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Fluid Mechanics — *Dynamic of thermo-MHD flows via a new approach*, by SALVATORE RIONERO, communicated on November 11, 2016.

This paper is dedicated to the memory of Professor Ennio De Giorgi

ABSTRACT. — Via a new approach, the dynamic of thermo-MHD flows in horizontal layers heated from below, in the Boussinesq scheme, is investigated. Denoting by m_0, E, P_r, P_m and $\bar{R}_C$ the thermal conduction solution, the L^2-energy of the nonlinear perturbations to m_0, the Prandtl number, the magnetic Prandtl number and the steady convection onset critical value of the Rayleigh number R^2, first of all we obtain, for any initial data, the ultimately boundedness of E, via the existence of L^2-absorbing sets. Successively we introduce a new approach for the E-longtime behaviour associated to the R^2-increasing. This new approach is based on an L^2-energy *linearization principle* and on a new way of analyzing and using the linear stability. As concerns the linearization principle, denoting by $\mathscr{E}$ the L^2-energy of the linear perturbations to m_0, it is shown that: "$\left(\dfrac{d\mathscr{E}}{dt}\right)_{(t=0)} < 0$ *for any initial data, implies* $\dfrac{dE}{dt} < 0$, $\forall t \in \mathbb{R}^+$".

In order to obtain $\left(\dfrac{d\mathscr{E}}{dt}\right)_{(t=0)} < 0$ *for any initial data*, denoting by $\mathtt{L}_n$ the linear operator governing the nth-Fourier component of perturbations, we introduce the characteristic values $\mathtt{I}_{jn}$, $(j = 1, 2, 3)$ of $\mathtt{L}_n$ via the $\mathtt{L}_n$-entries and obtain the equation $\lambda^3 - \mathtt{I}_{1n}\lambda^2 + \mathtt{I}_{2n}\lambda - \mathtt{I}_{3n} = 0$, governing the $\mathtt{L}_n$-eigenvalues. To this equation we apply the Hurwitz's Criterion guaranteeing that all the eigenvalues have negative real part. As matter of fact, the Hurwitz's Criterion, applied for each $n \in \mathbb{N}$, allows to obtain conditions necessary and sufficient for being $\left(\dfrac{d\mathscr{E}}{dt}\right)_{(t=0)} < 0$ *for any initial data*. Following this new approach, we show that *the unconditional nonlinear asymptotic stability of* m_0, *with respect to the* L^2*-energy norm, is guaranteed by the linear stability* and obtain – among other things – two conditions, in a very simple closed forms, guaranteeing the onset of oscillatory convection (*overstability laws*).

All the results, first obtained for the free-free boundary case, are successively generalized to the rigid-rigid, rigid-free and free-rigid boundary cases.

KEY WORDS: MHD, convection, unconditional stability, overstability, longtime behaviour

MATHEMATICS SUBJECT CLASSIFICATION: 76E25, 76E06, 35B35

Contents

1. Introduction

This paper is devoted to the dynamic of thermo-MHD flows in horizontal layers, heated from below, filled by an electrically conducting fluid under the action of a constant vertical magnetic field **H**. These flows – for their importance in Geophysics, Astrophysics and industrial applications – have attracted in the past as nowadays, the attention of many researchers (see [1]–[16] and the references therein).

Denoting by m_0, E, P_r, P_m and $\bar{R}_C$ the thermal conduction solution, the L^2-energy of the nonlinear perturbations to m_0, the Prandtl number, the magnetic Prandtl number and the critical value of the Rayleigh number for the onset of steady convection, our aim is to show that, *for any initial data*, the following properties hold.

Proposition 1.1. *The thermo-MHD flows in horizontal layers heated from below are ultimately bounded:*

$$\lim_{t\to\infty} E \leq E_\infty < \infty.$$

Proposition 1.2. *The conditions guaranteeing the linear stability of m_0 guarantee the unconditional nonlinear asymptotic stability with respect to the L^2-energy norm.*

Proposition 1.3 (First Overstability Law). *Let $P_r < P_m$. Then*

$$\bar{R}_C > \frac{27}{4}\pi^4 \frac{1+P_m}{P_m - P_r}, \tag{1.1}$$

guarantees the onset of oscillatory convection for $R^2 \in [\mathscr{R}_C, \overline{R}_C[$ *with*

$$\mathscr{R}_C = \frac{27}{4}\pi^4\left(1 + \frac{1}{P_m}\right) + \frac{P_r}{P_m}\overline{R}_C. \tag{1.2}$$

PROPOSITION 1.4 (Second Overstability Law). *Let* $P_r < P_m$. *Then*

$$P_r < \left[1 - \frac{1}{\pi^2(\pi^2 + Q^2)}\right]P_m, \tag{1.3}$$

guarantees oscillatory convection.

Property 1.1 is expected and is obtained by looking for L^2-absorbing sets (Section 3). Properties 1.2–1.4 – as far as we know – are new in the existing literature.

As concerns property 1.2, we recall that the onset of convection in the case at stake is named Magnetic Bénard Problem (MBP). The most important phenomenon in the MBP linear theory is the strong inhibiting effect of the magnetic field on the onset of convection. This effect – obtained theoretically in [15], [1], [2] and described extensively in [3], has been confirmed by the experiments [1], [9]. During the years, many efforts have been done in order to recover this effect in the nonlinear MHD theory (see [5]–[8], [10]–[14], [16]). This goal has been reached partially in 1988 in [13] and totally in 2003 in [16]. But it is to remark that both the partial and the total coincidence require a very big price. As matter of fact, both hold under very severe restrictions on the initial data (*conditional nonlinear asymptotic stability*). In Table 1 the restrictions on the radius r_ε of the basin of attraction found in [13], in the case of mercury, are recalled. The results of [13] and [16] – although of notable theoretical interest – appear weak from the applications point of view. In fact a stability condition holding only for initial perturbations of the order 10^{-13} appears of very small practical interest.

Table 1. Values of the radius r_ε of the basin of concentration given in [13] in the case of mercury. Q^2 is the Chandrasekhar number growing with $\mathbf{H}$ (see $(2.2)_2$)

R_C^2	Q^2	r_ε	R_C^2	Q^2	r_ε
625	0	10^{-16}	1000	10^2	4×10^{-12}
625	50	9×10^{-12}	1000	10^3	2.7×10^{-9}
625	10^2	8×10^{-11}	1000	10^4	4×10^{-7}
625	10^3	2.1×10^{-8}	1000	10^5	2.9×10^{-5}
625	10^4	1.4×10^{-6}	5000	10^3	1.2×10^{-12}
625	10^5	3×10^{-4}	5000	10^4	3.2×10^{-10}
1000	50	3.2×10^{-13}	5000	10^5	5.6×10^{-8}

In this paper we reconsider the MBP in the more general context of the nonlinear longtime behaviour of thermo-MHD flows. Denoting by R_C the R^2 critical value for the onset of instability given by linear MHD stability of m_0, our aim is to show that (Property 1.2)

$$(1.4) \qquad R^2 < R_C \Rightarrow \text{nonlinear unconditional asymptotic stability of } m_0.$$

This property is obtained via a new version of an L^2-Energy Linearization Principle (Section 4) obtained by showing that $\frac{dE}{dt} < 0, \ \forall t \in \mathbb{R}^+$ is implied by $\left(\frac{d\mathscr{E}}{dt}\right)_{(t=0)} < 0$, *for any initial data*, $\mathscr{E}$ being the L^2-energy of the linear perturbations to m_0. Based on the linearization principle, a new approach to the linear stability is introduced (Sections 5–6). Denoting by $\mathtt{L}_n$ the linear operator governing the n-th Fourier component of the linear perturbations to m_0, to the spectral equation

$$\lambda^3 - \mathtt{I}_{1n}\lambda^2 + \mathtt{I}_{2n}\lambda - \mathtt{I}_{3n} = 0,$$

of $\mathtt{L}_n - \mathtt{I}_{in}$, $(i = 1, 2, 3)$, being the characteristic values of $\mathtt{L}_n$ written via the $\mathtt{L}_n$ entries (Appendix 1.1) – the Hurwitz's Criterion (Appendix 1.2), guaranteeing that all the eigenvalues have negative real part, is hence applied in order to guarantee the consistency of the linearization principle (Section 6) and hence the consistency of property 1.2. The steady and oscillatory bifurcating critical values of R^2 are obtained in Sections 7–8. The Oscillatory Bifurcation Law (Property 1.2), given in Section 9, appears to be new and is a remarkable consequence of the introduced new approach to the linear stability. All the results are – in a first step – obtained for the free-free boundary conditions. Successively, in Section 10, rigid-rigid, free-rigid and rigid-free boundaries are considered. In the subsequent Section 11, the obtained results are summarized and perspectives of new works are considered. The paper ends with the Appendix (Section 12) where the $\mathtt{L}_n$ characteristic in terms of the $\mathtt{L}_n$ entries are obtained (Subsection 12.1) while in the subsection 12.2 the Hurwitz's Criterion is recalled. Finally the subsection 12.3 is devoted to the properties 1.2–1.4 for the classical Bénard problem in rotating layers.

2. Preliminaries

Let $\mathtt{L}$ be an infinite horizontal layer of a homogeneous, viscous, electrically conducting fluid, permeated by an imposed uniform magnetic field $\mathbf{H}$ normal to the layer, under the action of a vertical gravity field $\mathbf{g} = -g\mathbf{k}$, and in which a constant adverse temperature gradient $\beta > 0$ is maintained. Let $d > 0$, $\Omega_d = \mathbb{R}^2 \times (0, d)$ and $Oxyz$ be a Cartesian frame of reference with unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ respectively. We assume that the fluid is confined between the planes $z = 0$ and $z = d$, with assigned temperatures $\tilde{T}(x, y, 0) = \tilde{T}_0$, $\tilde{T}(x, y, d) = -\beta d + \tilde{T}_0$. Here we consider the rest state $m_0 = (\tilde{v}, \tilde{\mathbf{H}}, \tilde{T}, \tilde{p}) = (0, H\mathbf{k}, -\beta z + \tilde{T}_0, \tilde{p})$ (thermal conduction), which is a solution of the Oberbeck-Boussinesq approximation of the stationary non-isothermal equations of hydromagnetics (with appropriate boundary conditions, see [3] Chapter 4).

The (non-dimensional) equations for a perturbation $(\mathbf{u}, \mathbf{h}, \theta, p_1)$ to m_0 are:

$$
(2.1) \qquad \begin{cases} \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - P_m \mathbf{h} \cdot \nabla \mathbf{h} = -\nabla p_1 + R\theta \mathbf{k} + \Delta u + Q h_z, \\ \nabla \cdot \mathbf{u} = 0, \\ P_m(\mathbf{h}_t + \mathbf{u} \cdot \nabla \mathbf{h} - \mathbf{h} \cdot \nabla \mathbf{u}) = Q u_z + \Delta h, \\ \nabla \cdot \mathbf{h} = 0, \\ P_r(\theta_t + \mathbf{u} \cdot \nabla \theta) = Rw + \Delta \theta. \end{cases}
$$

in $\Omega_1 \times (0, \infty)$, $\Omega_1 = R^2 \times (0, 1)$.

In (2.1) the symbols are standard: $\mathbf{u} = (u, v, w)$, $\mathbf{h} = (h_1, h_2, h_3)$, θ and p_1 are the perturbations of the velocity, magnetic, temperature and pressure (incorporating the magnetic pressure) fields, respectively. The subscripts z and t denote partial derivatives, ∇ is the "nabla" operator and Δ is the Laplacian. The non-dimensional parameters R^2, Q^2, P_r and P_m are the Rayleigh, Chandrasekhar, Prandtl and magnetic Prandtl numbers. They are given by

$$
(2.2) \qquad R^2 = \frac{g\beta\alpha d^4}{\nu k}, \quad Q^2 = \frac{\mu H^2 d^2}{4\pi\rho\nu\mu}, \quad P_r = \frac{\nu}{k}, \quad P_m = \frac{\nu}{\eta},
$$

where the constants $g, \alpha, \rho, \mu, \eta, k$ and ν are the gravitational acceleration, the coefficient of volume expansion, the density, the magnetic permeability, the resistivity, the thermal diffusivity and the kinematic viscosity coefficients.

To the system (2.1) we add the *admissible initial conditions*

$$
(2.3) \qquad \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{h}(\mathbf{x}, 0) = \mathbf{h}_0(\mathbf{x}), \quad \theta(\mathbf{x}, 0) = \theta_0(\mathbf{x}),
$$

where $\mathbf{x} = (x, y, z) \in \Omega_1$, θ_0, $\mathbf{u}_0$, $\mathbf{h}_0$ are assigned initial fields with $\mathbf{u}_0$, $\mathbf{h}_0$ divergence-free. As concerns the boundaries, we shall assume that they are stress-free and electrically non-conducting, therefore, to the system (2.1)–(2.2), we add the following boundary conditions (see [3])

$$
(2.4) \qquad \begin{cases} \theta(x, y, z, t) = 0, \\ w(x, y, z, t) = u_z(x, y, z, t) = v_z(x, y, z, t) = 0, \\ h_1(x, y, z, t) = h_2(x, y, z, t) = h_z(x, y, z, t) = 0, \end{cases}
$$

$(x, y) \in \mathbb{R}^2$, $t > 0$ and $z = 0$, $z = 1$.

We assume that:

i) the perturbations $(\nabla\pi, u, v, w, \theta, h_1, h_2, h)$ are periodic in the x and y directions of periods $2\pi/a_x$, $2\pi/a_y$, respectively;
ii) $\Omega = [0, 2\pi/a_x] \times [0, 2\pi/a_y] \times [0, 1]$ is the periodicity cell;
iii) $u, v, w, \theta, h_1, h_2, h$ are such that together with all their first derivatives and second spatial derivatives are square integrable in Ω, $\forall t \in \mathbb{R}^+$ and can be expanded in a Fourier series uniformly convergent in Ω.

Let us denote by $\mathscr{A}(\Omega)$ the set of functions Ψ such that:

1) $\Psi : (\mathbf{x}, t) \in \Omega \times \mathbb{R}^+ \to \Psi(\mathbf{x}, t) \in \mathbb{R}$, $\Psi \in W^{2,2}(\Omega)$, $\forall t \in \mathbb{R}^+$, Ψ is periodic in the x and y directions of period $\frac{2\pi}{a_x}, \frac{2\pi}{a_y}$ respectively;
2) Ψ, together with all the first derivatives and second spatial derivatives, can be expanded in a Fourier series absolutely uniformly convergent in Ω, $\forall t \in \mathbb{R}^+$;
3) $(\Psi)_{z=0} = (\Psi)_{z=1} = 0$

and by $\mathscr{B}(\Omega)$ the set of the functions φ verifying 1)–2) and

1)′ $\left[\frac{\partial \varphi}{\partial z}\right]_{z=0} = \left[\frac{\partial \varphi}{\partial z}\right]_{z=1} = 0.$

Since the sequence $\{\sin n\pi z\}_{n\in\mathbb{N}}$, is a complete orthogonal system for $L^2(0,1)$ under the boundary conditions $[\Psi]_{z=0} = [\Psi]_{z=1} = 0$, by virtue of periodicity, it turns out that $\forall \Psi \in \mathscr{A}(\Omega)$, there exists a sequence $\{\tilde{\Psi}_n(x, y, t)\}$ such that

$$(2.5)\qquad \begin{cases} \Psi = \sum_{n=1}^{\infty} \tilde{\Psi}_n \sin n\pi z, \quad \frac{\partial \Psi}{\partial t} = \sum_{n=1}^{\infty} \frac{\partial \tilde{\Psi}_n}{\partial t} \sin n\pi z, \\ \Delta_1 \Psi = -a^2 \Psi, \quad \Delta \Psi = -\sum_{n=1}^{\infty} \xi_n \tilde{\Psi}_n \sin n\pi z, \end{cases}$$

with $\Delta_1 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ and

$$(2.6)\qquad \xi_n = a^2 + n^2\pi^2, \quad a^2 = a_x^2 + a_y^2,$$

the series appearing in (2.5) being absolutely uniformly convergent in Ω. Analogously, since the sequence $\{\cos n\pi z\}_{n\in\mathbb{N}}$, is a complete orthogonal system for $L^2(0,1)$ under the boundary conditions $\left[\frac{\partial \varphi}{\partial z}\right]_{z=0} = \left[\frac{\partial \varphi}{\partial z}\right]_{z=1} = 0$, by virtue of periodicity, it turns out that $\forall \varphi \in \mathscr{B}(\Omega)$, there exists a sequence $\{\tilde{\varphi}_n(x, y, t)\}$ such that

$$(2.7)\qquad \begin{cases} \varphi = \sum_{n=1}^{\infty} \tilde{\varphi}_n \cos n\pi z, \quad \frac{\partial \varphi}{\partial t} = \sum_{n=1}^{\infty} \frac{\partial \tilde{\varphi}_n}{\partial t} \cos n\pi z, \\ \Delta_1 \varphi = -a^2 \varphi, \quad \Delta \varphi = -\sum_{n=1}^{\infty} \xi_n \tilde{\varphi}_n \cos n\pi z. \end{cases}$$

Remark 2.1. We remark that uniqueness, existence and regularity theorems for the solutions of (2.1), (2.3)–(2.4) in L^2-subspaces can be found in [17], in the absence of thermal field, and in the absence of magnetic field. Existence theorems – in the presence of Hall and ion-slip effects – can be found in [20]. In [22] and references therein, uniqueness, existence and regularity properties of Boussinesq systems can be found.

3. L^2-ABSORBING SETS

We denote by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ respectively the norm and the scalar product in $L^2(\Omega)$ and introduce the L^2-energy of the perturbation field

$$E = \frac{1}{2}(\|\mathbf{u}\|^2 + P_m\|\mathbf{h}\|^2 + P_r\|\theta\|^2). \tag{3.1}$$

In view of (2.4) and $\nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{h} = 0$, one easily obtains

$$\begin{cases} \langle\mathbf{u}\cdot\nabla\mathbf{u},\mathbf{u}\rangle = -\frac{1}{2}\langle\mathbf{u},\nabla\cdot\mathbf{u}\rangle = 0,\ \langle\mathbf{u}\cdot\nabla\mathbf{h},\mathbf{h}\rangle = -\frac{1}{2}\langle\mathbf{h}^2,\nabla\cdot\mathbf{u}\rangle = 0, \\ \langle\mathbf{h}\cdot\nabla\mathbf{h},\mathbf{u}\rangle + \langle\mathbf{h}\cdot\nabla\mathbf{u},\mathbf{h}\rangle = -(\mathbf{h}\cdot\mathbf{u})\nabla\cdot\mathbf{h} = 0, \\ \langle\mathbf{h}_z,\mathbf{u}\rangle + \langle\mathbf{u}_z,\mathbf{h}\rangle = 0. \end{cases} \tag{3.2}$$

Therefore, setting

$$\mathcal{Q}_*(\mathbf{u},\mathbf{h},\theta) = 2R\langle w,\theta\rangle + \langle\mathbf{u},\Delta\mathbf{u}\rangle + \langle\mathbf{h},\Delta\mathbf{h}\rangle + \langle\theta,\Delta\theta\rangle, \tag{3.3}$$

by virtue of (2.1) and (3.2) one easily obtains that the time derivative $\frac{dE}{dt}$ of E along the solution of (2.1), (2.3)–(2.4), is given by

$$\frac{dE}{dt} = \mathcal{Q}_*(\mathbf{u},\mathbf{h},\theta). \tag{3.4}$$

In view of

$$\begin{cases} 2\langle w,\theta\rangle < \varepsilon\|\mathbf{u}\|^2 + \frac{1}{\varepsilon}\|\theta\|^2, \quad \varepsilon = \text{positive constant}, \\ \langle\mathbf{u},\Delta\mathbf{u}\rangle = -\|\nabla\mathbf{u}\|^2,\ \langle\mathbf{h},\Delta\mathbf{h}\rangle = -\|\nabla\mathbf{h}\|^2,\ \langle\theta,\Delta\theta\rangle = -\|\nabla\theta\|^2 \end{cases} \tag{3.5}$$

and the Poincaré inequality

$$\|\nabla f\|^2 \geq \pi^2\|f\|^2, \tag{3.6}$$

it follows that

$$\frac{dE}{dt} < \frac{R}{\varepsilon}\|\theta\|^2 - \pi^2[(1+R\varepsilon)\|\mathbf{u}\|^2 + \|\mathbf{h}\|^2 + \|\theta\|^2]. \tag{3.7}$$

On choosing $\varepsilon = \frac{1}{2R}$ one obtains

$$\frac{dE}{dt} < 2R^2\|\theta\|^2 - \pi^2\Big(\frac{1}{2}\|\mathbf{u}\|^2 + \|\mathbf{h}\|^2 + \|\theta\|^2\Big) \leq 2R^2\|\theta\|^2 - k\pi^2 E, \tag{3.8}$$

with

$$k < 2\min\Big(\frac{1}{P_m}, \frac{1}{P_r}\Big). \tag{3.9}$$

Let

$$2R^2\|\theta\|^2 < a = \text{const} > 0 \tag{3.10}$$

and set

$$b = k\pi^2, \tag{3.11}$$

then (3.8) becomes

$$\frac{dE}{dt} < a - bE \tag{3.12}$$

and via a standard procedure it follows that the following theorem holds.

Theorem 3.1. *The ball of radius*

$$E < (1+\eta)\frac{b}{a}, \tag{3.13}$$

centered at the origin of the L^2-phase space, for any $\eta > 0$ is an L^2-absorbing set.

Proof. It remains to show that there exist positive constants such that (3.10) holds. As matter of fact, in [17] it is shown that

$$\theta = \tilde{\theta} + \bar{\theta}$$

with

$$-1 \le \tilde{\theta} \le 1$$

and

$$\|\bar{\theta}\| \le \{\|(\theta - 1)_+\| + \|(\theta + 1)_-\|\}_{t=0} e^{-\pi^2 t}.$$

4. Energy Linearization Principle

To (2.1), (2.3)–(2.4), we associate the linear system

$$\begin{cases} \hat{\mathbf{u}}_t = R\hat{\theta}\mathbf{k} + Q\hat{\mathbf{h}}_z + \Delta\hat{\mathbf{u}}, \\ \nabla\cdot\hat{\mathbf{u}} = 0, \\ P_m\hat{\mathbf{h}}_t = Q\hat{\mathbf{u}}_z + \Delta\hat{\mathbf{h}}, \\ \nabla\cdot\hat{\mathbf{h}} = 0, \\ P_r\hat{\theta}_t = R\hat{w} + \Delta\hat{\theta}, \end{cases} \tag{4.1}$$

under the initial boundary conditions

$$\begin{cases} \hat{\mathbf{u}}(\mathbf{x},0) = \hat{\mathbf{u}}_0(\mathbf{x}),\ \hat{\mathbf{h}}(\mathbf{x},0) = \hat{\mathbf{h}}_0(\mathbf{x}),\ \hat{\theta}(\mathbf{x},0) = \hat{\theta}_0, \\ \nabla\cdot\hat{\mathbf{u}}_0 = 0,\ \nabla\cdot\hat{\mathbf{h}}_0 = 0 \end{cases} \tag{4.2}$$

$$\begin{cases} \hat{\theta}(x,y,z,t) = 0, \\ \hat{w}(x,y,z,t) = \hat{u}_z(x,y,z,t) = \hat{v}_z(x,y,z,t) = 0, \\ \hat{h}_1(x,y,z,t) = \hat{h}_2(x,y,z,t) = \hat{h}_z(x,y,z,t) = 0, \end{cases} \tag{4.3}$$

$(x,y) \in \mathbb{R}^2$, $t > 0$ and $z = 0$, $z = 1$.

Denoting by

$$\hat{E} = \frac{1}{2}(\|\hat{\mathbf{u}}\|^2 + P_m\|\hat{\mathbf{h}}\|^2 + P_r\|\hat{\theta}\|^2), \tag{4.4}$$

the $L^2(\Omega)$-energy of the solutions of (4.1)–(4.3) and by $\frac{d\hat{E}}{dt}$ the time derivative of $\hat{E}$ along the solution of (4.1)–(4.3), it follows that

$$\frac{d\hat{E}}{dt} = \hat{\mathscr{Q}}_*(\hat{\mathbf{u}},\hat{\mathbf{h}},\hat{\theta}), \tag{4.5}$$

with

$$\hat{\mathscr{Q}}_*(\hat{\mathbf{u}},\hat{\mathbf{h}},\hat{\theta}) = 2R\langle\hat{w},\hat{\theta}\rangle + \langle\hat{\mathbf{u}},\Delta\hat{\mathbf{u}}\rangle + \langle\hat{\mathbf{h}},\Delta\hat{\mathbf{h}}\rangle. \tag{4.6}$$

The following theorem holds (*linearization principle*).

THEOREM 4.1. *Let*

$$\left(\frac{d\hat{E}}{dt}\right)_{(t=0)} < 0, \tag{4.7}$$

for arbitrary initial data (4.2). *Then*

$$\frac{dE}{dt} < 0, \quad \forall t \geq 0. \tag{4.8}$$

PROOF. Let $(\mathbf{u},\mathbf{h},\theta)$ be a solution of (2.1), (2.3)–(2.4). Then, for any fixed $\tau > 0$ one has

$$\nabla\cdot\mathbf{u}(\mathbf{x},\tau) = \nabla\cdot\mathbf{h}(\mathbf{x},\tau) = 0, \quad \forall\mathbf{x}\in\Omega \tag{4.9}$$

and for $\mathbf{x}\in\Omega$ and $z = 0,1$ it follows that

$$\begin{cases} \theta(\mathbf{x},\tau) = 0, \\ w(\mathbf{x},\tau) = u_z(\mathbf{x},\tau) = v_z(\mathbf{x},\tau) = 0, \\ h_1(\mathbf{x},\tau) = h_2(\mathbf{x},\tau) = h_z(\mathbf{x},\tau) = 0, \end{cases} \tag{4.10}$$

$$
(4.11)\qquad [\mathcal{Q}_*(\mathbf{u},\mathbf{h},\theta)]_{(t=\tau)} = \begin{cases} \langle \mathbf{u}(\mathbf{x},\tau), R\theta(\mathbf{x},\tau)\mathbf{k} + \Delta\mathbf{u}(\mathbf{x},\tau)\rangle \\ \quad + \langle \mathbf{h}(\mathbf{x},\tau), \Delta\mathbf{h}(\mathbf{x},\tau)\rangle \\ \quad + \langle \theta(\mathbf{x},\tau), Rw(\mathbf{x},\tau) + \Delta\theta(\mathbf{x},\tau)\rangle. \end{cases}
$$

But in view of (4.9)–(4.10), one can choose

$$
(4.12)\qquad \hat{\mathbf{u}}(\mathbf{x},0) = \mathbf{u}(\mathbf{x},\tau), \quad \hat{\mathbf{h}}(\mathbf{x},0) = \mathbf{h}(\mathbf{x},\tau), \quad \hat{\theta}(\mathbf{x},0) = \theta(\mathbf{x},\tau)
$$

and hence via (4.6) one has

$$
(4.13)\qquad [\hat{\mathcal{Q}}_*(\hat{\mathbf{u}},\hat{\mathbf{h}},\hat{\theta})]_{(t=0)} = [\mathcal{Q}_*(\mathbf{u},\mathbf{h},\theta)]_{(t=\tau)}, \quad \forall\tau > 0.
$$

Then (4.7) and (4.13) imply (4.8) $\forall\tau > 0$. In other word, (4.7), *for any initial data*, implies the negative definiteness of $\hat{\mathcal{Q}}_*$. Then (4.8), in view of (4.13), immediately follows.

5. New approach to energy stability

A new approach to the nonlinear asymptotic unconditional energy stability – guaranteed by linear stability – is implied by Theorem 4.1. The consistency of (4.7) – *for any initial data* – requires that all the eigenvalues of (4.1)–(4.3) have negative real part. As matter of fact, let $\{\varphi_n^*\}$, $\{\bar{\varphi}_n\}$ and $\{\bar{\bar{\varphi}}_n\}$ be complex orthogonal sequences of functions verifying the boundary conditions of velocity, magnetic and temperature fields respectively. Then *any solution* of (4.1)–(4.3) can be written as Fourier series as follows

$$
(5.1)\qquad \begin{cases} \hat{\mathbf{u}} = \sum_{n=1}^{\infty} \hat{\mathbf{u}}_n,\ \hat{\mathbf{h}} = \sum_{n=1}^{\infty} \hat{\mathbf{h}}_n,\ \hat{\theta} = \sum_{n=1}^{\infty} \hat{\theta}_n, \\ \hat{\mathbf{u}}_n = \mathbf{u}_n^{(*)}\varphi_n^*,\ \hat{\mathbf{h}}_n = \bar{\mathbf{h}}_n\bar{\varphi}_n,\ \hat{\theta}_n = \bar{\bar{\theta}}_n\bar{\bar{\varphi}}_n. \end{cases}
$$

Let $\lambda_n = a_n + ib_n$ be an eigenvalue associated to the nth-component $(\hat{\mathbf{u}}_n, \hat{\mathbf{h}}_n, \hat{\theta}_n)$ of the solution $(\hat{\mathbf{u}}, \hat{\mathbf{h}}, \hat{\theta})$ of (4.1)–(4.3). Then

$$
(5.2)\qquad \begin{pmatrix} \hat{\mathbf{u}}_n \\ \hat{\mathbf{h}}_n \\ \hat{\theta}_n \end{pmatrix} = e^{(a_n+ib_n)t} \begin{pmatrix} \hat{\mathbf{u}}_n^{(0)}(\mathbf{x}) \\ \hat{\mathbf{h}}_n^{(0)}(\mathbf{x}) \\ \hat{\theta}_n^{(0)}(\mathbf{x}) \end{pmatrix}
$$

and, denoting by $f^{(c)}$ the complex conjugate of f, it follows that

$$
(5.3)\qquad \frac{1}{2}\frac{d}{dt}(\langle \hat{\mathbf{u}}_n, \hat{\mathbf{u}}_n^{(c)}\rangle + P_m\langle \hat{\mathbf{h}}_n, \hat{\mathbf{h}}_n^{(c)}\rangle + P_r\langle \hat{\theta}_n, \hat{\theta}_n^{(c)}\rangle) = a_n e^{2a_n t}(\|\hat{\mathbf{u}}_n^{(0)}\|^2 + P_m\|\hat{\mathbf{h}}_n^{(0)}\|^2 + P_r\|\hat{\theta}_n^{(0)}\|^2)
$$

with $(\hat{\mathbf{u}}_n^{(0)}, \hat{\mathbf{h}}_n^{(0)}, \hat{\theta}_n^{(0)})$ *initial data*. Therefore if and only if $a_n < 0$, $\forall n \in \mathbb{N}$ one has

$$\left(\frac{d\hat{E}_n}{dt}\right)_{(t=0)} < 0, \tag{5.4}$$

along the solutions $(\hat{\mathbf{u}}_n, \hat{\mathbf{h}}_n, \hat{\theta}_n)$ of (4.1)–(4.3) *for any initial data*, and hence (4.13), *for any initial data*, implies the following theorem.

THEOREM 5.1. *Conditions guaranteeing that all the eigenvalues of* (4.1)–(4.2) *have negative real part, guarantee linear stability and nonlinear unconditional asymptotic energy stability.*

REMARK 5.1. We remark that

1) the application to (4.1)–(4.3) of the Hurwitz's Criterion guaranteeing matrix eigenvalues have all real negative part [18], [19], appears to be the right way for getting the optimum unconditional nonlinear asymptotic stability threshold;
2) the strategy based on Theorem 5.1 is completely different from the traditional one [4]–[14], [16]. As matter of fact in the traditional nonlinear stability approach, it is request that $\mathscr{Q}_*$ is negative definite in the class $\mathscr{I}$ of the "*kinematically admissible perturbations*" which is nothing else that the class of initial data. For instance, setting

$$\mathscr{F} = \frac{2R\langle w, \theta\rangle}{\|\nabla\mathbf{u}\|^2 + \|\nabla\mathbf{h}\|^2 + \|\nabla\theta\|^2}, \tag{5.5}$$

one has

$$\frac{dE}{dt} = (\mathscr{F} - 1)(\|\nabla\mathbf{u}\|^2 + \|\nabla\mathbf{h}\|^2 + \|\nabla\theta\|^2) \tag{5.6}$$

and hence introducing the variational problem

$$m = \max_{\mathscr{I}} \mathscr{F}, \tag{5.7}$$

one has that

$$m < 1 \tag{5.8}$$

guarantees $\mathscr{Q}_* < 0$ in $\mathscr{I}$ and hence $\hat{\mathscr{Q}}_* < 0$.

In the new strategy it is request that (4.7) and hence $\hat{\mathscr{Q}}_* < 0$, is verified in a subspace of $\mathscr{I}$. In fact to each initial data belonging to $\mathscr{I}$, it is requested that (4.7) holds only on the solution of (4.1)–(4.3);
3) since

$$\left\langle \mathbf{u}, \frac{\partial\mathbf{h}}{\partial t}\right\rangle + \left\langle \mathbf{h}, \frac{\partial\mathbf{u}}{\partial z}\right\rangle = 0, \tag{5.9}$$

on the numerator of $\mathscr{F}$ does not appear any stabilizing contribution of the magnetic field (contribution found via (4.1)–(4.3)). Therefore in order to obtain – in the old approach of nonlinear stability – this effect, one has to introduce Liapunov functions much more complicated than the natural one given by E ([13], [16]);

4) we recall that the Energy Linearization Principle can be also obtained by the auxiliary system method introduced in [21].

6. Consistency of the new stability approach

In view of (4.1) it follows that

$$\begin{cases} \dfrac{\partial \hat{u}}{\partial t} = Q\dfrac{\partial \hat{h}_1}{\partial z} + \Delta \hat{u}, \\ P_m\dfrac{\partial \hat{h}_1}{\partial t} = Q\dfrac{\partial \hat{u}}{\partial z} + \Delta \hat{h}_1 \end{cases} \tag{6.1}$$

and hence (2.4) implies

$$\frac{1}{2}\frac{d}{dt}(\|\hat{u}\|^2 + P_m\|\hat{h}_1\|^2) = -(\|\nabla\hat{u}\|^2 + \|\nabla\hat{h}_1\|^2) < 0 \tag{6.2}$$

$$\frac{1}{2}\frac{d}{dt}(\|\hat{v}\|^2 + P_m\|\hat{h}_2\|^2) = -(\|\nabla\hat{v}\|^2 + \|\nabla\hat{h}_2\|^2) < 0 \tag{6.3}$$

for any initial data. Therefore, setting

$$\hat{\mathscr{E}} = \frac{1}{2}(\|\hat{w}\|^2 + P_m\|\hat{h}\|^2 + P_r\|\hat{\theta}\|^2), \tag{6.4}$$

it follows that (4.7) is implied by

$$\left(\frac{d\hat{\mathscr{E}}}{dt}\right)_{(t=0)} < 0 \tag{6.5}$$

for any initial data. In view of (5.1), (6.5) is implied by

$$\left(\frac{d\hat{\mathscr{E}}_n}{dt}\right)_{(t=0)} < 0 \tag{6.6}$$

for any initial data, with

$$\hat{\mathscr{E}}_n = \frac{1}{2}(\|\hat{w}_n\|^2 + P_m\|\hat{h}_n\|^2 + P_r\|\hat{\theta}_n\|^2), \tag{6.7}$$

$\dfrac{d\hat{\mathscr{E}}_n}{dt}$ being the time derivative along the solutions of (4.1)–(4.3).

The equations governing $(\hat{w}_n, \hat{h}_n, \hat{\theta}_n)$, disregarding the hats, are given by

$$(6.8) \qquad \begin{cases} \dfrac{\partial}{\partial t}\Delta w_n = R\Delta_1\theta_n + \Delta\Delta w_n + Q\Delta\dfrac{\partial h_n}{\partial z}, \\ P_m\dfrac{\partial}{\partial t}\Big(\dfrac{\partial h_n}{\partial z}\Big) = Q\dfrac{\partial^2 w_n}{\partial z^2} + \Delta\Big(\dfrac{\partial h_n}{\partial z}\Big), \\ P_r\dfrac{\partial}{\partial t}\theta_n = Rw_n + \Delta\theta_n \end{cases}$$

with $(6.8)_1$ third component of the double curl of $(4.1)_1$ and $(6.8)_2$ partial derivative $\dfrac{\partial}{\partial z}$ of the third component of $(4.1)_3$, under the boundary conditions

$$(6.9) \qquad w_n = \theta_n = \frac{\partial h_n}{\partial z} = 0, \quad \text{on } z = 0, 1.$$

Therefore, since $h_n \in \mathscr{B}(\Omega)$ and w_n, $\theta_n \in \mathscr{A}(\Omega)$, setting

$$(6.10) \qquad \begin{cases} w_n = \tilde{w}_n(x, y, t)\sin(n\pi z), \\ \theta_n = \tilde{\theta}_n(x, y, t)\sin(n\pi z), \\ h_n = \tilde{h}_n(x, y, t)\cos(n\pi z), \end{cases}$$

by virtue of (2.4)–(2.5), (6.8) can be written

$$(6.11) \qquad \begin{cases} \dfrac{\partial\tilde{w}_n}{\partial t} = -\xi_n\tilde{w}_n - Qn\pi\tilde{h}_n + \dfrac{Ra^2}{\xi_n}\tilde{\theta}_n, \\ \dfrac{\partial\tilde{h}_n}{\partial t} = \dfrac{n\pi Q}{P_m}\tilde{w}_n - \dfrac{\xi_n}{P_m}\tilde{h}_n, \\ \dfrac{\partial\tilde{\theta}_n}{\partial t} = \dfrac{R}{P_2}\tilde{w}_n - \dfrac{\xi_n}{P_r}\tilde{\theta}_n, \end{cases}$$

i.e.

$$(6.12) \qquad \frac{\partial}{\partial t}\begin{pmatrix}\tilde{w}_n\\ \tilde{\theta}_n\\ \tilde{h}_n\end{pmatrix} = \mathscr{L}_n\begin{pmatrix}\tilde{w}_n\\ \tilde{\theta}_n\\ \tilde{h}_n\end{pmatrix},$$

with

$$(6.13) \qquad \mathscr{L}_n = \begin{pmatrix} -\xi_n & -Qn\pi & \dfrac{Ra^2}{\xi_n} \\ \dfrac{n\pi Q}{P_m} & -\dfrac{\xi_n}{P_m} & 0 \\ \dfrac{R}{P_r} & 0 & -\dfrac{\xi_n}{P_r} \end{pmatrix}.$$

The characteristic values (invariants) of (6.13) are

$$(6.14)\qquad \begin{cases} \mathtt{I}_{1n} = -\left(1 + \dfrac{1}{P_r} + \dfrac{1}{P_m}\right)\xi_n = -\dfrac{(P_r P_m + P_r + P_m)\xi_n}{P_r P_m}, \\ \mathtt{I}_{2n} = \dfrac{1}{P_r \xi_n}\left[\dfrac{P_r}{P_m}\xi_n(\xi_n^2 + Q^2 n^2 \pi^2) + \left(\dfrac{1}{P_m} + 1\right)\xi_n^3 - R^2 a^2\right], \\ \mathtt{I}_{3n} = \dfrac{1}{P_r P_m}[R^2 a^2 - \xi_n(\xi_n^2 + Q^2 n^2 \pi^2)], \end{cases}$$

associated to the spectral equation

$$(6.15)\qquad \lambda^3 - \mathtt{I}_{1n}\lambda^2 + \mathtt{I}_{2n}\lambda - \mathtt{I}_{3n} = 0.$$

Then – in view of $\mathtt{I}_{1n} < 0$ – the Hurwitz's Criterion guarantees that if and only if

$$(6.16)\qquad \mathtt{I}_{3n} < 0, \quad \mathtt{I}_{2n} > 0, \quad \mathtt{I}_{1n}\mathtt{I}_{2n} < \mathtt{I}_{3n}, \quad \forall (a^2, n) \in \mathbb{R}^+ \times \mathbb{N},$$

the eigenvalues have negative real part (see Appendix 1.2). On setting

$$(6.17)\qquad R_{C_2} = \min_{(a^2,n)\in\mathbb{R}^+\times\mathbb{N}} \frac{\dfrac{P_r}{P_m}\xi_n(\xi_n^2 + Q^2 n^2 \pi^2) + \left(\dfrac{1}{P_m} + 1\right)\xi_n}{a^2}$$

$$= \min_{a^2\in\mathbb{R}^+} \frac{\dfrac{P_r}{P_m}(a^2 + \pi^2)[(a^2 + \pi^2)^2 + Q^2\pi^2] + \left(\dfrac{1}{P_m} + 1\right)(a^2 + \pi^2)}{a^2}$$

$$(6.18)\qquad R_{C_3} = \min_{(a^2,n)\in\mathbb{R}^+\times\mathbb{N}} \frac{\xi_n(\xi_n^2 + Q^2 n^2 \pi^2)}{a^2}$$

$$= \min_{a^2\in\mathbb{R}^+} \frac{(a^2 + \pi^2)[(a^2 + \pi^2)^2 + Q^2\pi^2]}{a^2},$$

it follows that the request

$$(6.19)\qquad R^2 < \min(R_{C_2}, R_{C_3}),$$

is necessary for inhibiting the onset of convection. Further

$$(6.20)\qquad R^2 < R_{C_3} \quad \Leftrightarrow \quad \mathtt{I}_3 < 0.$$

Then, setting $\mathtt{I}_i = \mathtt{I}_{i1}$, $(i = 1, 2, 3)$, convection occurs if

$$(6.21)\qquad \mathtt{I}_2 < 0, \quad \text{or} \quad \mathtt{I}_1\mathtt{I}_2 > \mathtt{I}_3.$$

In view of (6.20) and $\mathtt{I}_1 < 0$ it follows that

$$(6.22)\qquad \mathtt{I}_2 < 0 \quad \Rightarrow \quad \mathtt{I}_1\mathtt{I}_2 > \mathtt{I}_3,$$

i.e.

$$R^2 < R_{C_3}, \quad R^2 > R_{C_2}, \tag{6.23}$$

imply onset of convection.

7. Onset of steady convection

The onset of convection is named *steady* only if the passage from the stability to the instability happens through a steady solution and is characterized by the existence of a zero eigenvalue. Since the existence of a zero eigenvalue is guaranteed if and only if $\mathtt{I}_{3n} = 0$, it follows that the critical value $\bar{R}_C$ of R^2 for the onset of steady convection is given by R_{C_3}. The inequalities appearing in (6.16) have a special meaning. As concerns $(6.16)_1$, we remark that it is verified if and only if

$$R^2 < \bar{R}_C = \min_{(a^2,n)\in\mathbb{R}^+\times\mathbb{N}} \frac{\xi_n(\xi_n^2 + Q^2n^2\pi^2)}{a^2},$$

i.e. if and only if

$$R^2 < \bar{R}_c = \min_{a^2\in\mathbb{R}^+} \frac{(a^2+\pi^2)[(a^2+\pi^2)^2 + Q^2\pi^2]}{a^2}. \tag{7.1}$$

Taking into account that we have denoted by R^2 and Q^2 the non-dimensional numbers denoted by R and Q in [3], we remark that R_{C_3} agrees with the critical value of steady convection given in formula ((163), page 170) of [1] where – in table XLV – critical Rayleigh numbers and wave numbers for the onset of steady convection, for growing Q^2, are provided.

8. Onset of oscillatory bifurcation

Since the critical values R_{C_2} and R_{C_3} are obtained for $n = 1$, (6.15) reduces to

$$\lambda^3 - \mathtt{I}_1\lambda^2 + \mathtt{I}_2\lambda - \mathtt{I}_3 = 0, \tag{8.1}$$

with $\mathtt{I}_j = \mathtt{I}_{j1}$, $(j = 1, 2, 3)$. As concerns the onset of convection via oscillatory state (*Hopf bifurcation*), denoting by $\hat{i}$ the imaginary unit, an eigenvalue of type $\lambda = \hat{i}\mu$, with μ real constant, is a root of (8.1) iff

$$\mu(\mathtt{I}_2 - \mu^2) = 0, \quad \mu^2 = \frac{\mathtt{I}_3}{\mathtt{I}_1}, \tag{8.2}$$

i.e. μ has to be solution of

$$\mathtt{I}_1\mathtt{I}_2 = \mathtt{I}_3. \tag{8.3}$$

Therefore *only if the lowest positive root* $R^2 = \bar{\bar{R}}_C$ *of* (8.3) *verifies*

$$\bar{\bar{R}}_C < \bar{R}_C, \tag{8.4}$$

oscillatory convection occurs.

In view of (6.14), (8.3) becomes

$$\begin{aligned}\left(1+\frac{1}{P_r}+\frac{1}{P_m}\right)\left[\left(\frac{P_r}{P_m}+\frac{1}{P_m}+1\right)\frac{\xi_1^3}{a^2}+\pi^2Q^2\frac{P_r}{P_m}\frac{\xi_1}{a^2}-R^2\right]\\ =\frac{1}{P_m}\left[\frac{\xi_1^3+Q^2\pi^2\xi_1}{a^2}-R^2\right],\end{aligned} \tag{8.5}$$

Since $P_r \geq P_m$ implies $(\forall a^2)$

$$\begin{aligned}\left(1+\frac{1}{P_r}+\frac{1}{P_m}\right)\left[\left(\frac{P_r}{P_m}+\frac{1}{P_m}+1\right)\frac{\xi_1^3}{a^2}+\pi^2Q^2\frac{P_r}{P_m}\frac{\xi_1}{a^2}-R^2\right]\\ >\left(1+\frac{1}{P_r}+\frac{1}{P_m}\right)\frac{\xi_1^3+Q^2\pi^2\xi_1}{a^2}>\frac{\xi_1^3+Q^2\pi^2\xi_1}{a^2},\end{aligned} \tag{8.6}$$

one immediately recovers that the condition *necessary for the onset of overstability* (see [3] page 183, (233)): *overstability can occur only if* $P_r < P_m$. Further the following theorem holds.

Theorem 8.1. *Let* $P_r < P_m$. *then overstable convection occurs if and only if*

$$\bar{\bar{R}}_C = \min_{a^2\in\mathbb{R}^+}\frac{(\pi^2+a^2)[A(\pi^2+a^2)+B\pi^2Q^2]}{a^2} < \bar{R}_C, \tag{8.7}$$

with

$$A=\frac{(P_r+P_m+1)-\alpha}{P_m-\alpha},\quad B=\frac{P_r-\alpha}{P_m-\alpha},\quad \alpha=\frac{P_mP_r}{P_mP_r+P_r+P_m}. \tag{8.8}$$

Proof. As matter of fact, in view of (8.5), (8.8) one easily obtains

$$R^2=\frac{A\xi_1^3+B\pi^2Q^2\xi_1}{a^2} \tag{8.9}$$

and (8.7) immediately follows.

Remark 8.1. We remark that:

1) In view of

$$\alpha < \min(1, P_r, P_m) \tag{8.10}$$

and $P_r < P_m$ it follows that

$$A > 0, \quad 0 < B < 1. \tag{8.11}$$

Therefore the consistency of (8.7) is guaranteed for $A \simeq 1.$, i.e. by $\dfrac{P_r}{P_m} \ll 1$, since

$$\lim_{P_r \to 0} B = 0, \quad \lim_{P_m \to \infty} A = 1. \tag{8.12}$$

2) On letting $Q^2 \to 0$, one has

$$\lim_{Q^2 \to 0} \bar{\bar{R}}_C = A \min_{a^2 \in \mathbb{R}^+} \frac{\xi_1^3}{a^2}, \quad A = 1 + \frac{P_r + 1}{P_m - \alpha} > 1, \tag{8.13}$$

as expected, since in the Hydrodynamic Bénard Problem the principle of exchange of stability holds.

9. Oscillatory bifurcation laws

We remark that – in view of the analysis performed in Sections 7–8 – the new approach introduced for the linear stability, based on the Huwitz's Criterion applied to (6.15) with the characteristic $\mathtt{I}_{in}$ written via the $\mathtt{L}_n$ entries, appears to be less difficult than the traditional one, at least for the onset of oscillatory bifurcation (see [3], pp. 181–186). Further the conditions $\mathtt{I}_2 > 0$, $\mathtt{I}_3 < 0$, necessary for inhibiting the onset of instability, in view of

$$\begin{cases} \mathtt{I}_2 > 0, \ \forall a^2 \in \mathbb{R}^+ \Leftrightarrow R^2 < R_{C_2}, \\ \mathtt{I}_3 < 0, \ \forall a^2 \in \mathbb{R}^+ \Leftrightarrow R^2 < R_{C_3}, \\ \bar{R}_C = R_{C_3}, \end{cases} \tag{9.1}$$

allow to obtain – in simple algebraic closed forms – conditions guaranteeing oscillatory bifurcation (Properties 1.3–1.4 of Sect. 1).

Theorem 9.1. *Let $P_r < P_m$. Then*

$$\bar{R}_C > \frac{27}{4}\pi^4 \frac{1 + P_m}{P_m - P_r}, \tag{9.2}$$

guarantees the onset of oscillatory convection for $R^2 \in]\mathscr{R}_C, \bar{R}_C[$ with

$$\mathscr{R}_C = \frac{P_r}{P_m}\bar{R}_C + \frac{27}{4}\pi^4\left(1 + \frac{1}{P_m}\right). \tag{9.3}$$

Proof. On requiring

$$\frac{P_r}{P_m}\bar{R}_C + \left(\frac{1}{P_m} + 1\right)\frac{27}{4}\pi^4 < \bar{R}_C,$$

(9.2) immediately follows. But for any R^2 such that:

$$\frac{P_r}{P_m}\bar{R}_C + \left(\frac{1}{P_m} + 1\right)\frac{27}{4}\pi^4 < R^2 < \bar{R}_C,$$

since (6.19) is violated and m_0 is unstable and – in view of $R^2 < \bar{R}_C$ – the instability is of oscillatory type.

Theorem 9.2. *Let $P_r < P_m$. Then*

$$P_r < \left(1 - \frac{1}{\pi^2(\pi^2 + Q^2)}\right)P_m, \tag{9.4}$$

guarantees oscillatory convection.

Proof. In view of (6.17)–(6.18), it follows that

$$R_{C_2} < R_{C_3}, \tag{9.5}$$

is implied by

$$\frac{P_r}{P_m}\xi_1(\xi_1^2 + Q^2\pi^2) + \left(\frac{1}{P_m} + 1\right)\xi_1 < \xi_1(\xi_1^2 + Q^2\pi^2), \quad \forall a^2 \in \mathbb{R}^+ \tag{9.6}$$

i.e. by

$$\left(1 - \frac{P_r}{P_m}\right)[(a^2 + \pi^2)^2 + Q^2\pi^2] > \frac{1}{P_m} + 1, \quad \forall a^2 \in \mathbb{R}^+ \tag{9.7}$$

and (9.4) immediately follows.

Remark 9.1. We remark that:

1) As far as we know, the laws (9.2), (9.3) and (9.4) appear to be new in the existing literature. In view of similar difficulties and treatments, traditionally the onset of Hopf bifurcation in MBP is associated to the onset of bifurcation in the rotating classical Bénard Problem. In appendix we show that a law analogous to (9.2)–(9.3) holds also for the rotating classical Bénard Problem;
2) (9.3) gives an upper estimate of $\bar{\bar{R}}_C$.

10. Rigid-rigid, free-rigid, rigid-free boundary cases

The results obtained in the previous section are concerned with the free-free cases. But one easily realizes that the procedure used continues to hold also in the rigid-rigid, rigid-free and free-rigid boundary cases. For the sake of simplicity we refer to the rigid-rigid case. As matter of fact, in the rigid-rigid case, at the place of (2.4) one has to add to the boundary conditions (2.4) the following ones

$$u = v = \frac{\partial w}{\partial z} = 0, \quad z = 0, 1, \tag{10.1}$$

where $\dfrac{\partial w}{\partial z} = 0$ is implied by the request $\nabla \cdot \mathbf{u} = 0$ also on the boundaries. It follows that the basis $\{\sin n\pi z\}$ does not allow to satisfy $(10.1)_3$ and one has to choose the basis $\{f_n\}$ with f_n solutions of the Sturm–Liouville type problem ([ ], pages 634–637)

$$\begin{cases} \dfrac{d^4 f}{dz^4} = \alpha^4 f, & z \in]0, 1[, \\ f = \dfrac{df}{dz} = 0, & \text{on } z = 0, 1. \end{cases} \tag{10.2}$$

Then, either Section 3 and Section 4 – in which the Fourier expansion is not used – continue to hold since (2.4) continues to hold. As concerns sections 5–9, all the procedures continue to hold but one has to put f_n at the place of $\sin n\pi z$ and hence f_n' at the place of $\cos n\pi z$. In other words, (6.8) continues to hold and its new conditions depending on the basis $\{f_n\}$ will continue to guarantee the *unconditional nonlinear asymptotic stability of* m_0. In this way all the results of linear stability of the rigid-rigid case found in [3] are recovered as results *of unconditional nonlinear asymptotic stability*. This happens also for the other boundary conditions (rigid-free and free-rigid).

11. Discussion and future work

1) The paper concerns the dynamic of thermal MHD flows in horizontal layers heated from below and embedded in a constant transverse magnetic field;
2) the ultimately boundedness of the flows is guaranteed by the existence of L^2-absorbing sets;
3) an L^2-Energy Linearization Principle is obtained;
4) as consequence of the Energy Linearization Principle, a new approach to linear stability of the thermal conduction solution m_0 – based on the application to the spectral equation of the Routh–Hurwitz conditions guaranteeing that all eigenvalues have negative real part – is performed;
5) the relevant role played by the characteristic value of the linear operator at stake, is put in evidence;
6) the conditions of linear stability of m_0 are recovered as results guaranteeing the unconditional nonlinear asymptotic stability for any type of boundary conditions (free-free, free-rigid, rigid-free, rigid-rigid);
7) the steady and oscillatory Rayleigh critical values – in simple closed forms – are provided;

8) *the overstability laws* (Properties 1.3–1.4 of Sect. 2) (9.2)–(9.3) guaranteeing the onset of oscillatory convection (Hopf bifurcation) are obtained;
9) as concerns the future work, we confine ourselves to mentioning the case of thermo-MHD flows in rotating layers. In that case, to the right-hand side of (10.1), will appear a term of kind $\mathcal{T}\mathbf{u}\times\mathbf{k}$, $\mathcal{T}$ being a constant (depending on the rotation about z). Since $\langle\mathbf{u},\mathbf{u}\times\mathbf{k}\rangle = 0$, the Energy Linearization Principle continues to hold and also the subsequent procedure for the linear stability. But to (6.8) one has to add the equation governing the evolution of the third component of the curl of (2.1), when (2.1) is written with the presence of $\mathcal{T}\mathbf{u}\times\mathbf{k}$. Taking into account also the results concerning the thermo flows in rotating layers (see appendix), it appears that there exists the basis for looking for the generalization of the results obtained in the present paper to the thermo-MHD flows in rotating layers. In particular, to look for the validity of *overstability laws* analogous to Properties 1.3–1.4.

12. Appendix

12.1. Characteristic values (invariants) of a quadratic matrix via the matrix entries

For the sake of simplicity we confine ourselves to the quadratic matrix of order 3. Let $\mathtt{L} = \|a_{ih}\|$ $(i, j = 1, 2, 3)$ with real entries a_{ij} and let A, B, C be given by adding respectively the principal minor of order 1,2,3

$$(12.1)\qquad \begin{cases} A = a_{11} + a_{22} + a_{33}, \\ B = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \\ C = \det \mathtt{L}. \end{cases}$$

Then one easily verifies that the spectral equation of $\mathtt{L}$ is given by

$$(12.2)\qquad \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = -(\lambda^3 - A\lambda^2 + B\lambda - C) = 0.$$

On the other hand, denoting by $\lambda_1, \lambda_2, \lambda_3$ the roots of (12.2) – eigenvalues of $\mathtt{L}$ – one obtains

$$(12.3)\qquad (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3) = \lambda^3 - (\lambda_1+\lambda_2+\lambda_3)\lambda^2 + (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)\lambda - \lambda_1\lambda_2\lambda_3 = 0.$$

Since the characteristic values of $\mathtt{L}$ are given by

$$(12.4)\qquad \mathtt{I}_1 = \lambda_1+\lambda_2+\lambda_3, \quad \mathtt{I}_2 = \lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3, \quad \mathtt{I}_3 = \lambda_1\lambda_2\lambda_3,$$

it follows that

$$\lambda^3 - \mathtt{I}_1\lambda^2 + \mathtt{I}_2\lambda - \mathtt{I}_3 = 0 \tag{12.5}$$

and (12.2) gives

$$\begin{cases} \mathtt{I}_1 = a_{11} + a_{22} + a_{33},\ \mathtt{I}_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \\ \mathtt{I}_3 = \det \mathtt{L}_n. \end{cases} \tag{12.6}$$

Analogously the spectral equation of a matrix of order n is

$$\lambda^n - \mathtt{I}_1\lambda^{n-1} + \mathtt{I}_2\lambda^{n-2} + \cdots + (-1)^n\mathtt{I}_n = 0, \tag{12.7}$$

with $\mathtt{I}_m$, $(m = 1, 2, \ldots, n)$ characteristic values of $\|a_{ij}\|$ obtained by adding the principal minors of order m of $\|a_{ij}\|$.

12.2. Hurwitz's Criterion [18]–[19]

To the algebraic equation of n-th degree

$$a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0, \tag{12.8}$$

with $a_0, \ldots, a_n$ real numbers, one has to associate the matrix

$$\begin{pmatrix} a_1 & a_3 & a_5 & \cdots & 0 \\ a_0 & a_2 & a_6 & \cdots & 0 \\ 0 & a_1 & a_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{12.9}$$

and its diagonal mnors

$$\Delta_1 = a_1, \quad \Delta_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}, \ldots, \Delta_n = a_n\Delta_{n-1}. \tag{12.10}$$

Without loss of generality we *assume* $a_0 > 0$. Then the following properties hold:

1) a necessary condition for all roots of (12.8) have negative real part is that

$$a_i > 0, \quad \forall i \in \{1, 2, \ldots, n\}; \tag{12.11}$$

2) if one of the coefficient $a_1, \ldots, a_n$ is negative then some roots will have positive real part;

3) (Hurwitz's Criterion) if and only if

(12.12) $$\Delta_n > 0, \quad \forall n,$$

all the roots have negative real part.

In view of (12.2), it immediately follows that in the case $n = 3$ conditions

(12.13) $$\mathtt{I}_1 < 0, \quad \mathtt{I}_2 > 0, \quad \mathtt{I}_3 < 0,$$

are necessary for the roots have negative real part and

(12.14) $$\mathtt{I}_1 < 0, \quad \mathtt{I}_3 < 0, \quad \mathtt{I}_1\mathtt{I}_2 - \mathtt{I}_3 < 0,$$

are necessary and sufficient.

12.3. Two laws guaranteeing onset of oscillatory instability in rotating classical Bénard Problem

The equations governing the perturbations to the thermal conduction in a layer rotating about the axis z with uniform velocity ω, are (in the free-free case)

(12.15) $$\begin{cases} P_r^{-1}\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla\pi + \Delta\mathbf{u} + \mathscr{T}\mathbf{u}\times\mathbf{k} + R\theta\mathbf{k}, \\ \nabla\cdot\mathbf{u} = 0, \quad \text{in } \Omega \\ \theta_t + P_r\mathbf{u}\cdot\nabla\theta = Rw + \Delta\theta, \end{cases}$$

(12.16) $$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = \theta = 0, \quad \text{on } z = 0, 1,$$

with

(12.17) $$\mathscr{T}^2 = \frac{4\omega^2 d^4}{\nu^2}.$$

One easily verifies that, in view of $\langle \mathbf{u}, \mathbf{u}\times\mathbf{k}\rangle = 0$, the energy linearization principle continues to hold and hence, linearizing (12.15) and taking the z components of the curl and the double curl of (12.15), one obtains

(12.18) $$\begin{cases} P_r^{-1}\dfrac{\partial\zeta}{\partial t} = \Delta\zeta + \mathscr{T}\dfrac{\partial w}{\partial z}, \\[2mm] P_r^{-1}\dfrac{\partial\Delta w}{\partial t} = \Delta\Delta w - \mathscr{T}\dfrac{\partial\zeta}{\partial z} + R\Delta_1\theta, \\[2mm] \dfrac{\partial\theta}{\partial t} = Rw + \Delta\theta, \end{cases}$$

with $\zeta = (\mathrm{rot}\,\mathbf{u}) \cdot \mathbf{k}$. Setting $Z = \dfrac{\partial \zeta}{\partial z}$ the Fourier components of the perturbations have to verify the system [ ]

$$\text{(12.19)} \qquad \begin{cases} P_r^{-1} \dfrac{\partial \Delta w_n}{\partial t} = \Delta\Delta w_n - \mathscr{T} Z_n + R\Delta_1 \theta, \\ P_r^{-1} \dfrac{\partial Z_n}{\partial t} = \Delta Z_n + \mathscr{T} \dfrac{\partial^2 w_n}{\partial z^2}, \\ \dfrac{\partial \theta_n}{\partial t} = \Delta \theta_n + R w_n \end{cases}$$

i.e. – in view of $f_n = \bar{f}_n(x, y, t) \sin n\pi z$, $f \in \{w_n, Z_n, \theta_n\}$

$$\text{(12.20)} \qquad \frac{\partial}{\partial t} \begin{pmatrix} w_n \\ Z_n \\ \theta_n \end{pmatrix} = \mathtt{L}_n \begin{pmatrix} w_n \\ Z_n \\ \theta_n \end{pmatrix}$$

with

$$\text{(12.21)} \qquad \mathtt{L}_n = \begin{pmatrix} -P_r \xi_n & P_r \mathscr{T}/\xi_n & a^2 P_r R/\xi_n \\ -P_r \mathscr{T} n^2 \pi^2 & -P_r \xi_n & 0 \\ R & 0 & -\xi_n \end{pmatrix}.$$

The invariants of (12.21) are

$$\text{(12.22)} \qquad \begin{cases} \mathtt{I}_{1n} = -(2P_r + 1)\xi_n, \ \mathtt{I}_{2n} = \dfrac{a^2 P_r}{\xi_n} \left[\dfrac{(2 + P_r)\xi_n^3 + P_r \mathscr{T}^2 n^2 \pi^2}{a^2} - R^2 \right], \\ \mathtt{I}_{3n} = a^2 P_r^2 \left[R^2 - \dfrac{\xi_n^3 + \mathscr{T}^2 n^2 \pi^2}{a^2} \right] \end{cases}$$

and the spectral equation is

$$\text{(12.23)} \qquad \lambda^2 - \mathtt{I}_{1n} \lambda^2 + \mathtt{I}_{2n} \lambda - \mathtt{I}_{3n} = 0.$$

The critical value $\bar{R}_C$ of R^2 for the onset of steady convection is then

$$\bar{R}_C = \min_{(a^2, n) \in \mathbb{R}^+ \times \mathbb{N}} \frac{(a^2 + n^2\pi^2)^3 + \mathscr{T}^2 n^2 \pi^2}{a^2},$$

i.e.

$$\text{(12.24)} \qquad \bar{R}_C = \min_{a^2 \in \mathbb{R}^+} \frac{(a^2 + \pi^2)^3 + \mathscr{T}^2 \pi^2}{a^2},$$

while the critical value $\bar{\bar{R}}_C$ for the onset of oscillatory convection is the lowest positive root of

$$\mathtt{I}_1\mathtt{I}_2 - \mathtt{I}_3 = 0, \tag{12.25}$$

less than $\bar{R}_C$. In view of (12.22) for $n = 1$, (12.25) becomes

$$\frac{(2P_r+1)}{P_r}\left[R^2 - \frac{(2+P_r)\xi_1^3 + P_r\mathscr{T}^2\pi^2}{a^2}\right] = R^2 - \frac{\xi_1^3 + \mathscr{T}^2\pi^2}{a^2}$$

$$\begin{aligned}
\left(1+\frac{1}{P_r}\right)R^2 &= \frac{2P_r+1}{P_r}\,\frac{(2+P_r)\xi_1^3 + P_r\mathscr{T}^2}{a^2} - \frac{\xi_1^3 + \mathscr{T}^2\pi^2}{a^2} \\
&= \frac{2P_r+1}{P_r}\,\frac{(2+P_r)\xi_1^3}{a^2} + \frac{(2P_r+1)\mathscr{T}^2\pi^2}{a^2} - \frac{\xi_1^3 + \mathscr{T}^2\pi^2}{a^2} \\
&= \frac{\left[\dfrac{2P_r+1}{P_r}(2+P_r) - 1\right]\xi_1^3 + (2P_r+1-1)\mathscr{T}^2\pi^2}{a^2} \\
&= \frac{\left[\dfrac{2}{P_r}(2P_r+1) + 2P_r\right]\xi_1^3 + 2P_r\mathscr{T}^2\pi^2}{a^2} \\
&= \frac{\dfrac{2}{P_r}(2P_r+1)\xi_1^3 + 2P_r(\xi_1^3 + \mathscr{T}^2\pi^2)}{a^2}
\end{aligned}$$

i.e.

$$(1+P_r)R^2 = 2\,\frac{(2P_r+1)\xi_1^3 + P_r^2(\xi_1^3 + \mathscr{T}^2\pi^2)}{a^2} \tag{12.26}$$

and hence

$$\bar{\bar{R}}_C = \frac{2}{1+P_r}\min_{a^2\in\mathbb{R}^+}\frac{(2P_r+1)\xi_1^3 + P_r^2(\xi_1^3 + \mathscr{T}^2\pi^2)}{a^2}. \tag{12.27}$$

In [3] the request

$$\bar{\bar{R}}_C < \bar{R}_C, \tag{12.28}$$

for the onset of *overstability* is deeply analyzed and in Table XI of page 120 values of $\bar{\bar{R}}_C$ guaranteeing the onset of overstability for $P_r = 0.025$ and for various values of $\mathscr{T}^2$ are printout[1].

[1] We remark that the Taylor number $\mathscr{T}^2$ is denoted by $\mathscr{T}$ in [ ].

We now obtain two *overstability laws* analogous to (9.2), (9.4). Setting

$$R_{C_2} = \min_{a^2 \in \mathbb{R}^+} \frac{(2+P_r)\xi_1^3 + P_r\mathscr{T}^2\pi^2}{a^2}, \tag{12.29}$$

it follows that $\mathtt{I}_2 > 0$ if and only if $R^2 < R_{C_2}$.

On the other hand

$$R_{C_2} < P_r \min_{a^2 \in \mathbb{R}^+} \frac{\xi_1^3 + \mathscr{T}^2\pi^2}{a^2} + 2 \min_{a^2 \in \mathbb{R}^+} \frac{\xi_1^3}{a^2} = P_r\bar{R}_C + \frac{27}{2}\pi^4, \tag{12.30}$$

therefore

$$P_r\bar{R}_C + \frac{27}{2}\pi^4 < \bar{R}_C,$$

gives the *overstability law*.

THEOREM 12.1. *Let* $P_r < 1$. *Then*

$$\bar{R}_C > \frac{27\pi^4}{2(1-P_r)}, \tag{12.31}$$

guarantees the onset of oscillatory convection for $R^2 \in]\tilde{R}_C, \bar{R}_C[$, *with*

$$\tilde{R}_C = P_r\bar{R}_C + \frac{27\pi^4}{2}. \tag{12.32}$$

THEOREM 12.2. *Let* $P_r < 1$. *Then*

$$\frac{(1+P_r)\pi^3}{1-P_r} < \mathscr{T}^2, \tag{12.33}$$

guarantees oscillatory convection.

PROOF. Since R_{C_3} is given by (12.24), $R_{C_2} < R_{C_3}$ $\forall (a^2, n) \in \mathbb{R}^+ \times \mathbb{N}$ gives

$$(2+P_r)\xi_1^3 + P_r\mathscr{T}^2\pi^2 < \xi_1^3 + \mathscr{T}^2\pi^2 \tag{12.34}$$

and (12.33) immediately follows.

REMARK 12.1. We remark that, since $\bar{R}_C$ is an increasing function of $\mathscr{T}^2$, also P_r given by (12.31) is an increasing function of $\mathscr{T}^2$. Therefore in view of table VII of [3] where values of $\bar{R}_C$ are printout for increasing values of the Taylor number $\mathscr{T}^2$ (denoted by $\mathtt{T}$ in table VII of [3]) it easily follows that, since

$$\bar{R}_C(\mathscr{T}^2 = 10) = 6771 \times 10^2, \quad \frac{27\pi^4}{2\bar{R}_C} = 0.48,$$

it follows that for any $P_r < 0.52$ overstability occurs. If $P_r < 0.6$ it follows that since

$$\bar{R}_C(\mathscr{T}^2 = 100) = 0.396, \quad \frac{27\pi^4}{2\bar{R}_C} = 0.61,$$

overstability is guaranteed for $\mathscr{T}^2 \geq 100$.

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References

[1] S. Chandrasekhar, *On the inhibition of convection by a magnetic field*, Phil. Mag. Sc. Ser. 7, 43, 501–532 (1952).

[2] S. Chandrasekhar, *On the inhibition of convection by a magnetic field II*, Phil. Mag. Sc. Ser. 7, 45, 1177–1191 (1954).

[3] S. Chandrasekhar, *Hydrodynamic and hydromagnetic stability*, Dover, New York, (1981).

[4] J. Flavin - S. Rionero, *Qualitative estimates for partial differential equations*, An introduction. CRC Press; Boca Raton, 1996.

[5] D. D. Joseph, *Stability of fluid motions I, II*, Vol. 27–28 of Springer Tracts in Natural Philosophy, Springer, Berlin, 1976 5, 283–320 (1983).

[6] G. Mulone - S. Rionero, *On the stability of the magnetic Bénard problem with rotation*, Z. angew Math. Mech. 73, 35–45 (1993).

[7] G. Mulone - S. Rionero, *On the stability of the rotating Bénard problem*, Bull. Tech. Univ. Istanbul 47, 181–202 (1994).

[8] G. Sutton - A. Shermann, *Engineering Magnetihydrodynamic*, McGrow-Hill 1965.

[9] Y. Nakagawa, *An experiment on the inhibition of thermal convection by a magnetic field*, Nature 175, 417–419 (1955).

[10] S. Rionero, *Sulla stabilitá asintotica in media in magnetoidrodinamica*, Ann. Mat. Pura Appl. 76, 75–92 (1967).

[11] S. Rionero, *Metodi variazionali per la stabilitá asintotica in media in magnetoidrodinamica non isoterma*, Ricerche Mat. 16, 250–163 (1967).

[12] S. Rionero, *On magnetohydrodynamic stability*, Quaderni di Matematica 1, 347–376 (1997).

[13] G. Mulone - S. Rionero, *Non-linear stability analysis of the magnetic Bènard problem through the Lyapunov direct method*, Arch. Rational Mech. Anal., 103, 347–368, (1988).

[14] B. Straughan, *The Energy Method*, Stability, and Nonlinear Convection 2^{nd} edition, Appl. Math. Sci., 91, Springer-Verlag, Berlin, (2004).

[15] W. B. Thompson, *Thermal convection in a magnetic field*, Phil. Mag. Sc. Ser. 7, 42, 1417–1432 (1951).

[16] G. Mulone - S. Rionero, *Necessary and sufficient conditions for nonlinear stability in the Magnetic Bénard Problem*, Arch. Rational Mech. Anal. 166, 197–218 (2003).

[17] R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, Series in Applied Sciences, vool. 68, Springer, New York, 1997.

[18] D. R. Merkin, *Introduction to the theory of stability*, Texts in Applied Mathematics, 24, Springer-Verlag, New York, xx+319 pp., 1997.

[19] F. R. Gantmacher, *The theory of matrices*, vol. 2. AMS (Chelsea Publishing), Providence 2000.

[20] G. Mulone - V. A. Solonnikov, *On the solvability of some initial boundary value problems of magnetofluidmechanics with Hall and ion-slip effects*, Atti Acc. Naz. Lincei Cl. Sci. Fis. Mat. Natur. 6 (1955), 117–132.

[21] S. Rionero, *Heat and mass transfer by convection in multicomponent Navier-Stokes mixture: Absence of subcritical instabilities and global nonlinear stability via the Auxiliary System Method*, Rend. Lincei Mat. Appl., 25, (2014), 1–44.

[22] J. Fan - F. Li - G. Nakamura, *Regularity criteria for the Boussinesq system with temperature-dependent viscosity and thermal diffusivity in a bounded domain*, DCDS, v. 36, n. 9, 2016, DOI: 10.3934/dcds.2016012.

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