Rend. Lincei Mat. Appl. 27 (2016), 431–442
DOI 10.4171/RLM/742

Partial Differential Equations — *On an inverse problem in potential theory*, by Giovanni Cupini and Ermanno Lanconelli, communicated on 10 June 2016.

Abstract. — The Newtonian potential of a Euclidean ball B of $\mathbb{R}^n$ centered at x_0 is proportional, outside B, to the Newtonian potential of a mass concentrated at x_0. Vice-versa, as proved by Aharonov, Schiffer and Zalcman, if D is a bounded open set in $\mathbb{R}^n$, containing x_0, whose Newtonian potential is proportional, outside D, to the one of a mass concentrated at x_0, then D is a Euclidean ball with center x_0. In this paper we generalize this last result to more general measures and domains.

Key words: Harmonic function, Newtonian potential, inverse problem

Mathematics Subject Classification (primary; secondary): 31B05, 31B20; 35J05

1. Introduction

The Newtonian potential of a Euclidean ball B of $\mathbb{R}^n$ centered at x_0 is proportional, outside B, to the Newtonian potential of a mass concentrated at x_0. Vice-versa, if D is a bounded, open set in $\mathbb{R}^n$, containing x_0, having Newtonian potential proportional, outside D, to the one of a mass concentrated at x_0, then D is a Euclidean ball with center x_0. The first statement simply follows from the Gauss Mean Value property for the harmonic functions applied to the family of maps

$$B \ni y \mapsto \Gamma(y - x) \in \mathbb{R}, \quad x \notin B,$$

where Γ is the Newtonian kernel; i.e., the fundamental solution of the Laplace operator in $\mathbb{R}^n$. The second assertion is a theorem by Aharonov, Schiffer and Zalcman [2].

Generalization of these two problems to more general sets and measures are the problem (P) and its inverse (IP), described below.

(P) Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 3$, be a bounded open set and let $x_0 \in \Omega$. Does it exist a non-negative Radon measure μ, $\mu(\Omega^c) = 0$, such that

$$\Gamma_\mu(x) = \Gamma(x_0 - x) \quad \forall x \in \Omega^c? \tag{1.1}$$

Here Γ_μ denotes the Newtonian potential of μ; i.e.,

$$\Gamma_\mu : \mathbb{R}^n \to [0, \infty], \quad \Gamma_\mu(x) := \int_{\mathbb{R}^n} \Gamma(y - x)\, d\mu(y).$$

We will say that (Ω, μ, x_0) is a Γ-*triple* if Ω is a bounded open subset of $\mathbb{R}^n$, $x_0 \in \Omega$, μ is a non-negative Radon measure, $\mu(\Omega^c) = 0$, and (1.1) holds. Thus, (P) can be rephrased as follows: does it exist μ such that (Ω, μ, x_0) is a Γ-*triple*? Of course, a trivial solution is $\mu = \delta_{x_0}$, the Dirac measure at x_0. Not trivial solutions of (P) are basically given by the *volume densities with the mean value property* for harmonic functions constructed by Hansen-Netuka [9] and Aikawa [3], [4].

An inverse of (P) is:

(IP) Let (Ω, μ, x_0) and (D, ν, x_0) be Γ-triples, with $\mu \llcorner (\Omega \cap D) = \nu \llcorner (\Omega \cap D)$. Is it true that $\Omega = D$ and $\mu = \nu$?

In Section 2 we give an answer to this question, by using a variant of the notion of Γ-triple: we say that (Ω, μ, x_0) is a *strong* Γ-*triple* if (Ω, μ, x_0) is a Γ-triple,

$$\Gamma_\mu(x) < \Gamma(x_0 - x) \quad \forall x \in \Omega, \tag{1.2}$$

and

$$\Gamma_\mu \text{ is a real continuous function in } \mathbb{R}^n.$$

Our answer to question (IP) is the following theorem, the main result of this paper.

Theorem 1.1. *Let Ω and D be open bounded sets in $\mathbb{R}^n$ containing x_0. Assume that*

(i) (Ω, μ, x_0) *is a strong* Γ-*triple,*
(ii) (D, ν, x_0) *is a* Γ-*triple,*
(iii) $\mu \llcorner (\Omega \cap D) = \nu \llcorner (\Omega \cap D)$,
(iv) $\partial D \subseteq \operatorname{supp} \nu$,
(v) Ω *is a solid set.*

Then $D = \Omega$ and $\nu = \mu$.

The proof relies on the comparison between the potentials Γ_μ and Γ_ν and it is based on the weak and strong maximum principles for subharmonic functions.

We will exhibit examples to show that in Theorem 1.1 neither (iii) nor (iv) can be removed, see Examples 2.1 and 2.2. Moreover, the hypothesis that (Ω, μ, x_0) is a strong Γ-triple cannot be weakened by assuming that (Ω, μ, x_0) is simply a Γ-triple, see Example 2.3.

We notice that strong Γ-triples can be naturally defined on the Euclidean balls; indeed, by the Gauss Mean Value Theorem and by the Poisson-Jensen formula, it follows that

$$\left(B_r(x_0), \frac{1}{m(B_r(x_0))} m \llcorner B_r(x_0), x_0\right) \text{ is a strong } \Gamma\text{-triple.} \tag{1.3}$$

In Section 3 we will show that strong Γ-triples can be defined on every bounded, smooth and strongly star-shaped domain. Indeed, let d be any smooth homogeneous norm in $\mathbb{R}^n$ and denote $B_r^d(x_0)$ the d-balls of radius r centered at x_0; i.e.,

$$B_r^d(x_0) := \{y \in \mathbb{R}^n : d(y - x_0) < r\}.$$

Notice that every d-ball $B_r^d(x_0)$ is a smooth, strictly star-shaped domain with respect to x_0 and, vice-versa, every smooth, strictly star-shaped domain is a d-ball for a suitable homogeneous norm d.

Define

$$P_d : \partial B_1^d(0) \to \mathbb{R}, \quad P_d = -\frac{\partial G}{\partial \nu},$$

where G is the Green function of $B_1^d(0)$ with pole at the origin and ν is the outward normal to $B_1^d(0)$; moreover we let

$$m_d(y) := |\nabla d(y)| P_d\Big(\frac{y}{d(y)}\Big), \quad y \neq 0.$$

The following result holds.

Theorem 1.2. *Let $B_r^d(x_0)$ be a d-ball in $\mathbb{R}^n$ and define*

$$w_\alpha(y) := \frac{\alpha}{r^\alpha} \frac{m_d(y - x_0)}{(d(y - x_0))^{n-\alpha}}, \quad y \in B_r^d(x_0) \backslash \{x_0\}.$$

Let μ_α be the measure

$$\mu_\alpha := w_\alpha m \llcorner B_r^d(x_0). \tag{1.4}$$

If $\alpha > n - 2$, then $(B_r^d(x_0), \mu_\alpha, x_0)$ is a strong Γ-triple.

Moreover, if $\alpha > 0$, then μ_α is a measure with the mean value property for non-negative harmonic functions; i.e.,

$$u(x_0) = \int_{B_r^d(x_0)} u(y)\, d\mu_\alpha(y), \quad \forall u \in \mathscr{H}(B_r^d(x_0)),\ u \geq 0. \tag{1.5}$$

Notice that if d is the Euclidean norm then

$$m_d(y) = \frac{1}{n\omega_n} \quad \text{and} \quad w_\alpha(y) = \frac{\alpha}{n\omega_n r^\alpha} \frac{1}{|y - x_0|^{n-\alpha}};$$

thus, if $\alpha = n$, $\mu_n = \frac{1}{m(B_r(x_0))} m \llcorner B_r(x_0)$. Therefore (1.3) and the classical Gauss Mean Value Theorem for harmonic functions are particular cases of Theorem 1.2.

Theorem 1.1, together with Theorem 1.2, gives the following *d-spherical* symmetry result.

Theorem 1.3. *Let $D \subset \mathbb{R}^n$ be an open bounded set and $x_0 \in D$. Assume that for $\alpha > n-2$ and $c > 0$*

$$\Gamma(x_0 - x) = c \int_D \Gamma(y - x) \frac{m_d(y - x_0)}{(d(y - x_0))^{n-\alpha}} dy, \quad \forall x \notin D. \tag{1.6}$$

Then $c = \left(\int_D \frac{m_d(y-x_0)}{(d(y-x_0))^{n-\alpha}} dy \right)^{-1}$ and $D = B_r^d(x_0)$ with $r = \left(\frac{\alpha}{c}\right)^{\frac{1}{\alpha}}$.

Note that if $\alpha = n$ and d is the Euclidean norm, then Theorem 1.3 is Aharonov-Schiffer-Zalcman's Theorem in [2] quoted above. Actually, in the particular case of the Euclidean norm, in [2, Sect. 5] it is proved an analogous of Theorem 1.3 with more general radial densities.

From Theorem 1.3 we immediately obtain the following harmonic characterization of the d-balls.

Corollary 1.4. *Let $D \subset \mathbb{R}^n$ be an open bounded set and $x_0 \in D$. Assume that for $\alpha > n-2$ and $c > 0$*

$$u(x_0) = c \int_D u(y) \frac{m_d(y - x_0)}{(d(y - x_0))^{n-\alpha}} dy, \quad \forall u \in \mathscr{H}(B_r^d(x_0)),\ u \geq 0. \tag{1.7}$$

Then $c = \left(\int_D \frac{m_d(y-x_0)}{(d(y-x_0))^{n-\alpha}} dy \right)^{-1}$ and $D = B_r^d(x_0)$ with $r = \left(\frac{\alpha}{c}\right)^{\frac{1}{\alpha}}$.

Indeed, if $x \notin D$, the function $y \mapsto \Gamma(y - x)$ is non-negative and harmonic in D; therefore (1.7) implies (1.6). In the case $\alpha = n$ and d the Euclidean norm, this corollary gives a harmonic characterization of the Euclidean ball, a problem with a very long history, see Epstein [6], Epstein-Schiffer [7], Kuran [11].

We now describe the organization of our paper. Section 2 is devoted to the proof of Theorem 1.1 and to exhibit Examples 2.1–2.3. In Section 3 we prove Theorems 1.2 and 1.3.

All our results are based on general facts and tools in potential theory; therefore their generalization to general elliptic, parabolic and sub-elliptic settings seems possible. We plan to investigate this issue in forthcoming papers.

We note that the Aharonov-Schiffer-Zalcman's Theorem in [2] has been yet generalized to particular sub-elliptic settings in [12] and [1].

2. Proof of Theorem 1.1

Aim of this section is to prove Theorem 1.1. To this end, we recall that the support of a measure μ can be defined as follows:

$$\operatorname{supp} \mu := \{x \in \mathbb{R}^n : (A \text{ open set},\ x \in A) \Rightarrow \mu(A) > 0\}.$$

Proof of Theorem 1.1. We split the proof in several steps.

Step 1. We claim that $\Gamma_\mu \le \Gamma_\nu$ in $\mathbb{R}^n$.
We first prove that $\Gamma_\mu \le \Gamma_\nu$ in $\mathbb{R}^n \setminus D$.
By (i) and (ii) we have that

$$\Gamma_\mu(x) = \Gamma_\nu(x) = \Gamma(x_0 - x) \quad \forall x \in (D \cup \Omega)^c. \tag{2.1}$$

Since (D, ν, x_0) is a Γ-triple and (Ω, μ, x_0) is a strong Γ-triple, then

$$\Gamma_\mu(x) \le \Gamma(x_0 - x) = \Gamma_\nu(x) \quad \forall x \in \mathbb{R}^n \setminus D.$$

In particular, the above inequality holds on ∂D.
Let us prove that $\Gamma_\mu \le \Gamma_\nu$ in D.
By (iii) and $\mu = 0$ in Ω^c we get that, in a distributional sense,

$$\Delta(\Gamma_\mu - \Gamma_\nu) \ge 0 \quad \text{in } D.$$

By [5, Theorem 8.2.11], $\Gamma_\mu - \Gamma_\nu$ is a subharmonic function in D.
Since Γ_ν is lower semicontinuous and, by assumption (i), Γ_μ is continuous, then

$$\limsup_{y \to x} (\Gamma_\mu - \Gamma_\nu)(y) \le \Gamma_\mu(x) - \Gamma_\nu(x) \le 0 \quad \forall x \in \partial D.$$

By the maximum principle for subharmonic functions (see [5, Theorem 8.2.19 (ii)]) we get $\Gamma_\mu \le \Gamma_\nu$ in D. This concludes the proof of the claim. In particular, we have proved that $\Gamma_\mu \le \Gamma_\nu$ on $\partial\Omega$.

Step 2. Let us prove that $\partial D \setminus \overline{\Omega}$ is empty.
By contradiction, assume that there exists $x \in \partial D \setminus \overline{\Omega}$. Then there exists $r > 0$ such that $B_r(x) \subseteq \mathbb{R}^n \setminus \overline{\Omega}$. By definition of Γ-triple $\mu = 0$ in Ω^c; therefore $\mu(B_r(x)) = 0$. By assumption (iv) we get

$$\nu(B_r(x)) > 0. \tag{2.2}$$

On the other hand, in a distributional sense,

$$\Delta(\Gamma_\mu - \Gamma_\nu) = \nu \ge 0 \quad \text{in } B_r(x); \tag{2.3}$$

i.e., $\Gamma_\mu - \Gamma_\nu$ is a subharmonic function in $B_r(x)$.
By what previously proved in Step 1, $\Gamma_\mu - \Gamma_\nu \le 0$ in $B_r(x)$. Moreover, since $\partial D \setminus \overline{\Omega} \subseteq (\overline{\Omega} \cup D)^c$, by (2.1) we have that $\Gamma_\mu(x) - \Gamma_\nu(x) = 0$. Therefore, by the strong maximum principle for subharmonic functions (see in [5, Theorem 8.2.19 (i)]), $\Gamma_\mu - \Gamma_\nu = 0$ in $B_r(x)$. This implies $\Delta(\Gamma_\mu - \Gamma_\nu) = 0$ in $B_r(x)$, that is, by (2.3), $\nu(B_r(x)) = 0$. This is in contradiction with (2.2).

STEP 3. In this step we prove that $D \subseteq \Omega$.

We have

$$\mathbb{R}^n \backslash \overline{\Omega} = (D \cup D^c) \backslash \overline{\Omega} = (D \backslash \overline{\Omega}) \cup (\partial D \backslash \overline{\Omega}) \cup (\overline{D}^c \cap \overline{\Omega}^c) = (D \backslash \overline{\Omega}) \cup (\overline{D} \cup \overline{\Omega})^c.$$

By the boundedness of Ω and D, $(\overline{D} \cup \overline{\Omega})^c$ is not empty. Moreover, $D \backslash \overline{\Omega}$ and $(\overline{D} \cup \overline{\Omega})^c$ are open, disjoint sets. The set $\mathbb{R}^n \backslash \overline{\Omega}$ is connected by (v), then $D \backslash \overline{\Omega}$ must be empty. Therefore $D \subseteq \overline{\Omega}$. By (v) we have that $\operatorname{int} \overline{\Omega} = \Omega$, thus we obtain $D \subseteq \Omega$.

STEP 4. In this step we prove that $\Omega \subseteq D$.

We argue by contradiction; i.e., we assume that there exists $x \in \Omega \backslash D$. By Steps 1 and 3, $\Gamma_\mu \leq \Gamma_\nu$ and $D \subseteq \Omega$. Therefore, by (i), (iii) and (ii), we have

$$\Gamma(x_0 - x) > \Gamma_\mu(x) = \int_D \Gamma(y - x)\, d\mu(y) + \int_{\Omega \backslash D} \Gamma(y - x)\, d\mu(y)$$

$$\geq \int_D \Gamma(y - x)\, d\nu(y) = \Gamma_\nu(x) = \Gamma(x_0 - x).$$

This is an absurd.

We have so proved that $D = \Omega$ and, consequently, that $\mu = \nu$. □

Let us now exhibit examples to show that neither (iii) nor (iv) can be removed in Theorem 1.1.

EXAMPLE 2.1. *Assumption* (iii) *in Theorem* 1.1 *cannot be removed.*

For instance, if Ω *is the ball* $B_r(x_0)$ *and* D *is the ball* $B_{r'}(x_0)$, *with* $r \neq r'$, *and* $\mu = \frac{1}{m(B_r(x_0))} m \llcorner B_r(x_0)$ *and* $\nu = \frac{1}{m(B_{r'}(x_0))} m \llcorner B_{r'}(x_0)$ *then*

$$\mu \llcorner (\Omega \cap D) \neq \nu \llcorner (\Omega \cap D).$$

It is easy to prove that all the other assumptions of the theorem are satisfied. Indeed, (Ω, μ, x_0) *and* (D, ν, x_0) *are strong* Γ*-triples* (*see* (1.3)), $\operatorname{supp} \nu = \overline{D}$, *and* Ω *is a solid set.*

EXAMPLE 2.2. *Assumption* (iv) *in Theorem* 1.1 *cannot be removed.*

Indeed, consider $\Omega = B_r(x_0)$, $D = B_R(x_0)$, *with* $0 < r < R$. *Define* $\mu = \nu = \frac{1}{m(B_r(x_0))} m \llcorner B_r(x_0)$. *By* (1.3), (Ω, μ, x_0) *is a strong* Γ*-triple. Moreover, for every* $x \in D^c \subset \Omega^c$

$$\Gamma(x_0 - x) = \Gamma_\mu(x) = \frac{1}{m(B_r(x_0))} \int_{B_r(x_0)} \Gamma(y - x)\, dm(y)$$

$$= \int_{B_R(x_0)} \Gamma(y - x)\, d\nu(y) = \Gamma_\nu(x),$$

which implies that (D, ν, x_0) *is a* Γ*-triple. Of course* $\operatorname{supp}(\nu) = \overline{B_r(x_0)}$, *therefore* $\partial D \not\subseteq \operatorname{supp}(\nu)$.

In the next example we show that in (i) of Theorem 1.1 the assumption that the Γ-triple (Ω, μ, x_0) is strong cannot be removed.

Example 2.3. *Assumption* (i) *in Theorem* 1.1 *cannot be weakened asking* (Ω, μ, x_0) *be a* Γ*-triple.*

Indeed, consider $\Omega = B_R(x_0)$, $D = B_r(x_0)$, *with* $0 < r < R$. *Define* $\mu = \nu = \frac{1}{m(B_r(x_0))} m \llcorner B_r(x_0)$. *By* (1.3), (D, ν, x_0) *is a strong* Γ*-triple and* (Ω, μ, x_0) *is a* Γ*-triple. Moreover, for every* $x \in \Omega \cap D^c$

$$\Gamma(x_0 - x) = \Gamma_\nu(x) = \frac{1}{m(B_r(x_0))} \int_{B_r(x_0)} \Gamma(y - x)\, dm(y)$$

$$= \int_{B_R(x_0)} \Gamma(y - x)\, d\mu(y) = \Gamma_\mu(x),$$

which implies that (Ω, μ, x_0) *is not a strong* Γ*-triple. All the other assumptions* (ii)–(v) *hold true.*

3. Proof of Theorems 1.2 and 1.3

In this section we will exhibit strong Γ-triples on every d-ball

$$B_r^d(x_0) := \{y \in \mathbb{R}^n : d(y - x_0) < r\},$$

where d is any smooth homogeneous norm. Moreover, we will also prove the *d-spherical* symmetry result Theorem 1.3.

In the following we call Δ-*regular* every bounded open set for which the classical Dirichlet problem is solvable for any continuous boundary data.

To prove Theorem 1.2 we use an argument similar to the one in Aikawa's Theorem in [3].

Let Ω be a connected bounded open subset of $\mathbb{R}^n$, $n \geq 3$, and assume that there exists a family of open sets $(\Omega_t)_{0<t<T}$, $0 < T \leq \infty$, such that

(i) $\Omega = \bigcup_{0<t<T} \Omega_t$,
(ii) $\overline{\Omega_t} \subseteq \Omega_\tau$ if $0 < t < \tau < T$,
(iii) Ω_t is connected and Δ-regular for a.e. $t \in]0, T[$.

Let x_0 be a fixed point in Ω. For every non-negative and superharmonic function u in Ω we define

$$m_t(u)(x_0) := \int_{\partial\Omega_t} u(y)\, d\mu_{x_0}^{\Omega_t}(y) \tag{3.1}$$

where $\mu_{x_0}^{\Omega_t}$ denotes the *harmonic measure* of Ω_t at x_0.

The following result holds.

Lemma 3.1. *Let u be a superharmonic and non-negative function in Ω.*
Let $\varphi :]0, T[\to]0, \infty[$ be measurable and such that

$$\int_0^T \varphi(t)\,dt = 1. \tag{3.2}$$

Define

$$M(u)(x_0) := \int_0^T \varphi(t) m_t(u)(x_0)\,dt. \tag{3.3}$$

Then

(a) $u(x_0) \geq M(u)(x_0)$,
(b) $u(x_0) = M(u)(x_0)$ *if u is harmonic in Ω,*
(c) $u(x_0) > M(u)(x_0)$ *if $u(x_0) < \infty$ and $\Delta u \neq 0$ in Ω.*

Proof. If Ω_t is Δ-regular, we set

$$n_t(u)(x_0) := \int_{\Omega_t} G_{\Omega_t}(x_0, y)\,d\nu_u(y), \tag{3.4}$$

where $G_{\Omega_t}(x_0, \cdot)$ stands for the Green function of Ω_t with pole at x_0, and ν_u is the *Riesz measure* of u; i.e.,

$$\nu_u := -\Delta u \text{ in the weak sense of distributions.}$$

By Poisson-Jensen formula (see e.g. [10, Theorem 5.27], see also [5, Theorem 9.5.1]) and the assumptions on $(\Omega_t)_{0<t<T}$, we have

$$u(x_0) = m_t(u)(x_0) + n_t(u)(x_0) \quad \text{for a.e. } t \in]0, T[. \tag{3.5}$$

Since u is non-negative, then $m_t(u)(x_0) \geq 0$. Moreover, since $\Omega_t \subseteq \Omega_\tau$ if $t \leq \tau$, and $\nu_u \geq 0$, the function $t \mapsto n_t(u)$ is increasing and non-negative. By (3.5) and (3.2) we get

$$\begin{aligned} u(x_0) &= \int_0^T \varphi(t) m_t(u)(x_0)\,dt + \int_0^T \varphi(t) n_t(u)(x_0)\,dt \\ &=: M(u)(x_0) + N(u)(x_0). \end{aligned} \tag{3.6}$$

Since $N(u)(x_0) \geq 0$ and (3.6) hold, then (a) follows.

If u is harmonic in Ω then $N(u)(x_0) = 0$ and, by (3.6), (b) follows. Moreover, if $\Delta u \neq 0$ in Ω then $\nu_u \neq 0$ in Ω. Therefore, there exists $t_0 > 0$ such that $\nu_u(\Omega_{t_0}) > 0$. On the other hand $\nu_u(\Omega_t) \geq \nu_u(\Omega_{t_0})$ if $t \geq t_0$, and $G_{\Omega_t}(x_0, \cdot) > 0$ since Ω_t is connected. Then

$$n_t(u)(x_0) := \int_{\Omega_t} G_{\Omega_t}(x_0, y)\,d\nu_u(y) > 0 \quad \forall t \geq t_0,$$

so that

$$N(u)(x_0) := \int_0^T \varphi(t) n_t(u)(x_0)\, dt > 0.$$

Using this information in (3.6) together with the assumption $u(x_0) \in \mathbb{R}$, we immediately get (c). □

With the lemma above we can prove Theorem 1.2.

Proof of Theorem 1.2. We let $\Omega := B_r^d(x_0)$ and

$$\Omega_t := B_t^d(x_0) \quad 0 < t < r.$$

Then $(\Omega_t)_{0<t<r}$ satisfies conditions (i)–(iii) above. It is a standard fact that the measure $\mu_0^{B_1^d(0)}$ (see (3.1)) is such that

$$d\mu_0^{B_1^d(0)}(y) := P_d(y)\, d\sigma(y),$$

where

$$P_d : \partial B_1^d(0) \to \mathbb{R}, \quad P_d(y) := -\frac{\partial G}{\partial \nu}(0, y)$$

with $G(0, \cdot)$ the Green function of $B_1^d(0)$ with pole at 0, and ν the outward normal.

Then, since Δ is left translation invariant and homogeneous of degree two w.r.t. the dilation $y \mapsto \lambda y$, one has

$$d\mu_{x_0}^{B_t^d(x_0)}(y) := \frac{1}{t^{n-1}} P_d\Big(\frac{y - x_0}{t}\Big)\, d\sigma(y).$$

For $\alpha > 0$ the function

$$\varphi_\alpha :]0, r[\to]0, \infty[, \quad \varphi_\alpha(t) := \frac{\alpha}{r^\alpha} t^{\alpha - 1}$$

is non-negative and measurable, and

$$\int_0^r \varphi_\alpha(t)\, dt = 1.$$

Then, the operator M related to φ_α, see (3.3), takes the form

$$M(u)(x_0) = \frac{\alpha}{r^\alpha} \int_0^r \Big(\frac{1}{t^{n-\alpha}} \int_{d(y - x_0) = t} u(y) P_d\Big(\frac{y - x_0}{t}\Big)\, d\sigma(y) \Big)\, dt.$$

By the coarea formula, the right hand side is equal to

$$\frac{\alpha}{r^\alpha}\int_{B_r^d(x_0)} u(y)\frac{|\nabla d(y-x_0)|}{(d(y-x_0))^{n-\alpha}}P_d\Big(\frac{y-x_0}{d(y-x_0)}\Big)\,dy$$
$$=\frac{\alpha}{r^\alpha}\int_{B_r^d(x_0)} u(y)\frac{m_d(y-x_0)}{(d(y-x_0))^{n-\alpha}}\,dy.$$

Thus, keeping in mind the definition of μ_α, see (1.4), we have

$$M(u)(x_0)=\frac{\alpha}{r^\alpha}\int_{B_r^d(x_0)} u(y)\frac{m_d(y-x_0)}{(d(y-x_0))^{n-\alpha}}\,dy=\int_{B_r^d(x_0)} u(y)\,d\mu_\alpha(y).$$

By Lemma 3.1-(b), we obtain

$$u(x_0)=Mu(x_0)=\int_{B_r^d(x_0)} u(y)\,d\mu_\alpha(y) \tag{3.7}$$

for every $u\in\mathscr{H}(B_r^d(x_0))$, $u\geq 0$. We have so proved (1.5).

Using in (3.7) the family of functions

$$y\mapsto u_x(y):=\Gamma(y-x),\quad x\notin B_r^d(x_0),$$

which are non-negative and harmonic in $B_r^d(x_0)$, we get

$$\Gamma(x_0-x)=\int_{B_r^d(x_0)}\Gamma(y-x)\,d\mu_\alpha(y)=\Gamma_{\mu_\alpha}(x),\quad \forall x\notin B_r^d(x_0). \tag{3.8}$$

On the other hand, if $x\in B_r^d(x_0)\backslash\{x_0\}$ then

$$u_x \text{ is superharmonic in } B_r^d(x_0),\quad u_x(x_0)=\Gamma(x_0-x)<\infty\quad\text{and}\quad \Delta u_x=-\delta_x,$$

where δ_x is the Dirac measure at $\{x\}$.

Then, by Lemma 3.1-(c)

$$\Gamma(x_0-x)>\Gamma_{\mu_\alpha}(x)\quad \forall x\in B_r^d(x_0),\ x\neq x_0. \tag{3.9}$$

Moreover, keeping in mind that $\Gamma\in L^p_{\text{loc}}(\mathbb{R}^n)$ for every $p\in\left[1,\frac{n}{n-2}\right[$ while $\left(\frac{1}{d}\right)^{n-\alpha}\in L^q_{\text{loc}}(\mathbb{R}^n)$ for every $q\in\left[1,\frac{n}{n-\alpha}\right[$, then the potential

$$\mathbb{R}^n\ni x\mapsto\int_{B_r^d(x_0)}\Gamma(y-x)\,d\mu_\alpha(y)\quad\text{is continuous for every }\alpha>n-2,$$

see e.g. [8, Proposition 8.8]. In particular, (3.9) extends up to $x=x_0$. Then, also keeping in mind (3.8), $(B_r^d(x_0),\mu_\alpha,x_0)$ is a strong Γ-triple if $\alpha>n-2$. □

We turn to prove Theorem 1.3, which is a consequence of Theorems 1.1 and 1.2.

PROOF OF THEOREM 1.3. Condition (1.6) is equivalent to say that $(D, \nu_\alpha \llcorner D, x_0)$ is a Γ-triple, with

$$d\nu_\alpha(y) := \frac{c m_d(y - x_0)}{(d(y - x_0))^{n-\alpha}} dy.$$

Of course, $\mathrm{supp}(\nu_\alpha \llcorner D) = \overline{D} \supseteq \partial D$. Moreover, from (1.6) we get

$$1 = c \int_D \frac{\Gamma(y - x)}{\Gamma(x_0 - x)} \cdot \frac{m_d(y - x_0)}{(d(y - x_0))^{n-\alpha}} dy \quad \forall x \notin D.$$

Letting $|x|$ go to infinity we obtain that $c = \left(\int_D \frac{m_d(y-x_0)}{(d(y-x_0))^{n-\alpha}} dy \right)^{-1}$.

By Theorem 1.2 $(B_r^d(x_0), \mu_\alpha, x_0)$ is a strong Γ-triple for all $\alpha > n - 2$.

Choosing $r = \left(\frac{\alpha}{c}\right)^{\frac{1}{\alpha}}$ we obtain $\nu_\alpha \llcorner (B_r^d(x_0) \cap D) = \mu_\alpha \llcorner (B_r^d(x_0) \cap D)$. Taking also into account that $B_r^d(x_0)$ is a solid set we have that all the assumptions of Theorem 1.1 are satisfied with $\Omega = B_r^d(x_0)$. Therefore, the conclusion follows. □

ACKNOWLEDGMENTS. The first author has been supported by MIUR through the Project PRIN (2011) "Calcolo delle Variazioni" and by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

Basic Notation

$m :=$ Lebesgue measure in $\mathbb{R}^n$
$B_r(x_0) :=$ Euclidean ball in $\mathbb{R}^n$ with center x_0 and radius r
$\omega_n := m(B_1(0))$
$\Omega^c := \mathbb{R}^n \setminus \Omega$
$\overline{\Omega} :=$ closure of Ω
$\mathrm{int}\,\Omega :=$ interior of Ω
Ω solid set $:= \overline{\Omega}^c$ is connected and $\Omega = \mathrm{int}\,\overline{\Omega}$
$\Delta :=$ Laplace operator
$u : \Omega \to \mathbb{R}$ is harmonic in $\Omega :=$ u is smooth and $\Delta u = 0$ in Ω
$\mathscr{H}(\Omega) :=$ set of the harmonic functions in Ω

References

[1] B. ABBONDANZA - A. BONFIGLIOLI, *The Dirichlet problem and the inverse mean-value theorem for a class of divergence form operators*, J. Lond. Math. Soc. 87 (2013) 321–346.

[2] D. AHARONOV - M. M. SCHIFFER - L. ZALCMAN, *Potato kugel*, Israel J. Math. 40 (1981) 331–339.

[3] H. AIKAWA, *Integrability of superharmonic functions and subharmonic functions*, Proc. Amer. Math. Soc. 120 (1994) 109–117.

[4] H. Aikawa, *Densities with the mean value property for harmonic functions in a Lipschitz domain*, Proc. Amer. Math. Soc. 125 (1997) 229–234.

[5] A. Bonfiglioli - E. Lanconelli - F. Uguzzoni, *Stratified Lie Groups and Potential Theory for their sub-Laplacians*, Springer Monographs in Mathematics. Springer, Berlin, 2007.

[6] B. Epstein, *On the mean-value property of harmonic functions*, Proc. Amer. Math. Soc. 13 (1962) 830.

[7] B. Epstein - M. M. Schiffer, *On the mean-value property of harmonic functions*, J. Analyse Math. 14 (1965) 109–111.

[8] G. B. Folland, *Real Analysis. Modern techniques and their applications*, Pure and Applied Mathematics. John Wiley & Sons, Inc., New York, 1984.

[9] W. Hansen - I. Netuka, *Volume densities with the mean value property for harmonic functions*, Proc. Amer. Math. Soc. 123 (1995) 135–140.

[10] W. K. Hayman - P. B. Kennedy, *Subharmonic functions Vol. I*, London Mathematical Society Monographs, No. 9. Academic Press, London-New York, 1976.

[11] Ü. Kuran, *On the mean-value property of harmonic functions*, Bull. London Math. Soc. 4 (1972) 311–312.

[12] E. Lanconelli, *"Potato kugel" for sub-Laplacians*, Israel J. Math. 194 (2013) 277–283.

Received 16 March 2016,
and in revised form 22 April 2016.

Giovanni Cupini
Dipartimento di Matematica
Università di Bologna
Piazza di Porta S. Donato 5
40126 Bologna, Italy
giovanni.cupini@unibo.it

Ermanno Lanconelli
Dipartimento di Matematica
Università di Bologna
Piazza di Porta S. Donato 5
40126 Bologna, Italy
ermanno.lanconelli@unibo.it