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Mathematics — *On a model for the nexus between resource wealth and political regimes*, by Alberto Tesei, communicated on 10 June 2016.

Abstract. — We study in some detail a model proposed in [3], concerning the impact of natural resource rents on leader's policy. A major result of the analysis is that a reduction in resource rents can give rise to a political transition, from autocracy to democracy. It is also shown that incumbent leaders under the threat of a coup may decide not to make productive investments, if resource rents and probability of success of a coup are high. Both facts are in agreement with well-established empirical observations.

Key words: Political regimes, political transition, natural resource windfalls, natural resource curse

Mathematics Subject Classification: 91B02, 91A05

1. Introduction

It is widely accepted (in particular, in the recent literature on democratic development; see [2]) that institutions—namely, the way societies are organized—are important to determine the economic performance. However, in spite of their acknowledged importance, even in political economy papers institutions are often regarded as exogenously given, and little effort is made to understand why their structure varies across countries.

A more recent viewpoint stresses the endogenous character of institutions, as well as the interaction between political and economic institutions. In general terms, it is believed that political institutions determine the *de jure* political power, whereas economic institutions determine the distribution of resources, thus the *de facto* political power, and both sources of political power influence the further development of political institutions (see [1]). Ideally, this process should be rendered by a complete dynamical model with endogenous political and economic institutions, to be applied to different situations.

In this perspective, abundance of natural resources is often thought of as contributing to *negative* political and economic outcomes. This thesis is the amply discussed *natural resource curse* (e.g., see [9] and references therein).

As for economy, the resource curse thesis maintains that natural resource wealth is a hindrance to economic development. The case of Nigeria is often mentioned in this context: since the seventies of last century the fraction of people in this country living on less than 1 USD per day has gone from 36% to 70%, whereas in the same period the country exported oil worth around 10 billions

USD every year (a number of similar cases can be produced in favour of the thesis; e.g., see [7]).

As for politics, natural resource wealth is said to make autocracies stronger by increasing the value of staying in power and allowing the incumbent to spend on repressive activities. This is apparently true for oil-rich Saudi Arabia (as well as for other Middle-East countries), which is certainly a very stable and entrenched autocracy. However, this contrasts with the argument that resource wealth undermines regime stability, since it gives a financial incentive to challenge the incumbent. Impressive evidence for this second belief is provided by oil-rich Nigeria, Sudan and Venezuela, diamond-rich Angola and Sierra Leone, and rich-of-everything Democratic Republic of Congo, all of which are very unstable. At the same time, some of these countries are among the poorest in the world, thus in their case natural resources seem to be both a political and an economic curse. However, we cannot overlook stunning examples of "natural resource political and economic blessing" like Norway or US, which are both democratic and politically stable in spite of being oil-rich, or rich-of-everything.

Such a complicated situation could lead to infer that there is no actual relationship between abundance of natural resources and political outcomes. However, *natural resource windfalls provide evidence in the opposite direction*, displaying in various countries a strong correlation between the rise in the inter-

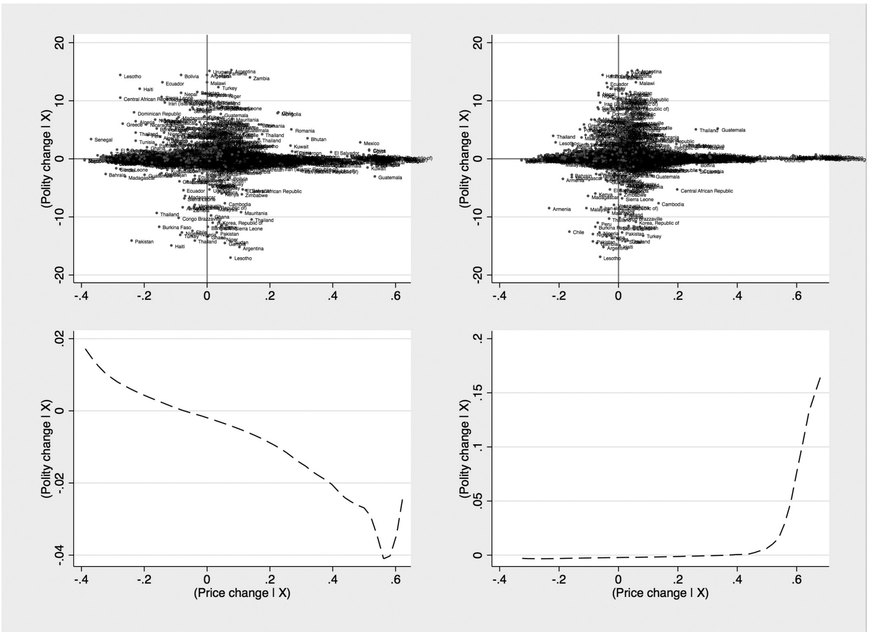

Figure 1. The figures on the left display the variation of the democracy index *Polity*2 versus the percentual variation of the international price of a major commodity for autocratic countries. The figures on the right do the same for democratic countries.

national price of some natural resource on one hand, and the rise of autocracy in the political regime on the other (see Figure 1 and [5], where specific impressive cases are discussed). The conclusion is that an abundance of natural resources does affect political outcomes, and yet a variety of outcomes is possible, depending on peculiar mechanisms underlying different countries (in this connection, see [7]). To identify possible mechanisms of this kind, through which the natural resource wealth operates, several political economy models have been recently formulated (in particular, see [3, 4, 5, 6, 8, 10] and references therein).

In this direction, let us resume the remark that an increase in resource revenue increases both the value for the incumbent of staying in power, and the likelihood of a challenge for the political control. In this situation, the most obvious behaviour of the incumbent is to strengthen his control through unproductive spending on security apparatuses. However, an alternative strategy is to make potential challengers less aggressive by productive investments, which improve the outside option offered by remunerative activities in the private sector.

It follows that both increased value of staying in power and increased likelihood of a challenge can have ambiguous effects on the leader spending strategy. Mutual influences expectedly arise when both effects are taken into account, leading to different outcomes. To make predictions we need to know "how productive productive investments are"—namely, how great is for potential challengers the profit of undertaking a private industrial activity. Also, we need to know how effective is the leader expenditure on repressive activities. The actual political outcome will essentially depend on these facts, as well as on the level of resource revenues and other country characteristics.

In this paper we study a simple model [3] where the above mechanisms are present, making a rigorous analysis of their mutual influences. In the model a leader controls the income from natural resources (represented by a parameter $\alpha \geq 0$), and decides how much of his budget to invest either in productive investments $I \geq 0$, or in repressive expenditure $C \geq 0$. No debt is allowed, hence the leader expenditure is subject to a budget constraint (which depends on α; see Section 2). On the other hand, a potential challenger decides whether to stage a coup and try to replace the leader, or to become an entrepreneur of the private sector.

This simple game gives rise to a fairly rich and intricate structure of possible situations. Different leader's spending strategies give rise to different policies, which range from the *democratic* (only productive investments, $C = 0$) to the *autocratic* (only funding of repressive activities, $I = 0$). Intermediate policies with both $I > 0$ and $C > 0$ can be ordered according to their *level of autocracy*, which increases as C increases and I decreases. Three political outcomes can arise, depending on the resource income α and on structural assumptions concerning the economy: *democracy*, *autocracy*, and *a fatalistic attitude* of the leader, who decides not to spend at all if avoiding a coup is beyond his reach. This lack of leader's investments in a highly risky situation is the explanation that the model provides for the natural resource curse.

Specifically, let us assume hereafter that the profit of undertaking a private industrial activity is sufficiently high, and the leader has a clearcut advantage in developing the industrial production (assumptions (H_0) and (H_4) below). Then (see Theorems 3.2(a) and 3.3(a)):

(R_0) there is a first threshold value α_* of the resource income, below which a coup is never attempted (namely, "the pie is too small"; see (A_1)). In this case democracy is the outcome.

If the resource income is above the threshold α_*, the situation is as follows. Consider first the simpler case in which the probability p of success of a coup is exogenously given, and the leader only makes productive investments (see Section 3). In this case the only policy to avoid a coup is to invest a minimal amount, which depends both on the resource income α and on the probability p. Then two cases are possible:

(R_1) If the profit of undertaking a private industrial activity is high (see (H_1)), for all values of α the outcome is democracy (Theorems 3.2(b) and 3.3(a)).

(R_2) If this profit is low (see (H_2)), a second threshold value $\alpha^* > \alpha_*$ of the resource income arises. Below this threshold (namely, when "the pie is not too big") the outcome is democracy as before (Theorems 3.2(b) and 3.3(a)). Above this value,
- the outcome is democracy as long as the probability p is small (Theorems 3.2(c) and 3.3(a)), whereas
- the leader takes a fatalistic attitude and stops spending at all, if p is great (Theorems 3.2(c) and 3.3(b)).

More possibilities arise, if the leader can lower the probability of success of a coup by spending on repressive activities (see Section 4). In this case:

(R_3) If the profit of becoming an entrepreneur is high, or if it is low but the resource income is not too high (i.e., if either (H_1) holds, or (H_2) holds and $\alpha < \alpha^*$), the situation is the same as before—namely, the outcome is democracy (Theorem 4.1).

(R_4) Otherwise (i.e., if (H_2) holds and $\alpha > \alpha^*$), the probability p and the efficiency δ of counter-insurgency structures play an important role to determine the political outcome. In fact, if p is low, the outcome is still democracy. If p is high, two scenarios are possible:
- if δ is low, the leader either does not make any investment, or chooses an intermediate policy (Theorem 4.1(a));
- if δ is high, the leader chooses an intermediate policy (Theorem 4.1(b)).

Importantly, the model predicts that *changes of the natural resource income can have effect on the political system* (see Theorem 4.2). In fact, in view of the above situation (see (R_1) and (R_3)), resource windfalls do not affect developed democracies—namely, democracies with a strong bias toward the economic development of the private sector. When such a bias is low and the probability of success of a coup is high, an increase of resource income can produce a change of the political system from democracy either to autocracy, or to a no-spending policy (see (R_2) and (R_4)), whereas no reverse transition is allowed. Therefore, such an increase does not produce political changes in established autocracies. These features of the model are in satisfactory agreement with observation: resource windfalls have little impact in Norway or Saudi Arabia, but change the political equilibrium in more unstable countries like Nigeria.[1]

To prove the above results boils down to maximize the leader's *utility function*, since this is the rationale behind his choice of the policy. Because of the budget constraint, the maximization must take place in suitable regions of the "phase space" $\{(I, C, p) \in \mathbb{R}^3 \mid I \geq 0, C \geq 0, p \in [0,1]\}$. Since these regions depend on α, and on δ in the case of endogenous p, a change of these parameters can change the location of the maximum points, thus giving rise to a change of policy. To address this point, in the endogenous case we must investigate the dependence on α of the *critical efficiency* of the counter-insurgency structures of the country and compare it with their actual efficiency δ (see Section 4). From the mathematical viewpoint, studying changes of policy can be regarded as a simple bifurcation problem.

Let us finally observe that the ideas underlying the model raise interesting questions, which in the author's opinion deserve a more refined mathematical modelling and deeper analysis.

2. The model

The model proposed in [3] can be described as follows. Consider a two-period economy. In both periods a population consisting of N agents is engaged in two good-producing activities:

a) the exploitation of natural resources (e.g., mineral extraction, or cultivating crops). The corresponding income is represented by an exogenous parameter A;
b) a *primitive,* small-scale activity (e.g., own-consumption agriculture, or providing artisanal services). Each agent can start this kind of activity. In every period $i = 1, 2$ the output of each agent's primitive activity is $\rho_S h_i$, where: (a) the quantity $h_i > 0$ represents essential conditions like infrastructures, efficient leader services, enforcement of law; (b) the exogenous parameter $\rho_S > 0$ captures skills and effort of each agent engaged in this activity.

Importantly, whereas the quantity h_1 is exogenous, h_2 depends on the investment $I \geq 0$ *per capita* made by the leader in period 1:

$$h_2 = h_1 + I. \tag{2.1}$$

[1] Analogous theoretical results have been obtained in [5] by a simpler model, where only repressive expenditure of the incumbent is considered. A rather different model proposed in [6] also shows that "booms based on resources exploited by the state tend to favor more dictatorial regimes".

A key feature of the model is that in period 2 a third kind of economic activity, called *industrial* activity, is possible. Industrial activity is more efficient and larger-scale than primitive activity. In view of its superiority, its output is of the form $y = \rho_L h_2$ with exogenous productivity $\rho_L > \rho_S$. We set $\rho := \rho_L - \rho_S > 0$.

Whereas the leader is the direct recipient of the income generated by natural resources, he also taxes the income generated by the private sector, either by primitive or by industrial activity, with an exogenous tax rate $\tau \in (0, 1)$. Reinforcing the assumption $\rho_L > \rho_S$, hereafter we always assume that

$$(H_0) \qquad (1-\tau)\rho - \tau\rho_S = (1-\tau)\rho_L - \rho_S > 0.$$

The meaning of assumption (H_0) will be discussed later (see Remark 2.1).

In contrast with the primitive activity, not every agent can undertake the industrial activity, since it needs managerial ability. For simplicity it is assumed that only one agent has such ability—namely, there is only one potential manager in the population. If he decides to start the industrial activity, as a monopsonist on the labor market he can hire the entire population. On the other hand, as explained below, he could instead decide to challenge the leader to seize power. No other possibility exists.

The game can now described as follows. In period 1 the leader is exogenously given. When period 2 begins, the talented agent has a choice between starting the industrial activity and staging a coup. If he decides to attempt a coup, this has a probability $p \in [0, 1]$ to succeed. If the coup fails, the coup leader incurs an exogenous cost $d > 0$ (e.g., imprisonment, exile or death). The leader of period 1 stays in power if the talented agent decides not to attempt a coup, or if a coup is attempted but fails. Otherwise, the talented agent becomes the new leader.

If the potential manager/challenger decides not to stage a coup, he starts the industrial activity by hiring the entire population. The overall output of this activity is $\rho_L h_2 N$, whereas he must spend $\rho_S h_2 N$ on wages (clearly, an agent's salary cannot be less than the utility he would have from starting himself a primary activity). Therefore in the no-coup case, after paying taxes, the profit of the talented agent in period 2 is

$$(2.2) \qquad (1-\tau)\rho(h_1 + I)N.$$

On the other hand, if he decides to attempt a coup, his utility is

$$(2.3) \qquad [A + \tau\rho_S(h_1 + I)N]p - d(1-p),$$

where the first term is the period 2 leader's profit in the coup case (see (2.7) below), weighted by the probability p of coup success, and the second is the coup cost, weighted by the probability $(1-p)$ of coup failure. In view of (2.2) and (2.3), a coup is attempted if and only if

$$\underbrace{[A + \tau\rho_S(h_1 + I)N]p - d(1-p)}_{\text{utility in the coup case}} > \underbrace{(1-\tau)\rho(h_1 + I)N}_{\text{profit of the industrial activity}}$$

—namely, if and only if

$$\alpha p - D(1-p) > [(1-\tau)\rho - p\tau\rho_S](h_1 + I), \tag{2.4}$$

where $\alpha := \frac{A}{N}$ is the income *per capita* of natural resources and $D := \frac{d}{N}$ denotes the normalized cost of the coup.

Let us now distinguish two cases.

(i) *Exogenous p:* the leader cannot affect the probability of success of a coup, which is exogenously given.

(ii) *Endogenous p:* the leader can lower this probability by investing a portion *per capita* $C \geq 0$ of his resources on counter-insurgency infrastructures (e.g., secret police, equipment of the army, infiltration of opposite groups). Therefore, the probability is some given function $p = p(C)$. A simple choice made below is

$$p(C) = p_0 - \delta C \tag{2.5}$$

(however, more general forms of the function $p(C)$ could be considered). The coefficients $p_0 \in (0,1]$ and $\delta > 0$ in (2.5) are exogenous parameters; the value of δ is a measure of the efficiency of counter-insurgency structures and plays an important role in the following analysis. If $C = 0$, we recover the exogenous case with $p = p_0$.

In both cases the mission of the leader consists in maximizing his overall profits.

Exogenous p. In this case the income of the leader is:
—in period 1,

$$A + (\tau\rho_S h_1 - I)N; \tag{2.6}$$

—in period 2,

$$\begin{cases} A + \tau\rho_S(h_1 + I)N & \text{if a coup takes place,} \\ A + \tau\rho_L(h_1 + I)N & \text{otherwise.} \end{cases} \tag{2.7}$$

By (2.6)–(2.7), the leader utility is

$$\begin{aligned} A + (\tau\rho_S h_1 - I)N + (1-\chi)[A + \tau\rho_L(h_1 + I)N] \\ + \chi[A + \tau\rho_S(h_1 + I)N](1-p), \end{aligned} \tag{2.8}$$

where

$$\chi := \begin{cases} 1 & \text{if a coup takes place,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.9}$$

Observe that the third term in (2.8) is the period 2 leader profit in the coup case, weighted by the probability $1 - p$ of failure of the coup.

It is assumed that no leader's debt is allowed. Then by (2.6) the leader investment I *per capita* is subject to the *budget constraint*

$$0 \leq I \leq \alpha + \tau\rho_S h_1. \tag{2.10}$$

Endogenous p. In this case, at the beginning of period 1 the leader decides how to divide his budget between investments I and counter-insurgency expenditure C. Then the period 1 leader income is

$$A + (\tau\rho_S h_1 - I - C)N, \tag{2.11}$$

whereas his income in period 2 is again given by (2.7). Hence his utility is

$$\begin{aligned} A + (\tau\rho_S h_1 - I - C)N + (1-\chi)[A + \tau\rho_L(h_1 + I)N] \\ + \chi[1 - p(C)]([A + \tau\rho_S(h_1 + I)N] \end{aligned} \tag{2.12}$$

with χ given by (2.9), to be maximized under the budget constraint

$$0 \leq I + C \leq \alpha + \tau\rho_S h_1. \tag{2.13}$$

REMARK 2.1. If (H_0) holds, for small values of α the profit of undertaking a private industrial activity is greater than the period 2 leader's profit in the coup case (see (2.2), (2.3) and (2.7)). Therefore, assumption (H_0) renders the fact that a higher profit from private industrial activity (which corresponds to a greater economic development) reduces the risk of a coup. Observe that assumption (H_0) can be read as a condition on τ,

$$\tau < 1 - \frac{\rho_S}{\rho_L}.$$

Hence for given ρ_S, ρ_L assumption (H_0) is satisfied if τ is low enough—namely, the leader could reduce the risk of a coup by lowering taxes. We shall not address this point since we regard τ as exogenously given (in this connection, see [3]).

Let us summarize for convenience the quantities which appear in the model:

- $\alpha \in [0, \infty)$ represents the natural resource revenue *per capita*;
- $p \in [0, 1]$ is the probability of success of a coup. In the endogenous case $p_0 \in (0, 1]$ is the probability of success in the absence of repressive expenditure;
- $\tau \in (0, 1)$ is the optimal tax rate;
- h_1, h_2 are the infrastructural conditions of period 1 and 2, respectively. They are linked by the equality $h_2 = h_1 + I$, $I \geq 0$ being the leader investment *per capita* in period 1;
- ρ_S, ρ_L are exogenous parameters related with the primitive, respectively the industrial production. By assumption, $\rho := \rho_L - \rho_S > 0$;
- $d > 0$ denotes the cost the coup leader incurs if the coup fails, and $D := \frac{d}{N}$ denotes the normalized cost;

- $C \geq 0$ is the leader expenditure *per capita* in period 1 on counter-insurgency structures, in the endogenous case;
- $\delta > 0$ is a measure of efficiency of the counter-insurgency structures.

3. Main results: Exogenous p

3.1. The coup region. By assumption (H_0) and condition (2.4) we have the following

Proposition 3.1. *A coup takes place if and only if:*

$$0 \leq I < \frac{\alpha p - D(1-p)}{(1-\tau)\rho - p\tau\rho_S} - h_1 =: I_c(\alpha, p) \quad (\alpha \in [0, \infty), p \in [0, 1]). \tag{3.1}$$

By Proposition 3.1 and the budget constraint (2.10), a coup is attempted if and only if inequality (3.1) is satisfied in the *budget region*

$$R_b(\alpha) := \{(I, p) \,|\, 0 \leq p \leq 1, 0 \leq I \leq \alpha + \tau\rho_S h_1\} \quad (\alpha \in [0, \infty)). \tag{3.2}$$

This suggest to define the *coup region* as the subdomain of $R_b(\alpha)$ where the period 1 investment is not sufficient to suppress the possibility of a coup:

$$R_c(\alpha) := \{(I, p) \in R_b(\alpha) \,|\, 0 \leq I < I_c(\alpha, p)\}. \tag{3.3}$$

For any $\alpha \in [0, \infty)$ we also consider the *no-coup region*

$$R_{nc}(\alpha) := R_b(\alpha) \backslash R_c(\alpha). \tag{3.4}$$

Theorem 3.2. (a) *The coup region $R_c(\alpha)$ is nonempty if and only if*

$$\alpha > \alpha_* := [(1-\tau)\rho - \tau\rho_S]h_1. \tag{A_1}$$

(b) *Let (A_1) hold, and either*

$$(1-\tau)\rho - \tau\rho_S > 1, \tag{H_1}$$

or

$$0 < (1-\tau)\rho - \tau\rho_S < 1 \tag{H_2}$$

and

$$\alpha \leq \alpha^* := \frac{1 + \tau\rho_S}{1 + \tau\rho_S - (1-\tau)\rho}\alpha_*. \tag{A_2}$$

Then

$$R_c(\alpha) = \{(I, p) \,|\, P_0(\alpha) < p \leq 1, 0 \leq I < I_c(\alpha, p)\}, \tag{3.5}$$

where

$$P_0(\alpha) := \frac{(1-\tau)\rho h_1 + D}{\alpha + \tau\rho_S h_1 + D}. \tag{3.6}$$

(c) *Let* (H_2) *hold, and*

$$\alpha > \alpha^*. \tag{A_3}$$

Then the coup region is

$$\begin{aligned} R_c(\alpha) = &\{(I,p) \mid P_0(\alpha) < p \le P_1(\alpha), 0 \le I < I_c(\alpha,p)\} \\ &\cup \{(I,p) \mid P_1(\alpha) < p \le 1, 0 \le I \le \alpha + \tau\rho_S h_1\}, \end{aligned} \tag{3.7}$$

where

$$P_1(\alpha) := \frac{(1-\tau)\rho[\alpha + (1+\tau\rho_S)h_1] + D}{(1+\tau\rho_S)(\alpha + \tau\rho_S h_1) + D}. \tag{3.8}$$

The coup region is represented in Figures 2–4 for the cases listed in Theorem 3.2.

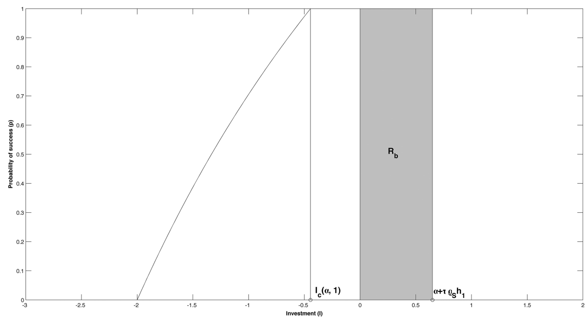

Figure 2. $\alpha < \alpha_*$: $R_c(\alpha) = \emptyset$.

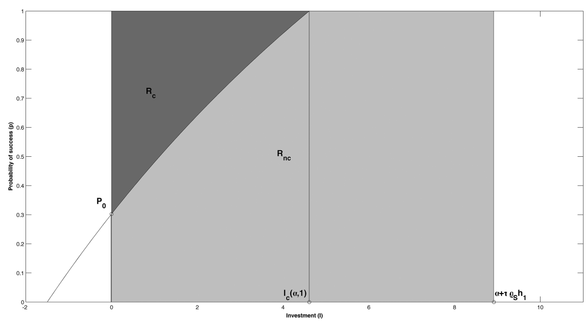

Figure 3. Either (H_1) and $\alpha > \alpha_*$, or (H_2) and $\alpha_* < \alpha < \alpha_*$.

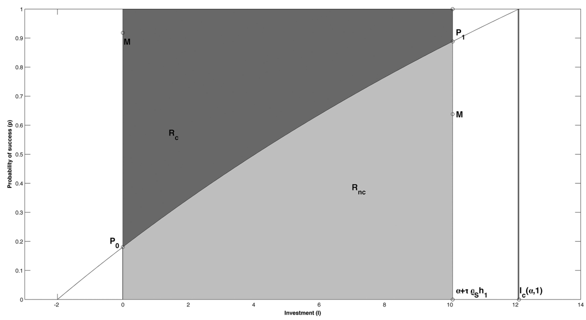

Figure 4. (H_2) and $\alpha > \alpha_*$.

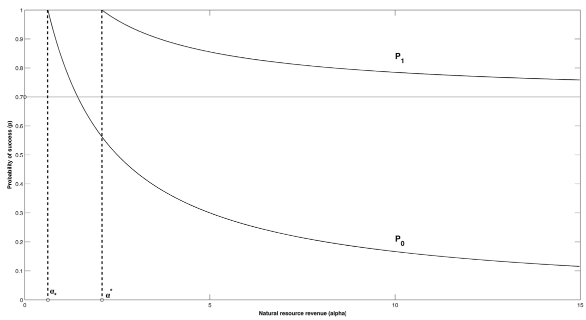

Figure 5. The functions $P_0(\cdot)$ and $P_1(\cdot)$ if (H_2) holds.

REMARK 3.1. It is immediately seen that the map $P_0(\cdot)$ defined in (3.6) is decreasing and convex on (α_*, ∞), $P_0(\alpha_*) = 1$ and $\lim_{\alpha\to\infty} P_0(\alpha) = 0$. Similarly, if (H_2) holds, the map $P_1(\cdot)$ defined in (3.8) decreasing and convex on (α^*, ∞), $P_1(\alpha^*) = 1$ and there holds

$$\lim_{\alpha\to\infty} P_1(\alpha) = \frac{(1-\tau)\rho}{1+\tau\rho_S} =: P_1(\infty). \tag{3.9}$$

Observe that $P_1(\infty) < 1$ if and only if (H_2) holds. The behaviour of the functions $P_0(\cdot)$, $P_1(\cdot)$ is represented in Figure 5.

Theorem 3.2 follows immediately from Lemmata 5.1–5.3 (see Subsection 5.1). Specifically, Lemma 5.1 shows that the coup region is nonempty if and only if the natural resource income is sufficiently high (i.e., if $\alpha > \alpha_*$), and no coup is attempted if its probability of success is too low (i.e., if $p \in [0, P_0(\alpha)]$; see Theorem 3.2(a),(b)).

If (A_1) holds and the leader wants to avoid a coup having probability of success $p > P_0(\alpha)$, the only possibility he has is to invest at least the amount $I_c(\alpha, p)$ in period 1. In view of the budget constraint (2.10), to check the feasibility of this policy we must compare $I_c(\alpha, p)$ with the maximal budget $\alpha + \tau\rho_S h_1$. In fact, the leader can afford making the minimal investment $I_c(\alpha, p)$ if and only if

$$I_c(\alpha, p) \le \alpha + \tau\rho_S h_1. \tag{3.10}$$

By Lemmata 5.2–5.3 the above inequality is satisfied in the following cases (see Figures 3–4):

(1) (H_1) holds, $\alpha > \alpha_*$ and $p \in [P_0(\alpha), 1]$;
(2) (H_2) holds, $\alpha_* < \alpha < \alpha^*$ and $p \in [P_0(\alpha), 1]$;
(3) (H_2) holds, $\alpha > \alpha^*$, and $p \in [P_0(\alpha), P_1(\alpha)]$.

Instead, (3.10) is not satisfied if (see Figure 4):

(4) (H_2) holds, $\alpha > \alpha^*$, and $p \in (P_1(\alpha), 1]$.

Hence Theorem 3.2(b),(c) follows.

In the following, every situation where inequality (3.10) is satisfied (i.e., if $\alpha < \alpha_*$, or $\alpha > \alpha_*$ and $p \in [0, P_0(\alpha))$, or in cases (1)–(3) above) will be referred to as Case A; otherwise (i.e., in the above case (4)) we shall speak of Case B. Therefore, if (H_1) holds, Case A covers all possiblilities, $\alpha \ge 0$ and $p \in [0, 1]$. Instead, if (H_2) holds, we are in Case A if $\alpha \ge 0$ and $p \in [0, P_1(\alpha)]$, in Case B otherwise (see Figure 5). We set

$$I_m(\alpha, p) := \min\{I_c(\alpha, p), \alpha + \tau\rho_S h_1\} = \begin{cases} I_c(\alpha, p) & \text{in Case } A, \\ \alpha + \tau\rho_S h_1 & \text{in Case } B. \end{cases} \tag{3.11}$$

By (3.11) and Theorem 3.2, whenever $\alpha > \alpha^*$ there holds

$$R_c(\alpha) = \{(I, p) \mid P_0(\alpha) \le p \le 1, 0 \le I < I_m(\alpha, p)\}. \tag{3.12}$$

A different description of the coup region can be given by observing that

$$I = I_c(\alpha, p) \quad \Leftrightarrow \quad p = P_c(\alpha, I) := \frac{(1-\tau)\rho(h_1 + I) + D}{\alpha + \tau\rho_S(h_1 + I) + D}. \tag{3.13}$$

There holds

$$\frac{\partial P_c}{\partial \alpha} = -\frac{P_c}{\alpha + \tau\rho_S(h_1 + I) + D} < 0, \tag{3.14}$$

$$\frac{\partial P_c}{\partial I} = \frac{\alpha(1-\tau)\rho + D[(1-\tau)\rho - \tau\rho_S]}{[\alpha + \tau\rho_S(h_1 + I) + D]^2} > 0, \tag{3.15}$$

$$\frac{\partial^2 P_c}{\partial I^2} = -\frac{2\tau\rho_S}{\alpha + \tau\rho_S(h_1 + I) + D}\,\frac{\partial P_c}{\partial I} < 0, \tag{3.16}$$

thus the map $P_c(\alpha, \cdot)$ is increasing and concave. Clearly,

$$P_c(\alpha, 0) = \frac{(1-\tau)\rho h_1 + D}{\alpha + \tau\rho_S h_1 + D} = P_0(\alpha) \tag{3.17}$$

and

$$P_c(\alpha, \alpha + \tau\rho_S h_1) = \frac{(1-\tau)\rho[\alpha + (1+\tau\rho_S)h_1] + D}{(1+\tau\rho_S)(\alpha + \tau\rho_S h_1) + D} = P_1(\alpha). \tag{3.18}$$

It follows from (3.14)–(3.15) that the size of the coup region $R_c(\alpha)$ monotonically increases with α. Moreover,

$$\lim_{\alpha\to\infty} P_c(\alpha, I) = 0 \quad \text{for any } I \in (0, \infty).$$

According to the above remarks, there holds

$$R_c(\alpha) = \{(I, p) \,|\, 0 \le I < I_m(\alpha, 1), P_c(\alpha, I) < p \le 1\} \tag{3.19}$$

with I_m defined by (3.11).

3.2. The maximization problem. Denote by U the leader (normalized) utility function. According to (2.8), there holds

$$U = U_c(\alpha, p, I) := (2-p)(\alpha + \tau\rho_S h_1) + [(1-p)\tau\rho_S - 1]I \quad \text{in } R_c(\alpha), \tag{3.20}$$

$$U = U_{nc}(\alpha, I) := 2\alpha + \tau(\rho_S + \rho_L)h_1 + (\tau\rho_L - 1)I \quad \text{in } R_{nc}(\alpha) \tag{3.21}$$

(observe that U_{nc} does not depend on p). By taking the budget constraint (2.10) into account, the maximization problem of the leader can be formulated as follows:

To find the maximum of the function U in the budget region $R_b(\alpha)$, for any given value of α and p.

For fixed values of the exogenous parameters α and p, the solution of the maximization problem only depends on the amount I of investment. Every choice of $I \in [0, \alpha\tau\rho_S h_1]$ can be regarded as a leader *policy* (in this respect, see Definition 4.1 below), and the policy adopted by the leader is the solution of the maximization problem.

The following result will be proven (see Figure 4, where the maximum points of the utility function are denoted by the letter M).

THEOREM 3.3. *Let*

$$\tau\rho_S < 1 < \tau\rho_L. \tag{H_4}$$

Then:

(a) *in Case A the leader uses all his resources to make investments;*
(b) *in Case B the leader makes no investment.*

The above result is easily understood by observing that $\tau\rho_L$ and $(1-p)\tau\rho_S$ are the return of one unit of period 1 leader's investment if no coup takes place, respectively if it does (see (2.8)), and $(1-p)\tau\rho_S < 1 < \tau\rho_L$ by (H_4). Therefore, if the leader can avoid a coup by investing, it is convenient for him to invest as much as possible since the return $\tau\rho_L$ is larger than 1. Instead, it is convenient for the leader not to invest at all if his budget does not allow him to exit the coup region, since the return $(1-p)\tau\rho_S$ is less than 1.

Let us complete the discussion by assuming that, instead of (H_4), there holds

$$(H_5) \qquad \tau(\rho_S+1) < \tau\rho_L < 1 \quad \Leftrightarrow \quad \tau\rho_L < 1,\, \rho > 1.$$

In this case the leader makes the minimal investment needed to avoid a coup, if he can afford it. In fact, as before it is convenient for him to exit the coup region since $(1-p)\tau\rho_S < \tau\rho_L$, yet investing more makes no sense since now the return $\tau\rho_L$ is less than 1.

THEOREM 3.4. *Let (H_5) be satisfied. Then:*

(a) *if $\alpha < \alpha_*$, or if $\alpha > \alpha_*$ and $p \in [0, P_0(\alpha))$, or in Case B the leader makes no investment;*
(b) *in every other situation pertaining to Case A the leader makes the minimal investment needed to avoid a coup.*

REMARK 3.2. By assumption (H_5) the quantities $\tau\rho_L$ and $(1-p)\tau\rho_S$ are "not too close", in the sense that their difference is always larger than τ. Under the weaker condition

$$\tau\rho_S < \tau\rho_L < 1,$$

if ρ_L is very close to ρ_S there is no clear-cut advantage for the leader in avoiding a coup. Then it is easily seen that in Case A, for α sufficiently large, there is a threshold value $\tilde{P}(\alpha) > P_0(\alpha)$ such that the leader only makes investments if $p \in (P_0(\alpha), \tilde{P}(\alpha))$ (in this connection, see [3]). We omit the details.

4. MAIN RESULTS: ENDOGENOUS p

In this section we assume that (A_1) holds. Moreover, we suppose that the leader can lower the probability of success of a coup by spending part of his resources on repression.

4.1. The coup region. By (2.13) the budget region is now

$$(4.1) \qquad R_b(\alpha) := \{(I, C, p) \,|\, 0 \le p \le 1, I \ge 0, C \ge 0, I + C \le \alpha + \tau\rho_S h_1\}.$$

To suppress the possibility of a coup having a probability of success $p_0 > P_0(\alpha)$, for any $I_0 \in [0, I_m(\alpha, p_0)]$ a repressive expenditure $C > 0$ is needed, such that

$$p_0 - \delta C = P_c(\alpha, I_0) \quad \Leftrightarrow \quad C = C(\alpha, \delta, I_0, p_0),$$

where

$$C(\alpha, \delta, I, p) := \frac{p - P_c(\alpha, I)}{\delta} \tag{4.2}$$

(see (3.19); observe that $C(\alpha, \delta, I_0, p_0) \geq 0$). Therefore, a coup takes place if and only if

$$0 \leq I \leq I_m(\alpha, p_0) \quad \text{and} \quad 0 \leq P_c(\alpha, I) < p_0 - \delta C \quad (p_0 \in [P_0(\alpha), 1]). \tag{4.3}$$

This leads to define the coup region as follows:

$$R_c(\alpha, \delta) := \{(I, C, p) \in R_b(\alpha) \mid P_0(\alpha) \leq p \leq 1, 0 \leq I < I_m(\alpha, p), \\ 0 \leq C < C(\alpha, \delta, I, p)\}. \tag{4.4}$$

By definition, $R_c(\alpha, \delta) \neq \emptyset$ and $R_c(\alpha, \delta) \subseteq R_b(\alpha)$. As before, we also consider the no-coup region $R_{nc}(\alpha, \delta) := R_b(\alpha) \backslash R_c(\alpha, \delta)$.

For every $p_0 \in [0, 1]$ the intersection of the budget region $R_b(\alpha)$ with the plane $p = p_0$ is a right-angled triangle in the plane (I, C, p_0), whose hypotenuse is the *budget line* $I + C = \alpha + \tau p_S h_1$. Hereafter we set $R_{b0}(\alpha) := R_b(\alpha) \cap \{p = 0\}$. The projection of $R_c(\alpha, \delta)$ on the coordinate plane $C = 0$ is the subset (3.12), whereas the projection on the coordinate plane $p = 0$ of the intersection $R_c(\alpha, \delta) \cap \{p = p_0\}$ is

$$R_c(\alpha, \delta, p_0) \\ := \{(I, C) \in R_{b0}(\alpha) \mid 0 \leq I \leq I_m(\alpha, p_0), 0 \leq C < C(\alpha, \delta, I, p_0)\}. \tag{4.5}$$

Set $R_{nc}(\alpha, \delta, p_0) := R_{b0}(\alpha) \backslash R_c(\alpha, \delta, p_0)$ $(p_0 \in [P_0(\alpha), 1])$. By abuse of language, in the following also $R_{b0}(\alpha)$, $R_c(\alpha, \delta, p_0)$ and $R_{nc}(\alpha, \delta, p_0)$ will be called budget region, coup region and no-coup region, respectively $(p_0 \in [0, 1])$.

By (3.15)–(3.16) and (4.2) there holds

$$\frac{\partial C}{\partial I} = -\frac{1}{\delta} \frac{\partial P_c}{\partial I} < 0, \quad \frac{\partial^2 C}{\partial I^2} = -\frac{1}{\delta} \frac{\partial^2 P_c}{\partial I^2} > 0,$$

hence the function $C(\alpha, \delta, \cdot, p)$ is decreasing and convex. The intersections of its graph with the axes $C = 0$ and $I = 0$ are the points $(I_c(\alpha, p_0), 0)$, respectively $(0, C_c(\alpha, \delta, p_0))$, where

$$C_c(\alpha, \delta, p) := C(\alpha, \delta, 0, p) = \frac{p - P_0(\alpha)}{\delta}$$

(see (3.13) and (3.17)).

4.2. How to avoid a coup? Let $p_0 \in [P_0(\alpha), 1]$ be fixed. Figures 6–10 represent the coup region $R_c(\alpha, \delta, p_0)$ in the following mutually exclusive cases:

$$\max\{I_c(\alpha, p_0), C_c(\alpha, \delta, p_0)\} < \alpha + \tau\rho_S h_1, \tag{4.6}$$

$$I_c(\alpha, p_0) < \alpha + \tau\rho_S h_1 < C_c(\alpha, \delta, p_0), \tag{4.7}$$

$$\min\{I_c(\alpha, p_0), C_c(\alpha, \delta, p_0)\} > \alpha + \tau\rho_S h_1, \tag{4.8}$$

$$C_c(\alpha, \delta, p_0) < \alpha + \tau\rho_S h_1 < I_c(\alpha, p_0). \tag{4.9}$$

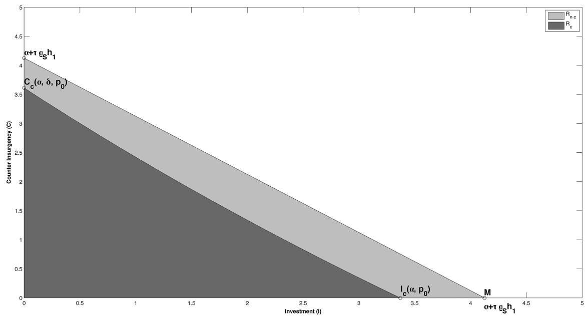

Figure 6. Condition (4.6).

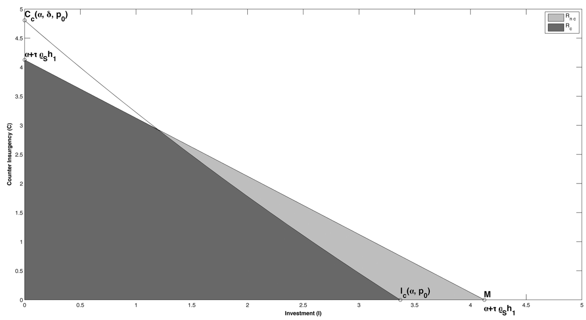

Figure 7. Condition (4.7).

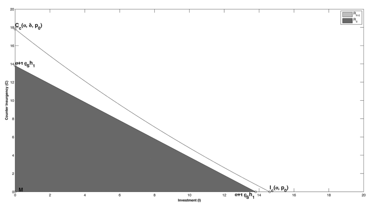

Figure 8. Condition (4.8): $R_{nc}(\alpha, \delta, p_0) = \emptyset$.

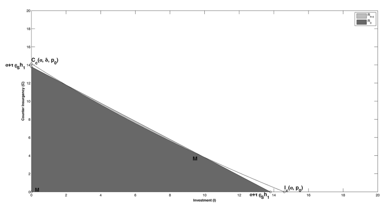

Figure 9. Condition (4.8): $R_{nc}(\alpha, \delta, p_0) \neq \emptyset$.

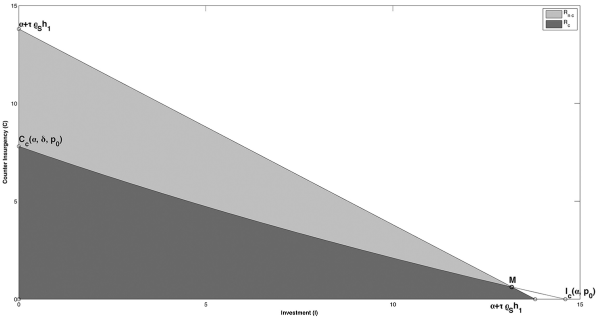

Figure 10. Condition (4.9).

Observe that (4.6) and (4.7) refer to Case A, whereas (4.8) and (4.9) refer to Case B. Moreover, the no-coup region $R_{nc}(\alpha, \delta, p_0)$ can be empty if (4.8) holds (see Figure 8), whereas it is nonempty in the other cases.

The above conditions (4.6)–(4.9) give rise to different leader's spending strategies and suggest a classification of possible policies. This is the content of the following definition.

Definition 4.1. Every point (I, C) in the budget region $R_{b0}(\alpha)$ is called a *policy*. We call *democratic* a policy $(I, 0)$ with $I > 0$, *autocratic* a policy $(0, C)$ with $C > 0$, and *intermediate* a policy (I, C) with $I > 0$, $C > 0$. A policy (I_1, C_1) is said to be *more autocratic* than a policy (I_2, C_2), if $I_1 < I_2$ and $C_1 > C_2$.

Suppose that the leader wants to avoid a coup with probability of success $p_0 > P_0(\alpha)$. Then in period 1 he has three possible spending strategies (see Figure 11):

(a) to spend the amount $I_c(\alpha, p_0)$ on productive investments and nothing on counter-insurgency;
(b) to spend the amount $C_c(\alpha, \delta, p_0)$ on counter-insurgency and nothing on investments;
(c) to spend the amount $I_0 \in (0, I_c(\alpha, p_0))$ on productive investments and the amount $C(\alpha, \delta, I_0, p_0) \in (0, C_c(\alpha, \delta, p_0))$ on counter-insurgency.

According to Definition 4.1, policy (a) is democratic, policy (b) autocratic, and policy (c) intermediate. Clearly, as I_0 ranges from 0 to $I_c(\alpha, p_0)$, there is a string of intermediate policies of type (c) which connects (a) to (b).

Before comparing the above policies between themselves, it is natural to ask whether they are feasible at all. Clearly, all of them are feasible if (4.6) holds, since the function $C(\alpha, \delta, \cdot, p)$ is convex (see Figure 6). Instead, only democracy and "weak autocracies" are possible if (4.7) holds, since in this case the cost of the autocratic policy exceeds the leader budget (see Figure 7). Similarly, only autocracy and "weak democracies" are possible if (4.9) holds, since the cost of

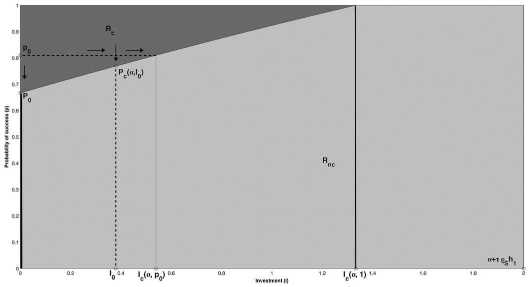

Figure 11. Different policies to exit the coup region.

the democratic policy exceeds the budget (see Figure 10). If (4.8) holds, neither democracy nor autocracy is feasible, yet intermediate policies of type (b) are possible since $C(\alpha, \delta, \cdot, p)$ is convex (see Figures 8–9).

The prevalence of one situation over the others can be interpreted as follows. Conditions (4.6)–(4.9) are distinguished by the following inequalities:

$$\begin{cases} I_c(\alpha, p_0) \gtrless \alpha + \tau\rho_S h_1, \\ C_c(\alpha, \delta, p_0) \gtrless \alpha + \tau\rho_S h_1. \end{cases}$$

For fixed p_0 the first inequality only depends on α, and determines whether we are in Case A or in Case B—namely, whether the leader can afford the democratic policy (in this connection, see (3.10) of the exogenous case). As for the second, obviously there holds

$$C_c(\alpha, \delta, p_0) = \frac{p_0 - P_0(\alpha)}{\delta} \gtrless \alpha + \tau\rho_S h_1 \quad \Leftrightarrow \quad \delta \lessgtr \frac{p_0 - P_0(\alpha)}{\alpha + \tau\rho_S h_1}.$$

This suggests to define the *critical efficiency*

$$\hat{\delta} \equiv \hat{\delta}(\alpha, p_0) := \frac{p_0 - P_0(\alpha)}{\alpha + \tau\rho_S h_1}, \tag{4.10}$$

to be compared with the actual efficiency δ of the leader counter-insurgency structures. Clearly,

$$\delta \lessgtr \hat{\delta}(\alpha, p_0) \quad \Leftrightarrow \quad C_c(\alpha, \delta, p_0) \gtrless \alpha + \tau\rho_S h_1.$$

The above inequalities show that the autocratic policy is not feasible if $\delta < \hat{\delta}(\alpha, p_0)$—namely, the leader cannot afford this policy, if the efficiency of his counter-insurgency structures is too low.

To sum up, the choice of the policy to avoid a coup having probability of success p_0 depends on the parameters α and δ through the sign of the difference $\delta - \hat{\delta}(\alpha, p_0)$, and *a change of this sign can give rise to a change in the leader policy*.

This leads to study the dependence of $\hat{\delta}(\alpha, p_0)$ on its arguments, as we do in Subsection 6.2. Similar arguments hold for the feasibility of intermediate policies, as well as to compare different policies between themselves (see below).

Let us point out that the above discussion only concerns the possibility of avoiding a coup, whereas the actual policy is chosen by maximizing the leader utility function (see Subsections 4.3 and 6.1 for the relationship between these two aspects). In fact, let $I_c(\alpha, p_0) < \alpha + \tau\rho_S h_1$, so that we are in Case A, and suppose that

$$(4.11)\quad \delta > \frac{p_0 - P_0(\alpha)}{I_c(\alpha, p_0)} > \frac{p_0 - P_0(\alpha)}{\alpha + \tau\rho_S h_1} \quad \Leftrightarrow \quad C_c(\alpha, \delta, p_0) < I_c(\alpha, p_0) < \alpha + \tau\rho_S h_1.$$

Then we are in case (4.6) where all policies are feasible, and, since $C_c(\alpha, \delta, p_0) < I_c(\alpha, p_0)$, *as for avoiding a coup* the autocratic policy (b) is more convenient than the democratic policy (a). The critical efficiency relativi to this comparison is (see the first inequality in (4.11)):

$$(4.12)\qquad \hat{\delta}(\alpha, p_0) := \frac{p_0 - P_0(\alpha)}{I_c(\alpha, p_0)}.$$

However, if (H_4) holds, the democratic policy is preferred since it maximizes the leader utility (see Theorem 4.1). This can be understood by observing that the utility function depends on the productive investment I, which produces higher revenues from industrial production in period 2 (see (4.15)).

Finally, iet us compare policies (a), (b) and (c). Since the cost of (c) is $I_0 + C(\alpha, \delta, I_0, p_0)$, comparing it with (a) gives the inequality

$$I_c(\alpha, p_0) \gtreqless I_0 + C(\alpha, \delta, I_0, p_0) = I_0 + \frac{p_0 - P_c(\alpha, I_0)}{\delta} \quad \Leftrightarrow \quad \delta \gtreqless \hat{\delta}(\alpha, I_0, p_0),$$

where now the relevant critical efficiency is

$$(4.13)\qquad \hat{\delta}(\alpha, I_0, p_0) := \frac{p_0 - P_c(\alpha, I_0)}{I_c(\alpha, p_0) - I_0}.$$

Therefore, to avoid a coup every intermediate policy is more convenient than democracy, if $\delta > \hat{\delta}(\alpha, I_0, p_0)$—namely, if the efficiency of the counter-insurgency structures is "sufficiently high". Observe that the first inequality in (4.11) is a particular case of (4.13) with $I_0 = 0$. Similarly, comparing (c) with the autocratic policy (b) gives

$$C_c(\alpha, \delta, p_0) \gtreqless I_0 + \frac{p_0 - P_c(\alpha, I_0)}{\delta} \quad \Leftrightarrow \quad \delta \lesseqgtr \hat{\delta}(\alpha, I_0, p_0) := \frac{P_c(\alpha, I_0) - P_0(\alpha)}{I_0},$$

which can be regarded as a particular case of (4.13) with $p_0 = P_0(\alpha)$, $I_c(\alpha, p_0) = 0$.

In general, let us compare two policies such that $0 \le I_1 \le I_2 \le I_m(\alpha, p_0)$, thus $C(\alpha, \delta, I_1, p_0) \ge C(\alpha, \delta, I_2, p_0)$. Their cost to avoid a coup is $I_i + C(\alpha, \delta, I_i, p_0)$ $(i = 1, 2)$, hence

$$I_1 + \frac{p_0 - P_c(\alpha, I_1)}{\delta} \gtrless I_2 + \frac{p_0 - P_c(\alpha, I_2)}{\delta} \quad \Leftrightarrow \quad \delta \lesseqgtr \frac{P_c(\alpha, I_2) - P_c(\alpha, I_1)}{I_2 - I_1}.$$

The last inequality determines whether the more autocratic policy 1 is more convenient than policy 2 to avoid a coup. Observe that, since $P_c(\alpha, \cdot)$ is concave (see (3.16)),

$$\frac{\partial P_c}{\partial I}(\alpha, I_1) \ge \frac{P_c(\alpha, I_2) - P_c(\alpha, I_1)}{I_2 - I_1} \ge \frac{\partial P_c}{\partial I}(\alpha, I_2).$$

Therefore, if $\delta \ge \frac{\partial P_c}{\partial I}(\alpha, I_1)$, the more autocratic policy 1 is more convenient than every policy 2; similarly, if $\delta \le \frac{\partial P_c}{\partial I}(\alpha, I_2)$, the more democratic policy 2 is more convenient than every policy 1.

4.3. The maximization problem. In view of (2.12), the utility function is

$$U = U_c(\alpha, p_0, I, C) := \alpha + \tau\rho_S h_1 - I - C + [1 - (p_0 - \delta C)](\alpha + \tau\rho_S h_1 + \tau\rho_S I) \quad \text{in } R_c(\alpha, \delta, p_0), \tag{4.14}$$

$$U = U_{nc}(\alpha, I, C) := \alpha + \tau\rho_S h_1 - I - C + (\alpha + \tau\rho_L h_1 + \tau\rho_L I) \quad \text{in } R_{nc}(\alpha, \delta, p_0). \tag{4.15}$$

Observe that both U_c and U_{nc} can be regarded as defined in the whole of $R_{b0}(\alpha)$, and by (4.14)–(4.15) there holds $U_c(I_0, C_0) \le U_{nc}(I_0, C_0)$ for all $(I_0, C_0) \in R_{b0}(\alpha)$.

THEOREM 4.1. *Let* (H_4) *be satisfied. Then in Case A the leader chooses a democratic policy and uses all his resources to make investments. In Case B,*

(a) *if* (4.8) *holds, the leader either chooses an intermediate policy, or does not make any investment;*
(b) *if* (4.9) *holds and* $\delta(\alpha + \tau\rho_S h_1) > 1$, *the leader chooses an intermediate policy* $(\bar{I}, \bar{C})$. *The amounts* $\bar{I} \in (0, I_c(\alpha, p_0))$ *and* $\bar{C} \in (0, C_c(\alpha, \delta, p_0))$ *spent on productive investments, respectively on counter-insurgency, satisfy the system*

$$\begin{cases} C = C(\alpha, \delta, I, p_0) \\ C + I = \alpha + \tau\rho_S h_1. \end{cases} \tag{4.16}$$

The content of Theorem 4.1 is depicted in Figures 6–10, where the maximum points of the utility function are denoted by the letter M. The situation where (H_5) holds, which can be similarly investigated, is omitted.

4.4. Changes of policy. Let us now examine whether the leader policy changes, depending on changes of the exogenous parameters α and p_0. The solution of the maximization problem being known in each situation (4.6)–(4.9) (see Theorem 4.1), this amounts to study transitions between different situations as α and p_0 vary. In turn, these transitions depend on the behaviour of the critical efficiency. Since we want to compare the three quantities $C_c(\alpha, \delta, p_0)$, $I_c(\alpha, p_0)$ and $\alpha + \tau\rho_S h_1$, we shall study the general expression

$$\hat{\delta}(\alpha, p_0) = \frac{p_0 - P_0(\alpha)}{I_m(\alpha, p_0)}, \tag{4.17}$$

which reduces to (4.10) in Case A and to (4.12) in Case B. This gives the following result.

THEOREM 4.2. (a) *If (H_1) holds, for any $\alpha \geq 0$ and $p_0 \in [0, 1]$ the outcome is democracy. The same is true for any $\alpha \geq 0$ and $p_0 \in [0, P_1(\infty)]$, if (H_2) holds.*

(b) *Let (H_2) hold, and let $p_0 \in (P_1(\infty), 1]$. Then:*
  (i) *if $\delta > \hat{\delta}(\alpha^*, 1)$, as α increases from 0 to ∞ the leader switches from democracy to an intermediate policy. The change of policy occurs beyond some value $\alpha^{**} = \alpha^{**}(\delta) \geq \alpha^*$, which decreases as δ increases;*
  (ii) *if $\delta < \hat{\delta}(\alpha^*, 1)$ as α increases from 0 to ∞ the leader switches from democracy to an intermediate policy, possibly not spending at all in a finite interval of values of $\alpha > \alpha^*$. The values where the changes of policy occur decrease as δ increases.*

5. EXOGENOUS p: PROOFS

5.1. The coup region. The following lemmata provide the proof of Theorem 3.2.

LEMMA 5.1. (a) *There exists $p \in [0, 1]$ such that $I_c(\alpha, p) > 0$ if and only if (A_1) holds.*

(b) *Let (A_1) hold. No coup takes place if $p \in [0, P_0(\alpha)]$, with $P_0(\alpha)$ defined by* (3.6).

PROOF. By assumption (H_0), from (3.1) we get

$$\frac{\partial I_c}{\partial p} = \frac{\alpha(1-\tau)\rho + D[(1-\tau)\rho - \tau\rho_S]}{[(1-\tau)\rho - p\tau\rho_S]^2} > 0. \tag{5.1}$$

Then $I_c(\alpha, p) > 0$ for some $p \in [0, 1]$ if and only if $I_c(\alpha, 1) > 0$. Since

$$I_c(\alpha, 1) > 0 \quad \Leftrightarrow \quad \frac{\alpha}{(1-\tau)\rho - \tau\rho_S} > h_1 \quad \Leftrightarrow \quad \alpha > [(1-\tau)\rho - \tau\rho_S]h_1,$$

claim (a) follows. Claim (b) is immediate, by observing that $I_c(\alpha, p) > 0$ if and only if $p > P_0(\alpha)$. □

LEMMA 5.2. *Let* (A_1) *hold, and let either* (H_1)*, or* (H_2) *and* (A_2) *be satisfied. Then for every* $p \in [0,1]$ *inequality* (3.10) *is satisfied.*

PROOF. By assumption (A_1) there holds $I_c(\alpha, 1) > 0$. On the other hand,

$$I_c(\alpha, 1) = \frac{\alpha}{(1-\tau)\rho - \tau\rho_S} - h_1 \leq \alpha + \tau\rho_S h_1$$

$$\Leftrightarrow \quad \alpha \frac{1 + \tau\rho_S - (1-\tau)\rho}{(1-\tau)\rho - \tau\rho_S} \leq (1 + \tau\rho_S)h_1,$$

which is satisfied if either (H_1), or (H_2) and (A_2) hold true. By (5.1), in either case it follows that

$$I_c(\alpha, p) \leq I_c(\alpha, 1) \leq \alpha + \tau\rho_S h_1$$

for every $p \in [0,1]$. □

LEMMA 5.3. *Let assumptions* (H_2) *and* (A_3) *be satisfied. Then*

$$\begin{cases} 0 \leq I_c(\alpha, p) \leq \alpha + \tau\rho_S h_1 & \text{if } p \in [P_0(\alpha), P_1(\alpha)], \\ \alpha + \tau\rho_S h_1 < I_c(\alpha, p) & \text{if } p \in (P_1(\alpha), 1], \end{cases} \tag{5.2}$$

with $P_1(\alpha)$ *defined by* (3.8).

PROOF. By studying the inequality

$$I_c(\alpha, p) > \alpha + \tau\rho_S h_1$$

under the present assumptions, we obtain plainly (5.2). It is immediately seen that

$$\frac{dP_1}{d\alpha} = \frac{-(1-\tau)\rho(1+\tau\rho_S)h_1 + D\{[(1-\tau)\rho - \tau\rho_S)] - 1\}}{[(1+\tau\rho_S)(\alpha + \tau\rho_S h_1) + D]^2} < 0 \tag{5.3}$$

by assumption (H_2), and equality (3.9) holds. □

5.2. The maximization problem. Let us prove Theorems 3.3 and 3.4.

PROOF OF THEOREM 3.3. By (3.21) and the second inequality in assumption (H_4) there holds

$$\frac{\partial U_{nc}}{\partial I} = \tau\rho_L - 1 > 0, \tag{5.4}$$

whereas by (3.20) and the first inequality in the same assumption

$$\frac{\partial U_c}{\partial I} = (1-p)\tau\rho_S - 1 < 0. \tag{5.5}$$

Moreover,

$$\frac{\partial U_c}{\partial p} = -\alpha - \tau\rho_S(h_1 + I) < 0. \tag{5.6}$$

If the coup region is empty, by (5.4) the maximum of $U = U_{nc}$ is attained at $I = \alpha + \tau\rho_S h_1$,

$$\max_{0 \le I \le \alpha + \tau\rho_S h_1} U_{nc}(\alpha, I) = U_{nc}(\alpha, \alpha + \tau\rho_S h_1) = \alpha + \tau\rho_L[\alpha + (1 + \tau\rho_S)h_1]; \tag{5.7}$$

the same holds if $\alpha > \alpha_*$ and $p \in [0, P_0(\alpha))$. In all other situations pertaining to Case A (see Subsection 3.1), by (5.5) for every fixed $p > P_0(\alpha)$ the maximum of $U_c(\alpha, p, \cdot)$ is attained at the point $(0, p)$. Moreover, by (5.6)

$$U_c(\alpha, p, 0) \le U_c(\alpha, P_0(\alpha), 0) \quad (p > P_0(\alpha)). \tag{5.8}$$

It is easily seen that

$$U_c(\alpha, P_0(\alpha), 0) < U_{nc}(\alpha, \alpha + \tau\rho_S h_1). \tag{5.9}$$

In fact, observe that

$$\rho_L - \rho_S > \rho_S(1 - \tau\rho_L),$$

since $\rho_L - \rho_S > 0$, whereas $1 - \tau\rho_L < 0$ by assumption (H_4). Rearranging the terms of the above inequality and multiplying by τh_1 we obtain

$$\tau\rho_L(1 + \tau\rho_S)h_1 > 2\tau\rho_S h_1. \tag{5.10}$$

On the other hand, there holds

$$\alpha(1 + \tau\rho_L) > 2\alpha > [2 - P_0(\alpha)]\alpha, \tag{5.11}$$

since by (H_4) there holds $\tau\rho_L > 1$. Adding inequalities (5.10) and (5.11) we obtain

$$[2 - P_0(\alpha)](\alpha + \tau\rho_S h_1) < \alpha + \tau\rho_L[\alpha + (1 + \tau\rho_S)h_1],$$

namely inequality (5.9).

From (5.8)–(5.9) and the above remarks claim (a) follows. Concerning (b), let $p \in [P_1(\alpha), 1]$. By (5.5) the maximum of $U = U_c$ is attained at $I = 0$, hence claim (b) follows. This completes the proof. □

PROOF OF THEOREM 3.4. Under assumption (H_5) both derivatives $\frac{\partial U_c}{\partial I}$ and $\frac{\partial U_{nc}}{\partial I}$ are negative (see (5.4)–(5.5)). Then claim (a) immediately follows. As for (b), we must compare the values

$$\max_{0 \le I \le I_c(\alpha, p)} U_c(\alpha, p, I) = U_c(\alpha, p, 0) = (2 - p)(\alpha + \tau\rho_S h_1)$$

and

$$\max_{I_c(\alpha,p)\le I\le \alpha+\tau\rho_S h_1} U_{nc}(\alpha, I) = U_{nc}(\alpha, I_c(\alpha, p))$$
$$= 2\alpha + \tau(\rho_S + \rho_L)h_1 + (\tau\rho_L - 1)I_c(\alpha, p).$$

Let us prove that the inequality

$$\mathcal{U}(\alpha, p) := U_{nc}(\alpha, I_c(\alpha, p)) - U_c(\alpha, p, 0) > 0 \tag{5.12}$$

is satisfied for $p \in [P_0(\alpha), 1]$ in subcases (1)–(2), and for $p \in [P_0(\alpha), P_1(\alpha)]$ in subcase (3) of Case A (see Subsection 3.1).

To this purpose, observe that

$$\begin{aligned} \mathcal{U}(\alpha, P_0(\alpha)) &= U_{nc}(\alpha, 0) - U_c(\alpha, P_0(\alpha), 0) \\ &= \tau\rho h_1 + P_0(\alpha)(\alpha + \tau\rho_S h_1) > 0, \end{aligned} \tag{5.13}$$

and

$$\frac{\partial^2 \mathcal{U}}{\partial p^2} = (\tau\rho_L - 1)\frac{\partial^2 I_c}{\partial p^2}$$
$$= (\tau\rho_L - 1)\frac{2\tau\rho_S\{\alpha(1-\tau)\rho + D[(1-\tau)\rho - \tau\rho_S]\}}{[(1-\tau)\rho - p\tau\rho_S]^3} < 0.$$

Hence inequality (5.12) follows, if there holds $\mathcal{U}(\alpha, 1) > 0$ in subcases (1)–(2), respectively $\mathcal{U}(\alpha, P_1(\alpha)) > 0$ in subcase (3).

It is easily seen that

$$\mathcal{U}(\alpha, 1) > 0 \quad \Leftrightarrow \quad h_1 > \frac{1-\rho}{(1-\tau)\rho - p\tau\rho_S},$$

which holds true since $\rho > 1$ (see (H_5)). This settles (1)–(2). As for (3), since $I_c(\alpha, P_1(\alpha)) = \alpha + \tau\rho_S h_1$, it is easy to check that

$$\mathcal{U}(\alpha, P_1(\alpha)) > 0 \quad \Leftrightarrow \quad \tau\rho h_1 > [1 - \tau\rho_L - P_1(\alpha)](\alpha + \tau\rho_S h_1).$$

A lengthy calculation shows that $1 - \tau\rho_L - P_1(\alpha) < 0$ since $\rho > 1$. Hence the result follows. □

6. Endogenous p: Proofs

6.1. The maximization problem. Let us prove Theorem 4.1.

Proof of Theorem 4.1. From (4.15) we get

$$\frac{\partial U_{nc}}{\partial C} = -1, \tag{6.14}$$

and by the second inequality in assumption (H_4)

$$\frac{\partial U_{nc}}{\partial I} = \tau\rho_L - 1 > 0. \tag{6.15}$$

Hence, if either (4.6) or (4.7) holds (see Figures 6–7),

$$\max_{R_{nc}} U = U_{nc}(\alpha, \alpha + \tau\rho_S h_1, 0). \tag{6.16}$$

If $\alpha < \alpha_+$, or $\alpha > \alpha_+$ and $p_0 \in [0, P_0(\alpha))$ there holds $I_m(\alpha, p_0) = I_c(\alpha, p_0) < 0$, thus the coup region $R_c(\alpha, \delta, p_0)$ is empty (see (4.5)) and the claim concerning Case A follows from (6.16).

On the other hand, by (4.14) and the first inequality in assumption (H_4) there holds

$$\frac{\partial U_c}{\partial I} = -1 + [1 - (p_0 - \delta C)]\tau\rho_S < -1 + \tau\rho_S < 0, \tag{6.17}$$

whereas

$$\frac{\partial U_c}{\partial C} = -1 + \delta(\alpha + \tau\rho_S h_1 + \tau\rho_S I). \tag{6.18}$$

By (6.17) the maximum of U_c is attained on the axis $I = 0$. Then by (6.18) the maximum point of U_c is determined by the sign of the derivative

$$\frac{\partial U_c}{\partial C}(\alpha, p_0, 0, C) = -1 + \delta(\alpha + \tau\rho_S h_1). \tag{6.19}$$

Let $p_0 \in [P_0(\alpha), 1]$, and let (4.7) hold. Then

$$\alpha + \tau\rho_S h_1 < \frac{p_0 - P_0(\alpha)}{\delta} < \frac{1}{\delta}, \tag{6.20}$$

whence by (6.19)

$$\frac{\partial U_c}{\partial C}(\alpha, p_0, 0, C) < 0 \quad \Rightarrow \quad \max_{R_c} U = U_c(\alpha, p_0, 0, 0). \tag{6.21}$$

Therefore,

$$\begin{aligned} \max_{R_c} U &= U_c(\alpha, p_0, 0, 0) \\ &\le U_{nc}(\alpha, 0, 0) \le U_{nc}(\alpha, \alpha + \tau\rho_S h_1, 0) = \max_{R_{nc}} U; \end{aligned} \tag{6.22}$$

here use of (6.15), (6.16) and (6.21) and the remark following (4.15) has been made.

If (4.6) holds, the only difference with respect to the previous case is that

$$\max_{R_c} U = U_c(\alpha, p_0, 0, \hat{C}) \tag{6.23}$$

with some $\hat{C} \in [0, \alpha + \tau\rho_S h_1]$, possibly $\hat{C} > 0$. Arguing as before and using (6.14)–(6.15) gives (see Figure 6):

$$\begin{aligned}\max_{R_c} U = U_c(\alpha, p_0, 0, \hat{C}) &\leq U_{nc}(\alpha, 0, \hat{C}) \\ &\leq U_{nc}(\alpha, C^{-1}(\alpha, \delta, \hat{C}, p), \hat{C}) \leq U_{nc}(\alpha, \alpha + \tau\rho_S h_1, 0) = \max_{R_{nc}} U.\end{aligned} \tag{6.24}$$

By (6.22) and (6.24) the statement concerning Case A follows.

Concerning Case B, suppose first that (4.8) holds. It is easily checked that

$$\max_{R_c} U = U_c(\alpha, p_0, 0, 0).$$

as in the case of condition (4.7) (see (6.17)–(6.21)). On the other hand, either $R_{nc}(\alpha, \delta, p_0) = \emptyset$ as in Figure 8, or the budget line and the curvilinear boundary have two intersections (I_1, C_1), (I_2, C_2) with $0 < I_1 < I_2 < \alpha + \tau\rho_S h_1$, $0 < C_2 < C_1 < \alpha + \tau\rho_S h_1$ (see Figure 9). In the latter case by (6.14)–(6.15) there holds

$$\max_{R_{nc}} U = U_{nc}(\alpha, I_2, C_2).$$

If $\max_{R_c} U > \max_{R_{nc}} U$ the leader does not spend at all, whereas he chooses the intermediate policy (I_2, C_2) in the opposite case. Hence the first statement concerning Case B follows.

Finally, suppose that (4.9) holds and $\delta(\alpha + \tau\rho_S h_1) > 1$. In this case by (6.14)–(6.15) there holds

$$\max_{R_{nc}} U = U_{nc}(\alpha, \bar{I}, \bar{C}), \tag{6.25}$$

with $(\bar{I}, \bar{C})$ given by (4.16) (see Figure 10). On the other hand, since by assumption $\delta(\alpha + \tau\rho_S h_1) > 1$, by (6.19) there holds

$$\frac{\partial U_c}{\partial C}(\alpha, p_0, 0, C) > 0.$$

From (6.17) and the above inequality we get

$$\max_{R_c} U = U_c(\alpha, p_0, 0, C_c(\alpha, \delta, p_0)). \tag{6.26}$$

Arguing as in the proof of (6.24), from (6.25)–(6.26) we obtain

$$\max_{R_c} U = U_c(\alpha, p_0, 0, C_c(\alpha, \delta, p_0))$$
$$\leq U_{nc}(\alpha, 0, C_c(\alpha, \delta, p_0)) \leq U_{nc}(\alpha, \bar{I}, \bar{C}) = \max_{R_{nc}} U.$$

This proves the second statement concerning Case B, thus the result follows. □

6.2. Changes of policy. Let us now study the dependence on α and p_0 of the critical efficiency $\hat{\delta}(\alpha, p_0)$ given by (4.17). This is the content of the following lemmata.

LEMMA 6.1. *For any fixed $\alpha > \alpha_*$ the function $\hat{\delta}(\alpha, \cdot)$ is decreasing in Case A and increasing in Case B.*

PROOF. It suffices to observe that in Case A from (3.1) and (3.6) we obtain plainly

$$\hat{\delta}(\alpha, p_0) = \frac{p_0 - P_0(\alpha)}{I_c(\alpha, p_0)} = \frac{(1-\tau)\rho - p_0 \tau \rho_S}{\alpha + \tau \rho_S h_1 + D} > 0, \tag{6.27}$$

whereas in Case B there holds $\hat{\delta}(\alpha, p_0) = \frac{p_0 - P_0(\alpha)}{\alpha + \tau \rho_S h_1}$. □

LEMMA 6.2. *Let $\alpha > \alpha_*$ and $p_0 \in (P_0(\alpha), 1]$ be fixed. Then the function $\hat{\delta}(\cdot, p_0)$ is decreasing in (α, ∞).*

PROOF. In Case A, from (6.27) we get

$$\frac{\partial \hat{\delta}}{\partial \alpha} = -\frac{(1-\tau)\rho - p_0 \tau \rho_S}{(\alpha + \tau \rho_S h_1 + D)^2} < 0,$$

whence the claim follows in this case. Concerning Case B, observe that by (4.10)

$$\begin{aligned} \frac{\partial \hat{\delta}}{\partial \alpha} &= -\frac{p_0 - P_0(\alpha)}{(\alpha + \tau \rho_S h_1)^2} - \frac{P_0'(\alpha)}{\alpha + \tau \rho_S h_1} \\ &= -\frac{1}{\alpha + \tau \rho_S h_1} \left[\hat{\delta} - \frac{(1-\tau)\rho h_1 + D}{(\alpha + \tau \rho_S h_1 + D)^2} \right]. \end{aligned} \tag{6.28}$$

From the first equality of (6.28) and (3.6) we obtain

$$\begin{aligned} \frac{\partial \hat{\delta}}{\partial \alpha} &= -\frac{1}{(\alpha + \tau \rho_S h_1)^2} \left[p_0 - \frac{(1-\tau)\rho h_1 + D}{\alpha + \tau \rho_S h_1 + D} \right] + \frac{1}{\alpha + \tau \rho_S h_1} \frac{(1-\tau)\rho h_1 + D}{(\alpha + \tau \rho_S h_1 + D)^2} \\ &= \frac{-p_0(\alpha + \tau \rho_S h_1)^2 + 2a(\alpha + \tau \rho_S h_1) + aD}{(\alpha + \tau \rho_S h_1)^2 (\alpha + \tau \rho_S h_1 + D)^2}, \end{aligned}$$

where

$$a := (1-\tau)\rho h_1 + D(1-p_0).$$

It follows immediately that

$$(6.29)\qquad \frac{\partial\hat{\delta}}{\partial\alpha} < 0 \quad \text{if and only if} \quad \alpha + \tau\rho_S h_1 > x_+ := \frac{a + \sqrt{a(a+p_0 D)}}{p_0}.$$

We shall prove that

$$(6.30)\qquad \frac{\partial\hat{\delta}}{\partial\alpha}(\alpha^*, p_0) \le 0.$$

Then by (6.29)–(6.30) there holds $\alpha^* + \tau\rho_S h_1 \ge x_+$, whence $\alpha + \tau\rho_S h_1 > x_+$ for any $\alpha > \alpha^*$. By (6.29), this implies that $\frac{\partial\hat{\delta}}{\partial\alpha}(\alpha, p_0) < 0$ for any $\alpha > \alpha^*$, thus the conclusion follows.

It remains to prove (6.30). Since

$$\frac{\partial\hat{\delta}}{\partial\alpha}(\alpha^*, p_0) = \lim_{\alpha\to(\alpha^*)^+} \frac{\hat{\delta}(\alpha, p_0) - \hat{\delta}(\alpha^*, p_0)}{\alpha - \alpha^*},$$

the claim follows if we show that

$$(6.31)\qquad \hat{\delta}(\alpha, p_0) \le \hat{\delta}(\alpha^*, p_0) \quad \text{for any } \alpha \ge \alpha^*.$$

To this purpose, observe preliminarily that for any $\alpha \ge \alpha^*$

$$(6.32)\qquad \alpha + \tau\rho_S h_1 + D \ge \alpha^* + \tau\rho_S h_1 + D = \frac{(1-\tau)\rho h_1}{1 + \tau\rho_S - (1-\tau)\rho} + D > (1-\tau)\rho h_1 + D,$$

since by (H_2) there holds $0 < 1 + \tau\rho_S - (1-\tau)\rho < 1$. Also observe that

$$(6.33)\qquad \hat{\delta}(\alpha^*, 1) = \frac{1 - P_0(\alpha^*)}{\alpha^* + \tau\rho_S h_1} = \frac{\alpha^* + \tau\rho_S h_1 - (1-\tau)\rho}{(\alpha^* + \tau\rho_S h_1)(\alpha^* + \tau\rho_S h_1 + D)} = \frac{[(1-\tau)\rho - \tau\rho_S](\alpha^* + \tau\rho_S h_1)}{(\alpha^* + \tau\rho_S h_1)(\alpha^* + \tau\rho_S h_1 + D)} = \frac{(1-\tau)\rho - \tau\rho_S}{\alpha^* + \tau\rho_S h_1 + D}.$$

From (6.32)–(6.33) we get for any $\alpha \ge \alpha^*$

$$(6.34)\qquad \frac{(1-\tau)\rho h_1 + D}{(\alpha + \tau\rho_S h_1 + D)^2} \le \frac{1}{\alpha^* + \tau\rho_S h_1 + D} = \frac{\hat{\delta}(\alpha^*, 1)}{(1-\tau)\rho - \tau\rho_S}.$$

Plugging the above inequality into the second equality of (6.28) gives for any $\alpha \geq \alpha^*$

$$\frac{\partial}{\partial \alpha}\left\{\hat{\delta} - \frac{\hat{\delta}(\alpha^*, 1)}{(1-\tau)\rho - \tau\rho_S}\right\} \leq -\frac{1}{\alpha + \tau\rho_S h_1}\left\{\hat{\delta} - \frac{\hat{\delta}(\alpha^*, 1)}{(1-\tau)\rho - \tau\rho_S}\right\}, \tag{6.35}$$

whence plainly (6.30) follows. This completes the proof. □

LEMMA 6.3. *If* $\alpha \in (\alpha_*, \alpha^*)$, *there holds* $\hat{\delta}(\alpha, p_0) > \hat{\delta}(\alpha^*, 1)$ *for any* $p_0 \in (P_0(\alpha), 1]$. *If* $\alpha > \alpha^*$, *there holds* $\hat{\delta}(\alpha, p_0) < \hat{\delta}(\alpha^*, 1)$ *for any* $p_0 \in (P_1(\alpha), 1]$.

PROOF. By Lemma 6.2 there holds $\hat{\delta}(\alpha, 1) > \hat{\delta}(\alpha^*, 1)$ for any $\alpha \in (\alpha_*, \alpha^*)$, whereas by Lemma 6.1, Case A there holds $\hat{\delta}(\alpha, p_0) > \hat{\delta}(\alpha, 1)$ for any $p_0 \in (P_0(\alpha), 1]$. Hence the first claim follows. Similarly, by Lemma 6.2 there holds $\hat{\delta}(\alpha^*, 1) > \hat{\delta}(\alpha, 1)$ for any $\alpha > \alpha^*$, whereas by Lemma 6.1, Case B there holds $\hat{\delta}(\alpha, p_0) < \hat{\delta}(\alpha, 1)$ for any $p_0 \in (P_1(\alpha), 1]$. This proves the second claim, thus the conclusion follows. □

Let us mention the following consequence of Lemmata 6.1–6.3.

PROPOSITION 6.4. (i) *Let* $\delta > \hat{\delta}(\alpha^*, 1)$. *Then for any* $\alpha > \alpha^*$ *and* $p_0 \in (P_1(\alpha), 1]$ *there holds* $\delta > \hat{\delta}(\alpha, p_0)$. *Moreover, there exists a unique couple* $\underline{\alpha} = \underline{\alpha}(\delta) \in (\alpha_*, \alpha^*)$, $\underline{p}_0 = \underline{p}_0(\delta) \in (P_0(\underline{\alpha}), 1]$ *such that* $\delta > \hat{\delta}(\alpha, p_0)$ *for any* $\alpha \in (\underline{\alpha}, \alpha^*)$ *and* $p_0 \in (\underline{p}_0, 1]$. *The functions* $\underline{\alpha} = \underline{\alpha}(\delta)$ *and* $\underline{p}_0 = \underline{p}_0(\delta)$ *are nonincreasing.*

(ii) *Let* $\delta < \hat{\delta}(\alpha^*, 1)$. *Then for any* $\alpha \in (\alpha_*, \alpha^*)$ *and* $p_0 \in (P_0(\alpha), 1]$ *there holds* $\delta < \hat{\delta}(\alpha, p_0)$. *Moreover, there exists a unique* $\bar{\alpha} = \bar{\alpha}(\delta) > \alpha^*$ *such that* $\delta > \hat{\delta}(\alpha, p_0)$ *for any* $\alpha > \bar{\alpha}$ *and* $p_0 \in (P_1(\alpha), 1]$. *The function* $\bar{\alpha} = \bar{\alpha}(\delta)$ *is decreasing.*

PROOF. (i) By Lemma 6.3, for any $\alpha > \alpha^*$ and $p_0 \in (P_1(\alpha), 1]$ there holds $\hat{\delta}(\alpha, p_0) < \hat{\delta}(\alpha^*, 1)$. Hence

$$\delta > \hat{\delta}(\alpha^*, 1) \quad \Rightarrow \quad \delta > \hat{\delta}(\alpha, p_0) \quad \text{for any } \alpha > \alpha^*,\ p_0 \in (P_1(\alpha), 1].$$

This proves the first statement of claim (i).

Further, let $\alpha_1 \in (\alpha_*, \alpha^*)$ and $p_1 \in (P_0(\alpha_1), 1]$, thus by Lemma 6.3 there holds $\hat{\delta}(\alpha_1, p_1) > \hat{\delta}(\alpha^*, 1)$. If $\delta > \hat{\delta}(\alpha_1, p_1)$, there holds $\delta > \hat{\delta}(\alpha, p_0)$ for any $\alpha \in (\alpha_1, \alpha^*)$ and $p_0 \in (p_1, 1]$ (see Lemmata 6.1, Case A, and 6.2). On the other hand, if $\hat{\delta}(\alpha^*, 1) < \delta < \hat{\delta}(\alpha_1, p_1)$, there exists a unique $\underline{\alpha} = \underline{\alpha}(\delta) \in (\alpha_1, \alpha^*)$, $\underline{p}_0 = \underline{p}_0(\delta) \in (p_1, 1)$ such that $\delta > \hat{\delta}(\alpha, p_0)$ for any $\alpha \in (\underline{\alpha}, \alpha^*)$ and $\underline{p}_0 \in (p_1, 1]$. In either case, the second statement of claim (i) follows with $\underline{\alpha}(\delta) \in [\alpha_1, \alpha^*)$, $\underline{p}_0 \in [p_1, 1]$. Clearly, $\underline{\alpha}$ and $\underline{p}_0$ are nonincreasing functions of δ. Hence claim (i) follows.

(ii) By Lemma 6.3, for any $\alpha \in (\alpha_*, \alpha^*)$ and $p_0 \in (P_0(\alpha), 1]$ there holds $\hat{\delta}(\alpha, p_0) > \hat{\delta}(\alpha^*, 1)$. Hence

$$\delta < \hat{\delta}(\alpha^*, 1) \quad \Rightarrow \quad \delta < \hat{\delta}(\alpha, p_0) \quad (\alpha \in (\alpha_*, \alpha^*), p_0 \in (P_0(\alpha), 1]).$$

Further, for any $\alpha > \alpha^*$ the map $\hat{\delta}(\alpha, \cdot)$ is increasing in $(P_1(\alpha), 1]$ (see Lemma 6.1, Case B), hence $\hat{\delta}(\alpha, p_0) < \hat{\delta}(\alpha, 1)$ for any $p_0 \in (P_1(\alpha), 1]$. Since the map $\hat{\delta}(\cdot, 1)$ is decreasing (see Lemma 6.2), $\delta < \hat{\delta}(\alpha^*, 1)$ by assumption and by (4.10)

$$\hat{\delta}(\alpha, 1) \to 0 \quad \text{as } \alpha \to \infty,$$

there exists a unique $\bar{\alpha} = \bar{\alpha}(\delta) > \alpha^*$ such that

$$\delta > \hat{\delta}(\alpha, 1) \quad \text{for any } \alpha > \bar{\alpha} \quad \Rightarrow \quad \delta > \hat{\delta}(\alpha, p_0) \quad \text{for any } \alpha > \bar{\alpha},\ p_0 \in (P_1(\alpha), 1].$$

Clearly, $\bar{\alpha}$ is a nonincreasing function of δ. This proves claim (ii), thus the result follows. □

Now we can prove Theorem 4.2.

Proof of Theorem 4.2. Whenever we are in Case A, the maximum of the leader utility function is attained at $(\alpha + \tau\rho_S h_1, 0)$ (see Theorem 4.1), thus the policy is democratic. Hence claim (a) follows.

Let (H_2) hold and $p_0 \in (P_1(\infty), 1]]$. Since $P_1(\cdot)$ is decreasing, there exists a unique $\tilde{\alpha} \geq \alpha^*$ such that $p_0 = P_1(\tilde{\alpha})$, thus $p_0 \in (P_1(\alpha), 1]$ if and only if $\alpha > \tilde{\alpha}$. It follows that we are in Case A if $\alpha \leq \tilde{\alpha}$, and in Case B if $\alpha > \tilde{\alpha}$. In particular, no change of policy occurs below $\tilde{\alpha}$.

Let $\alpha > \tilde{\alpha}$. If $\delta > \hat{\delta}(\alpha^*, 1)$, by Proposition 6.4(i) there holds

$$\delta > \hat{\delta}(\alpha, p_0) = \frac{p_0 - P_0(\alpha)}{\alpha + \tau\rho_S h_1} \quad \Leftrightarrow \quad C_c(\alpha, \delta, p_0) < \alpha + \tau\rho_S h_1, \tag{6.36}$$

since $p_0 \in (P_1(\alpha), 1]$. On the other hand, there holds $I_c(\alpha, p_0) > \alpha + \tau\rho_S h_1$ since we are in Case B, thus condition (4.9) is satisfied. If $\delta(\alpha^* + \tau\rho_S h_1) > 1$, there holds $\delta(\alpha + \tau\rho_S h_1) > 1$ since $\alpha > \tilde{\alpha} \geq \alpha^*$. Instead, if

$$\hat{\delta}(\alpha^*, 1) < \delta < \frac{1}{\alpha^* + \tau\rho_S h_1},$$

there holds

$$\delta(\alpha + \tau\rho_S h_1) > 1 \quad \text{for any } \alpha > \frac{1}{\delta} - \tau\rho_S h_1.$$

Therefore, for any $\alpha > \alpha^{**} := \max\left\{\tilde{\alpha}, \frac{1}{\delta} - \tau\rho_S h_1\right\}$ both conditions (4.9) and $\delta(\alpha + \tau\rho_S h_1) > 1$ are satisfied. Then by Theorem 4.1 the maximum of the leader utility function is attained at the intermediate policy $(\bar{I}, \bar{C})$ given by (4.16). Hence the leader policy changes beyond the value α^{**}, which is a nonincreasing function of δ. This proves claim (b)(i).

Let $\alpha > \tilde{\alpha}$ and $\delta < \hat{\delta}(\alpha^*, 1)$. Then, since $p_0 \in (P_1(\alpha), 1]$, by Proposition 6.4(ii) there exists a unique $\bar{\alpha} = \bar{\alpha}(\delta) > \tilde{\alpha}$ (decreasing with respect to δ) such that inequalities (6.36) hold true if and only if $\alpha > \bar{\alpha}$. Namely, if $\alpha > \bar{\alpha}$ condition (4.9) holds, whereas condition (4.8) is satisfied if $\tilde{\alpha} < \alpha < \bar{\alpha}$. Arguing as above shows that for any $\alpha > \bar{\bar{\alpha}} := \max\{\bar{\alpha}, \frac{1}{\delta} - \tau\rho_S h_1\}$ both conditions (4.9) and $\delta(\alpha + \tau\rho_S h_1) > 1$ are satisfied. Therefore, for all $\alpha > \bar{\bar{\alpha}}$ the leader chooses the intermediate policy $(\bar{I}, \bar{C})$ where the maximum of his utility function is attained (see Theorem 4.1). On the other hand, as long as (4.8) holds, by Theorem 4.1 the leader either chooses an intermediate policy, or does not spend at all. Hence the no-spending policy is only adopted in some possibly empty subset of the interval $(\tilde{\alpha}, \bar{\bar{\alpha}})$. Since both $\tilde{\alpha}$ and $\bar{\bar{\alpha}}$ decrease when δ increases, claim (b)(ii) follows. This completes the proof. □

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