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Partial Differential Equations — *On the existence and blow-up of solutions for a mean field equation with variable intensities*, by Tonia Ricciardi, Ryo Takahashi, Gabriella Zecca and Xiao Zhang, communicated on 14 April 2016.

Abstract. — We study an elliptic problem with exponential nonlinearities describing the statistical mechanics equilibrium of point vortices with variable intensities. For suitable values of the physical parameters we exclude the existence of blow-up points on the boundary, we prove a mass quantization property and we apply our analysis to the construction of minimax solutions.

Key words: Mean field equation, blow-up solutions, turbulent Euler flow

Mathematics Subject Classification: 35J91, 35B44, 35J20

1. Introduction and main result

Motivated by the theory of hydrodynamic turbulence as developed by Onsager [8, 17], we consider the problem:

$$\begin{cases} -\Delta u = \lambda\left(\dfrac{e^u}{\int_\Omega e^u} + \sigma\gamma\dfrac{e^{\gamma u}}{\int_\Omega e^{\gamma u}}\right) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda, \sigma > 0$, $\gamma \in [-1, 1)$ and $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain. Problem (1.1) is derived by statistical mechanics arguments under a "deterministic" assumption on the point vortex intensities [3, 27]. More precisely, the equation derived in [27] is given by

$$\begin{cases} -\Delta u = \tilde{\lambda}\displaystyle\int_{[-1,1]} \frac{\alpha e^{\alpha u}}{\int_\Omega e^{\alpha u}\,dx}\mathcal{P}(d\alpha) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where u is the stream function of the two-dimensional flow, $\mathcal{P}$ is a Borel probability measure defined on the interval $[-1, 1]$ describing the point vortex intensity distribution and $\tilde{\lambda} > 0$ is a constant related to the inverse temperature. In the special case $\mathcal{P}(d\alpha) = \mathcal{P}_\gamma(d\alpha)$, where

$$\mathcal{P}_\gamma(d\alpha) = \tau\delta_1(d\alpha) + (1-\tau)\delta_\gamma(d\alpha), \tag{1.3}$$

and $\delta_1(d\alpha)$, $\delta_\gamma(d\alpha)$ denote the Dirac measures concentrated at the points $1, \gamma \in [-1, 1]$, respectively, and $\tau \in (0, 1]$, problem (1.2) takes the form

$$\begin{cases} -\Delta u = \tilde{\lambda}\Big(\tau \frac{e^u}{\int_\Omega e^u\, dx} + (1-\tau)\gamma \frac{e^{\gamma u}}{\int_\Omega e^{\gamma u}\, dx}\Big) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

Setting

$$\lambda = \tilde{\lambda}\tau, \quad \sigma = \frac{1-\tau}{\tau}, \tag{1.5}$$

problem (1.4) reduces to (1.1).

We observe that taking $\gamma = -1$ in problem (1.1) we obtain the sinh-Poisson type problem derived in [19]:

$$\begin{cases} -\Delta u = \lambda\Big(\frac{e^u}{\int_\Omega e^u} - \sigma \frac{e^{-u}}{\int_\Omega e^{-u}}\Big) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

which has received a considerable interest in recent years, see [1, 10, 12, 13, 16, 20] and the references therein. In particular, the blow-up analysis for (1.6) has been clarified by geometrical arguments involving constant mean curvature surfaces in [13]. However, such an approach seems difficult to extend to our case.

For $\sigma = 0$ problem (1.1) reduces to the standard mean field problem

$$\begin{cases} -\Delta u = \lambda \frac{e^u}{\int_\Omega e^u\, dx} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

which has been extensively analyzed in view of its connections to differential geometry, physics and biology, see, e.g., [11]. However, even in the "positive case" $\gamma \in (0, 1)$, problem (1.1) does not necessarily exhibit the properties of a perturbation of (1.7). This fact may be seen, for example, by considering the optimal constant for the Moser-Trudinger inequality associated to (1.1), see [21, 29] or the proof of Lemma 3.2 below. In this respect, problem (1.1) significantly differs from its "stochastic" version derived in [14] and recently analyzed in [18, 22, 23, 24, 25, 26]. In fact, our aim in this article is to determine suitable smallness conditions for $|\gamma|$ *and* σ (see (1.8)–(1.9) below) which ensure that the nonlinearity $e^{\gamma u}$ may indeed be treated as "lower-order" with respect to the "principal" term e^u. In particular, under such smallness conditions we prove the mass quantization for blow-up solution sequences, we derive an improved Moser-Trudinger inequality and we consequently obtain an existence result for solutions in the supercritical range $\lambda > 8\pi$.

More precisely, for every fixed γ satisfying $0 < |\gamma| < 1/2$ let

$$\sigma_\gamma := \frac{1 - 2|\gamma|}{2\gamma^2} \tag{1.8}$$

and

$$\lambda_{\sigma,\gamma} := \min\left\{\frac{16\pi}{1 + 2\sigma\gamma^2}, \frac{4\pi}{|\gamma|(1 + |\gamma|\sigma)}\right\}. \tag{1.9}$$

Our main result is the following.

THEOREM 1.1. *Assume that $\mathbb{R}^2\backslash\Omega$ has a bounded component containing at least one interior point. Fix $0 < |\gamma| < 1/2$ and $0 < \sigma < \sigma_\gamma$. Then, there exists a solution to Problem* (1.1) *for every $\lambda \in (8\pi, \lambda_{\sigma,\gamma})$.*

We note that $8\pi < \lambda_{\gamma,\sigma} < 16\pi$ whenever $0 < |\gamma| < 1/2$ and $\sigma \in (0, \sigma_\gamma)$, see Lemma 3.3 below.

Finally, we remark that problem (1.1) shares some similarity in structure with Liouville systems and Toda-type systems. Indeed, setting $v_1 = G * e^u$, $v_2 = G * e^{\gamma u}$, $a^{11} = \lambda \Big/ \int_\Omega e^u$, $a^{12} = \lambda\sigma\gamma \Big/ \int_\Omega e^{\gamma u}$, $a^{21} = \gamma\lambda \Big/ \int_\Omega e^u$, $a^{22} = \gamma^2\sigma\lambda \Big/ \int_\Omega e^{\gamma u}$, we obtain $u = \lambda v_1 \Big/ \int_\Omega e^u\,dx + \lambda\sigma\gamma v_2 \Big/ \int_\Omega e^{\gamma u}$ and problem (1.1) takes the form $-\Delta v_i = \exp\{\sum_{j=1,2} a^{ij} v_j\}$, $i = 1, 2$, which is a system of Liouville type, as analyzed in [4, 6]. On the other hand, setting $w_1 = u$, $w_2 = \gamma u$, $b^{11} = \lambda \Big/ \int_\Omega e^{w_1}$, $b^{12} = \lambda\sigma\gamma \Big/ \int_\Omega e^{w_2}$, $b^{21} = \lambda\gamma \Big/ \int_\Omega e^{w_1}$, $b^{22} = \lambda\sigma\gamma^2 \Big/ \int_e^{w_2}$, we obtain the system $-\Delta w_i = \sum_{j=1,2} b^{ij} e^{w_j}$ $j = 1, 2$, which has a "Toda-like" structure when $\gamma < 0$, see [1] and the references therein. However, Theorem 1.1 does not follow directly from the results for systems of Liouville and Toda type mentioned above, due to the substantially different assumptions for the coefficients (a^{ij}) and (b^{ij}), $i, j = 1, 2$.

This note is organized as follows. In Section 2 we use Brezis-Merle estimates [2] to exclude the existence of blow-up points on the boundary and to derive a mass quantization property, for suitably small values of γ and σ. We note that the exclusion of boundary blow-up points could also be derived by extending the argument in [22]. Here, we provide a simple *ad hoc* proof which exploits the smallness assumptions on $|\gamma|$ and σ. In Section 3 we derive an improved Moser-Trudinger type inequality. We prove Theorem 1.1 by suitably adapting an argument in [7] and by applying the blow-up results derived in Section 2.

Notation

Henceforth, all integrals are taken with respect to the Lebesgue measure. We may omit the integration variables if they are clear from the context. We denote by C a general constant whose actual value may vary from line to line.

2. Blow-up results

In this section we show that for suitably small values of $|\gamma|$ and σ the blow-up analysis for problem (1.1) is similar to the blow-up analysis for the standard mean field equation (1.7). We first exclude the existence of boundary blow-up points. Then, we prove a mass quantization property.

More precisely, let (u_n, λ_n) be a solution sequence for (1.1) with $\lambda_n \to \lambda_0 \geq 0$. We define

$$\mathcal{S}_\pm = \{x_0 \in \overline{\Omega} : \exists x_n \to x_0 \text{ such that } u_n(x_n) \to \pm\infty\}$$

and we set $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_-$.

2.1. Boundary blow-up exclusion

In the case where $\gamma > 0$ the boundary blow-up is readily excluded in view of the moving plane argument in [9], p. 223. Therefore, throughout this subsection, we consider the "asymmetric sinh-case" of (1.1), namely

$$\begin{cases} -\Delta u = \lambda\Big(\dfrac{e^u}{\int_\Omega e^u} - \sigma|\gamma|\dfrac{e^{-|\gamma|u}}{\int_\Omega e^{-|\gamma|u}}\Big) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

We make the following assumption:

$$\lambda(1 + \sigma|\gamma|) < \frac{4\pi}{|\gamma|}. \tag{2.2}$$

In this subsection we show the following.

Proposition 2.1. *Let (u_n, λ_n) be a solution sequence for problem (2.1) with $\lambda_n \to \lambda_0 \geq 0$ and assume that λ_0 satisfies (2.2). Then, $\mathcal{S} \cap \partial\Omega = \emptyset$.*

We first reduce problem (2.1) to a mean field type problem with smooth weight function. Let $G = G(x, y)$ be the Green's function defined for $x, y \in \Omega$ by

$$\begin{cases} -\Delta G(\cdot, y) = \delta_y & \text{in } \Omega \\ G(\cdot, y) = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $u := u_+ - u_-$, where

$$u_+ = G * \lambda\frac{e^u}{\int_\Omega e^u}$$

$$u_- = G * \lambda\sigma|\gamma|\frac{e^{-|\gamma|u}}{\int_\Omega e^{-|\gamma|u}}.$$

We observe that

$$\begin{cases} -\Delta u_+ = \lambda \dfrac{h(x)e^{u_+}}{\int_\Omega h(x)e^{u_+}} & \text{in } \Omega \\ u_+ = 0 & \text{on } \partial\Omega, \end{cases}$$

where $h(x) = e^{-u_-}$ satisfies $\|h\|_{C^{1,\alpha}(\bar{\Omega})} \leq C$, $h \equiv 1$ on $\partial\Omega$. In fact, we have

$$\begin{cases} -\Delta u_- = \lambda\sigma|\gamma| \dfrac{e^{-|\gamma|u}}{\int_\Omega e^{-|\gamma|u}} & \text{in } \Omega \\ u_- = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda\sigma|\gamma| \frac{e^{-|\gamma|u}}{\int_\Omega e^{-|\gamma|u}}$ is L^q-bounded for some $q > 1$. To see this fact, recall from [2] that if u satisfies:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

for some $f \in L^1(\Omega)$, then for any small $\delta > 0$ we have

$$\int_\Omega \exp\left\{ \frac{(4\pi - \delta)}{\|f\|_{L^1(\Omega)}} |u| \right\} \leq \frac{4\pi^2}{\delta} (\operatorname{diam} \Omega)^2. \tag{2.3}$$

Hence, by elliptic estimates, $\|u_-\|_{L^\infty(\Omega)} \leq C$. Now we write the equation for u_- in the form

$$\begin{cases} -\Delta u_- = \lambda\sigma|\gamma| \dfrac{e^{|\gamma|u_- - |\gamma|u_+}}{\int_\Omega e^{-|\gamma|u}} & \text{in } \Omega \\ u_- = 0 & \text{on } \partial\Omega \end{cases} \tag{2.4}$$

and we observe that since $u_+ \geq 0$ we have $e^{-|\gamma|u_+} \leq 1$. Hence, the right hand side in (2.4) is $L^\infty(\Omega)$-bounded. It follows that $\|u_-\|_{W^{2,p}(\Omega)} \leq C$ for every $p \in (1, +\infty)$. In particular, $\|u_-\|_{C^{1,\alpha}(\bar{\Omega})} \leq C$.

Proof of Proposition 2.1. We adapt an argument of [9] p. 223 to our case. Let $x_0 \in \partial\Omega$ and let D_r be a closed disc touching $\overline{\Omega}$ only at x_0. For convenience we assume $D_r = D(0, r)$ and $x_0 = (r, 0)$. Then, the inversion mapping $x \mapsto y = r^2 x/|x|^2$ fixes the boundary of D_r and maps Ω to a region $y(\overline{\Omega})$ contained inside D_r. Setting $v(y) = u^+(x)$, recalling that

$$D_x = \frac{r^2}{|y|^4} \begin{pmatrix} -y_1^2 + y_2^2 & -2y_1y_2 \\ -2y_1y_2 & y_1^2 - y_2^2 \end{pmatrix}, \quad D_x^T D_x = \frac{r^4}{|y|^4} I, \tag{2.5}$$

where I denotes the identity mapping, we obtain the following equation for v:

$$\Delta v + \rho \frac{r^4}{|y|^4} h(x(y)) e^v = 0 \quad \text{in } y(\overline{\Omega}).$$

CLAIM. For every $x_0 \in \partial\Omega$ there exist $r(x_0) > 0$, $\delta(x_0) > 0$ and $\delta'(x_0) > 0$ such that the function

$$H(y) = \frac{r^4}{|y|^4} h(x(y))$$

is decreasing in y_1-direction in the set

$$\left\{ \frac{r(x_0)}{1+\delta(x_0)} \le |y| \le r(x_0),\, y_1 > 0, |y_2| \le \delta'(x_0) \right\} \subset y(\overline{\Omega}),$$

provided that $r \le r(x_0)$.

PROOF OF CLAIM. In view of (2.5) we compute:

$$\partial_{y_1} \frac{1}{|y|^4} = -\frac{4y_1}{|y|^6}$$

$$\partial_{y_1} h(x(y)) = \frac{r^2}{|y|^4} \{ (\partial_{x_1} h(x(y)))(-y_1^2 + y_2^2) + (\partial_{x_2} h(x(y)))(-2y_1 y_2) \}$$

and therefore

$$\frac{1}{r^4} \partial_{y_1} H(y)$$

$$= \frac{1}{|y|^6} \left\{ -4y_1 h(x(y)) + \frac{r^2}{|y|^2} [(\partial_{x_1} h(x(y)))(-y_1^2 + y_2^2) + (\partial_{x_2} h(x(y)))(-2y_1 y_2)] \right\}$$

$$= \frac{1}{|y|^6} \left\{ -y_1 \left[4h(x(y)) + \frac{r^2}{|y|^2} (\partial_{x_1} h) y_1 \right] + \frac{r^2}{|y|^2} y_2^2 (\partial_{x_1} h) + \frac{r^2}{|y|^2} (\partial_{x_2} h)(-2y_1 y_2) \right\}.$$

We estimate, for $|y| \ge r/(1+\delta)$, $|y_2| < \delta'$:

$$\left| \frac{r^2}{|y|^2} (\partial_{x_1} h) y_1 \right| \le (1+\delta)^2 \|h\|_{C^1(\overline{\Omega})} r$$

$$\left| \frac{r^2}{|y|^2} y_2^2 (\partial_{x_1} h) \right| \le (1+\delta)^2 \|h\|_{C^1(\overline{\Omega})} (\delta')^2$$

$$\left| \frac{r^2}{|y|^2} (\partial_{x_2} h)(-2y_1 y_2) \right| \le 2(1+\delta)^2 \|h\|_{C^1(\overline{\Omega})} r\delta'.$$

By choosing $r = r(x_0)$ sufficiently small, we achieve

$$4h(x(y)) + \frac{r^2}{|y|^2}(\partial_{x_1} h)y_1 \geq 2.$$

Then, for δ' sufficiently small, we have

$$\begin{aligned}&-y_1\left[4h(x(y)) + \frac{r^2}{|y|^2}(\partial_{x_1})y_1\right] + \frac{r^2}{|y|^2}y_2^2(\partial_{x_1}h) + \frac{r^2}{|y|^2}(\partial_{x_2}h)(-2y_1y_2)\\ &\quad\leq -2y_1 + (1+\delta)^2\|h\|_{C^1(\bar{\Omega})}(\delta')^2 + 2(1+\delta)^2\|h\|_{C^1(\bar{\Omega})}r\delta'\\ &\quad\leq -\frac{r}{2} < 0.\end{aligned}$$

Now the argument in [9] concludes the proof. □

2.2. Mass quantization

In view of Proposition 2.1 we have $\mathscr{S} \cap \partial\Omega = \emptyset$. Therefore, by local blow-up results from [15, 23] we know that setting

$$\mu_1(dx) := \lambda\frac{e^u}{\int_\Omega e^u}dx, \quad \mu_\gamma(dx) := \lambda\frac{e^{\gamma u}}{\int_\Omega e^{\gamma u}}dx$$

we have

$$\begin{aligned}\mu_1(dx) &\overset{*}{\rightharpoonup} \sum_{p\in\mathscr{S}} m_1(p)\delta_p(dx) + r_1(x)\,dx\\ \mu_\gamma(dx) &\overset{*}{\rightharpoonup} \sum_{p\in\mathscr{S}} m_\gamma(p)\delta_p(dx) + r_\gamma(x)\,dx.\end{aligned} \tag{2.6}$$

LEMMA 2.1. *At every fixed $p \in \mathscr{S}$ we have the quadratic identity:*

$$8\pi(m_1(p) + \sigma m_\gamma(p)) = (m_1(p) + \sigma\gamma m_\gamma(p))^2. \tag{2.7}$$

PROOF. We recall from [15] that if $(u_k, \tilde{\lambda}_k)$ is a solution sequence for (1.4) with

$$\tilde{\lambda}\frac{e^{u_k}}{\int_\Omega e^{u_k}} \overset{*}{\rightharpoonup} \sum_{p\in\mathscr{S}}\tilde{m}_1(p)\delta_p(dx) + \tilde{r}_1(x), \quad \tilde{\lambda}\frac{e^{\gamma u_k}}{\int_\Omega e^{\gamma u_k}} \overset{*}{\rightharpoonup} \sum_{p\in\mathscr{S}}\tilde{m}_\gamma(p)\delta_p(dx) + \tilde{r}_\gamma(x),$$

where $\delta_p(dx)$ denotes the Dirac mass centered at $p \in \Omega$, then the following relation holds:

$$8\pi(\tau\tilde{m}_1(p) + (1-\tau)\tilde{m}_\gamma(p)) = (\tau\tilde{m}_1(p) + (1-\tau)\gamma\tilde{m}_\gamma(p))^2,$$

for every $p \in \mathscr{S}$. In view of (1.5) we have $\tau\tilde{m}_1(p) = m_1(p)$ and $(1-\tau)\tilde{m}_\gamma(p) = \sigma m_\gamma(p)$. Hence, we derive the asserted identity. Alternatively, we may derive identity (2.7) by applying the Pohozaev identity in a standard way. □

Lemma 2.2. *Let u_n be a solution sequence for* (1.1). *For any $\gamma \in [-1,1]$ we have*

$$\int_\Omega e^{\gamma u_n} \geq c_0 > 0.$$

Proof. If $\gamma > 0$, we have $u_n \geq 0$ in Ω by the maximum principle and therefore

$$\int_\Omega e^{\gamma u_n} \geq |\Omega| > 0.$$

Therefore, we assume $\gamma < 0$. We note that since $\|u_n\|_{W_0^{1,q}(\Omega)} \leq C$ for any $q \in [1,2)$, there exists $u_0 \in W_0^{1,q}(\Omega)$ such that $u_n \to u_0$ weakly in $W_0^{1,q}(\Omega)$, strongly in $L^p(\Omega)$ for any $p \geq 1$ and a.e. in Ω. In view of Fatou's lemma, we derive

$$\liminf_{n\to\infty} \int_\Omega e^{\gamma u_n} \geq \int_\Omega e^{\gamma u_0} > 0. \qquad \square$$

Proposition 2.2 (Mass quantization). *Let (u_n, λ_n) be a solution sequence for* (1.1) *with $\lambda_n \to \lambda_0$. Assume that $|\gamma| < 1/2$ and $\sigma \in (0, \sigma_\gamma)$, where σ_γ is defined in* (1.8). *Moreover, assume that*

$$8\pi < \lambda_0 < \frac{4\pi}{|\gamma|(1+|\gamma|\sigma)}. \tag{2.8}$$

Then, we have $m_\gamma(p) = 0$ and consequently $m_1(p) = 8\pi$, $r_1 \equiv 0$ and $\lambda_0 \in 8\pi\mathbb{N}$.

Proof. Throughout this proof we omit the subscript n. Similarly as above, in view of (2.3) with $f = \lambda\left(\frac{e^u}{\int_\Omega e^u} + \sigma\gamma\frac{e^{\gamma u}}{\int_\Omega e^{\gamma u}}\right)$, $\|f\|_{L^1(\Omega)} \leq \lambda(1+\sigma|\gamma|)$, we have that $\|e^{\gamma u}\|_{L^q(\Omega)}$ is bounded if $1 < q < 4\pi/[\lambda|\gamma|(1+\sigma|\gamma|)]$. The existence of such a $q > 1$ follows from (2.8). Moreover, since by assumption we have $\lambda > 8\pi$, we derive that necessarily

$$8\pi < \frac{4\pi}{|\gamma|(1+\sigma|\gamma|)}.$$

This inequality holds in view of the assumption $\sigma \in (0, \sigma_\gamma)$. Therefore we have that $1 < 4\pi/[\lambda|\gamma|(1+\sigma|\gamma|)]$ and $\|e^{\gamma u}\|_{L^q(\Omega)}$ is bounded for some $q > 1$.

Hence, $m_\gamma(p) = 0$. Now (2.7) implies $m_1(p) = 8\pi$. We decompose $u = w_1 + w_2$, with

$$\begin{cases} -\Delta w_1 = \lambda \dfrac{e^u}{\int_\Omega e^u} & \text{in } \Omega \\ w_1 = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta w_2 = \sigma\gamma\lambda \dfrac{e^{\gamma u}}{\int_\Omega e^{\gamma u}} & \text{in } \Omega \\ w_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, setting

$$\phi = \sigma\gamma\lambda \frac{e^{\gamma u}}{\int_\Omega e^{\gamma u}}$$

we have $\|\phi\|_{L^q(\Omega)} \leq C$ for some $q > 1$ and therefore $\|w_2\|_{L^\infty(\Omega)} \leq C$. It follows that $e^u = he^{w_1}$ with $h = e^{w_2} \geq e^{\inf_\Omega w_2} \geq e^{-C(\lambda)} > 0$. Moreover,

$$w_1 \to G * (m_1(p)\delta_p + r_1) = 4 \log \frac{1}{|x-p|} + \omega + G * r_1$$

with ω smooth in the closure of a neighbourhood U of p. Therefore, by Fatou's lemma:

$$\liminf \int_\Omega e^u \geq \int_\Omega \liminf e^u \geq e^{-C(\lambda)} \int_U e^\omega \frac{dx}{|x-p|^4} = +\infty.$$

Hence $r_1 \equiv 0$ since u is locally uniformly bounded in $\Omega \backslash \mathscr{S}$.

Now, the first equation in (2.6) implies the mass quantization $\lambda \in 8\pi\mathbb{N}$. □

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1 by suitably adapting a variational argument due to [7]. The variational functional for Problem (1.1) is given by:

$$J_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \lambda \ln \int_\Omega e^u\, dx - \lambda\sigma \ln \int_\Omega e^{\gamma u}\, dx.$$

3.1. *An improved Moser-Trudinger inequality*

We observe that the standard well-known improved Moser-Trudinger inequality [5] readily implies an improved inequality for J_λ. For any fixed $a_0, d_0 > 0$ we consider the set

$$\mathscr{A}_{a_0,d_0} := \left\{ u \in H_0^1(\Omega) : \exists \Omega_1, \Omega_2 \subset \Omega \text{ s.t. } \begin{array}{l} \text{(i) } \mathrm{dist}(\Omega_1, \Omega_2) \geq d_0 \\ \text{(ii) } \dfrac{\int_{\Omega_i} e^u \, dx}{\int_\Omega e^u \, dx} \geq a_0, \ i = 1, 2 \end{array} \right\}.$$

LEMMA 3.1 (Improved Moser-Trudinger Inequality). *The functional J_λ is bounded from below on $\mathscr{A}_{a_0,d_0}$ if*

$$\lambda < \frac{16\pi}{1 + 2\sigma\gamma^2}. \tag{3.1}$$

PROOF. For $\epsilon \in (0, 1)$ to be fixed later we decompose:

$$\begin{aligned} J_\lambda(u) &= (1-\epsilon)\left\{\frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \frac{\lambda}{1-\epsilon} \ln \int_\Omega e^u \right\} \\ &\quad + \frac{\epsilon}{\gamma^2}\left\{\frac{1}{2}\int_\Omega |\nabla \gamma u|^2 \, dx - \frac{\lambda\sigma\gamma^2}{\epsilon} \ln \int_\Omega e^{\gamma u} \right\} \\ &:= K^1(u) + K^\gamma(u) \end{aligned} \tag{3.2}$$

In view of the Improved Moser-Trudinger inequality, the functional K^1 is bounded below on $\mathscr{A}_{a_0,d_0}$ if

$$\frac{\lambda}{1-\epsilon} < 16\pi. \tag{3.3}$$

On the other hand, the functional K^γ is bounded below on $H_0^1(\Omega)$ if

$$\frac{\lambda\sigma\gamma^2}{\epsilon} \leq 8\pi. \tag{3.4}$$

Considering (3.3) and (3.4) we can take a suitable $\epsilon \in (0, 1)$ satisfying

$$\epsilon < 1 - \frac{\lambda}{16\pi} \quad \text{and} \quad \epsilon \geq \frac{\lambda\sigma\gamma^2}{8\pi}$$

if

$$\frac{\lambda\sigma\gamma^2}{8\pi} < 1 - \frac{\lambda}{16\pi}$$

which is equivalent to (3.1). □

REMARK 3.1. Actually, we expect boundedness below of J_λ for all $\lambda \in (8\pi, 16\pi)$.

LEMMA 3.2. *For every $0 < |\gamma| < 1/2$ and for every $0 < \sigma < (1 - 2|\gamma|)/(2\gamma^2) = \sigma_\gamma$, the functional J_λ is bounded below on $H_0^1(\Omega)$ if and only if $\lambda \le 8\pi$.*

PROOF. We rewrite

$$(3.5)\qquad J_\lambda(u) = \tilde{J}_{\tilde{\lambda}}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \tilde{\lambda}\tau \ln \int_\Omega e^u\,dx - \tilde{\lambda}(1-\tau)\ln\int_\Omega e^{\gamma u}\,dx.$$

where:

$$\sigma = \frac{1-\tau}{\tau} \quad \text{and} \quad \tilde{\lambda} = \frac{\lambda}{\tau} \quad \tau \in (0,1].$$

We use a result from [21] for the functionals of the form

$$J_{\tilde{\lambda}}^{\mathscr{P}}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \tilde{\lambda}\int_I \left(\log\int_\Omega e^{\alpha u}\,dx\right)\mathscr{P}(d\alpha),$$

$u \in H_0^1(\Omega)$. Note that $J_{\tilde{\lambda}}^{\mathscr{P}}$ is the Euler-Lagrange funtional for problem (1.2). In view of Theorem 4 in [21] (see also [28]) we have that $\tilde{J}_{\tilde{\lambda}}$ is bounded below if and only if $\tilde{\lambda} \le \bar{\lambda}^{\mathscr{P}}$ where

$$\bar{\lambda}^{\mathscr{P}} = 8\pi \inf\left\{\frac{\mathscr{P}(K_\pm)}{(\int_{K_\pm}\alpha\mathscr{P}(d\alpha))^2}, K_\pm \subset I_\pm \cap \operatorname{supp}\mathscr{P}\right\},$$

$I_+ = [0,1]$, $I_- = [-1,0)$ and $\mathscr{P} = \mathscr{P}_\gamma$ is defined by (1.3), i.e., $\mathscr{P}_\gamma(d\alpha) = \tau\delta_1(d\alpha) + (1-\tau)\delta_\gamma(d\alpha)$.

Assume $\gamma \ge 0$. In this case, we have

$$\tau\frac{\mathscr{P}_\gamma(K)}{(\int_K \alpha\mathscr{P}_\gamma(d\alpha))^2} = \begin{cases} 1, & \text{if } K = \{1\} \\ \frac{\tau}{\gamma^2(1-\tau)} = \frac{1}{\sigma\gamma^2}, & \text{if } K = \{\gamma\} \\ \frac{\tau}{(\tau+\gamma(1-\tau))^2} = \frac{1+\sigma}{(1+\sigma\gamma)^2}, & \text{if } K = \{\gamma, 1\}. \end{cases}$$

Hence, if $\gamma > 0$ we have $\tau\bar{\lambda}^{\mathscr{P}} = 8\pi$ whenever $0 < \sigma \le \frac{1-2\gamma}{2\gamma^2} = \sigma_\gamma$.

Analogously, for $\gamma < 0$ we have

$$\tau\frac{\mathscr{P}_\gamma(K)}{(\int_K \alpha\mathscr{P}_\gamma(d\alpha))^2} = \begin{cases} 1, & \text{if } K = \{1\} \\ \frac{\tau}{\gamma^2(1-\tau)} = \frac{1}{\sigma\gamma^2}, & \text{if } K = \{\gamma\}. \end{cases}$$

Hence, if $\gamma < 0$ we have that $\tau\bar{\lambda}^{\mathscr{P}} = 8\pi$ if $0 < \sigma < \sigma_\gamma$. □

LEMMA 3.3. *Let $0 < |\gamma| < 1/2$ and let $0 < \sigma < \sigma_\gamma$, where σ_γ is defined in* (1.8). *Then, we have $8\pi < \lambda_{\gamma,\sigma} \le 16\pi$, where $\lambda_{\gamma,\sigma}$ is defined in* (1.9).

PROOF. The upper bound is clear. Therefore, we only prove the lower bound $\lambda_{\gamma,\sigma} > 8\pi$. We readily check that

$$\frac{16\pi}{1+2\sigma\gamma^2} > 8\pi \quad \text{if and only if} \quad \sigma < \frac{1}{2\gamma^2}$$

and

$$\frac{4\pi}{|\gamma|(1+\sigma|\gamma|)} > 8\pi \quad \text{if and only if} \quad \sigma < \sigma_\gamma = \frac{1-2|\gamma|}{2\gamma^2} < \frac{1}{2\gamma^2}.$$

The claim follows. □

For every $u \in H_0^1(\Omega)$ we consider the measure:

$$\mu_u = \frac{e^u}{\int_\Omega e^u\,dx}dx \in \mathscr{M}(\Omega)$$

and the corresponding "center of mass":

$$\bar{x}_\mu(u) = \int_\Omega x\,d\mu_u \in \mathbb{R}^2.$$

LEMMA 3.4. *Let $\lambda \in \left(8\pi, \frac{16\pi}{1+2\sigma\gamma^2}\right)$ and let $\{u_n\} \subset H_0^1(\Omega)$ be a sequence such that $J_\lambda(u_n) \to -\infty$ and $\bar{x}_\mu(u_n) \to x_0 \in \mathbb{R}^2$. Then, $x_0 \in \overline{\Omega}$ and*

$$\mu_{u_n} \rightharpoonup \delta_{x_0} \quad \textit{weakly* in } \mathscr{C}(\overline{\Omega})'.$$

PROOF. For every fixed $r > 0$ we denote by $\mathscr{Q}_n(r)$ the concentration function of μ_n, i.e.

$$\mathscr{Q}_n(r) = \sup_{x\in\Omega}\int_{B(x,r)\cap\Omega}\mu_n.$$

For every n, there exists $\tilde{x}_n \in \overline{\Omega}$ such that

$$\mathscr{Q}_n(r/2) = \int_{B(\tilde{x}_n,r/2)\cap\Omega}\mu_n.$$

Upon taking a subsequence, we have that $\tilde{x}_n \to \tilde{x}_0 \in \overline{\Omega}$.

Now, let us set

$$\Omega_1^n = B(\tilde{x}_n, r/2) \cap \Omega \quad \text{and} \quad \Omega_2^n = \Omega \backslash B(\tilde{x}_n, r),$$

so that

$$\operatorname{dist}(\Omega_1^n, \Omega_2^n) \geq r/2.$$

Since $J_\lambda(u_n) \to -\infty$ and since $\lambda < \frac{16\pi}{1+2\sigma\gamma^2}$, in view of (3.2) necessarily we have $K^1(u_n) \to -\infty$. Therefore, in view of the standard Improved Moser-Trudinger inequality [5], we conclude that

$$\min\{\mu_n(\Omega_1^n), \mu_n(\Omega_2^n)\} \to 0.$$

In particular, $\min\{\mathcal{Q}_n(r/2), 1 - \mathcal{Q}_n(r)\} \le \min(\mu_n(\Omega_1^n), \mu_n(\Omega_2^n)) \to 0$.

On the other hand, for every fixed $r > 0$ let $k_r \in \mathbb{N}$ be such that Ω is covered by k_r balls of radius $r/2$. Then, $1 = \mu_n(\Omega) \le k_r Q_n(r/2)$, so that $Q_n(r/2) \ge k_r^{-1}$ for every n. We conclude that necessarily $\mathcal{Q}_n(r) \to 1$ as $n \to \infty$. Since $r > 0$ is arbitrary, we derive in turn that $1 - \mathcal{Q}_n(r/2) = \mu_n(\Omega \backslash B(\tilde{x}_n, r/2)) \to 0$ as $n \to \infty$. That is, $\mu_{u_n} \rightharpoonup \delta_{\tilde{x}_0}$. It follows that $\bar{x}_\mu(u_n) = \displaystyle\int_\Omega x\, d\mu_n \to \tilde{x}_0 = x_0 \in \bar{\Omega}$, as asserted. □

At this point, in order to prove Theorem 1.1, we shall adapt a construction in [7]. Let $\Gamma_1 \subset \Omega$ be a non-contractible curve which exists since Ω is non-simply connected. Let $\mathbb{D} = \{(r,\theta) : 0 \le r < 1, 0 \le \theta < 2\pi\}$ be the unit disc. Define

$$\mathscr{D}_\lambda := \left\{ h \in C(\mathbb{D}, H_0^1(\Omega)) \;\middle|\; \begin{array}{l} \text{(i) } \lim\limits_{r\to 1} \sup\limits_{\theta\in[0,2\pi)} J_\lambda(h(r,\theta)) = -\infty \\ \text{(ii) } \bar{x}_\mu(h(r,\theta)) \text{ can be extended continuously to } \bar{\mathbb{D}} \\ \text{(iii) } \bar{x}_\mu(h(1,\cdot)) \text{ is one-to-one from } \partial\mathbb{D} \text{ onto } \Gamma_1 \end{array} \right\}$$

LEMMA 3.5. *For every $\lambda \in (8\pi, 16\pi)$ the set $\mathscr{D}_\lambda$ is non-empty.*

PROOF. Let $\gamma_1(\theta) : [0, 2\pi) \to \Gamma_1$ be a parametrization of Γ_1 and let $\varepsilon_0 > 0$ be sufficiently small so that $B(\gamma_1(\theta), \varepsilon_0) \subset \Omega$. Let $\varphi_\theta(x) = \varepsilon_0^{-1}(x - \gamma_1(\theta))$ so that $\varphi_\theta(B(\gamma_1(\theta), \varepsilon_0)) = B(0,1)$. We define "truncated Green's function":

$$V_r(X) = \begin{cases} 4\log\dfrac{1}{1-r} & \text{for } X \in B(0, 1-r) \\ 4\log\dfrac{1}{|X|} & \text{for } X \in B(0,1)\backslash B(0,1-r) \end{cases}$$

and

$$v_{r,\theta}(x) = \begin{cases} 0 & \text{for } x \in \Omega\backslash B(\gamma_1(\theta), \varepsilon_0)) \\ V_r(\varphi_\theta(x)) & \text{for } x \in B(\gamma_1(\theta), \varepsilon_0). \end{cases}$$

We set

$$h(r,\theta)(x) = v_{r,\theta}(x), \quad x \in \Omega. \tag{3.6}$$

The function h defined in (3.6) satisfies $h \in \mathscr{D}_\lambda$. To see that h verifies the (i)-condition it is sufficient to note that

$$\int_\Omega e^{\gamma h}\,dx \geq |\Omega| - \pi\varepsilon_0 > 0.$$

Then the claim follows by [7]. □

Define

$$c_\lambda := \inf_{h \in \mathscr{D}_\lambda} \sup_{(r,\theta)\in\mathbb{D}} J_\lambda(h(r,\theta)). \tag{3.7}$$

We shall prove that c_λ defines a critical value for J_λ using the Struwe Monotonicity Trick contained in [26], Proposition 4.1.

In view of Lemma 3.5, we have $c_\lambda < +\infty$.

Lemma 3.6. *For any $\lambda \in (8\pi, \lambda_{\gamma,\sigma})$, $c_\lambda > -\infty$.*

Proof. Denote by B a bounded component of $\mathbb{R}^2 \backslash \Omega$ with at least an interior point and such that Γ_1 encloses B. By continuity and by the (iii)-property defining $\mathscr{D}_\lambda$, we have $\bar{x}_\mu(h(\mathbb{D})) \supset B$ for all $h \in \mathscr{D}_\lambda$. By contradiction, assume that $c_\lambda = -\infty$. Then, there exists a sequence $\{h_n\} \subset \mathscr{D}_\lambda$ such that $\sup_{(r,\theta)\in\mathbb{D}} J_\lambda(h_n(r,\theta)) \to -\infty$. Let x_0 be an interior point of B. For every n we take $(r_n, \theta_n) \in \mathbb{D}$ such that $\bar{x}_\mu(h_n(r_n,\theta_n)) = x_0$. In view of Lemma 3.4, it should be $x_0 \in \mathring{B} \cap \overline{\Omega} = \emptyset$, a contradiction. □

At this point we set

$$\mathscr{G}(u) = \ln \int_\Omega e^u\,dx + \sigma \ln \int_\Omega e^{\gamma u}\,dx \tag{3.8}$$

so that our functional (3.5) takes the form

$$J_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \lambda \mathscr{G}(u).$$

Lemma 3.7. *For $8\pi < \lambda_1 \leq \lambda_2 < 16\pi$, we have $\mathscr{D}_{\lambda_1} \subseteq \mathscr{D}_{\lambda_2}$.*

Proof. It is sufficient to note that whenever $J_\lambda(u) \leq 0$ it is $\mathscr{G}(u) \geq 0$, with $\mathscr{G}$ given by (3.8) and this implies that

$$J_{\lambda_1}(u) \geq J_{\lambda_2}(u) \quad \text{for } 8\pi < \lambda_1 \leq \lambda_2 < 16\pi \quad \text{if } J_{\lambda_1}(u) \leq 0.$$

Hence, $\mathscr{D}_{\lambda_1} \subseteq \mathscr{D}_{\lambda_2}$ for every $8\pi < \lambda_1 \leq \lambda_2 < 16\pi$. □

LEMMA 3.8. *The function* $\mathcal{G} : H_0^1(\Omega) \to \mathbb{R}$ *defined by* (3.8) *satisfies:*

1) $\mathcal{G} \in \mathcal{C}^2(H_0^1(\Omega); \mathbb{R})$
2) $\mathcal{G}'$ *is compact*
3) $\langle \mathcal{G}''(u)\varphi, \varphi \rangle \geq 0$ *for every* $u, \varphi \in H_0^1(\Omega)$, *where* $\langle \cdot , \cdot \rangle$ *is the* L^2*-inner product.*

PROOF. For every $u, \varphi \in H_0^1(\Omega)$ we have:

$$\mathcal{G}'(u)\varphi = \frac{\int_\Omega \varphi e^u \, dx}{\int_\Omega e^u \, dx} + \sigma \frac{\int_\Omega \gamma \varphi e^{\gamma u} \, dx}{\int_\Omega e^{\gamma u} \, dx}$$

and therefore the compactness of $\mathcal{G}'$ follows by the compactness of the Moser-Trudinger embedding. Moreover, for every $u, \varphi \in H_0^1(\Omega)$ we have, using the Schwarz inequality,

$$\begin{aligned}\langle \mathcal{G}''(u)\varphi, \varphi \rangle = {} & \frac{1}{\left(\int_\Omega e^u \, dx\right)^2} \left[\left(\int_\Omega e^u \varphi^2 \, dx \right) \left(\int_\Omega e^u \, dx \right) - \left(\int_\Omega e^u \varphi \, dx \right)^2 \right] \\ & + \frac{\gamma^2 \sigma}{\left(\int_\Omega e^{\gamma u} \, dx\right)^2} \left[\left(\int_\Omega e^{\gamma u} \varphi^2 \, dx \right) \left(\int_\Omega e^{\gamma u} \, dx \right) - \left(\int_\Omega e^{\gamma u} \varphi \, dx \right)^2 \right] \geq 0.\end{aligned}$$

□

Now we are able to prove the following.

PROPOSITION 3.1. *Let* $\sigma > 0$ *and assume that* (1.8) *holds. For almost every* $\lambda \in (8\pi, \lambda_{\gamma,\sigma})$, $c_\lambda > -\infty$ *given by* (3.7) *is a saddle-type critical value for* J_λ.

PROOF OF PROPOSITION 3.1. In view of Lemma 3.8, Lemma 3.5, Lemma 3.6 and Lemma 3.7, we may apply the well known Struwe's monotonicity trick to derive the existence of the desired critical value. See [7] or [26], Proposition 4.1 with $\mathcal{H} = H_0^1(\Omega)$, $V = \mathbb{D}$, $A = -\infty$ and $\mathcal{F}_\lambda = \mathcal{D}_\lambda$. □

PROOF OF THEOREM 1.1 (Completion by blow-up results). We fix $\lambda_0 \in (8\pi, \lambda_{\gamma,\sigma})$. In view of Proposition 3.1 there exists $\lambda_n \to \lambda_0$ such that problem (1.1) with $\lambda = \lambda_n$ admits a solution u_n. By the blow up analysis as stated in Proposition 2.2, we have the compactness of solution sequences. Therefore, up to subsequences, we obtain that $u_n \to u_0$ with u_0 a solution to (1.1) with $\lambda = \lambda_0$. □

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