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Mathematical Analysis — *Sharp geometric quantitative estimates*, by Flavia Giannetti, communicated on November 11, 2016.[1]

Abstract. — Let $E \subset B \subset \mathbb{R}^n$ be closed, bounded, convex sets. The monotonicity of the surface areas tells us that

$$\mathcal{H}^{n-1}(\partial E) \le \mathcal{H}^{n-1}(\partial B). \tag{0.1}$$

We give quantitative estimates from below of the difference $\delta(E;B) = \mathcal{H}^{n-1}(\partial B) - \mathcal{H}^{n-1}(\partial E)$ in the cases $n = 2$ and $n = 3$. As an application, considered a decomposition of a closed and bounded set into a number k of convex pieces, we deduce an estimate from below of the minimal number of convex components that may exist.

Key words: Convex sets, surface areas, Hausdorff distance

Mathematics Subject Classification: 52A20

1. Introduction

Let $E \subset B \subset \mathbb{R}^n$ be bounded, convex sets. It is known (see [1], [2]) that the monotonicity of the surface areas holds, i.e.

$$\mathcal{H}^{n-1}(\partial E) \le \mathcal{H}^{n-1}(\partial B) \tag{1.1}$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure and therefore $\mathcal{H}^{n-1}(\partial E)$ is the measure of the boundary of E, the surface area of E for short.

In the last years, some papers have been devoted to the study of a quantitative estimate from below of the difference of the surface areas $\delta(E;B) = \mathcal{H}^{n-1}(\partial B) - \mathcal{H}^{n-1}(\partial E)$. The first contribution in this direction has been given, in the planar case $n = 2$, by M. La Civita and F. Leonetti in [7] where the deficit $\delta(E;B)$ between the measures of the boundaries of E and B is expressed in terms of the diameter of the set B and the Hausdorff distance $h(E,B)$ between E and B.

For the reader's convenience, recall that

$$\mathrm{dist}(x,E) = \inf_{y \in E} |x - y|$$

[1] This paper is related to a talk given by F. Giannetti at "XXVI Convegno Nazionale di Calcolo delle Variazioni" which took place in Levico Terme on January 18–22, 2016.

and that

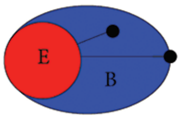
$$h(E,B) = \max\left\{\sup_{x\in B} \operatorname{dist}(x,E), \sup_{y\in E} \operatorname{dist}(y,B)\right\}.$$

In particular, for E, B closed, bounded and such that $E \subset B$, it holds that

$$h(E,B) = \max_{x\in B} \min_{y\in E} |x-y|. \tag{1.2}$$

Since for E convex, we have that $\partial E = \partial(\bar{E})$, we also assume E, B closed without loss of generality and therefore (1.2) holds.

It turns out that, if $b \in B$ and $P(b) \in E$ are such that $|b - P(b)| = h(E,B)$, one has that $b \in B \backslash E$ and $P(b)$ is the projection of b on the closed convex set E (see figure below).



The quantitative version of (1.1) which has been shown in [7] states as

Theorem 1.1. *For $\emptyset \neq E \subsetneq B \subset \mathbb{R}^2$, convex, closed and bounded, it holds*

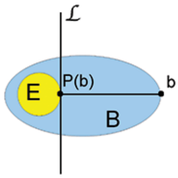
$$\mathscr{H}^1(\partial E) + \frac{|h(E,B)|^2}{\sqrt{(\operatorname{diam} B)^2 + |h(E,B)|^2}} \leq \mathscr{H}^1(\partial B). \tag{1.3}$$

Observe that in (1.3), $\mathscr{H}^1$ is the 1-dimensional Hausdorff measure, thus $\mathscr{H}^1(\partial E)$ is the length of the boundary of E, i.e. its perimeter.

In the next section we will show an improvement and an extension of the Theorem 1.1, contained respectively in the two recent papers [3] and [4], and that have been object of my talk at the XXVI *Convegno Nazionale di Calcolo delle Variazioni* held in Levico Terme. The last section will be devoted to an application of the result in [3] which has been proven in a forthcoming paper (see [5]).

2. The main results

In the improved version of the estimate given in Theorem 1.1, established in [3], the deficit $\delta(E;B)$ is expressed in terms of the Hausdorff distance $h(E,B)$ and in terms of the measure of the section of B orthogonal to the line segment which realizes the Hausdorff distance between E and B. Note that the minimality of the projection tells us that $P(b) \in \partial E$ and the line through $P(b)$ orthogonal to $b - P(b)$ is a supporting one for the convex set E (see the next figure).



The result obtained in [3] states as:

THEOREM 2.1. *Let E and B be two closed bounded convex sets, with $\emptyset \neq E \subsetneq B \subset \mathbb{R}^2$, then*

$$\mathcal{H}^1(\partial E) + \frac{4|h(E,B)|^2}{2\sqrt{\left(\frac{\mathcal{H}^1(B\cap\mathcal{L})}{2}\right)^2 + |h(E,B)|^2} + \mathcal{H}^1(B\cap\mathcal{L})} \leq \mathcal{H}^1(\partial B) \tag{2.1}$$

where, $b \in B$, $P(b) \in E$ are such that $|b - P(b)| = h(E,B)$ and $\mathcal{L}$ is the line orthogonal to the vector $b - P(b)$ through $P(b)$.

Note that Theorem 2.1 is actually an improvement of Theorem 1.1 since, as one can easily check, the deficit $\delta(E;B)$ in (2.1) is greater than the one in (1.3).

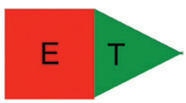
The main idea in both the theorems, was to approximate the difference $\mathcal{H}^1(\partial B) - \mathcal{H}^1(\partial E)$ from below with the perimeter of a triangle contained in $B\setminus E$ and having the basis on the line segment $\mathcal{L}$, see also (4.1) in [6]. In order to have better than in [7], we estimated the length of the basis of the triangle by means of the measure $\mathcal{H}^1(B\cap\mathcal{L})$ instead of using the diameter of the set B. This choice allowed us, in some sense, to preserve the information of the position of E with respect to B and, at the same time, to refine the estimate of the deficit successfully.

Indeed, we point out that the result in Theorem 2.1 revealed to be sharp as shown by the following example:

EXAMPLE 2.1. *Let E be a square of side l, T the triangle of base length l and height h and $B = E \cup T$ as in the figure below. Easy calculations give*

$$\mathcal{H}^1(\partial E) = 4l, \quad \mathcal{H}^1(\partial B) = 3l + 2\sqrt{\left(\frac{l}{2}\right)^2 + h^2}$$
$$h(E,B) = h, \quad \mathcal{H}^1(B\cap\mathcal{L}) = l$$

and therefore the equality in (2.1) occurs.



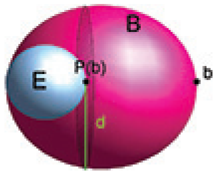
Once obtained the sharp estimate in Theorem 2.1, we faced the analogous problem in the case $n = 3$ (see [4]) obtaining the following

THEOREM 2.2. *For $\emptyset \neq E \subsetneq B \subset \mathbb{R}^3$ closed, convex and bounded sets, it holds that*

$$\mathscr{H}^2(\partial E) + \frac{\pi d |h(E,B)|^2}{\sqrt{d^2 + |h(E,B)|^2} + d} \leq \mathscr{H}^2(\partial B), \tag{2.2}$$

where $b \in B$ and $P(b) \in E$ are such that $|b - P(b)| = h(E,B)$, $S^+ = \{x \in \mathbb{R}^3 : \langle b - P(b), P(b) - x \rangle \geq 0\}$ and $d = \text{dist}(P(b), \partial B \cap \partial S^+)$.

Note that $b \in B \backslash E$ and $P(b)$ is the projection of b on the closed convex set E. Moreover, $P(b) \in \partial E$ and the plane ∂S^+ through $P(b)$ orthogonal to $b - P(b)$ is a supporting one for the convex set E (see the next figure).



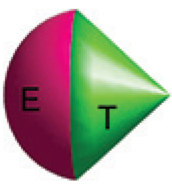
In the proof of Theorem 2.2 we were inspired by the successful idea we had in the planar case to approximate from below the deficit $\delta(E,B)$ between the perimeters of the convex sets with the perimeter of a triangle contained in $B \backslash E$ and having the basis on the segment supporting E. More precisely, arguing in a similar way, we estimated from below the difference of the surfaces of E and B with the surface of a cone contained in $B \backslash E$ and having the basis on the plane ∂S^+ supporting E.

Also in this case, the argument led to success and the obtained result revealed to be sharp as shown by the next example.

EXAMPLE 2.2. *Let E be the half ball of radius d, C the maximum circle contained in ∂E, T the cone with base C and height h and $B = E \cup T$ as in the figure below. Easy calculations give*

$$\mathscr{H}^2(\partial E) = 3\pi d^2, \quad \mathscr{H}^2(\partial B) = 2\pi d^2 + \pi d \sqrt{d^2 + h^2} \quad h(E,B) = h$$

and therefore the equality in (2.2) occurs.



Very recently, the problem to estimate the difference between the anisotropic (Wulff) perimeters of two convex sets in the n-dimensional case has been faced by G. Stefani [8].

3. An application

Let us consider a closed bounded set $E \subset \mathbb{R}^n$ and a decomposition of E of the form

$$E = \bigcup_{i=1}^{k} E_i,$$

where $\{E_i\}_{i=1,\dots,k}$ is a family of closed convex sets. Since it is clear that such decomposition is not unique, it is legitimate to search for an estimate from below of the minimal number $k_{\min}$ of the convex components of E that may exist. The first result of this type has been given in [7], where it is shown that $k_{\min}$ cannot be less than the ratio between the measure of ∂E and the measure of $\partial(\mathrm{co}(E))$), where $\mathrm{co}(E)$ is the convex hull of the set E. More precisely, in [7] the authors proved the following inequality

$$\text{upper integer part}\left(\frac{\mathscr{H}^{n-1}(\partial E)}{\mathscr{H}^{n-1}(\partial(\mathrm{co}(E)))}\right) \le k_{\min}. \tag{3.1}$$

The key tool in the proof of such estimate is the monotonicity property of the perimeter in (1.1).

Very recently, we were able to improve the inequality (3.1), at least in the planar case, using the quantitative estimate (2.1) applied to some convex components E_i and to the convex hull $\mathrm{co}(E)$ of E. More precisely, in the forthcoming paper [5], we established the following result

Theorem 3.1. *Let us consider a bounded closed set $\emptyset \neq E \subset \mathbb{R}^2$. Assume that there exist $q \in \mathbb{N}_0$, $c \in (0,1)$ and $p \in \mathbb{N}$ such that, for every family $\{E_i\}_{i=1,\dots,k}$ of closed convex sets such that $E = \bigcup_{i=1}^k E_i$ there exist $E_{i_1}, \dots, E_{i_p}$ such that*

$$h(E_{i_j}, \mathrm{co}(E)) \ge c\,\mathrm{diam}(\mathrm{co}(E)) \quad \forall j = 1, \dots, p \tag{3.2}$$

and

$$q\mathscr{H}^1(\partial(\mathrm{co}(E))) - \mathscr{H}^1(\partial E) < \frac{4c^2 p}{1+\sqrt{1+4c^2}}\,\mathrm{diam}(\mathrm{co}(E)). \tag{3.3}$$

Then

$$q + 1 \le k_{\min}. \tag{3.4}$$

Remark 3.1. Observe that the convex hull of a compact set is closed. Hence estimate (2.1) can be applied to E_i and $\mathrm{co}(E)$.

REMARK 3.2. Starting from the observation that each convex component of E has an Hausdorff distance from the convex hull of E less than the diameter of the convex hull itself, our main idea was to assume that for a finite number $p \in \mathbb{N}$ of such components, it holds that

$$c \operatorname{diam}(\operatorname{co}(E)) \leq h(E_i, \operatorname{co}(E)) \leq \operatorname{diam}(\operatorname{co}(E))$$

for a constant $0 < c < 1$. It is worth pointing out that p and c are independent of the decomposition of the set E and that the case $p = 0$ corresponds to the trivial case of E convex.

In the same paper we gave some examples showing that the estimate of $k_{\min}$ in Theorem 3.1 is sharp, since for them in (3.4) equality occurs.

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