



**Partial Differential Equations** — *The equality case in a Poincaré–Wirtinger type inequality*, by BARBARA BRANDOLINI, FRANCESCO CHIACCHIO, DAVID KREJČIŘÍK and CRISTINA TROMBETTI, communicated on 10 June 2016.

ABSTRACT. — It is known that, for any convex planar set  $\Omega$ , the first non-trivial Neumann eigenvalue  $\mu_1(\Omega)$  of the Hermite operator is greater than or equal to 1. Under the additional assumption that  $\Omega$  is contained in a strip, we show that  $\mu_1(\Omega) = 1$  if and only if  $\Omega$  is any strip. The study of the equality case requires, among other things, an asymptotic analysis of the eigenvalues of the Hermite operator in thin domains.

KEY WORDS: Hermite operator, Neumann eigenvalues, thin strips

MATHEMATICS SUBJECT CLASSIFICATION: 35P15, 35P20, 35B25

### 1. INTRODUCTION

Let  $\Omega \subset \mathbb{R}^2$  be a convex domain and let us denote by  $\gamma$  and  $dm_\gamma$  the standard Gaussian function and measure in  $\mathbb{R}^2$  respectively, that is

$$\gamma(x, y) := \exp\left(-\frac{x^2 + y^2}{2}\right) \quad \text{and} \quad dm_\gamma := \gamma(x, y) \, dx \, dy.$$

In this paper we consider the following Neumann eigenvalue problem for the Hermite operator

$$(1.1) \quad \begin{cases} -\operatorname{div}(\gamma \nabla u) = \mu \gamma u & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $\mathbf{n}$  stands for the outward normal to  $\partial\Omega$ . As usual, we understand (1.1) as a spectral problem for the self-adjoint operator  $T$  in the Hilbert space  $L_\gamma^2(\Omega) := L^2(\Omega, dm_\gamma)$  associated with the quadratic form  $t[u] := \|\nabla u\|^2$ ,  $D(t) := H_\gamma^1(\Omega)$ . Here  $\|\cdot\|$  denotes the norm in  $L_\gamma^2(\Omega)$  and

$$H_\gamma^1(\Omega) := \{u \in L_\gamma^2(\Omega) \mid \nabla u \in L_\gamma^2(\Omega)\}$$

is a weighted Sobolev space equipped with the norm  $\sqrt{\|\cdot\|^2 + \|\nabla \cdot\|^2}$ . Since the embedding  $H_\gamma^1(\Omega) \hookrightarrow L_\gamma^2(\Omega)$  is compact (see e.g. [8], [14], [17]), the spectrum of  $T$  is purely discrete. We arrange the eigenvalues of  $T$  in a non-decreasing sequence  $\{\mu_n(\Omega)\}_{n=0}^{+\infty}$  where each eigenvalue is repeated according to its multiplicity. The

first eigenfunction of (1.1) is clearly a constant with eigenvalue  $\mu_0(\Omega) = 0$  for any  $\Omega$ . We shall be interested in the first non-trivial eigenvalue  $\mu_1(\Omega)$  of (1.1), which admits the following variational characterisation

$$(1.2) \quad \mu_1(\Omega) = \min \left\{ \frac{\int_{\Omega} |\nabla u|^2 dm_{\gamma}}{\int_{\Omega} u^2 dm_{\gamma}} : u \in H_{\gamma}^1(\Omega) \setminus \{0\}, \int_{\Omega} u dm_{\gamma} = 0 \right\}.$$

A classical Poincaré-Wirtinger type inequality which goes back to Hermite (see for example [13, Chapter II, p. 91 ff]) states that

$$(1.3) \quad \mu_1(\mathbb{R}^2) = 1$$

and therefore

$$\int_{\mathbb{R}^2} \left( u - \int_{\mathbb{R}^2} u dm_{\gamma} \right)^2 dm_{\gamma} \leq \int_{\mathbb{R}^2} |\nabla u|^2 dm_{\gamma}, \quad \forall u \in H_{\gamma}^1(\mathbb{R}^2).$$

If  $\Omega$  is any convex subset of  $\mathbb{R}^2$ , it is possible to prove that

$$(1.4) \quad \mu_1(\Omega) \geq 1$$

using various different techniques. For instance, in [9], among other things, the authors consider smooth densities of the type  $e^{-V}$ , with  $D^2V \geq Id$ , which applies to the Gaussian measure restricted to a convex set, by standard approximation arguments. A different approach, from optimal transportation theory, is contained in [11]. More recently, an improved inequality has been obtained for bounded sets. In [5] (see also [2]) the authors prove that if  $\Omega$  is a bounded, convex set then

$$(1.5) \quad \mu_1(\Omega) \geq \mu_1\left(-\frac{d(\Omega)}{2}, \frac{d(\Omega)}{2}\right)$$

where  $d(\Omega)$  is the diameter of  $\Omega$  and, here and throughout,  $\mu_1(a, b)$  will denote the first nontrivial eigenvalue of the Sturm–Liouville problem

$$(1.6) \quad \begin{cases} -(\gamma_1 v')' = \mu \gamma_1 v & \text{in } (a, b), \\ v'(a) = v'(b) = 0, \end{cases}$$

with  $-\infty \leq a < b \leq +\infty$  and

$$\gamma_1(x) := \exp\left(-\frac{x^2}{2}\right).$$

Again, we understand (1.6) as a spectral problem for a self-adjoint operator with compact resolvent in  $L^2_{\gamma_1}((a, b))$ . It is well known (see for instance [13, p. 328]) that

$$(1.7) \quad \mu_1(a, b) \geq 1 \quad \text{with} \quad \mu_1(a, b) = 1 \quad \text{if and only if } (a, b) = \mathbb{R}.$$

An alternative way to gain (1.4) consists into passing to the limit in (1.5) as  $d(\Omega)$  goes to infinity (see [6]).

Inequality (1.4) is sharp in the sense that equality sign holds when  $\Omega$  is any two-dimensional strip (by a two-dimensional strip we mean, up to rotations and translations, a set in the form  $\mathbb{R} \times I$ , where  $I$  is any open interval in  $\mathbb{R}$ ). It is natural to ask if the strips are the unique domains for which the equality in (1.4) is achieved.

We provide a partial answer to the uniqueness question via the following theorem, which is the main result of this paper.

**THEOREM 1.1.** *Let  $\Omega$  be a convex subset of  $S_{y_1, y_2} := \{(x, y) \in \mathbb{R}^2 : y_1 < y < y_2\}$  for some  $y_1, y_2 \in \mathbb{R}, y_1 < y_2$ . If  $\mu_1(\Omega) = 1$ , then  $\Omega$  is a strip.*

Inequality (1.5) is a Payne-Weinberger type inequality for the Hermite operator. We recall that the classical Payne-Weinberger inequality states that the first nontrivial eigenvalue of the Neumann Laplacian in a bounded convex set  $\Omega$ ,  $\mu_1^\Delta(\Omega)$ , satisfies the following bound

$$(1.8) \quad \mu_1^\Delta(\Omega) \geq \frac{\pi^2}{d(\Omega)^2},$$

where  $\pi^2/d(\Omega)^2$  is the first nontrivial Neumann eigenvalue of the one-dimensional Laplacian in  $(-d(\Omega)/2, d(\Omega)/2)$  (see [20]). The above estimate is the best bound that can be given in terms of the diameter alone in the sense that  $\mu_1^\Delta(\Omega)d(\Omega)^2$  tends to  $\pi^2$  for a parallepiped all but one of whose dimensions shrink to zero (see [18, 22]).

Estimate (1.4) is sharp, not only asymptotically, since the equality sign is achieved when  $\Omega$  is any strip  $S$ . Indeed, it is straightforward to verify that  $\mu_1(S) = \mu_1(\mathbb{R}) = 1$  for any strip  $S$ . Hence the question faced in Theorem 1.1 appears quite natural.

The paper is organised as follows. Section 2 contains the proof of Theorem 1.1. The latter consists in various steps. We firstly deduce from (1.4) that any optimal set must be unbounded; then we show that it is possible to split an optimal set  $\Omega$  getting two sets that are still optimal and have Gaussian area  $m_\gamma(\Omega)/2$ . Repeating this procedure we obtain a sequence of thinner and thinner, optimal sets  $\Omega_k$  and we finally prove that there exists  $a \in \overline{\mathbb{R}}$  such that  $\mu_1(\Omega_k)$  converges as  $k \rightarrow +\infty$  to  $\mu_1(a, +\infty)$ , which is strictly greater than 1 unless  $a = -\infty$ . This circumstance implies that  $\Omega$  contains a straight line, and hence  $\Omega$  is a strip.

The convergence of  $\mu_1(\Omega_k)$  to  $\mu_1(a, +\infty)$  follows by a more general result established in Section 3, where we actually prove a convergence of *all* eigenvalues of  $T$  in thin domains to eigenvalues of a one-dimensional problem (see Theorem 3.1). We also establish certain convergence of eigenfunctions. We believe that the convergence results are of independent interest, since our method of proof differs from known techniques in the case of the Neumann Laplacian in thin domains [3, 4, 19, 21].

For optimisation results related to the present work, we refer the interested reader to [7, 15, 17, 10, 12].

### 2. PROOF OF THEOREM 1.1

The main ingredient in our proof of Theorem 1.1 is the following lemma, which tells us that cutting the optimiser of (1.4) in two convex, unbounded sets with equal Gaussian area, we again get two optimisers.

LEMMA 2.1. *Let  $\Omega$  be a convex subset of  $S_{y_1, y_2}$  with  $\mu_1(\Omega) = 1$  and suppose that  $\Omega$  is not a strip. Let  $\bar{y} \in (y_1, y_2)$  be such that the straight line  $\{y = \bar{y}\}$  divides  $\Omega$  into two convex subsets with equal Gaussian area  $m_\gamma(\Omega)$ . Then*

$$\mu_1(\Omega \cap \{y < \bar{y}\}) = \mu_1(\Omega \cap \{y > \bar{y}\}) = 1.$$

PROOF. Let  $u$  be an eigenfunction of (1.1) corresponding to  $\mu_1(\Omega)$ . By (1.2), we know that  $\int_{\Omega} u \, dm_\gamma = 0$  and

$$1 = \frac{\int_{\Omega} |\nabla u|^2 \, dm_\gamma}{\int_{\Omega} u^2 \, dm_\gamma}.$$

For each  $\alpha \in [0, 2\pi]$  there is a unique straight line  $r_\alpha$  orthogonal to  $(\cos \alpha, \sin \alpha)$  such that it divides  $\Omega$  into two convex sets  $\Omega'_\alpha, \Omega''_\alpha$  with equal Gaussian measure.

Let  $I(\alpha) := \int_{\Omega'_\alpha} u \, dm_\gamma$ . Since  $I(\alpha) = -I(\alpha + \pi)$ , by continuity there is  $\bar{\alpha}$  such that  $I(\bar{\alpha}) = 0$ . Now we claim that  $r_{\bar{\alpha}}$  is parallel to the  $x$ -axis. Note firstly that  $\Omega'_{\bar{\alpha}}$  and  $\Omega''_{\bar{\alpha}}$  are obviously convex and by (1.5), (1.7) and (1.4) we have

$$(2.1) \quad \mu_1(\Omega'_{\bar{\alpha}}) \geq 1, \quad \mu_1(\Omega''_{\bar{\alpha}}) \geq 1.$$

Moreover, it is immediate to verify that

$$\begin{aligned} 1 = \mu_1(\Omega) &= \frac{\int_{\Omega'_{\bar{\alpha}}} |\nabla u|^2 \, dm_\gamma + \int_{\Omega''_{\bar{\alpha}}} |\nabla u|^2 \, dm_\gamma}{\int_{\Omega'_{\bar{\alpha}}} u^2 \, dm_\gamma + \int_{\Omega''_{\bar{\alpha}}} u^2 \, dm_\gamma} \\ &\geq \min \left\{ \frac{\int_{\Omega'_{\bar{\alpha}}} |\nabla u|^2 \, dm_\gamma}{\int_{\Omega'_{\bar{\alpha}}} u^2 \, dm_\gamma}, \frac{\int_{\Omega''_{\bar{\alpha}}} |\nabla u|^2 \, dm_\gamma}{\int_{\Omega''_{\bar{\alpha}}} u^2 \, dm_\gamma} \right\}, \end{aligned}$$

with equality holding if and only if

$$\frac{\int_{\Omega'_{\bar{\alpha}}} |\nabla u|^2 \, dm_\gamma}{\int_{\Omega'_{\bar{\alpha}}} u^2 \, dm_\gamma} = \frac{\int_{\Omega''_{\bar{\alpha}}} |\nabla u|^2 \, dm_\gamma}{\int_{\Omega''_{\bar{\alpha}}} u^2 \, dm_\gamma}.$$

Without loss of generality we can assume that

$$\min \left\{ \frac{\int_{\Omega'_z} |\nabla u|^2 dm_y}{\int_{\Omega'_z} u^2 dm_y}, \frac{\int_{\Omega''_z} |\nabla u|^2 dm_y}{\int_{\Omega''_z} u^2 dm_y} \right\} = \frac{\int_{\Omega'_z} |\nabla u|^2 dm_y}{\int_{\Omega'_z} u^2 dm_y}.$$

Finally, (2.1) ensures that

$$(2.2) \quad 1 = \mu_1(\Omega) = \mu_1(\Omega'_z) = \mu_1(\Omega''_z).$$

Now we want to show that both  $\Omega'_z$  and  $\Omega''_z$  are unbounded, and hence  $r_z$  is parallel to the  $x$ -axis. Suppose by contradiction that, for instance,  $\Omega'_z$  is bounded. In such a case (1.5) yields

$$\mu_1(\Omega'_z) \geq \mu_1 \left( -\frac{d(\Omega'_z)}{2}, \frac{d(\Omega'_z)}{2} \right).$$

Taking into account (2.2) and (1.7), we get that

$$\mu_1 \left( -\frac{d(\Omega'_z)}{2}, \frac{d(\Omega'_z)}{2} \right) = 1$$

that is  $d(\Omega'_z) = +\infty$ , which is a contradiction. □

**PROOF OF THEOREM 1.1.** By contradiction, let us assume that  $\Omega \subset S_{y_1, y_2}$  is a convex domain different from a strip and  $\mu_1(\Omega) = 1$ . Let us denote

$$\Omega = \{(x, y) \in \mathbb{R}^2 : y_1 < y < y_2, p(y) < x\},$$

where  $p$  is a convex, non-trivial function. From (1.5) and (1.7) it follows that  $\Omega$  is necessarily unbounded. By employing a separation of variables, we also deduce from (1.5) and (1.7) that  $\Omega$  cannot be a semi-strip. Finally, we may assume that  $\inf\{x : \exists y \in [y_1, y_2], (x, y) \in \Omega\}$  is finite (otherwise, we would have the finite supremum, which can be transferred to our situation by a reflection of the coordinate system).

Repeating the procedure described in the above lemma, since at any step we are dividing into two convex subsets with equal Gaussian area, we can obtain a sequence of unbounded convex domains

$$(2.3) \quad \begin{aligned} \Omega_{\epsilon_k} &:= \{(x, y) \in \mathbb{R}^2 : y_0 < y < d_k, p(y) < x\} \\ &= \{(x, y) \in \Omega : y_0 < y < d_k\} \end{aligned}$$

such that

$$\mu_1(\Omega_{\epsilon_k}) = 1, \quad \epsilon_k := d_k - y_0 \xrightarrow[k \rightarrow +\infty]{} 0.$$

Here the point  $y_0$  is chosen in such a way that  $p'(y_0) \neq 0$ , which is always possible because the situation of semi-strips has been excluded. Without loss of

generality (reflecting again the coordinate system if necessary), we may in fact assume

$$(2.4) \quad p'(y_0) > 0,$$

so that  $p$  is increasing on  $[y_0, d_k]$  whenever  $k$  is sufficiently large. Applying now a more general convergence result for eigenvalues in thin Neumann domains that we shall establish in the following section (Theorem 3.1), we have

LEMMA 2.2.  $\lim_{k \rightarrow \infty} \mu_1(\Omega_{\epsilon_k}) = \mu_1(p^{-1}(y_0), +\infty)$ .

Since  $\mu_1(\Omega_{\epsilon_k})$  equals 1 for every  $k$ , we conclude that

$$\mu_1(p^{-1}(y_0), +\infty) = 1.$$

However, from (1.7), we then deduce that  $p^{-1}(y_0) = -\infty$ , which contradicts our assumptions from the beginning of the proof. In other words,  $\Omega$  contains a straight line and the theorem immediately follows.  $\square$

It thus remains to establish Lemma 2.2.

### 3. EIGENVALUE ASYMPTOTICS IN THIN STRIPS

In this section we establish Lemma 2.2 as a consequence of a general result about convergence of *all* eigenvalues of  $T$  in thin domains of the type (2.3).

3.1. *The geometric setting.* Let  $f : [0, +\infty) \rightarrow [0, +\infty)$  be a concave non-decreasing continuous non-trivial function such that  $f(0) = 0$  (the case  $f(0) > 0$  is actually much easier to deal with). Given a positive number  $\epsilon < \sup f$ , we put

$$f_\epsilon(x) := \min\{\epsilon, f(x)\}$$

and define an unbounded domain

$$\Omega_\epsilon := \{(x, y) \in \mathbb{R}^2 : 0 < x, 0 < y < f_\epsilon(x)\}.$$

Clearly, (2.3) can be cast into this form after identifying  $f = p^{-1}$  in a small neighbourhood of zero and a translation. However, keeping in mind that the problem (1.1) is not translation-invariant, we accordingly change the definition of the Gaussian weight throughout this section

$$\gamma(x, y) := \exp\left(-\frac{(x_0 + x)^2 + (y_0 + y)^2}{2}\right).$$

Here  $y_0$  is primarily thought as the point from (2.3) and  $x_0$  is then such that  $(x_0, y_0) \in \Omega_{\epsilon_k}$ . For the results established in this section, however,  $x_0$  and  $y_0$  can

be thought as arbitrary real numbers. For our method to work, it is only important to assume (2.4), which accordingly transfers to

$$(3.1) \quad f'(0) < +\infty.$$

3.2. *The analytic setting and main result.* Keeping the translation we have made in mind, instead of (1.1) we equivalently consider the eigenvalue problem

$$(3.2) \quad \begin{cases} -\operatorname{div}(\gamma \nabla u) = \mu \gamma u & \text{in } \Omega_\varepsilon, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial \Omega_\varepsilon. \end{cases}$$

We understand (3.2) as a spectral problem for the self-adjoint operator  $T_\varepsilon$  in the Hilbert space  $L^2_\gamma(\Omega_\varepsilon)$  associated with the quadratic form  $t_\varepsilon[u] := \|\nabla u\|_\varepsilon^2$ ,  $\mathbf{D}(t_\varepsilon) := H^1_\gamma(\Omega_\varepsilon)$ . Here  $\|\cdot\|_\varepsilon$  denotes the norm in  $L^2_\gamma(\Omega_\varepsilon)$ . We arrange the eigenvalues of  $T_\varepsilon$  in a non-decreasing sequence  $\{\mu_n(\Omega_\varepsilon)\}_{n \in \mathbb{N}}$  where each eigenvalue is repeated according to its multiplicity. In this paper we adopt the convention  $0 \in \mathbb{N}$ . We are interested in the behaviour of the spectrum as  $\varepsilon \rightarrow 0$ , particularly  $\mu_1(\Omega_\varepsilon)$  because of Lemma 2.2.

It is expectable that the eigenvalues will be determined in the limit  $\varepsilon \rightarrow 0$  by the one-dimensional problem

$$(3.3) \quad \begin{cases} -(\gamma_0 u')' = v \gamma_0 u & \text{in } (0, +\infty), \\ u'(0) = 0, \end{cases}$$

where

$$\gamma_0(x) := \gamma(x, 0) = \exp\left(-\frac{(x_0 + x)^2 + y_0^2}{2}\right).$$

Again, we understand (3.3) as a spectral problem for the self-adjoint operator  $T_0$  in the Hilbert space  $L^2_{\gamma_0}((0, +\infty))$  associated with the quadratic form  $t_0[u] := \|\nabla u\|_0^2$ ,  $\mathbf{D}(t_0) := H^1_{\gamma_0}((0, +\infty))$ , where  $\|\cdot\|_0$  denotes the norm in  $L^2_{\gamma_0}((0, +\infty))$ . As above, we arrange the eigenvalues of  $T_0$  in a non-decreasing sequence  $\{v_n\}_{n \in \mathbb{N}}$  where each eigenvalue is repeated according to its multiplicity. By construction, for each  $n \in \mathbb{N}$ ,  $v_n$  coincides with the eigenvalue  $\mu_n(x_0, +\infty)$  defined in (1.6).

In this section we prove the following convergence result.

**THEOREM 3.1.** *Let  $f : [0, +\infty) \rightarrow [0, +\infty)$  be a concave non-decreasing continuous non-trivial function such that  $f(0) = 0$ . Assume in addition (3.1). Then*

$$\forall n \in \mathbb{N}, \quad \mu_n(\Omega_\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} v_n.$$

We shall also establish certain convergence of eigenfunctions of  $T_\varepsilon$  to eigenfunctions of  $T_0$ .

Clearly, Lemma 2.2 is the case  $n = 1$  of this general theorem.

The rest of this section is devoted to a proof of Theorem 3.1.

3.3. *From the moving to a fixed domain.* Our main strategy is to map  $\Omega_\varepsilon$  into a fixed strip  $\Omega$ . We introduce a refined mapping in order to effectively deal with the singular situation  $f(0) = 0$ .

Let

$$a_\varepsilon := \inf f_\varepsilon^{-1}(\{\varepsilon\}).$$

By the definition of  $f_\varepsilon$  and since  $f$  is non-decreasing,  $a_\varepsilon \rightarrow 0$  as  $\varepsilon \rightarrow 0$  and  $f_\varepsilon(x) = \varepsilon$  for all  $x > a_\varepsilon$ . If  $f(0) > 0$ , then there exists  $\varepsilon_0 > 0$  such that  $a_\varepsilon = 0$  for all  $\varepsilon \leq \varepsilon_0$ . On the other hand, if  $f(0) = 0$ , then  $a_\varepsilon > 0$  for all  $\varepsilon > 0$ . The troublesome situation is the latter, to which we have restricted from the beginning. In this case, we introduce an auxiliary function

$$g_\varepsilon(s) := \begin{cases} a_\varepsilon s + a_\varepsilon & \text{if } s \in [-1, 0), \\ s + a_\varepsilon & \text{if } s \in [0, +\infty). \end{cases}$$

Since we are interested in the limit  $\varepsilon \rightarrow 0$ , we may henceforth assume

$$(3.4) \quad \varepsilon \leq 1 \quad \text{and} \quad a_\varepsilon \leq 1.$$

Define  $\varepsilon$ -independent sets

$$\Omega_- := (-1, 0) \times (0, 1), \quad \Omega_+ := (0, +\infty) \times (0, 1), \quad \Omega := (-1, +\infty) \times (0, 1).$$

The mapping

$$(3.5) \quad \mathcal{L}_\varepsilon : \Omega \rightarrow \Omega_\varepsilon : \{(s, t) \mapsto \mathcal{L}_\varepsilon(s, t) := (g_\varepsilon(s), f_\varepsilon(g_\varepsilon(s))t)\}$$

represents a  $C^{0,1}$ -diffeomorphism between  $\Omega$  and  $\Omega_\varepsilon$  ( $f$  is differentiable almost everywhere, as it is supposed to be concave). In this way, we obtain a convenient parametrisation of  $\Omega_\varepsilon$  via the coordinates  $(s, t) \in \Omega$  whose Jacobian is

$$(3.6) \quad j_\varepsilon(s, t) = g'_\varepsilon(s)f'_\varepsilon(g_\varepsilon(s)).$$

Note that the Jacobian is independent of  $t$  and singular at  $s = -1$ . Now we reconsider (3.2) in  $\Omega$ . With the notation

$$\gamma_\varepsilon(s, t) := (\gamma \circ \mathcal{L}_\varepsilon)(s, t) = \exp\left(-\frac{[x_0 + g_\varepsilon(s)]^2 + [y_0 + f_\varepsilon(g_\varepsilon(s))t]^2}{2}\right),$$

introduce the unitary transform

$$U_\varepsilon : L^2_\gamma(\Omega_\varepsilon) \rightarrow L^2_{\gamma_\varepsilon j_\varepsilon/\varepsilon}(\Omega) : \{u \mapsto \sqrt{\varepsilon}u \circ \mathcal{L}_\varepsilon\}.$$

Here, in addition to the change of variables (3.5), we also make an irrelevant scaling transform (so that the renormalised Jacobian  $j_\varepsilon/\varepsilon$  is 1 in  $\Omega_+$ ). The



operators  $H_\varepsilon := U_\varepsilon T_\varepsilon U_\varepsilon^{-1}$  and  $T_\varepsilon$  are isospectral. By definition,  $H_\varepsilon$  is associated with the quadratic form  $h_\varepsilon[\psi] := t_\varepsilon[U_\varepsilon^{-1}\psi]$ ,  $D(h_\varepsilon) := U_\varepsilon D(t_\varepsilon)$ .

PROPOSITION 3.1. *Assume (3.1). Then*

$$(3.7) \quad h_\varepsilon[\psi] = \int_\Omega \left[ \left( \frac{\partial_s \psi}{g'_\varepsilon} - \frac{f'_\varepsilon \circ g_\varepsilon}{f_\varepsilon \circ g_\varepsilon} t \partial_t \psi \right)^2 + \frac{(\partial_t \psi)^2}{(f_\varepsilon \circ g_\varepsilon)^2} \right] \gamma_\varepsilon g'_\varepsilon \frac{f_\varepsilon \circ g_\varepsilon}{\varepsilon} ds dt,$$

$$(3.8) \quad D(h_\varepsilon) \subset H^1_{\gamma_\varepsilon j_\varepsilon / \varepsilon}(\Omega).$$

Here we have started to simplify the notation by suppressing arguments of the functions.

PROOF. The space  $\mathcal{D}_\varepsilon := C^1_0(\mathbb{R}^2) \upharpoonright \Omega_\varepsilon$  is a core of  $t_\varepsilon$ . The transformed space  $\mathcal{D} := U_\varepsilon \mathcal{D}_\varepsilon$  is a subset of  $C^1_0(\mathbb{R}^2) \upharpoonright \Omega$  consisting of Lipschitz continuous functions on  $\Omega$  which belong to  $C^1(\overline{\Omega_-}) \oplus C^1(\overline{\Omega_+})$  (we do not have  $C^1$  globally, because  $g_\varepsilon$  and  $f_\varepsilon$  are not smooth). For any  $\psi \in \mathcal{D}$ , it is easy to check (3.7); this formula extends to all  $\psi$  from the domain

$$D(h_\varepsilon) = \overline{\mathcal{D}}^{\|\cdot\|_{h_\varepsilon}}, \quad \|\cdot\|_{h_\varepsilon} := \sqrt{h_\varepsilon[\cdot] + \|\cdot\|^2},$$

where  $\|\cdot\|$  denotes the norm of  $L^2_{\gamma_\varepsilon j_\varepsilon / \varepsilon}(\Omega)$ . Let  $\psi \in \mathcal{D}$ . Using elementary estimates, we easily check

$$(3.9) \quad h_\varepsilon^-[\psi] \leq h_\varepsilon[\psi]$$

where

$$\begin{aligned} h_\varepsilon^-[\psi] := & \delta \int_\Omega \left( \frac{\partial_s \psi}{g'_\varepsilon} \right)^2 \gamma_\varepsilon g'_\varepsilon \frac{f_\varepsilon \circ g_\varepsilon}{\varepsilon} ds dt \\ & + \left( 1 - \frac{\delta}{1 - \delta} \|f'_\varepsilon\|_\infty^2 \right) \int_\Omega \frac{(\partial_t \psi)^2}{(f_\varepsilon \circ g_\varepsilon)^2} \gamma_\varepsilon g'_\varepsilon \frac{f_\varepsilon \circ g_\varepsilon}{\varepsilon} ds dt \end{aligned}$$

with any  $\delta \in (0, 1)$ . Note that  $f'_\varepsilon$  is bounded under the assumption (3.1) and the concavity. For any  $\varepsilon > 0$ , we can choose  $\delta$  so small that  $h_\varepsilon^-[\psi]$  is composed of a sum of two non-negative terms ( $\delta$  can be made independent of  $\varepsilon$  if we restrict the latter to a fixed bounded interval, say  $(0, 1]$ , see (3.4), because  $\|f'_\varepsilon\|_{L^\infty((0,1))} \leq \|f'\|_{L^\infty((0,1))}$ , but this assumption is not needed for the property we are proving). Using that  $g'_\varepsilon$  is bounded for any fixed  $\varepsilon$  and the estimate  $f_\varepsilon \circ g_\varepsilon \leq \varepsilon$ , we thus deduce from (3.9) that there is a positive constant  $c_{\varepsilon, \delta}$  (again, this constant can be made independent of  $\varepsilon$  if  $\varepsilon \leq 1$ ) such that

$$c_{\varepsilon, \delta} \|\psi\|_{H^1_{\gamma_\varepsilon j_\varepsilon / \varepsilon}(\Omega)}^2 \leq \|\psi\|_{h_\varepsilon}.$$

This proves (3.8) because  $\mathcal{D}$  is dense in  $H^1_{\gamma_\varepsilon j_\varepsilon / \varepsilon}(\Omega)$ . □

3.4. *The eigenvalue equation.* Recall that we denote the eigenvalues of  $T_\varepsilon$  (and hence  $H_\varepsilon$ ) by  $\mu_n(\Omega_\varepsilon)$  with  $n \in \mathbb{N}$  ( $= \{0, 1, \dots\}$ ). The  $(n + 1)^{\text{th}}$  eigenvalue can be characterised by the Rayleigh-Ritz variational formula

$$(3.10) \quad \mu_n(\Omega_\varepsilon) = \inf_{\substack{\dim \mathfrak{L}_n = n+1 \\ \mathfrak{L}_n \subset \mathcal{D}(h_\varepsilon)}} \sup_{\psi \in \mathfrak{L}_n} \frac{h_\varepsilon[\psi]}{\|\psi\|^2}.$$

PROPOSITION 3.2. *For any  $n \in \mathbb{N}$ , there exists a positive constant  $C_n$  such that for all  $\varepsilon \leq 1$ ,*

$$\mu_n(\Omega_\varepsilon) \leq C_n.$$

PROOF. Assuming  $\varepsilon \leq 1$ , we have the following two-sided  $\varepsilon$ - and  $t$ -independent bound

$$(3.11) \quad \gamma_-(s) \leq \gamma_\varepsilon(s, t) \leq \gamma_+(s)$$

valid for every  $(s, t) \in \Omega_+$  with

$$\begin{aligned} \gamma_-(s) &:= \exp\left(-\frac{(|x_0| + s + 1)^2 + (|y_0| + 1)^2}{2}\right), \\ \gamma_+(s) &:= \exp\left(-\frac{(-|x_0| + s)^2 - 2|y_0|}{2}\right). \end{aligned}$$

Using in addition that  $g'_\varepsilon = 1$  and  $f_\varepsilon \circ g_\varepsilon = \varepsilon$  in  $\Omega_+$ , we obviously have

$$\forall \psi \in C_0^\infty((0, +\infty)) \otimes \{1\}, \quad \frac{h_\varepsilon[\psi]}{\|\psi\|^2} \leq \frac{\int_{\Omega_+} (\partial_s \psi)^2 \gamma_+(s) \, ds \, dt}{\int_{\Omega_+} \psi^2 \gamma_-(s) \, ds \, dt}.$$

It then follows from (3.10) that the inequality of the proposition holds with the numbers

$$C_n := \inf_{\substack{\dim \mathfrak{L}_n = n+1 \\ \mathfrak{L}_n \subset C_0^\infty((0, +\infty))}} \sup_{\psi \in \mathfrak{L}_n} \frac{\int_0^{+\infty} \psi'(s)^2 \gamma_+(s) \, ds}{\int_0^{+\infty} \psi(s)^2 \gamma_-(s) \, ds},$$

which are actually eigenvalues of the one-dimensional operator  $-\gamma_-^{-1} \partial_s \gamma_+ \partial_s$  in  $L^2_{\gamma_-}((0, +\infty))$ , subject to Dirichlet boundary conditions.  $\square$

Let us now fix  $n \in \mathbb{N}$  and abbreviate the  $(n + 1)^{\text{th}}$  eigenvalue of  $H_\varepsilon$  by  $\mu_\varepsilon := \mu_n(\Omega_\varepsilon)$ . We denote an eigenfunction corresponding to  $\mu_\varepsilon$  by  $\psi_\varepsilon$  and normalise it to 1 in  $L^2_{\gamma_\varepsilon j_\varepsilon / \varepsilon}(\Omega)$ , i.e.,

$$(3.12) \quad \|\psi_\varepsilon\| = 1$$

for every admissible  $\varepsilon > 0$ .

The weak formulation of the eigenvalue equation  $H_\varepsilon \psi_\varepsilon = \mu_\varepsilon \psi_\varepsilon$  reads

$$(3.13) \quad \forall \phi \in D(h_\varepsilon), \quad h_\varepsilon(\phi, \psi_\varepsilon) = \mu_\varepsilon(\phi, \psi_\varepsilon),$$

where  $(\cdot, \cdot)$  stands for the inner product in  $L^2_{\gamma_\varepsilon j_\varepsilon / \varepsilon}(\Omega)$  and  $h_\varepsilon(\cdot, \cdot)$  denotes the sesquilinear form corresponding to  $h_\varepsilon[\cdot]$ , that is  $\forall \phi \in D(h_\varepsilon)$

$$(3.14) \quad \int_\Omega \left[ \left( \frac{\partial_s \psi_\varepsilon}{g'_\varepsilon} - \frac{f'_\varepsilon \circ g_\varepsilon}{f_\varepsilon \circ g_\varepsilon} t \partial_t \psi_\varepsilon \right) \left( \frac{\partial_s \phi}{g'_\varepsilon} - \frac{f'_\varepsilon \circ g_\varepsilon}{f_\varepsilon \circ g_\varepsilon} t \partial_t \phi \right) + \frac{(\partial_t \psi_\varepsilon)}{(f_\varepsilon \circ g_\varepsilon)} \frac{(\partial_t \phi)}{(f_\varepsilon \circ g_\varepsilon)} \right] \gamma_\varepsilon g'_\varepsilon \frac{f_\varepsilon \circ g_\varepsilon}{\varepsilon} ds dt = \mu_\varepsilon \int_\Omega \psi_\varepsilon \phi \gamma_\varepsilon g'_\varepsilon \frac{f_\varepsilon \circ g_\varepsilon}{\varepsilon} ds dt.$$

3.5. *What happens in  $\Omega_+$ .* Using  $|t| \leq 1$ , we easily verify

$$(3.15) \quad \forall (s, t) \in \Omega_+, \quad \gamma_\varepsilon(s, t) \geq \rho_\varepsilon(s) \gamma_0(s),$$

where the function

$$\rho_\varepsilon(s) := \exp\left(-\frac{a_\varepsilon^2 + 2|x_0|a_\varepsilon + \varepsilon^2 + 2|y_0|\varepsilon}{2}\right) \exp(-a_\varepsilon s)$$

is converging pointwise to 1 as  $\varepsilon \rightarrow 0$ .

Choosing  $\phi = \psi_\varepsilon$  as a test function in (3.13) and using (3.15) together with Proposition 3.2 and (3.12), we obtain

$$(3.16) \quad \int_{\Omega_+} (\partial_s \psi_\varepsilon)^2 \rho_\varepsilon \gamma_0 ds dt + \int_{\Omega_+} \frac{(\partial_t \psi_\varepsilon)^2}{\varepsilon^2} \rho_\varepsilon \gamma_0 ds dt \leq h_\varepsilon[\psi_\varepsilon] = \mu_\varepsilon \|\psi_\varepsilon\|^2 \leq C.$$

Here and in the sequel, we denote by  $C$  a generic constant which is independent of  $\varepsilon$  and may change its value from line to line. Writing

$$(3.17) \quad \psi_\varepsilon(s, t) = \varphi_\varepsilon(s) + \eta_\varepsilon(s, t),$$

where

$$(3.18) \quad \int_0^1 \eta_\varepsilon(s, t) dt = 0 \quad \text{for a.e. } s \in (0, +\infty),$$

we deduce from the second term on the left hand side of (3.16)

$$(3.19) \quad \pi^2 \int_{\Omega_+} \eta_\varepsilon^2 \rho_\varepsilon \gamma_0 ds dt \leq \int_{\Omega_+} (\partial_t \eta_\varepsilon)^2 \rho_\varepsilon \gamma_0 ds dt \leq C \varepsilon^2.$$

Differentiating (3.18) with respect to  $s$ , we may write

$$\int_{\Omega_+} (\partial_s \psi_\varepsilon)^2 \rho_\varepsilon \gamma_0 \, ds \, dt = \int_{\Omega_+} \varphi_\varepsilon'^2 \rho_\varepsilon \gamma_0 \, ds \, dt + \int_{\Omega_+} (\partial_s \eta_\varepsilon)^2 \rho_\varepsilon \gamma_0 \, ds \, dt$$

and putting this decomposition into (3.16), we get from the first term on the left hand side

$$(3.20) \quad \int_0^{+\infty} \varphi_\varepsilon'^2 \rho_\varepsilon \gamma_0 \, ds \leq C, \quad \int_0^{+\infty} (\partial_s \eta_\varepsilon)^2 \rho_\varepsilon \gamma_0 \, ds \leq C.$$

At the same time, from (3.12) using (3.15), we obtain

$$(3.21) \quad \int_{\Omega_+} \varphi_\varepsilon^2 \rho_\varepsilon \gamma_0 \, ds \, dt + \int_{\Omega_+} \eta_\varepsilon^2 \rho_\varepsilon \gamma_0 \, ds \, dt = \int_{\Omega_+} \psi_\varepsilon^2 \rho_\varepsilon \gamma_0 \, ds \, dt \leq \|\psi_\varepsilon\|^2 = 1,$$

where the first equality employs (3.18). Consequently,

$$(3.22) \quad \int_0^{+\infty} \varphi_\varepsilon^2 \rho_\varepsilon \gamma_0 \, ds \leq 1.$$

Finally, employing the first inequality from (3.20) and (3.22), we get

$$(3.23) \quad \int_0^{+\infty} (\sqrt{\rho_\varepsilon} \varphi_\varepsilon)^2 \gamma_0 \, ds \leq C.$$

From (3.22) and (3.23), we see that  $\{\sqrt{\rho_\varepsilon} \varphi_\varepsilon\}_{\varepsilon>0}$  is a bounded family in  $H_{\gamma_0}^1((0, +\infty))$  and therefore precompact in the weak topology of this space. Let  $\varphi_0$  be a weak limit point, i.e. for a decreasing sequence of positive numbers  $\{\varepsilon_i\}_{i \in \mathbb{N}}$  such that  $\varepsilon_i \rightarrow 0$  as  $i \rightarrow +\infty$ ,

$$(3.24) \quad \sqrt{\rho_{\varepsilon_i}} \varphi_{\varepsilon_i} \xrightarrow{i \rightarrow +\infty} \varphi_0 \quad \text{in } H_{\gamma_0}^1((0, +\infty)).$$

Since  $H_{\gamma_0}^1((0, +\infty))$  is compactly embedded in  $L_{\gamma_0}^2((0, +\infty))$ , we may assume

$$(3.25) \quad \sqrt{\rho_{\varepsilon_i}} \varphi_{\varepsilon_i} \xrightarrow{i \rightarrow +\infty} \varphi_0 \quad \text{in } L_{\gamma_0}^2((0, +\infty)).$$

3.6. *What happens in  $\Omega_-$ .* Here  $\gamma_\varepsilon$  can be estimated from below just by an  $\varepsilon$ -independent positive number, e.g.,

$$(3.26) \quad \forall (s, t) \in \Omega_-, \quad \gamma_\varepsilon(s, t) \geq \exp\left(-\frac{(|x_0| + 1)^2 + |y_0| + 1}{2}\right).$$

On the other hand, we need a lower bound to  $f_\varepsilon$ . Employing that  $f$  is concave and non-decreasing, we can use

$$(3.27) \quad \forall s \in (-1, 0), \quad f_\varepsilon(g_\varepsilon(s)) \geq \varepsilon(s + 1).$$

Recall also that  $g'_\varepsilon = a_\varepsilon$  on  $(-1, 0)$ .

Choosing  $\phi = \psi_\varepsilon$  as a test function in (3.13) and using (3.26) and (3.27), we obtain

$$(3.28) \quad \int_{\Omega_-} \left( \frac{\partial_s \psi_\varepsilon}{a_\varepsilon} - \frac{f'_\varepsilon \circ g_\varepsilon}{f_\varepsilon \circ g_\varepsilon} t \partial_t \psi_\varepsilon \right)^2 a_\varepsilon(s+1) ds dt + \int_{\Omega_-} \frac{(\partial_t \psi_\varepsilon)^2}{(f_\varepsilon \circ g_\varepsilon)^2} a_\varepsilon(s+1) ds dt \leq C.$$

Assume (3.1). Using elementary estimates as in the proof of Proposition 3.1, this inequality implies

$$(3.29) \quad \delta \int_{\Omega_-} \left( \frac{\partial_s \psi_\varepsilon}{a_\varepsilon} \right)^2 a_\varepsilon(s+1) ds dt + \left( 1 - \frac{\delta}{1-\delta} \|f'_\varepsilon\|_\infty^2 \right) \int_{\Omega_-} \frac{(\partial_t \psi_\varepsilon)^2}{(f_\varepsilon \circ g_\varepsilon)^2} a_\varepsilon(s+1) ds dt \leq C$$

with any  $\delta \in (0, 1)$ . We can choose  $\delta$  (independent of  $\varepsilon$  due to (3.4)) so small that the left hand side of (3.29) is composed of a sum of two non-negative terms. Using in addition  $f_\varepsilon \circ g_\varepsilon \leq \varepsilon$ , we thus deduce from (3.29)

$$\frac{1}{a_\varepsilon} \int_{\Omega_-} (\partial_s \psi_\varepsilon)^2 (s+1) ds dt + \frac{a_\varepsilon}{\varepsilon^2} \int_{\Omega_-} (\partial_t \psi_\varepsilon)^2 (s+1) ds dt \leq C.$$

Moreover, it follows from (3.1) and the convexity bound

$$(3.30) \quad \forall s \geq 0, \quad f(s) \leq f'(0)s$$

that

$$(3.31) \quad \varepsilon \leq f'(0)a_\varepsilon.$$

Hence

$$(3.32) \quad \int_{\Omega_-} |\nabla \psi_\varepsilon|^2 (s+1) ds dt \leq Ca_\varepsilon.$$

Now we write ( $\varphi_\varepsilon$  is constant!)

$$(3.33) \quad \psi_\varepsilon(s, t) = \varphi_\varepsilon + \eta_\varepsilon(s, t),$$

where

$$(3.34) \quad \int_{\Omega_-} \eta_\varepsilon(s, t)(s+1) ds dt = 0.$$

We state the following explicit result (although positivity of the minimum would be sufficient for our purposes).

LEMMA 3.1. *Writing  $w(s, t) := s + 1$ , we have*

$$\min \left\{ \frac{\int_{\Omega_-} |\nabla \eta|^2 w}{\int_{\Omega_-} \eta^2 w} : u \in H_w^1(\Omega_-) \setminus \{0\}, \int_{\Omega_-} \eta w = 0 \right\} = \pi^2.$$

PROOF. The minimum equals the first non-trivial eigenvalue  $\mu_1$  of the operator  $-w^{-1} \operatorname{div}(w \nabla)$  in  $L_w^2(\Omega_-)$ , subject to Neumann boundary conditions. By separation of variables,  $\mu_1$  coincides with the minimum between the first non-trivial eigenvalue of the operator  $-(s + 1)^{-1} \partial_s((s + 1) \partial_s)$  in  $L^2((0, 1), (s + 1) ds)$ , subject to Neumann boundary conditions, and the first non-trivial eigenvalue of the Laplacian  $-\partial_t^2$  in  $L^2((0, 1), dt)$ , subject to Neumann boundary conditions. (We remark that the former operator is the radial component of the Laplacian in the unit disk centred at  $(-1, 0)$ .) By solving these one-dimensional eigenvalue problems explicitly in terms of special functions, we know that the first non-trivial eigenvalues are given by  $j_{1,1}^2$  and  $\pi^2$ , respectively. Here  $j_{1,1} \approx 3.83$  is the first positive zero of the Bessel function  $J_1$  (see [1, Sec. 9]). Since,  $j_{1,1} > \pi$ , we get the desired claim.  $\square$

With help of this lemma, we deduce from (3.32)

$$(3.35) \quad \pi^2 \int_{\Omega_-} \eta_\varepsilon^2 (s + 1) ds dt \leq \int_{\Omega_-} |\nabla \eta_\varepsilon|^2 (s + 1) ds dt \leq C a_\varepsilon.$$

At the same time, from (3.12) using (3.26) and (3.27), we obtain

$$(3.36) \quad \int_{\Omega_-} \varphi_\varepsilon^2 a_\varepsilon (s + 1) ds dt + \int_{\Omega_-} \eta_\varepsilon^2 a_\varepsilon (s + 1) ds dt = \int_{\Omega_-} \psi_\varepsilon^2 a_\varepsilon (s + 1) ds dt \leq C,$$

where the first equality employs (3.34). Consequently, recalling that  $\varphi_\varepsilon$  is constant,

$$(3.37) \quad \varphi_\varepsilon^2 a_\varepsilon \leq C \quad \text{on } \Omega_-.$$

3.7. *The limiting eigenvalue equation in  $\Omega_+$ .* Now we consider (3.13) for the sequence  $\{\varepsilon_i\}_{i \in \mathbb{N}}$  and a test function  $\phi(s, t) = \varphi(s)$ , where  $\varphi \in C_0^\infty(\mathbb{R})$  is such that  $\varphi' = 0$  on  $[-1, 0]$ , and take the limit  $i \rightarrow +\infty$ .

We shall need a lower bound analogous to the upper bound (3.31). From the fundamental theorem of calculus, we deduce

$$(3.38) \quad \forall s \in [0, a_\varepsilon], \quad f(s) \geq \left( \operatorname{ess\,inf}_{(0, a_\varepsilon)} f' \right) s.$$

Note that the infimum cannot be zero unless  $f$  is trivial (we assume from the beginning  $\varepsilon < \sup f$  and that  $f$  is non-decreasing) and that it converges to  $f'(0) > 0$  as  $\varepsilon \rightarrow 0$ . Consequently, for all sufficiently small  $\varepsilon$ , we have

$$(3.39) \quad \varepsilon \geq \frac{1}{2} f'(0) a_\varepsilon.$$

At the same time, in analogy with (3.15), we have

$$(3.40) \quad \forall (s, t) \in \Omega_+, \quad \gamma_\varepsilon(s, t) \leq c_\varepsilon \rho_\varepsilon(s) \gamma_0(s),$$

where

$$c_\varepsilon := \exp\left(\frac{2|x_0|a_\varepsilon + \varepsilon^2 + 2|y_0|\varepsilon}{2}\right)$$

is converging to 1 as  $\varepsilon \rightarrow 0$ .

We first look at the right hand side of (3.13). Using the decompositions (3.17) and (3.33), we have

$$\begin{aligned} (\varphi, \psi_\varepsilon) &= \int_{\Omega_-} \varphi \varphi_\varepsilon \gamma_\varepsilon a_\varepsilon \frac{f_\varepsilon \circ g_\varepsilon}{\varepsilon} ds dt + \int_{\Omega_-} \varphi \eta_\varepsilon \gamma_\varepsilon a_\varepsilon \frac{f_\varepsilon \circ g_\varepsilon}{\varepsilon} ds dt \\ &\quad + \int_{\Omega_+} \varphi \varphi_\varepsilon \gamma_\varepsilon ds dt + \int_{\Omega_+} \varphi \eta_\varepsilon \gamma_\varepsilon ds dt. \end{aligned}$$

Estimating  $\gamma_\varepsilon \leq 1$  and using (3.30) and (3.39), we get

$$\begin{aligned} \left| \int_{\Omega_-} \varphi \eta_\varepsilon \gamma_\varepsilon a_\varepsilon \frac{f_\varepsilon \circ g_\varepsilon}{\varepsilon} ds dt \right| &\leq \frac{a_\varepsilon^2}{\varepsilon} f'(0) \int_{\Omega_-} |\varphi| |\eta_\varepsilon| (s+1) ds dt \\ &\leq 2a_\varepsilon \int_{\Omega_-} |\varphi| |\eta_\varepsilon| (s+1) ds dt, \end{aligned}$$

where the right hand side tends to zero as  $\varepsilon \rightarrow 0$  due to the Schwarz inequality and (3.35). At the same time, recalling that  $\varphi_\varepsilon$  is constant in  $\Omega_-$ ,

$$\begin{aligned} \left| \int_{\Omega_-} \varphi \varphi_\varepsilon \gamma_\varepsilon a_\varepsilon \frac{f_\varepsilon \circ g_\varepsilon}{\varepsilon} ds dt \right| &\leq \frac{a_\varepsilon^2}{\varepsilon} f'(0) \int_{\Omega_-} |\varphi| |\varphi_\varepsilon| (s+1) ds dt \\ &\leq 2a_\varepsilon |\varphi_\varepsilon| \int_{\Omega_-} |\varphi| (s+1) ds dt, \end{aligned}$$

where the right hand side tends to zero as  $\varepsilon \rightarrow 0$  due to (3.37). Using (3.40), we also get

$$\left| \int_{\Omega_+} \varphi \eta_\varepsilon \gamma_\varepsilon ds dt \right| \leq c_\varepsilon \int_{\Omega_+} |\varphi| |\eta_\varepsilon| \rho_\varepsilon \gamma_0 ds dt \leq c_\varepsilon \sqrt{\int_{\Omega_+} \varphi^2 \gamma_0 ds dt} \sqrt{\int_{\Omega_+} \eta_\varepsilon^2 \rho_\varepsilon \gamma_0 ds dt},$$

where the right hand side tends to zero as  $\varepsilon \rightarrow 0$  due to (3.19). Finally, we write

$$\int_{\Omega_+} \varphi \varphi_{\varepsilon_i} \gamma_{\varepsilon_i} ds dt = \int_{\Omega_+} \varphi \varphi_{\varepsilon_i} \sqrt{\rho_{\varepsilon_i}} \gamma_0 ds dt + \int_{\Omega_+} \varphi \varphi_{\varepsilon_i} \sqrt{\rho_{\varepsilon_i}} \gamma_0 \left( \frac{\gamma_{\varepsilon_i}}{\sqrt{\rho_{\varepsilon_i}} \gamma_0} - 1 \right) ds dt.$$

Here the first term on the right hand side converges to  $\int_{\Omega_+} \varphi \varphi_0 \gamma_0 ds dt$  as  $i \rightarrow +\infty$  due to (3.24), while the second term vanishes in the limit because of

$$\begin{aligned} & \left| \int_{\Omega_+} \varphi \varphi_{\varepsilon_i} \sqrt{\rho_{\varepsilon_i}} \gamma_0 \left( \frac{\gamma_{\varepsilon_i}}{\sqrt{\rho_{\varepsilon_i}} \gamma_0} - 1 \right) ds dt \right| \\ & \leq \sqrt{\int_{\Omega_+} \varphi^2 \gamma_0 \left( \frac{\gamma_{\varepsilon_i}}{\sqrt{\rho_{\varepsilon_i}} \gamma_0} - 1 \right)^2 ds dt} \sqrt{\int_{\Omega_+} \varphi_{\varepsilon_i}^2 \rho_{\varepsilon_i} \gamma_0 ds dt}. \end{aligned}$$

Indeed the second term on the right hand side is bounded by (3.21), while first term tends to zero as  $i \rightarrow +\infty$  by the dominated convergence theorem. Summing up,

$$(3.41) \quad \lim_{i \rightarrow +\infty} (\varphi, \psi_{\varepsilon_i}) = \int_0^{+\infty} \varphi \varphi_0 \gamma_0 ds.$$

Employing that the test function  $\varphi$  is constant on  $[-1, 0]$  and the decomposition (3.17), we have

$$h_\varepsilon(\varphi, \psi_\varepsilon) = \int_{\Omega_+} \varphi' \varphi'_\varepsilon \gamma_\varepsilon ds dt + \int_{\Omega_+} \varphi' \partial_s \eta_\varepsilon \gamma_\varepsilon ds dt.$$

Here the first term on the right hand side can be treated in the same way as above with the conclusion

$$\int_{\Omega_+} \varphi' \varphi'_\varepsilon \gamma_\varepsilon ds dt \xrightarrow{i \rightarrow +\infty} \int_{\Omega_+} \varphi' \varphi'_0 \gamma_0 ds dt = \int_0^{+\infty} \varphi' \varphi'_0 \gamma_0 ds,$$

while we integrate by parts to handle the second term,

$$\int_{\Omega_+} \varphi' \partial_s \eta_\varepsilon \gamma_\varepsilon ds dt = - \int_{\Omega_+} \varphi'' \eta_\varepsilon \gamma_\varepsilon ds dt - \int_{\Omega_+} \varphi' \eta_\varepsilon \partial_s \gamma_\varepsilon ds dt.$$

Notice that the boundary terms vanish because  $\varphi$  has a compact support in  $\mathbb{R}$  and  $\varphi'(0) = 0$ . As above, the first term on the right hand side vanishes as  $\varepsilon \rightarrow 0$  due to (3.19). Similarly, using  $\partial_s \gamma_\varepsilon(s, t) = -\gamma_\varepsilon(s, t)(x_0 + s + a_\varepsilon)$  for all  $(s, t) \in \Omega_+$  and (3.40), we have

$$\begin{aligned} \left| \int_{\Omega_+} \varphi' \eta_\varepsilon \partial_s \gamma_\varepsilon ds dt \right| & \leq c_\varepsilon \int_{\Omega_+} |\varphi'| |\eta_\varepsilon| \rho_\varepsilon \gamma_0 (x_0 + s + a_\varepsilon) ds dt \\ & \leq c_\varepsilon \sqrt{\int_{\Omega_+} \varphi'^2 \gamma_0 (x_0 + s + a_\varepsilon)^2 ds dt} \sqrt{\int_{\Omega_+} \eta_\varepsilon^2 \rho_\varepsilon \gamma_0 ds dt}, \end{aligned}$$



where the right hand side tends to zero as  $\varepsilon \rightarrow 0$  due to (3.19). Summing up,

$$(3.42) \quad \lim_{i \rightarrow +\infty} h_{\varepsilon_i}(\varphi, \psi_{\varepsilon_i}) = \int_0^{+\infty} \varphi' \varphi_0' \gamma_0 \, ds.$$

Since the set of functions  $\varphi \in C_0^\infty(\mathbb{R})$  satisfying  $\varphi'(0) = 0$  is a core for the form domain of the operator  $T_0$ , we conclude from (3.42) and (3.41) that  $\varphi_0$  belongs to  $D(T_0)$  and solves the one-dimensional problems

$$(3.43) \quad \begin{aligned} T_0 \varphi_0 &= \mu_0^+ \varphi_0, & \mu_0^+ &:= \limsup_{i \rightarrow +\infty} \mu_{\varepsilon_i}, \\ T_0 \varphi_0 &= \mu_0^- \varphi_0, & \mu_0^- &:= \liminf_{i \rightarrow +\infty} \mu_{\varepsilon_i}. \end{aligned}$$

If  $\varphi_0 \neq 0$  on  $(0, +\infty)$ , then  $\mu_0^\pm$  must coincide with some eigenvalues of  $T_0$ . It remains to check that indeed  $\varphi_0 \neq 0$  on  $(0, +\infty)$ .

3.8. *The limiting problem in  $\Omega_-$ : a crucial step.* Define

$$\Omega'_- := (-1/2, 0) \times (0, 1), \quad \Omega'_+ := (0, 1/2) \times (0, 1), \quad \Omega' := (-1/2, 1/2) \times (0, 1).$$

From (3.32) and (3.36), we respectively have

$$(3.44) \quad \int_{\Omega'_-} |\nabla \psi_\varepsilon|^2 \, ds \, dt \leq 2C a_\varepsilon, \quad \int_{\Omega'_-} \psi_\varepsilon^2 \, ds \, dt \leq \frac{2C}{a_\varepsilon}.$$

At the same time, denoting  $m_0 := \min_{[0, 1/2]} \gamma_0$  and assuming  $\varepsilon \leq 1$ , from (3.16) and (3.21), we respectively get

$$(3.45) \quad \int_{\Omega'_+} |\nabla \psi_\varepsilon|^2 \, ds \, dt \leq \frac{C}{m_0 \rho_\varepsilon(1/2)}, \quad \int_{\Omega'_+} \psi_\varepsilon^2 \, ds \, dt \leq \frac{1}{m_0 \rho_\varepsilon(1/2)}.$$

Consequently,  $\psi_\varepsilon \in H^1(\Omega')$  for any  $\varepsilon \leq 1$  (although, in principle,  $\|\psi_\varepsilon\|_{H^1(\Omega')}$  might not be uniformly bounded in  $\varepsilon$ ).

It follows that the boundary values  $\psi_\varepsilon(0-, t)$  and  $\psi_\varepsilon(0+, t)$  exist in the sense of traces in  $\Omega'_-$  and  $\Omega'_+$ , respectively, and they must be equal as functions of  $t$  in  $L^2((0, 1))$ . Using the decompositions (3.17) and (3.33), we therefore have, for almost every  $t \in (0, 1)$ ,

$$\begin{aligned} [\varphi_\varepsilon(0-) - \varphi_\varepsilon(0+)]^2 &= [\eta_\varepsilon(0+, t) - \eta_\varepsilon(0-, t)]^2 \leq 2[\eta_\varepsilon(0+, t)]^2 + 2[\eta_\varepsilon(0-, t)]^2 \\ &\leq 2C \int_0^{1/2} ([\eta_\varepsilon(s, t)]^2 + [\partial_s \eta_\varepsilon(s, t)]^2) \, ds \\ &\quad + 2C \int_{-1/2}^0 ([\eta_\varepsilon(s, t)]^2 + [\partial_s \eta_\varepsilon(s, t)]^2) \, ds. \end{aligned}$$

Here  $C$  is a positive constant coming from the Sobolev embedding theorem  $H^1((0, 1/2)) \hookrightarrow C^0([0, 1/2])$  applied to  $s \mapsto \eta_\varepsilon(0_\pm, t)$  for almost every  $t \in (0, 1)$ , which is justified by  $\eta_\varepsilon \in H^1(\Omega'_\pm)$  and Fubini's theorem. Recall that  $\varphi_\varepsilon$  is constant on  $(-1, 0)$  and  $\varphi_\varepsilon \in H^1((0, 1/2)) \hookrightarrow C^0([0, 1/2])$ ; more specifically, the first inequality of (3.20) and (3.22) respectively yield

$$(3.46) \quad \int_0^{1/2} \varphi_\varepsilon'^2 ds \leq \frac{C}{m_0 \rho_\varepsilon(1/2)}, \quad \int_0^{1/2} \varphi_\varepsilon^2 ds \leq \frac{1}{m_0 \rho_\varepsilon(1/2)}.$$

Integrating with respect to  $t$  above, we deduce

$$[\varphi_\varepsilon(0-) - \varphi_\varepsilon(0+)]^2 \leq 2C \int_{\Omega'_+} [\eta_\varepsilon^2 + (\partial_s \eta_\varepsilon)^2] ds dt + 2C \int_{\Omega'_-} [\eta_\varepsilon^2 + (\partial_s \eta_\varepsilon)^2] ds dt.$$

Applying (3.19), the second inequality of (3.20) and (3.35), we may write

$$(3.47) \quad [\varphi_\varepsilon(0-) - \varphi_\varepsilon(0+)]^2 \leq C,$$

where  $C$  is a constant (different from the above) independent of  $\varepsilon$ , provided that (3.4) holds. Finally, applying (3.46) and the Sobolev embedding  $H^1((0, 1/2)) \hookrightarrow C^0([0, 1/2])$ , we deduce from (3.47) the following improvement upon (3.37)

$$(3.48) \quad \varphi_\varepsilon^2 \leq C \quad \text{on } \Omega_-.$$

3.9. As  $\varepsilon \rightarrow 0$  only  $\Omega_+$  matters: convergence of eigenvalues and eigenfunctions. Estimate (3.48) provides a crucial information whose significance consists in that what happens in  $\Omega_-$  is insignificant.

PROPOSITION 3.3. *One has*

$$\|\psi_{\varepsilon_i}\| \xrightarrow{i \rightarrow +\infty} \|\varphi_0\|_{L^2_{\gamma_0}((0, +\infty))}.$$

PROOF. We have

$$\begin{aligned} \|\psi_\varepsilon\|^2 &= \int_{\Omega_-} \varphi_\varepsilon^2 \gamma_\varepsilon a_\varepsilon \frac{f_\varepsilon \circ g_\varepsilon}{\varepsilon} ds dt + \int_{\Omega} \eta_\varepsilon^2 \gamma_\varepsilon a_\varepsilon \frac{f_\varepsilon \circ g_\varepsilon}{\varepsilon} ds dt \\ &\quad + \int_{\Omega_-} 2\varphi_\varepsilon \eta_\varepsilon \gamma_\varepsilon a_\varepsilon \frac{f_\varepsilon \circ g_\varepsilon}{\varepsilon} ds dt \\ &\quad + \int_{\Omega_+} \varphi_\varepsilon^2 \gamma_\varepsilon ds dt + \int_{\Omega_+} \eta_\varepsilon^2 \gamma_\varepsilon ds dt + \int_{\Omega_+} 2\varphi_\varepsilon \eta_\varepsilon \gamma_\varepsilon ds dt. \end{aligned}$$

The right hand side of the first line together with the mixed term on the second line goes to zero as  $\varepsilon \rightarrow 0$ . Indeed, recalling (3.30), (3.39) and  $\gamma_\varepsilon \leq 1$ ,

$$\int_{\Omega_-} \varphi_\varepsilon^2 \gamma_\varepsilon a_\varepsilon \frac{f_\varepsilon \circ g_\varepsilon}{\varepsilon} ds dt \leq 2a_\varepsilon \varphi_\varepsilon^2 \int_{\Omega_-} (s+1) ds dt \xrightarrow{\varepsilon \rightarrow 0} 0$$

due to (3.48);

$$\int_{\Omega_-} \eta_\varepsilon^2 \gamma_\varepsilon a_\varepsilon \frac{f_\varepsilon \circ g_\varepsilon}{\varepsilon} ds dt \leq 2a_\varepsilon \int_{\Omega_-} \eta_\varepsilon^2 (s+1) ds \xrightarrow{\varepsilon \rightarrow 0} 0$$

due to (3.35); and the mixed term goes to zero by the Schwarz inequality. Similarly, recalling (3.40),

$$\int_{\Omega_+} \eta_\varepsilon^2 \gamma_\varepsilon ds dt \leq c_\varepsilon \int_{\Omega_+} \eta_\varepsilon^2 \rho_\varepsilon \gamma_0 ds dt \xrightarrow{\varepsilon \rightarrow 0} 0$$

due to (3.19); while the Schwarz inequality yields

$$\left| \int_{\Omega_+} 2\varphi_\varepsilon \eta_\varepsilon \gamma_\varepsilon ds dt \right| \leq 2c_\varepsilon \sqrt{\int_{\Omega_+} \eta_\varepsilon^2 \rho_\varepsilon \gamma_0 ds dt} \sqrt{\int_{\Omega_+} \varphi_\varepsilon^2 \rho_\varepsilon \gamma_0 ds dt} \xrightarrow{\varepsilon \rightarrow 0} 0,$$

where the second square root is bounded in  $\varepsilon$  due to (3.22). Finally, we write

$$\int_{\Omega_+} \varphi_{\varepsilon_i}^2 \gamma_{\varepsilon_i} ds dt = \int_{\Omega_+} \varphi_{\varepsilon_i}^2 \rho_{\varepsilon_i} \gamma_0 ds dt + \int_{\Omega_+} \varphi_{\varepsilon_i}^2 (\gamma_{\varepsilon_i} - \rho_{\varepsilon_i} \gamma_0) ds dt$$

and observe that the first term on the right hand side tends to the desired result  $\|\varphi_0\|_{L^2_{\gamma_0}((0, +\infty))}^2$  as  $i \rightarrow +\infty$  by the strong convergence (3.25), while the second term vanishes in the limit. In more detail,

$$\begin{aligned} \left| \int_{\Omega_+} \varphi_{\varepsilon_i}^2 (\gamma_{\varepsilon_i} - \rho_{\varepsilon_i} \gamma_0) ds dt \right| &= \left| \int_{\Omega_+} (\varphi_{\varepsilon_i}^2 \rho_{\varepsilon_i} - \varphi_0^2 + \varphi_0^2) \left( \frac{\gamma_{\varepsilon_i}}{\rho_{\varepsilon_i}} - \gamma_0 \right) ds dt \right| \\ &\leq \int_{\Omega_+} |\varphi_{\varepsilon_i}^2 \rho_{\varepsilon_i} - \varphi_0^2| (c_{\varepsilon_i} \gamma_0 + \gamma_0) ds dt \\ &\quad + \int_{\Omega_+} \varphi_0^2 \left( \frac{\gamma_{\varepsilon_i}}{\rho_{\varepsilon_i}} - \gamma_0 \right) ds dt, \end{aligned}$$

where the first term after the inequality tends to zero as  $i \rightarrow +\infty$  by the strong convergence again, while the second term vanishes by the dominated convergence theorem.  $\square$

It follows from Proposition 3.3 that  $\varphi_0 \neq 0$ , so that it is indeed an eigenfunction of  $T_0$  due to (3.43). In particular,  $\mu_0^+ = \mu_0^-$ .

Now, let  $\hat{\psi}_\varepsilon$  be a normalised eigenfunction corresponding to possibly another eigenvalue  $\hat{\mu}_\varepsilon := \mu_m(\varepsilon)$ . Again, we use the decompositions (3.17) and (3.33) and distinguish the individual components by hat. In the same way as we proved Proposition 3.3, we can establish

PROPOSITION 3.4. *One has*

$$(\psi_{\varepsilon_i}, \hat{\psi}_{\varepsilon_j}) \xrightarrow{i, j \rightarrow +\infty} (\varphi_0, \hat{\varphi}_0)_{L^2_{\gamma_0}((0, +\infty))}.$$

If  $m \neq n$ , then  $(\psi_{\varepsilon_i}, \hat{\psi}_{\varepsilon_j}) = 0$  and thus  $(\varphi_0, \hat{\varphi}_0)_{L^2_{\gamma_0}((0, +\infty))} = 0$ . Hence  $\varphi_0$  and  $\hat{\varphi}_0$  correspond to distinct eigenvalues of  $T_0$ . In particular,  $\varphi_0$  is an eigenfunction corresponding to the  $(n + 1)^{\text{th}}$  eigenvalue  $v_n$  of  $T_0$ . Since we get this result for any weak limit point of  $\{\varphi_\varepsilon\}_{\varepsilon > 0}$ , we have the convergence results actually in  $\varepsilon \rightarrow 0$  (no need to pass to subsequences).

This completes the proof of Theorem 3.1.

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