)}|f(y)|^{p(y)}\,\mathrm{d}y\\&\le CI_{p(\cdot),\lambda(\cdot)}(f)\le C\end{aligned}$$

uniformly, by the hypothesis $I_{p(\cdot),\lambda(\cdot)}(f)\le 1$. Regarding the other term in (33) we obtain

$$\| |x-\cdot|^{\alpha(x)+\beta(x)-n+\frac{\lambda(x)s'}{p(x)}}\|_{L^{\left(\frac{p(\cdot)}{s'}\right)'}(U\backslash B(x,2^jr))}\le(2^jr)^{\alpha(x)+\beta(x)-\frac{s'(n-\lambda(x))}{p(x)}}$$

when $\left(\alpha(x)+\beta(x)+\frac{\lambda(x)s'}{p(x)}\right)\frac{p(x)}{s'}<n$.

Gathering all the above estimates, we obtain

$$\int_{|x-y|>r}\frac{\Omega(x-y)f(y)}{|x-y|^{n-\alpha(x)}}\,\mathrm{d}y\le C\|\Omega\|_{L^s(\mathbb{S}^{n-1})}r^{\alpha(x)-\frac{n-\lambda(x)}{p(x)}}\left(\sum_{j=1}^{\infty}(2^j)^{\alpha(x)+\beta(x)-\frac{s'(n-\lambda(x))}{p(x)}}\right)^{\frac{1}{s'}},$$

whenever $\left(\alpha(x)+\beta(x)+\frac{\lambda(x)s'}{p(x)}\right)\frac{p(x)}{s'}<n$, which also implies the convergence of the series. Since $\inf_{x\in U}\left[\beta(x)-\frac{\alpha(x)}{s-1}\right]>0$ and $\sup_{x\in U}[\lambda(x)+\alpha(x)p(x)]<n$ there exists $\beta(x)$ that satisfies $\left(\alpha(x)+\beta(x)+\frac{\lambda(x)s'}{p(x)}\right)\frac{p(x)}{s'}<n$. $\square$

4. Theorems for the variable exponent Lebesgue spaces

A classical result regarding rough maximal operator states that if Ω satisfies (14) and $\Omega \in L^1(\mathbb{S}^{n-1})$ that M_Ω is of type (p, p) for $1 < p \le \infty$. The classical proof is based upon the rotation method which is not well suited for the case of variable exponents.

We use the pointwise estimates obtained in Section 3 to obtain boundedness results regarding rough operators in the framework of variable exponent spaces.

Theorem 4.1. *Let Ω satisfy* (14)*, $\Omega \in L^s(\mathbb{S}^{n-1})$, $s \ge (p')_+$, $\frac{p}{s'} \in \mathfrak{P}(U)$. Then the operator M_Ω is bounded in the space $L^{p(\cdot)}(U)$.*

Proof. By (15) we have

$$\|M_\Omega f\|_{L^{p(\cdot)}(U)} \le C\|(M(|f|^{s'}))^{\frac{1}{s'}}\|_{L^{p(\cdot)}(U)}.$$

By the property (2) we then get

$$\begin{aligned} \|M_\Omega f\|_{L^{p(\cdot)}(U)} &\le C\|M(|f|^{s'})\|^{\frac{1}{s'}}_{L^{\frac{p(\cdot)}{s'}}(U)} \\ &\le C\|\,|f|^{s'}\|^{\frac{1}{s'}}_{L^{\frac{p(\cdot)}{s'}}(U)} \\ &= C\|f\|_{L^{p(\cdot)}(U)} \end{aligned}$$

where the second inequality comes from the fact that $\frac{p}{s'} \in \mathfrak{P}(U)$. □

We can now show the boundedness of the fractional maximal rough operator, namely we have the following:

Theorem 4.2. *Let Ω satisfy* (14)*, $\Omega \in L^{\frac{sn}{n-\alpha}}(\mathbb{S}^{n-1})$, $s \ge \left[\left(\left(1-\frac{\alpha}{n}\right)q\right)'\right]_+$ and $\frac{(1-\alpha/n)q}{s'} \in \mathfrak{P}(U)$. Then the operator $M_{\Omega,\alpha}$ is $(L^{p(\cdot)}(U) \to L^{q(\cdot)}(U))$-bounded.*

Proof. Let $f \in L^{p(\cdot)}(U)$ such that $I_{p(\cdot)}(f) = 1$. Then

$$\begin{aligned} \|M_{\Omega,\alpha} f\|_{L^{q(\cdot)}(U)} &\le \|[M_{|\Omega|^{\frac{n}{n-\alpha}}}(|f(\cdot)|^{\frac{p(\cdot)}{q(\cdot)}\frac{n}{n-\alpha}})]^{1-\alpha/n}\|_{L^{q(\cdot)}(U)} \\ &= \|M_{|\Omega|^{\frac{n}{n-\alpha}}}(|f(\cdot)|^{\frac{p(\cdot)}{q(\cdot)}\frac{n}{n-\alpha}})\|^{1-\alpha/n}_{L^{(1-\alpha/n)q(\cdot)}(U)} \\ &\le \|\,|f(\cdot)|^{\frac{p(\cdot)}{q(\cdot)}\frac{n}{n-\alpha}}\|^{1-\alpha/n}_{L^{(1-\alpha/n)q(\cdot)}(U)} \\ &= \|\,|f(\cdot)|^{\frac{p(\cdot)}{q(\cdot)}}\|_{L^{q(\cdot)}(U)} \\ &\le 1 \end{aligned}$$

where the first inequality comes from the pointwise inequality (18), the second one from the Theorem 4.1 and the last one from the fact that $I_{q(\cdot)}(|f|^{\frac{p(\cdot)}{q(\cdot)}}) =$

$I_{p(\cdot)}(f)=1$ and the definition of variable exponent Lebesgue norm. The general case follows from the homogeneity of the fractional maximal rough operators. □

THEOREM 4.3. *Let Ω satisfy* (14), $\Omega\in L^s(\mathbb{S}^{n-1})$, $s\geq (p')_+$, $\frac{p}{s'}\in\mathfrak{P}(U)$. *Then the operator $M^\#_\Omega$ is bounded in the space $L^{p(\cdot)}(U)$.*

PROOF. The proof follows from the pointwise estimate (19)

$$|M^\#_\Omega f(x)|\leq M_\Omega f(x)+Mf(x)\frac{\|\Omega\|_{L^1(\mathbb{S}^{n-1})}}{n},$$

the fact that M is bounded whenever $p\in\mathfrak{P}(U)$ (this is valid under the assumption $\frac{p}{s'}\in\mathfrak{P}(U)$, see [21, Theorem 4.37 or Theorem 3.38]) and Theorem 4.1. □

We now want to show the validity of a Sobolev type theorem for the rough fractional integral operator $I^{\alpha(x)}_\Omega$, namely.

THEOREM 4.4. *Let U be a bounded open set, $\Omega\in L^s(\mathbb{S}^{n-1})$, $1<s<\infty$, $\inf_{x\in U}\alpha(x)>0$, $\sup_{x\in U}\alpha(x)p(x)<n$, $s\geq(p')_+$ and $p\in\mathbb{P}^{\log}(U)$. Then the rough fractional operator $I^{\alpha(x)}_\Omega$ is $(L^{p(\cdot)}(U)\to L^{q(\cdot)}(U))$-bounded, where $\frac{1}{q(x)}=\frac{1}{p(x)}-\frac{\alpha(x)}{n}$.*

PROOF. For arbitrary fixed $r>0$, from Corollary 3.5 and Lemma 3.7 we have the following pointwise inequality

$$|I^{\alpha(x)}_\Omega(f)(x)|\leq C\|\Omega\|_{L^s(\mathbb{S}^{n-1})}(r^{\alpha(x)}(M(|f|^{s'}\chi_{B(x,r)})(x))^{\frac{1}{s'}}+r^{\alpha(x)-\frac{n}{p(x)}}\|f\|_{p(\cdot)}).$$

Taking

$$r=\left(\frac{\|f\|_{p(\cdot)}}{(M|f|^{s'})^{1/s'}}\right)^{\frac{p(x)}{n}}$$

we obtain

$$|I^{\alpha(x)}_\Omega(f)(x)|\leq C\|\Omega\|_{L^s(\mathbb{S}^{n-1})}[(M|f|^{s'})^{1/s'}]^{1-\frac{\alpha(x)p(x)}{n}}\|f\|_{p(\cdot)}^{\frac{\alpha(x)p(x)}{n}}. \tag{34}$$

To show the boundedness of $I^{\alpha(x)}_\Omega$ in the variable exponent Lebesgue space, from (4), it is enough to show that

$$I_{q(\cdot)}(I^{\alpha(x)}_\Omega f)\leq C$$

for all functions f in the unit sphere, i.e. $\|f\|_{L^{p(\cdot)}(U)}=1$, and the general result follows from homogeneity. From the pointwise inequality (34) it is enough to show the boundedness of

$$I_{q(\cdot)}([(M|f|^{s'})^{1/s'}]^{1-\frac{\alpha(x)p(x)}{n}})=I_{p(\cdot)}((M|f|^{s'})^{1/s'}). \tag{35}$$

To bound the right-hand side in (35) it is enough, from (4), to bound the norm, i.e. to bound

$$\|(M|f|^{s'})^{1/s'}\|_{L^{p(\cdot)}(U)}.$$

From (2) and the fact that $s > (p')_+$ implies that $p/s' > 1$, we get

$$\begin{aligned}\|(M|f|^{s'})^{1/s'}\|_{L^{p(\cdot)}(U)} &= \|(M|f|^{s'})\|^{1/s'}_{L^{\frac{p(\cdot)}{s'}}(U)} \\ &\le \|\,|f|^{s'}\|^{1/s'}_{L^{\frac{p(\cdot)}{s'}}(U)} \\ &= \|f\|_{L^{p(\cdot)}(U)}\end{aligned}$$

which is bounded since $\|f\|_{L^{p(\cdot)}(U)} = 1$. □

Corollary 4.5. *Let U be a bounded open set, $\Omega \in L^s(\mathbb{S}^{n-1})$, $1 < s < \infty$, $\inf_{x \in U} \alpha(x) > 0$, $\sup_{x \in U} \alpha(x)p(x) < n$, $s \ge (p')_+$ and $p \in \mathbb{P}^{\log}(U)$. Then the rough fractional operator $M_{\Omega,\alpha(x)}$ is $(L^{p(\cdot)}(U) \to L^{q(\cdot)}(U))$-bounded, where $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n}$ and*

$$M_{\Omega,\alpha(x)}f(x) := \sup_{r>0} \frac{1}{r^{n-\alpha(x)}} \int_{|y|<r} |\Omega(y)|\,|f(x-y)|\,\mathrm{d}y.$$

Proof. Use Theorem 4.4 and the inequality

$$M_{\Omega,\alpha(x)}f \le C_n I_\Omega^{\alpha(x)} f,$$

which follows from the definitions of the operators. □

5. Boundedness results in variable exponent Morrey spaces

We can now state the boundedness results for rough operators in the framework of variable exponent Morrey spaces.

Theorem 5.1. *Let U be an open bounded set in $\mathbb{R}^n$, $0 \le \lambda(x) \le \lambda_+ < n$, $s \ge (p')_+$, $\Omega \in L^s(\mathbb{S}^{n-1})$, $1 < s < \infty$, $p \in \mathbb{P}^{\log}(U)$. Then the rough maximal operator M_Ω is bounded in the space $L^{p(\cdot),\lambda(\cdot)}(U)$.*

Proof. By (15) we have

$$\|M_\Omega f\|_{L^{p(\cdot),\lambda(\cdot)}(U)} \le C\|(M|f|^{s'})^{\frac{1}{s'}}\|_{L^{p(\cdot),\lambda(\cdot)}(U)}.$$

By the property (12) we then get

$$\begin{aligned}\|M_\Omega f\|_{L^{p(\cdot),\lambda(\cdot)}(U)} &\le C\|M|f|^{s'}\|^{\frac{1}{s'}}_{L^{\frac{p(\cdot)}{s'},\lambda(\cdot)}(U)}\\ &\le C\|\,|f|^{s'}\|^{\frac{1}{s'}}_{L^{\frac{p(\cdot)}{s'},\lambda(\cdot)}(U)}\\ &= C\|f\|_{L^{p(\cdot),\lambda(\cdot)}(U)}\end{aligned}$$

where we used Theorem 2.4 in the second inequality. □

Theorem 5.2. *Let U be a bounded open set, Ω satisfy* (14), *$\Omega \in L^{\frac{sn}{n-\alpha}}(\mathbb{S}^{n-1})$, $s \ge \left[\left((1-\frac{\alpha}{n})q\right)'\right]_+$ and $\frac{(1-\alpha/n)q}{s'} \in \mathbb{P}^{\log}(U)$. Then the rough fractional maximal operator $M_{\Omega,\alpha}$ is $(L^{p(\cdot),\lambda(\cdot)}(U) \to L^{q(\cdot),\lambda(\cdot)}(U))$-bounded.*

Proof. Analyzing the proof of inequality (18) we obtain a similar result for the case of variable exponent Morrey spaces, namely

$$M_{\Omega,\alpha}f(x) \le C[M_{|\Omega|^{\frac{n}{n-\alpha}}}(|f(\cdot)|^{\frac{p(\cdot)}{q(\cdot)}\frac{n}{n-\alpha}}(x))]^{1-\frac{\alpha}{n}}[I_{p(\cdot),\lambda(\cdot)}(f)]^{\frac{\alpha}{n}},$$

where now C depends on the diameter of U. The rest of the proof now follows closely the same lines as the proof of Theorem 4.2. □

Theorem 5.3. *Let Ω satisfy* (14), *$\Omega \in L^s(\mathbb{S}^{n-1})$, $s \ge (p')_+$, $p \in \mathbb{P}^{\log}(U)$. Then the operator $M^{\#}_\Omega$ is bounded in the space $L^{p(\cdot),\lambda(\cdot)}(U)$.*

Proof. As in the case of Theorem 4.3, the proof follows from the pointwise inequality (19). □

Theorem 5.4. *Let U be a bounded open set, $\Omega \in L^s(\mathbb{S}^{n-1})$, $1 < s < \infty$, $\inf_{x\in U}\alpha(x) > 0$, $\sup_{x\in U}[\lambda(x) + \alpha(x)p(x)] < n$, $s \ge (p')_+$ and $p \in \mathbb{P}^{\log}(U)$. Then the rough fractional operator $I^{\alpha(x)}_\Omega$ is $(L^{p(\cdot),\lambda(\cdot)}(U) \to L^{q(\cdot),\lambda(\cdot)}(U))$-bounded, where $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n-\lambda(x)}$.*

Proof. Gathering (24) and (32), we obtain the pointwise estimate

$$|I^{\alpha(x)}_\Omega(f)(x)| \le C\|\Omega\|_{L^s(\mathbb{S}^{n-1})}(r^{\alpha(x)}(M|f|^{s'})^{\frac{1}{s'}} + r^{\alpha(x)-\frac{n-\lambda(x)}{p(x)}}).$$

Taking

$$r = [(M|f|^{s'})^{\frac{1}{s'}}]^{-\frac{p(x)}{n-\lambda(x)}}$$

we arrive at

$$|I^{\alpha(x)}_\Omega(f)(x)| \le C\|\Omega\|_{L^s(\mathbb{S}^{n-1})}[(M|f|^{s'})^{\frac{1}{s'}}]^{\frac{p(x)}{q(x)}}. \tag{36}$$

By (36) we obtain

$$I_{q(\cdot),\lambda(\cdot)}(I^{\alpha(x)}_\Omega(f)) \le I_{p(\cdot),\lambda(\cdot)}((M|f|^{s'})^{\frac{1}{s'}}),$$

and now the proof follows the same lines as in Theorem 4.4. □

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REFERENCES

[1] R. ABOULAICH - S. BOUJENA - E. EL GUARMAH, *Sur un modèle non-linéaire pour le débruitage de l'image*, C. R., Math., Acad. Sci. Paris, 345(8):425–429, 2007.

[2] R. ABOULAICH - D. MESKINE - A. SOUISSI, *New diffusion models in image processing*, Comput. Math. Appl., 56(4):874–882, 2008.

[3] E. ACERBI - G. MINGIONE, *Regularity results for electrorheological fluids: The stationary case*, C. R., Math., Acad. Sci. Paris, 334(9):817–822, 2002.

[4] E. ACERBI - G. MINGIONE, *Regularity results for stationary electrorheological fluids*, Arch. Ration. Mech. Anal., 164(3):213–259, 2002.

[5] H. AL-QASSEM - A. AL-SALMAN, *l^p-boundedness of a class of singular integral operators with rough kernels*, Turk. J. Math., 25:519–533, 2001.

[6] H. AL-QASSEM - L. CHENG - Y. PAN, *On certain rough integral operators and extrapolation*, J. Ineq. Pure Appl. Math., 10:15 pp, 2009.

[7] A. AL-SALMAN, *Rough oscillatory singular integral operators-II*, Turk. J. Math., 27:565–579, 2003.

[8] A. AL-SALMAN, *Maximal operators with rough kernels on product domains*, J. Math. Anal. Appl., 311(1):338–351, 2005.

[9] A. AL-SALMAN, *On a class of singular integral operators with rough kernels*, Canad. Math. Bull., 4(1):3–10, 2006.

[10] A. ALMEIDA - J. HASANOV - S. SAMKO, *Maximal and potential operators in variable exponent Morrey spaces*, Georgian Math. J., 15(2):195–208, 2008.

[11] S. N. ANTONTSEV - J. F. RODRIGUES, *On stationary thermo-rheological viscous flows*, Ann. Univ. Ferrara, Sez. VII, Sci. Mat., 52(1):19–36, 2006.

[12] A. S. BALAKISHIYEV - V. S. GULIYEV - F. GURBUZ - A. SERBETCI, *Sublinear operators with rough kernel generated by Calderón–Zygmund operators and their commutators on generalized local Morrey spaces*, J. Inequal. Appl., 2015:18, 2015.

[13] P. BLOMGREN - T. F. CHAN - P. MULET - L. VESE - W. L. WAN, *Variational PDE models and methods for image processing*, In Numerical analysis 1999. Proceedings of the 18th Dundee biennial conference, Univ. of Dundee, GB, June 29th–July 2nd, 1999, pages 43–67. Boca Raton, FL: Chapman & Hall/CRC, 2000.

[14] E. M. BOLLT - R. CHARTRAND - S. ESEDOḠLU - P. SCHULTZ - K. R. VIXIE, *Graduated adaptive image denoising: Local compromise between total variation and isotropic diffusion*, Adv. Comput. Math., 31(1–3):61–85, 2009.

[15] A. P. CALDERÓN - A. ZYGMUND, *On singular integrals*, Amer. J. Math., 78:289–309, 1956.

[16] S. CHANILLO - D. WATSON - R. WHEEDEN, *Some integral and maximal operators related to starlike sets*, Studia Math., 107(3):223–255, 1993.

[17] L.-K. CHEN - H. LIN, *A maximal operator related to a class of singular integrals*, Illinois J. Math., 34(1):296–321, 1990.

[18] Y. CHEN - W. GUO - Q. ZENG - Y. LIU, *A nonstandard smoothing in reconstruction of apparent diffusion coefficient profiles from diffusion weighted images*, Inverse Probl. Imaging, 2(2):205–224, 2008.

[19] Y. Chen - S. Levine - M. Rao, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math., 66(4):1383–1406, 2006.

[20] D. Cruz-Uribe - A. Fiorenza - J. M. Martell - C. Pérez, *The boundedness of classical operators on variable L^p spaces*, Ann. Acad. Sci. Fenn., Math., 31(1):239–264, 2006.

[21] D. V. Cruz-Uribe - A. Fiorenza, *Variable Lebesgue spaces. Foundations and harmonic analysis*, New York, NY: Birkhäuser/Springer, 2013.

[22] L. Diening - P. Harjulehto - P. Hästö - M. Růžička, *Lebesgue and Sobolev spaces with variable exponents*, Berlin: Springer, 2011.

[23] Y. Ding, *Weak type bounds for a class of rough operators with power weights*, Proc. Amer. Math. Soc., 125(10):2939–2942, 1997.

[24] Y. Ding - Chin-Cheng Lin, *A class of maximal operators with rough kernel on product spaces*, Illinois J. Math., 45(2):545–557, 2001.

[25] Y. Ding - S. Lu, *Weighted norm inequalities for fractional integral operators with rough kernel*, Can. J. Math., 50(1):29–39, 1999.

[26] J. Duoandikoetxea, *Weighted norm inequalities for homogeneous singular integrals*, Proc. Amer. Math. Soc., 336(2):869–880, 1993.

[27] M. Giaquinta, *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, Annals of mathematics studies. Princeton University Press, 1983.

[28] P. Harjulehto - P. Hästö - Ú. V. Lê - M. Nuortio, *Overview of differential equations with non-standard growth*, Nonlinear Anal., Theory, Methods Appl., Ser. A, Theory Methods, 72(12):4551–4574, 2010.

[29] G. Hu - S. Lu - D. Yang, *Boundedness of rough singular integral operators on homogeneous Herz spaces*, J. Austral. Math. Soc. (Series A), 66(2):201–223, 1999.

[30] V. Kokilashvili - A. Meskhi, *Boundedness of maximal and singular operators in Morrey spaces with variable exponent*, Armen. J. Math., 1(1):18–28, 2008.

[31] V. Kokilashvili - A. Meskhi - H. Rafeiro - S. Samko, *Integral Operators in Non-Standard Function Spaces. Volume 1: Variable Exponent Lebesgue and Amalgam Spaces, volume 248 of Operator Theory: Advances and Applications*, Birkhäuser Basel, 2016.

[32] V. Kokilashvili - A. Meskhi - H. Rafeiro - S. Samko, *Integral Operators in Non-Standard Function Spaces. Volume 2: Variable Exponent Hölder, Morrey-Campanato and Grand Spaces, volume 249 of Operator Theory: Advances and Applications*, Birkhäuser Basel, 2016.

[33] G. Mingione, *Regularity of minima: an invitation to the dark side of the calculus of variations*, Appl. Math., Praha, 51(4):355–425, 2006.

[34] B. Muckenhoupt - R. L. Wheeden, *Weighted norm inequalities for singular and fractional integrals*, Trans. Amer. Math. Soc., 161:261–274, 1971.

[35] L. Pick - A. Kufner - O. John - S. Fucík, *Function Spaces, 1*, Number vol. 1 in De Gruyter Series in Nonlinear Analysis and Applications. De Gruyter, 2012.

[36] H. Rafeiro - N. Samko - S. Samko, *Morrey–Campanato spaces: an overview*, In Operator theory, pseudo-differential equations, and mathematical physics. The Vladimir Rabinovich anniversary volume, pages 293–323. Basel: Birkhäuser/Springer, 2013.

[37] M. Růžička, *Electrorheological fluids: modeling and mathematical theory*, Berlin: Springer, 2000.

[38] S. G. Samko, *Convolution and potential type operators in $L^{p(x)}(\mathbb{R}^n)$*, Integral Transforms Spec. Funct., 7(3–4):261–284, 1998.

[39] P. Sjögren - F. Soria, *Rough maximal functions and rough singular integral operators applied to integrable radial functions*, Rev. Mat. Iberoamericana, 13(1):691–701, 1997.

[40] E. M. Stein, *Harmonic Analysis: real-variable methods, orthogonality and oscillatory integrals*, Princeton Univ. Press, Princeton, NJ, 1993.

[41] T. Wunderli, *On time flows of minimizers of general convex functionals of linear growth with variable exponent in BV space and stability of pseudosolutions*, J. Math. Anal. Appl., 364(2):591–598, 2010.

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Humberto Rafeiro
Pontificia Universidad Javeriana
Departamento de Matemáticas, Facultad de Ciencias
Cra. 7 No. 43-82
Bogotá, Colombia
silva-h@javeriana.edu.co

Stefan Samko
Universidade do Algarve
Department of Mathematics, Faculty of Sciences
8005-139 Faro, Portugal
ssamko@ualg.pt