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Partial Differential Equations — *Viscosity solutions for junctions: well posedness and stability*, by Pierre-Louis Lions and Panagiotis Souganidis, communicated on 11 November 2016.

Abstract. — We introduce a notion of state-constraint viscosity solutions for one dimensional "junction"-type problems for Hamilton–Jacobi equations with non convex coercive Hamiltonians and study its well-posedness and stability properties. We show that viscosity approximations either select the state-constraint solution or have a unique limit, and we introduce another type of approximation by fattening the domain. We also make connections with existing results for convex equations and discuss extensions to time and/or multi-dimensional problems.

Key words: Viscosity solutions, junctions, Hamilton–Jacobi equations

Mathematics Subject Classification: 35F21, 49L25, 35B51, 49L20

1. The problem and the notion of solution

We introduce a notion of state-constraint viscosity solutions for one dimensional junction-type problems for non convex Hamilton–Jacobi equations and study its well-posedness (comparison principle and existence). We also investigate the stability properties of small diffusion approximations satisfying a Kirchoff property at the junction. We show that such approximations either converge to the state-constraint solution or have a unique limit. We also introduce a new type of approximations by "fattening" the junction, which, under some assumptions on the behavior of the Hamiltonian's at the junction, also yield the state-constraint solution. In addition we present a new and very simple proof for the uniqueness solution of the junction solutions introduced for quasi-convex problems by Imbert and Monneau [5]. Finally, we discuss extensions to time dependent and/or multi-dimensional problems.

For simplicity and due to the space limitation we concentrate here on one-dimensional time independent problems. Our results, however, extend with some additional technicalities, to time dependent as well as multi-dimensional "stratified" problems. Proofs as well extensions to multi-dimensional problems will appear in [9].

We emphasize that our results do not require any convexity conditions on the Hamiltonians contrary to all the previous literature that is based on the control theoretical interpretation of the problem and, hence, require convexity. Among the long list of references on this topic with convex Hamiltonians, in addition to [5], we refer to Barles, Briani and Chasseigne [1, 2], Barles and Chasseigne [3], Bressan and Hong [4] and Imbert and Nguen [6].

We consider a K-junction problem in the domain $I := \bigcup_{i=1}^{K} I_i$ with junction $\{0\}$, where, for $i = 1, \ldots, K$, $I_i := (-a_i, 0)$ and $a_i \in [-\infty, 0)$. We work with functions $u \in C(\bar{I}; \mathbb{R})$ and, for $x = (x_1, \ldots, x_K) \in \bar{I}$, we write $u_i(x_i) = u(0, \ldots, x_i, \ldots, 0)$. When possible, to simplify the writing, we drop the subscript on u_i and simply write $u(x_i)$. We also use the notation u_{x_i} and $u_{x_i x_i}$ for the first and second derivatives of u_i in x_i. Finally, to avoid unnecessarily long statements, we do not repeat, unless needed, that $i = 1, \ldots, K$.

For the Hamiltonians $H_i \in C(\mathbb{R} \times I; \mathbb{R})$ we assume that, for each i,

(1) $\quad H_i$ is coercive, that is $H_i(p_i, x_i) \to \infty$ as $|p_i| \to \infty$ uniformly on $\bar{I}_i$.

Next we present the definitions of the state-constraint sub- and super-solutions.

Definition 1.1. (i) $u \in C(\bar{I}; \mathbb{R})$ is a state-constraint sub-solution to the junction problem if

$$u_i + H_i(u_{x_i}, x_i) \leq 0 \quad \text{in } I_i \text{ for each } i. \tag{2}$$

(ii) $u \in C(\bar{I}; \mathbb{R})$ is a state-constraint super-solution to the junction problem if

$$u_i + H_i(u_{x_i}, x_i) \geq 0 \quad \text{in } I_i \text{ for each } i, \tag{3}$$

and

$$u(0) + \max_{1 \leq i \leq K} H_i(u_{x_i}, 0) \geq 0. \tag{4}$$

(iii) $u \in C(\bar{I}; \mathbb{R})$ is a solution if it is both sub- and super-solution.

The super-solution inequality at the junction is interpreted in the viscosity sense, that is if, for $\phi \in C^1(I) \cap C^{0,1}(\bar{I})$, $u - \phi$ has a (strict local) minimum at $x = 0$, then $u(0) + \max_{1 \leq i \leq K} H_i(\phi_{x_i}(0), 0) \geq 0$.

We remark that, for the sake of brevity, we are not precise about the boundary conditions at the end points a_i, which may be of any kind (Dirichlet, Neumann or state-constraint) that yields comparison for solutions in each I_i.

We also note that, without much difficulty, it is possible to study more than one junctions, since, as it will become apparent from the proofs below, the "influence" of the each junction is "local".

Finally, we denote by $u^{sc,i} \in C(\bar{I}_i)$ the unique constraint-solution to $w + H_i(w_{x_i}, x_i) = 0$ on $\bar{I}_i$.

2. The main results

We begin with the well-posedness of the state-constraint solution of the junction problem.

THEOREM 2.1. *Assume* (1).

(i) *If* $v, u \in C(\bar{I})$ *are respectively sub- and super-constraint solutions to the junction problem, then* $v \leq u$ *on* $\bar{I}$.
(ii) *There exists a unique state-constraint solution* $\hat{u}$ *of the junction problem.*
(iii) $\hat{u}(0) = \min_{1 \leq i \leq K} u^{sc,i}(0)$.

Since it is classical in the theory of viscosity solutions that the comparison principle yields via Perron's existence method, here we will not discuss this any further.

The second result is about the stability properties of "viscous" approximations to the junction problem. We begin with the formulation and the well-posedness of solutions to second-order uniformly elliptic equations on junctions satisfying a possibly nonlinear Neumann (Kirchoff-type) condition.

We assume that the functions $F_i := F(X_i, p_i, u_i, x_i)$ and $G := G(p_1, \dots, p_K, u)$ are continuous and, uniformly with respect to all the other arguments,

$$(5) \quad \begin{cases} F_i \text{ is strictly decreasing in } X_i, \text{ nonincreasing in } u_i, \text{ and coercive in } p_i; \\ G \text{ is strictly increasing with respect to the } p_i\text{'s and nonincreasing with} \\ \text{respect to } u, \end{cases}$$

and consider the problem

$$(6) \qquad \begin{cases} F_i(u_{x_i x_i}, u_{x_i}, x_i, u_i, x_i) = 0 & \text{in } I_i \text{ for each } i, \\ G(u_{x_1}, \dots, u_{x_K}, u) = 0 & \text{at } x = 0. \end{cases}$$

THEOREM 2.2. *Assume* (5). *Then* (6) *has a unique solution* $\hat{u} \in C^2(I) \cap C^{1,1}(\bar{I})$.

The meaning of the Neumann condition at the junction is that G quantifies the "amount" of the diffusion that goes into each direction as well as stays at 0.

We consider next, for each $\varepsilon > 0$, the problem

$$(7) \qquad \begin{cases} -\varepsilon u^{\varepsilon}_{x_i x_i} + u^{\varepsilon}_i + H_i(u^{\varepsilon}_{x_i}, x_i) = 0 & \text{in } I_i, \\ \sum_{i=1}^{K} u^{\varepsilon}_{x_i} = 0 & \text{at } x = 0, \end{cases}$$

which, in view of Theorem 2.2, has a unique solution $u^{\varepsilon} \in C^2(I) \cap C^{1,1}(\bar{I})$, that, in addition, is bounded in $C^{0,1}(\bar{I})$ with a bound independent of the ε; the uniform in ε bound is an easy consequence of the assumed coercivity of the Hamiltonian's.

We remark that the particular choice of the Neumann condition plays no role in the sequel and results similar to the ones stated below will also hold true for other, even nonlinear, conditions at the junction.

We are interested in the behavior, as $\varepsilon \to 0$, of the u^{ε}'s and, in particular, in the existence of a unique limit and its relationship to the constraint solution of the first-order junction problem.

THEOREM 2.3. *Assume* (1). *Then* $u := \lim_{\varepsilon\to 0} u^\varepsilon$ *exists and either* $u = \hat{u}$ *or* $u(0) < \hat{u}(0)$, $u_{x_i}(0^-)$ *exists for all i's and* $\sum_{i=1}^K u_{x_i}(0^-) = 0$.

A consequence of Theorem 2.3 is that, in principle, the junction problem has a unique state-constraint solution and a possible continuum of solutions obtained as limits of problems like (7) with other type of possibly degenerate second order terms and different Neumann conditions.

Under some additional assumptions it is possible to show that we always have $\hat{u} = \lim_{\varepsilon\to 0} u^\varepsilon$. Indeed suppose that, for each i,

(8) H_i has no flat parts and finitely many minima at $p_{i,1}^0 \le \cdots \le p_{i,K_i}^0$;

note that the assumption that H_i has no flat parts can be easily removed by a density argument, while, at the expense of some technicalities, it is not necessary to assume that there are only finitely minima.

THEOREM 2.4. *Assume* (1), (8) *and* $\sum_{i=1}^K p_{i,K_i}^0 \le 0$. *Then* $\hat{u} = \lim_{\varepsilon\to 0} u^\varepsilon$.

A particular case that (8) holds is when the H_i's are quasi-convex and coercive. Then, for each i, there exists single minimum point at p_i^0, and the condition above reduces to $\sum_{i=1}^K p_i^0 \le 0$. On the other hand, if $\sum_{i=1}^K p_i^0 > 0$, we have examples showing that $\hat{u} > \lim_{\varepsilon\to 0} u^\varepsilon$.

3. SKETCH OF PROOFS

The proof of Theorem 2.2 is standard so we omit it and we present the one of Theorem 2.1.

PROOF. It follows from (1) that v is Lipschitz continuous. In view of the comments in the previous section about the boundary conditions at the a_i's, here we assume that $v(0) - u(0) = \max_{\bar{I}}(u - v) > 0$ and we obtain a contradiction.

To conclude we adapt the argument introduced in Soner [10] to study state-constraint problems and we consider, for each i, $\varepsilon > 0$ and some $\delta = O(\varepsilon)$, a maximum point $(\bar{x}_i, \bar{y}_i) \in \bar{I}_i \times \bar{I}_i$ (over $\bar{I}_i \times \bar{I}_i$) of $(x_i, y_i) \to v(x_i) - u(y_i) - \frac{1}{2\varepsilon}(\bar{x}_i - \bar{y}_i + \delta)^2$.

It follows that, as $\varepsilon \to 0$, $\bar{x}_i, \bar{y}_i \to 0$, and the role of the δ above is to guarantee that, for all i, $\bar{x}_i < 0$ even if $\bar{y}_i = 0$.

If, for some j, $\bar{y}_j < 0$, we find, using the uniqueness arguments for state-constraint viscosity solutions in $\bar{I}_j$, a contradiction to $v(0) - u(0) > 0$.

It follows that we must have $\bar{y}_i = 0$ for all $i = 1, \ldots, K$, that is, $y \to v(y) + \frac{1}{2\varepsilon}\sum_i(\bar{x}_i - y_i + \delta)^2$ has a minimum at 0. Since v is a super-solution, (4) yields $v(0) + \max_{1\le i\le K} H_i\left(\frac{\bar{x}_i+\delta}{\varepsilon}, 0\right) \ge 0$ and, hence, for some j, $v_j(0) + H_j\left(\frac{\bar{x}_j+\delta}{\varepsilon}, 0\right) \ge 0$.

On the other hand, since $\bar{x}_j < 0$, we also have $u_j(\bar{x}_j) + H_j\left(\frac{\bar{x}_j+\delta}{\varepsilon}, \bar{x}_j\right) \le 0$.

Combining the last two inequalities we find, after letting $\varepsilon \to 0$, that we must have $u(0) = u_j(0) \le v_j(0) = v(0)$, which again contradicts the assumption.

The existence of a unique solution $\hat{u}$ follows from the comparison and Perron's method.

For the third claim first we observe that, since $\hat{u}$ is a viscosity sub-solution in each I_i, the comparison of state-constraint solutions yields that, for each i, $\hat{u} \leq u^{sc,i}$ on $\bar{I}_i$, and, hence, $\hat{u}(0) \leq \min_{1\leq i\leq K} u^{sc,i}(0)$.

For the equality, we need to show that, for some j, $u^{sc,j}(0) \leq \hat{u}(0)$. This follows by repeating the proof of the comparison above. □

To study the limiting behavior of the u^ε's, we investigate in detail the properties of solutions to the Dirichlet problem in each of the intervals I_i. For notational simplicity we omit next the dependence on i and we consider, for each $c \in \mathbb{R}$, the boundary value problem

$$u_c + H(u_{c,x}, x) = 0 \quad \text{in } I := (-a, 0) \text{ and } u(0) = c, \tag{9}$$

and we denote by u^{sc} the solution of the corresponding state constraint problem in I. Note that, since the real issue is the behavior near 0, again we do not specify any boundary condition at a, which can be either Dirichlet or Neumann or state constrain so that (9) is well defined. Finally, as we already mentioned earlier, we use (8) only to avoid technicalities.

PROPOSITION 3.1. *Assume that H satisfies* (1) *and* (8)*. Then, for every $c < u^{sc}(0)$,* (9) *has a unique solution $u_c \in C^{0,1}(\bar{I})$. Moreover, $u_{c,x}(0^-)$ exists and $u_c(0^-) + H(u_{c,x}(0^-), 0) = 0$. In addition, both $u_c(0^-)$ and $u_{c,x}(0^-)$ are nondecreasing in c, and $u_{c,x}(0^-)$ belongs to the decreasing part of H except for finitely many values of c.*

PROOF. The existence of solutions to (9) is immediate from Perron's method, since, for any $\lambda > 0$, $u^{sc} - \lambda$ is a sub-solution, while the coercivity of the H easily yields a super-solution. The Lipschitz continuity of the solution is an immediate consequence of the coercivity of H. The existence of $u_{c,x}(0^-)$ and the fact the equation is satisfied at 0 follow either along the lines of Jensen and Souganidis [7], which studied the detailed differentiability properties of viscosity solutions in one dimension, or a technical lemma stated without proof after the end of the ongoing one. The claimed monotonicity of $u_c(0^-)$ follows from the comparison principle, while the monotonicity of $u_{c,x}(0^-)$ is a consequence of the fact that, for any $c \neq c'$, the maximum of $u_c - u_{c'}$ is attained at $x = 0$. The last assertion results from the nondecreasing properties of $u_c(0^-)$ and $u_{c,x}(0^-)$ and the fact that $u_c(0^-) + H(u_{c,x}(0^-), 0) = 0$. □

The technical lemma that can be used in the above proof in place of [7] is stated next without a proof.

LEMMA 3.2. *Assume that $u \in C^{0,1}(\bar{I})$ solves $u + H(u_x, x) \leq 0$ (resp. $u + H(u_x, x) \geq 0$) in I and let $\bar{p} := \limsup_{x\to 0^-} \frac{u(x)-u(0)}{x}$ and $\underline{p} := \liminf_{x\to 0^-} \frac{u(x)-u(0)}{x}$. Then $u(0) + H(\bar{p}, 0) \leq 0$ (resp. $u(0) + H(\underline{p}, 0) \geq 0$.)*

We state next without a proof a well known fact which characterizes the possible limits of the uniform in ε Lipschitz continuous solutions u^ε to (7).

LEMMA 3.3. *Assume* (1). *Any subsequential limit u of the u^ε is a viscosity sub-solution to*

$$
\begin{cases} u + H_i(u_{x_i}, x_i) \leq 0 \quad \text{in } I_i \text{ for each } i, \\ \min\left[\sum_{i=1}^K u_{x_i}, u(0) + \min\limits_{1 \leq i \leq K} H_i(u_{x_i}, 0)\right] \leq 0 \quad \text{at } x = 0, \end{cases} \tag{10}
$$

and a viscosity super-solution to

$$
\begin{cases} u + H_i(u_{x_i}, x_i) \geq 0 \quad \text{in } I_i \text{ for each } i, \\ \max\left[\sum_{i=1}^K u_{x_i}, u(0) + \max\limits_{1 \leq i \leq K} H_i(u_{x_i}, 0)\right] \geq 0 \quad \text{at } x = 0. \end{cases} \tag{11}
$$

Recall that the inequalities at $x = 0$ must be interpreted in the viscosity sense. For example, if, for some $\phi \in C^{0,1}(\bar{I})$, $u - \phi$ has a maximum at 0, then $\min\left[\sum_{i=1}^d \phi_{x_i}(0^-), u(0) + \min\limits_{1 \leq i \leq K} H_i(\phi_{x_i}(0), 0)\right] \leq 0$.

Proposition 3.4 below refines the behavior of any u satisfying (10) and (11). The proof of Theorem 2.3 is then immediate.

PROPOSITION 3.4. *Assume* (1) *and* (8).

(i) *If u is continuous solution to* (10) *and* (11) *and $u(0) < \hat{u}(0)$, then* $\sum_{i=1}^d u_{x_i}(0^-) = 0$.
(ii) *The problem* (10) *and* (11) *has at most one solution on $u \in C^{0,1}(\bar{I})$ such that $u(0) < \hat{u}(0)$.*

PROOF. (i) Proposition 3.1 yields that, for each i, the $u_{x_i}(0^-)$'s exist and belong to the decreasing part of the H_i and $u(0) + H_i(u_{x_i}(0^-), 0) = 0$. It follows that there exists some small $\lambda > 0$ such that $u(0) + H_i(u_{x_i}(0^-) + \lambda, 0) < 0$ and $u(0) + H_i(u_{x_i}(0^-) - \lambda, 0) > 0$.

Choose $\phi^\pm \in C^{0,1}(\bar{I})$ such that $\phi^\pm_{x_i}(0^-) = u_{x_i}(0^-) \pm \lambda$. It follows that 0 is a local max and min of $u - \phi^-$ and $u - \phi^+$ respectively. Then (10) and (11) and the choice of $\phi^\pm$ yield the inequalities

$$
\begin{cases} \min\left[\sum_{i=1}^K \phi^-_{x_i}(0^-), u(0) + \min\limits_{1 \leq i \leq K} H_i(\phi^-_{x_i}(0^-), 0)\right] \\ \qquad = \min\left[\sum_{i=1}^K u_{x_i}(0^-) - \lambda K, u(0) + \min\limits_{1 \leq i \leq K} H_i(u_{x_i}(0^-) - \lambda, 0)\right] \leq 0, \end{cases}
$$

and

$$\begin{cases} \max\left[\sum_{i=1}^{K} \phi_{x_i}^{+}(0^-), u(0) + \max\limits_{1\le i\le K} H_i(\phi_{x_i}^{+}(0^-), 0)\right] \\ \qquad = \max\left[\sum_{i=1}^{K} u_{x_i}(0^-) + \lambda K, u(0) + \max\limits_{1\le i\le K} H_i(u_{x_i}(0^-) + \lambda, 0)\right] \ge 0. \end{cases}$$

It follows from the choice of λ that $\sum_{i=1}^{K} u_{x_i}(0^-) - \lambda \le 0 \le \sum_{i=1}^{K} u_{x_i}(0^-) + \lambda K$, and, hence, letting $\lambda \to 0$ yields the claim.

(ii) If u, v are two continuous solutions to (10) and (11), the Kirchoff condition established above implies that, for some small $\delta >$, $u(x) - v(x) - \delta \sum_{i=1}^{d} x_i$ cannot have a maximum at 0. The claim then follows from standard viscosity solutions arguments. $\square$

Theorem 2.4 is now immediate from the first claim in Proposition 3.4.

4. Some observations

We present another way to approximate the constrained solution of the junction based on "fattening" $\bar{I}$. To simplify the notation we assume that $K = 2$.

For $\varepsilon > 0$, let I_ε be an open neighborhood of $\bar{I}$ in $\mathbb{R}^2$ of size ε, that is $\bar{I} \subset I_\varepsilon$ and $\operatorname{diam} I_\varepsilon \le \varepsilon$, consider the coercive Hamiltonian $H : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ and the state-constraint problem

$$\begin{cases} u^\varepsilon + H(Du^\varepsilon, x) \le 0 & \text{in } I_\varepsilon, \\ u^\varepsilon + H(Du^\varepsilon, x) \ge 0 & \text{on } \bar{I}_\varepsilon, \end{cases} \tag{12}$$

where $Dv := (v_{x_1}, v_{x_2})$ and $x := (x_1, x_2)$. The coercivity of H yields Lipschitz bounds so that, along subsequences, $u^\varepsilon \to u$.

Define

$$H_1(p_1, x_1) := \min_{p_2 \in \mathbb{R}} H(p_1, p_2, x_1, 0) \text{ and } H_2(p_2, x_2) := \min_{p_1 \in \mathbb{R}} H(p_1, p_2, 0, x_2).$$

Theorem 4.1. *Any limit u of the solutions u^ε to (12) is a solution to $u + H_1(u_{x_1}, x_1) = 0$ in I_1 and $u + H_2(u_{x_2}, x_2) = 0$ in I_2, and if, for some $\phi \in C^1(\mathbb{R}^2)$, $u - \phi$ has local minimum at 0, then $u + H(\phi_{x_1}(0), \phi_{x_2}(0), 0) \ge 0$.*

Proof. The proof of the second claim is immediate. Here we concentrate on the first part and, since the arguments are similar, we take $i = 1$.

For some $\phi \in C^1(I_1)$, let $\bar{x}_1 \in I_1$ be a local minimum of $u(x_1, 0) - \phi(x_1)$. It is immediate that, for all $p_2 \in \mathbb{R}$, $u^\varepsilon(x_1, x_2) - \phi(x_1) - p_2 x_2$ has a minimum at $(\bar{x}_1^\varepsilon, \bar{x}_2^\varepsilon)$ and, as $\varepsilon \to 0$, $\bar{x}_1^\varepsilon \to x_1$ and $\bar{x}_2^\varepsilon \to 0$. It follows from (12) that $u(\bar{x}_1, 0) + H(\phi(\bar{x}_1, p_2, \bar{x}_1, 0) \ge 0$, and, since p_2 is arbitrary, $u(\bar{x}_1, 0) + H_1(\phi(\bar{x}_1, \bar{x}_1) \ge 0$.

The sub-solution property follows from the fact that $u^\varepsilon + H_1(u^\varepsilon_{x_1}, x_1) \le u^\varepsilon + H(u^\varepsilon_{x_1}, u^\varepsilon_{x_1}, x_1, 0)$. $\square$

The following proposition is a consequence of Theorem 4.1.

PROPOSITION 4.2. *If* $H(p_1, p_2, x_1, x_2) = \max(H_1(p_1, x_1), H_2(p_2, x_2))$, *then the* $\lim_{\varepsilon\to 0} u^\varepsilon$ *exists and is the state-constraint solution to the junction problem.*

In general, however, it is not true that $H(p_1, p_2, x_1, x_2) = \max(H_1(p_1, x_1), H_2(p_2, x_2))$. Indeed, if $H(p_1, p_2) = p_1^2 + 10p_2^2$, then $H_1(p_1) = p_1^2$ and $H_2(p_2) = 10p_2^2$ and $p_1^2 + 10p_2^2 \neq \max(p_1^2, 10p_2^2)$.

Next we use the arguments of the proof of the uniqueness of the state-constraint solutions to give a new and very simple proof of the comparison result established in [5] for a notion of limited flux junction solutions, which are "parametrized" by their values at 0. As in the rest of this paper we concentrate on the time-independent problem.

The notion of solution introduced in [5] requires the Hamiltonian's to be, in addition to coercive, quasiconvex and the condition at the junction involves the nondecreasing part of the Hamiltonians. To simplify the presentation, here we assume that each Hamiltonian H_i is convex and has no flat parts.

If $p_i^0 = \operatorname{argmin} H_i$, [5] uses the auxiliary Hamiltonians $H_i^-(p_i, 0) := H_1(p_1, 0)$ if $p_i \le p_i^0$ and $H_i^-(p_i, 0) := H_i(p_i^0, 0)$ if $p_i \ge p_i^0$, to define, for any $A \in \mathbb{R}$, the A-flux limiter $H_A(p) := \max\Big(A, \max_{1\le i\le K} H_i^-(p_i, 0)\Big)$.

The following definition was introduced in [5].

DEFINITION 4.3. An A-flux limited sub (respectively super)-solution to junction problem is a viscosity sub(respectively super)-solution, for each i, to $u + H_i(u_{x_i}, x_i)$ in I_i and $u + H_A(u_{x_1}, \dots, u_{x_K})$ at $x = 0$.

We remark that, in addition to the severe restriction of convexity, the A-flux limited solutions are classified essentially by their values at the origin and not the Kirchoff-type Neumann solution we use here, which is more natural for the interpretation of the solution.

Motivated by the control theoretic interpretation of the problem [5] constructed a rather elaborate test function to deal with the case that points coming up in the uniqueness proof are at the origin.

Here we present a rather simple proof for this uniqueness. To simplify the arguments we consider continuous solutions and prove the following.

PROPOSITION 4.4. *Let* u, v *be continuous* A*-flux limited sub- and super-solutions respectively. Then* $u \le v$ *on* $\bar{I}$.

PROOF. The first observation is that $u(0) \le -A$. Indeed, for $\varepsilon > 0$ small, consider a test function $\phi \in C^1(I) \cap C^{0,1}(\bar{I})$ such that $\phi_i(x_i) = -x_i/\varepsilon$. It is easy to see that $u - \phi$ attains a local maximum in a neighborhood of 0 at some point $\bar{X} := (\bar{x}_1, \dots, \bar{x}_K)$. If $\bar{X} \in I_i$ for some i, then $u(\bar{X}) + H_i(-1/\varepsilon, \bar{X}) \le 0$, which is not possible if ε is sufficiently small since H_i is coercive. Hence $\bar{X} = 0$ and the definition yields $u(0) + A \le u(0) + H_A(D\phi(0), 0) \le 0$.

For the comparison we follow the proof or Theorem 2.1 and recall that we only need to consider the case that the maximum of the "doubled" function is achieved for all i's at some $(\bar{x}_i, 0)$ with $\bar{x}_i < 0$.

The definition of the A-flux limited super-solution then yields $v(0) + H_A\big(\frac{\bar{x}_1+\delta}{\varepsilon}, \ldots, \frac{\bar{x}_K+\delta}{\varepsilon}, 0\big) \geq 0$.

If $H_A\big(\frac{\bar{x}_1+\delta}{\varepsilon}, \ldots, \frac{\bar{x}_K+\delta}{\varepsilon}, 0\big) = A$, then $v(0) + A \geq 0$, that is $v(0) \geq -A \geq u(0)$, and we may conclude.

If $H_A\big(\frac{\bar{x}_1+\delta}{\varepsilon}, \ldots, \frac{\bar{x}_K+\delta}{\varepsilon}, 0\big) = \max\limits_{1\leq i\leq K} H_i^-\big(\frac{\bar{x}_1+\delta}{\varepsilon}, \ldots, \frac{\bar{x}_K+\delta}{\varepsilon}, 0\big)$, then

$$
\begin{aligned}
v(0) + \max_{1\leq i\leq d} H_i\Big(\frac{\bar{x}_1+\delta}{\varepsilon}, \ldots, \frac{\bar{x}_d+\delta}{\varepsilon}, 0\Big) \\
\geq v(0) + \max_{1\leq i\leq d} H_i^-\Big(\frac{\bar{x}_1+\delta}{\varepsilon}, \ldots, \frac{\bar{x}_K+\delta}{\varepsilon}, 0\Big) \geq 0,
\end{aligned}
$$

and we may finish as in the proof of Theorem 2.1. □

We conclude with a proposition, which we state without a proof, that provides information about the location of the possible elements of the superdifferential at the junction of a sub-solution in I. An immediate consequence is that in the quasi-convex studied in [5], there is no need to use in advance the decreasing parts of the Hamiltonians in order to define the flux-limited solution at the junction.

PROPOSITION 4.5. *Assume that $u \in C(\bar{I})$ solves $u + H(u_x, x) \leq 0$ in I. Then either u is the state-constraint solution in $\bar{I}$ or $\limsup\limits_{x\to 0^-} \frac{u(x)-u(0)}{x} \leq \bar{P}$, where*

$$
\bar{P} := \inf\{z \in \mathbb{R} : H(z,0) \leq H(p,0) \textit{ for all } z \leq p\}.
$$

5. EXTENSIONS

A first extension of our results is about time dependent junction problems.

DEFINITION 5.1. (i) $u \in C(\bar{I} \times [0,T]; \mathbb{R})$ is a state-constraint sub-solution to the junction problem if

$$
u_{i,t} + H_i(u_{x_i}, x_i) \leq 0 \quad \text{in } I_i \times (0,T] \text{ for each } i. \tag{13}
$$

(ii) $u \in C(\bar{I} \times [0,T]; \mathbb{R})$ is a state-constraint super-solution if

$$
\begin{cases} u_{i,t} + H_i(u_{x_i}, x_i) \geq 0 \quad \text{in } I_i, \times (0,T] \text{ for each } i, \text{ and} \\ \max\limits_{1\leq i\leq K}(u_{i,t} + H_i(u_{x_i}, 0)) \geq 0. \end{cases} \tag{14}
$$

(iii) $u \in C(\bar{I} \times [0,T]; \mathbb{R})$ is a solution if it is both sub- and super-solution.

As for the time independent problems discussed earlier the super-solution inequality at the junction is interpreted in the viscosity sense, that is if, for $\phi \in C^1(I \times (0,T]) \cap C^{0,1}(\bar{I} \times [0,T])$, $u - \phi$ has a (local) minimum at $(0, t_0)$ with $t_0 \in (0,T]$, then $\max\limits_{1\leq i\leq K}[\phi_t(0,t_0) + H_i(\phi_{x_i}(0,t_0), 0)] \geq 0$.

The uniqueness of solutions as well the simple proof of the uniqueness of flux-limited solutions to the time dependentent junction problem follow after some easy modifications of the arguments presented in the previous sections. The convergence of the Kirchoff second-order approximations require some additional arguments. The details are given in [9].

Other possible generalizations to the so-called "stratified" problems were discussed by the first author in [8] and will be also presented in [9].

The following example is a typical problem. Consider the domain $\Sigma := \Sigma_1 \cup \Sigma_2$ with $\Sigma_1 := (-\infty, 0) \times \mathbb{R} \times \{0\}$ and $\Sigma_2 := \{0\} \times \{0\} \times (-\infty, 0)$ and the coercive nonlinearities F and H. The equation is:

$$\begin{cases} F(u_z, z) + u = 0 \quad \text{in } \Sigma_2, \\ H(u_x, u_y, x, y) + u = 0 \quad \text{in } \Sigma_1, \\ H(u_x, u_y, x, y) + u \geq 0 \quad \text{on } \partial\Sigma_1, \\ \min(H(u_x, u_y, x, y) + u, F(u_z, z) + u) \geq 0 \quad \text{at } \{0\} \times \{0\} \times \{0\}. \end{cases}$$

A more general multi-dimensional example, always for coercive nonlinearities, in the domain $\Sigma := \{(x, y) \in \mathbb{R}^{K+d} : x_i \leq 0\}$ is

$$\begin{cases} H_i(u_{x_i}, D_y u, x_i, y) + u_i = 0 \quad \text{in } (-\infty, 0) \times \mathbb{R}^d, \\ \max\limits_{1 \leq i \leq K} H_i(u_{x_i}, D_y u, 0, y) + u \geq 0 \quad \text{in } \{0\} \times \mathbb{R}^d. \end{cases}$$

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