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Partial Differential Equations — *Comparing monotonicity formulas for electrostatic potentials and static metrics*, by Virginia Agostiniani and Lorenzo Mazzieri, communicated on November 11, 2016.[1]

Abstract. — In this note we survey and compare the monotonicity formulas recently discovered by the authors in [1] and [2] in the context of classical potential theory and in the study of static metrics, respectively. In both cases we discuss the most significant implications of the monotonicity formulas in terms of sharp analytic and geometric inequalities. In particular, we derive the classical Willmore inequality for smooth compact hypersurfaces embedded in Euclidean space and the Riemannian Penrose inequality for static Black Holes with connected horizon.

Key words: Elliptic boundary value problems, electrostatic capacity, static metrics, Willmore inequality, Riemannian Penrose inequality

Mathematics Subject Classification: 35B06, 53C21, 83C57, 35N25

1. Introduction

In this paper, we consider two elliptic boundary value problems in exterior domains. The first one comes form classical potential theory, whereas the other arises in the study of static vacuum Einstein metrics with horizons in general relativity. In both cases we introduce some relevant integral quantities defined on the level sets of the solutions, and we prove that they are monotone along the flow of the level sets. As a consequence of this fact, we derive sharp analytic and geometric inequalities, whose equality cases are characterized in terms of the rotational symmetry of the solutions or, equivalently, in terms of the spherical symmetry of the boundary. For the complete proofs of the statements, we refer the reader to [1], [2] and [3].

1.1. Electrostatic potentials: setting of the problem, basic properties of the solutions and statement of the main result

We consider the electrostatic potential due to a charged body, modeled by a bounded domain Ω with smooth boundary. The potential is defined as the solution u of the following problem in the exterior domain

[1] This paper is related to a talk given by L. Mazzieri at "XXVI Convegno Nazionale di Calcolo delle Variazioni" which took place in Levico Terme on January 18–22, 2016.

$$(1.1)\qquad \begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \\ u = 0 & \text{on } \partial\Omega, \\ u(x) \to 1 & \text{as } |x| \to \infty. \end{cases}$$

It is worth pointing out that the most common convention for the formulation for this problem would be the one in which $u = 1$ at $\partial\Omega$ and $u \to 0$ at infinity. The reason for adopting a different convention lies in the fact that we want to stress the analogies between this problem and problem (1.7) below. Throughout the paper we assume that $\partial\Omega$ is a regular level set of u. We also observe that by the strong maximum principle, the solution u to (1.1) takes values in $[0, 1)$.

To fix the notation, we recall that the electrostatic capacity of the charged body Ω is defined as

$$\mathrm{Cap}(\Omega) = \inf\left\{ \frac{1}{(n-2)|\mathbb{S}^{n-1}|} \int_{\mathbb{R}^n} |\mathrm{D}w|^2 \,\mathrm{d}\mu \,\middle|\, w \in C_c^\infty(\mathbb{R}^n), w \equiv 1 \text{ in } \Omega \right\}.$$

In terms of the potential u, it can be computed, for every $t \in [0, 1)$, as

$$(1.2)\quad \mathrm{Cap}(\Omega) = \frac{1}{(n-2)|\mathbb{S}^{n-1}|} \int_{\partial\Omega} |\mathrm{D}u| \,\mathrm{d}\sigma = \frac{1}{(n-2)|\mathbb{S}^{n-1}|} \int_{\{u=t\}} |\mathrm{D}u| \,\mathrm{d}\sigma,$$

where the last equality is an easy consequence of the Divergence Theorem. On the other hand, the capacity of Ω can be used to describe the asymptotic expansion of u at infinity (see [15])

$$(1.3)\qquad u = 1 - \mathrm{Cap}(\Omega)|x|^{2-n} + o_2(|x|^{2-n}), \quad \text{as } |x| \to +\infty,$$

where the shorthand notation $o_2(|x|^{2-n})$ means that the reminder together with its first and second derivatives are infinitesimal if compared to $|x|^{2-n}$, $|x|^{1-n}$, and $|x|^{-n}$, respectively. It is worth noticing that the model solutions to problem (1.1) are the ones where the remainder terms in the above expansion are identically equal to zero. In this case, Ω is a ball of radius $[\mathrm{Cap}(\Omega)]^{1/(n-2)}$.

In contrast with the already observed constancy of the function $t \mapsto \int_{\{u=t\}} |\mathrm{D}u| \,\mathrm{d}\sigma$, we introduce, for $p \geq 0$, the functions

$$(1.4)\quad U_p : [0, 1) \to \mathbb{R}, \quad \text{given by } t \mapsto U_p(t) = \left[\frac{\mathrm{Cap}(\Omega)}{1-t}\right]^{\frac{(p-1)(n-1)}{(n-2)}} \int_{\{u=t\}} |\mathrm{D}u|^p \,\mathrm{d}\sigma.$$

Using expansion (1.3), one can easily compute the limit

$$(1.5)\qquad \lim_{t \to 1^-} U_p(t) = [\mathrm{Cap}(\Omega)]^p (n-2)^p |\mathbb{S}^{n-1}|,$$

that yields a natural extension of the functions $t \mapsto U_p(t)$ to the compact interval $[0, 1]$.

Before proceeding, it is worth noticing that the functions $t \mapsto U_p(t)$ are well defined, since the integrands are globally bounded and the level sets of u have finite hypersurface area. This follows from the results in [11] combined with the properness of u. Moreover, it is not difficult to check that they are constant if the potential u is rotationally symmetric. The content of our main result is that in general these functions are non increasing and as soon as they have a critical point they are constant, with the corresponding potential u rotationally symmetric and $\partial\Omega$ isometric to a $(n-1)$-dimensional sphere of constant curvature.

THEOREM 1.1 (Monotonicity-Rigidity Theorem for Electrostatic Potentials). *Let u be a solution to problem* (1.1) *and let $U_p : [0,1) \to \mathbb{R}$ be the function defined in* (1.4). *Then, the following properties hold true.*

(i) *For every $p \geq 1$, the function U_p is continuous.*
(ii) *For every $p \geq 2 - 1/(n-1)$, the function U_p is differentiable and the derivative satisfies for every $t \in [0,1)$ the inequality*

$$U_p'(t) = -(p-1)\left[\frac{\mathrm{Cap}(\Omega)}{1-t}\right]^{\frac{(p-1)(n-1)}{(n-2)}} \times \int_{\{u=t\}} |\mathrm{D}u|^{p-1}\left[\mathrm{H} - \left(\frac{n-1}{n-2}\right)|\mathrm{D}\log(1-u)|\right] \mathrm{d}\sigma \leq 0, \tag{1.6}$$

where H *is the mean curvature of the level set $\{u = t\}$. Moreover, if there exists $t \in (0,1]$ such that $U_p'(t) = 0$, then u is rotationally symmetric.*

It is worth pointing out that here and throughout the paper the mean curvature H of the level sets of u is computed with respect to the exterior unit normal vector $\nu = \mathrm{D}u/|\mathrm{D}u|$. We also notice that under the hypothesis of the above theorem, formula (1.6) implies the non existence of minimal level sets of u and in particular the non existence of smooth minimal compact hypersurfaces in $\mathbb{R}^n$. Further comments on the above statement will be given in Section 2 below, where the main consequences will also be discussed.

We conclude this section by noticing that the threshold value $p = 2 - 1/(n-1)$, that shows up at point (ii) of the above theorem, is closely related to the method employed for proving the monotonicity statement. More precisely, going through the argument presented in [1], it is not hard to realize that the reason for such a threshold comes from the use of a refined version of the Kato inequality, available for harmonic functions. It would be interesting to see if besides this technical reason there is also a physical motivation of this fact.

1.2. Static vacuum Einstein metrics: setting of the problem, basic properties of the solutions and statement of the main result

We consider an asymptotically flat n-dimensional Riemannian manifold (M, g), $n \geq 3$, with one end and with a nonempty, smooth, connected, compact boundary

∂M. We assume that there exists a function $v \in \mathscr{C}^\infty(M)$ such that the triple (M, g, v) satisfies the system

$$\begin{cases} v \,\mathrm{Ric} = \mathrm{D}^2 v & \text{in } M, \\ \Delta v = 0 & \text{in } M, \\ v = 0 & \text{on } \partial M, \\ v(x) \to 1 & \text{as } |x| \to +\infty, \end{cases} \tag{1.7}$$

where Ric, D, and Δ represent the Ricci tensor, the Levi–Civita connection, and the Laplace–Beltrami operator of the metric g, respectively. To clarify the meaning of the last condition in (1.7) we refer the reader to the precise definition of asymptotically flat Riemannian manifold, which is given a few lines below. Here we just observe that, outside a given compact set, our manifold is diffeomorphic to the exterior of a ball in $\mathbb{R}^n$. Such a diffeomoerphism naturally induces coordinates $(x^1, \ldots, x^n)$ and consequently their Euclidean norm $|x|$. Hence, the last condition in (1.7) really means that the function v is approaching the value 1 at infinity.

In the rest of the paper the metric g and the function v will be referred to as *static (vacuum Einstein) metric* and *static potential*, respectively, whereas the triple (M, g, v) will be called a *static solution*. A classical computation shows that if (M, g, v) satisfies (1.7), then the Lorentzian metric $\gamma = -v^2\, dt \otimes dt + g$ satisfies the *vacuum Einstein equations*

$$\mathrm{Ric}_\gamma = 0 \quad \text{in } \mathbb{R} \times (M \setminus \partial M).$$

To complete the picture, we observe that, as a consequence of the system (1.7), the scalar curvature R of g is identically equal to zero. Moreover, the boundary ∂M is a totally geodesic hypersurface embedded in M, and the function $|\mathrm{D}v|$ is constant on ∂M. It is also worth noticing that, since v is a non constant harmonic function in M and the boundary ∂M is assumed to be regular, the Hopf Lemma implies that $|\mathrm{D}v| > 0$ on ∂M and the Strong Maximun Principle implies that v takes values in $[0, 1)$.

To further specify our assumptions, we recall from [7] that a solution (M, g, v) to (1.7) is said to be *asymptotically flat* if there exists a compact set $K \subset M$ and a diffeomorphism $x = (x^1, \ldots, x^n) : M \setminus K \to \mathbb{R}^n \setminus B$ such that the metric g and the static potential v satisfy the following asymptotic expansions.

(i) In the coordinates induced by the diffeomorphism x the metric g can be expressed in $M \setminus K$ as

$$g = g_{\alpha\beta}\, dx^\alpha \otimes dx^\beta,$$

and the components satisfy the decay conditions

$$g_{\alpha\beta} = \delta_{\alpha\beta} + \eta_{\alpha\beta}, \quad \text{with } \eta_{\alpha\beta} = o_2(|x|^{\frac{2-n}{2}}), \text{ as } |x| \to +\infty, \tag{1.8}$$

for every $\alpha, \beta \in \{1, \ldots, n\}$.

(ii) In the same coordinates, the static potential v can be written as

$$v = 1 - m|x|^{2-n} + o_2(|x|^{2-n}), \quad \text{as } |x| \to +\infty, \tag{1.9}$$

for some positive real number $m > 0$.

Without entering into the details, it is worth mentioning that expansion (1.9) in (ii), which represents the counterpart of (1.3) in this relativistic context, can be deduced from (i) using $\Delta v = 0$. Moreover, we recall (see for instance [7] and [16]) that the coefficient m that shows up in expansion (1.9) coincides with the ADM mass $m_{ADM}(M, g)$ of the asymptotically flat manifold (M, g) and can be computed in terms of v as

$$\begin{aligned} m = m_{ADM}(M, g) &= \frac{1}{(n-2)|\mathbb{S}^{n-1}|} \int_{\partial M} |\mathrm{D}v| \, \mathrm{d}\sigma \\ &= \frac{1}{(n-2)|\mathbb{S}^{n-1}|} \int_{\{v=t\}} |\mathrm{D}v| \, \mathrm{d}\sigma, \end{aligned} \tag{1.10}$$

where the last equality is an easy consequence of the Divergence Theorem.

By far, the most important solution to system (1.7) obeying the above decay conditions is the so called *Schwarzschild solution*. To describe it, we consider, for a fixed $m > 0$, the manifold with boundary M given by the exterior domain $\mathbb{R}^n \setminus \{|x| < (2m)^{1/(n-2)}\}$ in the n-dimensional Euclidean space, so that $\partial M = \{|x| = (2m)^{1/(n-2)}\}$. The static metric g and the static potential v corresponding to the Schwarzschild solution are then given by

$$g = \frac{d|x| \otimes d|x|}{(1 - 2m|x|^{2-n})} + |x|^2 g_{\mathbb{S}^{n-1}} \quad \text{and} \quad v = \sqrt{1 - 2m|x|^{2-n}}, \tag{1.11}$$

respectively. In dimension $n = 3$, it is known by the work of Israel [14], Robinson [17], and Bunting and Masood-Ul-Alam [7] (where the boundary of M is a priori allowed to have several connected components) that (1.11) is the only static solution which is asymptotically flat with ADM mass equal to $m > 0$. This is the content of the so called Black Hole Uniqueness Theorem (see [10, 12, 18] for a comprehensive description of the subject). As a byproduct of our analysis, we will recover this result (see Theorem 2.11 below) and we will discuss some geometric conditions under which the same statement holds true in every dimension (see Theorem 2.12 below).

In analogy with the case of the electrostatic potentials, we introduce, for $p \geq 0$, the functions

$$V_p : [0, 1) \to \mathbb{R}, \quad \text{given by } t \mapsto V_p(t) = \left[\frac{2m}{1-t^2}\right]^{\frac{(p-1)(n-1)}{(n-2)}} \int_{\{v=t\}} |\mathrm{D}v|^p \, \mathrm{d}\sigma, \tag{1.12}$$

where (M, g, v) is a static solution. Using expansion (1.9), one can easily compute the limit

$$\lim_{t\to 1^-} V_p(t) = m^p(n-2)^p|\mathbb{S}^{n-1}|, \tag{1.13}$$

that yields a natural extension of the functions $t \mapsto V_p(t)$ to the compact interval $[0, 1]$. Again, it is easy to check that these functions are constant if computed on a Schwarzschild solution. More in general, they are non increasing and the monotonicity is strict unless both the static metric g and the static potential v are rotationally symmetric. This is the content of the following theorem.

Theorem 1.2 (Monotonicity-Rigidity Theorem for Static Metrics). *Let (M, g, v) be an asymptotically flat solution to problem (1.7) with ADM mass equal to $m > 0$ and let $V_p : [0, 1) \to \mathbb{R}$ be the function defined in (1.12). Then, the following properties hold true.*

(i) *For every $p \geq 1$, the function V_p is continuous.*
(ii) *For every $p \geq 2 - 1/(n-1)$, the function V_p is differentiable and the derivative satisfies for every $t \in [0, 1)$ the inequality*

$$V_p'(t) = -(p-1)\left[\frac{2m}{1-t^2}\right]^{\frac{(p-1)(n-1)}{(n-2)}} \times \int_{\{v=t\}} |\mathrm{D}v|^{p-1}\left[\mathrm{H} - \left(\frac{n-1}{n-2}\right)|\mathrm{D}\log(1-v^2)|\right] d\sigma \leq 0, \tag{1.14}$$

where H is the mean curvature of the level set $\{v = t\}$, computed with respect to the normal $\mathrm{D}v/|\mathrm{D}v|$. Moreover, if there exists $t \in (0, 1)$ such that $V_p'(t) = 0$, then (M, g, v) is isometric to a Schwarzschild solution with ADM mass equal to $m > 0$.

(iii) *For every $p \geq 2 - 1/(n-1)$, $V_p'(0) = 0$ and $V_p''(0) = \lim_{t\to 0^+} V_p'(t)/t$ satisfies the inequality*

$$V_p''(0) = -\left(\frac{p-1}{2}\right)(2m)^{\frac{(p-1)(n-1)}{(n-2)}} \times \int_{\partial M} |\mathrm{D}v|^{p-2}\left[\mathrm{R}^{\partial M} - 4\left(\frac{n-1}{n-2}\right)|\mathrm{D}v|^2\right] d\sigma \leq 0, \tag{1.15}$$

where $\mathrm{R}^{\partial M}$ is the scalar curvature of the metric $g_{\partial M}$ induced by g on ∂M. Moreover, if $V_p''(0) = 0$, then (M, g, v) is isometric to a Schwarzschild solution with ADM mass equal to $m > 0$.

Remark 1. We point out that points (ii) and (iii) above are proven only for $p \geq 3$ in the reference [2]. However, a straightforward adaptation of the arguments presented in [1] yields the desired conclusion.

Some comments are in order about the analogies and the differences between point (ii) in Theorem 1.1 and points (ii) and (iii) in Theorem 1.2. As already observed, the monotonicity formula in Theorem 1.1 implies at once the non existence of compact minimal hypersurfaces sitting inside the Euclidean space. In contrast with this, we have that the boundary of a static solution (M, g, v) is always totally geodesic. In particular, the condition $V_p'(0) = 0$ is always fulfilled and does not force any rigidity. For these reasons, the relevant condition at ∂M becomes $V_p''(0) = 0$, as described in point (iii) of the above statement. According to formulæ (1.6) and (1.15), it will become extremely evident in the next section the perfect parallelism between the roles played by

$$\frac{\mathrm{H}}{n-1} \quad \text{on } \partial\Omega \qquad \text{and} \qquad \frac{\mathrm{R}^{\partial M}}{(n-1)(n-2)} \quad \text{on } \partial M$$

in the setting of problem (1.1) and problem (1.7), respectively. Here we just observe that for rotationally symmetric solutions both quantities are constant and coincide with $[\mathrm{Cap}(\Omega)]^{-1/(n-2)}$ and $(2m)^{-1/(n-2)}$, respectively.

2. Consequences of the Monotonicity-Rigidity Theorems

In this section we discuss the most relevant consequences of Theorem 1.1 and Theorem 1.2, emphasizing the analogies and the differences between the two cases. A first bunch of corollaries is deduced by exploiting the local features of the monotonicity, namely the sign of the derivative of U_p and V_p, whereas the most geometric conclusions will follow from the global aspects of the monotonicity, namely the comparison between the values of U_p and V_p at the boundaries and at infinity. In the latter case, we will take advantage of formulæ (1.5) and (1.13).

2.1. Consequences of the local aspects of the monotonicity

We start with a couple of integral inequalities, which follows directly from formulæ (1.6), (1.14) and (1.15). The equality case characterizes the rotationally symmetric solutions.

For the electrostatic potential we have:

Theorem 2.1. *Let u be a solution to problem* (1.1). *Then, for every $p \geq 2 - 1/(n-1)$ and every $t \in [0, 1)$, the inequality*

$$\int_{\{u=t\}} \left|\frac{\mathrm{D}\log(1-u)}{n-2}\right|^p \mathrm{d}\sigma \leq \int_{\{u=t\}} \left|\frac{\mathrm{D}\log(1-u)}{n-2}\right|^{p-1} \frac{\mathrm{H}}{n-1}\,\mathrm{d}\sigma \tag{2.1}$$

holds true, where H *is the mean curvature of the level set $\{u = t\}$. Moreover, the equality is fulfilled for some $t_0 \in [0, 1)$ if and only if u is rotationally symmetric.*

To emphasize the analogy with the subsequent Theorem 2.3, we observe that for $t = 0$ the above inequality reduces to

$$\int_{\partial\Omega} \left| \frac{\mathrm{D}u}{n-2} \right|^p \mathrm{d}\sigma \leq \int_{\partial\Omega} \left| \frac{\mathrm{D}u}{n-2} \right|^{p-1} \frac{\mathrm{H}}{n-1} \mathrm{d}\sigma, \tag{2.2}$$

where H is the mean curvature of $\partial\Omega$.

For the static metrics we have:

THEOREM 2.2. *Let (M, g, v) be an asymptotically flat solution to problem* (1.7) *with ADM mass equal to $m > 0$. Then, for every $p \geq 2 - 1/(n-1)$ and every $t \in [0, 1)$, the inequality*

$$\int_{\{v=t\}} \left| \frac{\mathrm{D}\log(1-v^2)}{n-2} \right|^p \mathrm{d}\sigma \leq \int_{\{v=t\}} \left| \frac{\mathrm{D}\log(1-v^2)}{n-2} \right|^{p-1} \frac{\mathrm{H}}{n-1} \mathrm{d}\sigma \tag{2.3}$$

holds true, where H *is the mean curvature of the level set $\{v = t\}$. Moreover, the equality is fulfilled for some $t_0 \in (0, 1)$ if and only if (M, g, v) is isometric to a Schwarzschild solution with ADM mass equal to $m > 0$.*

THEOREM 2.3. *Let (M, g, v) be an asymptotically flat solution to problem* (1.7) *with ADM mass equal to $m > 0$. Then, for every $p \geq 2 - 1/(n-1)$, it holds*

$$\int_{\partial M} \left| \frac{2\mathrm{D}v}{n-2} \right|^p \mathrm{d}\sigma \leq \int_{\partial M} \left| \frac{2\mathrm{D}v}{n-2} \right|^{p-2} \frac{\mathrm{R}^{\partial M}}{(n-1)(n-2)} \mathrm{d}\sigma, \tag{2.4}$$

where $\mathrm{R}^{\partial M}$ denotes the scalar curvature of the metric induced by g on ∂M. Moreover, the equality holds if and only if (M, g, v) is isometric to a Schwarzschild solution with ADM mass equal to $m > 0$.

Applying the Hölder inequality to the right hand side of (2.2) and (2.4), we obtain geometric upper bounds for the L^p-norm of the normal derivative of the solutions at the boundary. Combining these facts with formulæ (1.2) and (1.10), we easily deduce geometric upper bounds for both the electrostatic capacity and the mass.

For the electrostatic potential we have:

COROLLARY 2.4. *Let u be a solution to problem* (1.1). *Then, for every $p \geq 2 - 1/(n-1)$ the inequality*

$$\left\| \frac{\partial u}{\partial \nu} \right\|_{L^p(\partial\Omega)} \leq (n-2) \left\| \frac{\mathrm{H}}{n-1} \right\|_{L^p(\partial\Omega)} \tag{2.5}$$

holds true, where H *is the mean curvature of $\partial\Omega$ and ν is the unit normal vector of $\partial\Omega$ pointing toward the interior of $\mathbb{R}^n \setminus \overline{\Omega}$. Moreover, the equality is fulfilled if* *and only if u is rotationally symmetric. Finally, letting $p \to +\infty$ in the previous inequality, one has that*

$$\max_{\partial\Omega}\left|\frac{\partial u}{\partial \nu}\right| \le (n-2)\,\max_{\partial\Omega}\left|\frac{\mathrm{H}}{n-1}\right|. \tag{2.6}$$

COROLLARY 2.5. *Let $\Omega \subset \mathbb{R}^n$, $n \ge 3$, be a bounded domain with smooth boundary. Then, for every $p \ge 2 - 1/(n-1)$, the inequality*

$$\mathrm{Cap}(\Omega) \le \frac{|\partial\Omega|}{|\mathbb{S}^{n-1}|}\left(\fint_{\partial\Omega}\left|\frac{\mathrm{H}}{n-1}\right|^p \mathrm{d}\sigma\right)^{1/p} \tag{2.7}$$

holds true, where H is the mean curvature of $\partial\Omega$. Moreover, the equality is fulfilled for some $p \ge 2 - 1/(n-1)$ if and only if Ω is a round ball. Finally, letting $p \to +\infty$ in the previous inequality, one has that

$$\mathrm{Cap}(\Omega) \le \frac{|\partial\Omega|}{|\mathbb{S}^{n-1}|}\max_{\partial\Omega}\left|\frac{\mathrm{H}}{n-1}\right|. \tag{2.8}$$

Moreover, the equality is fulfilled if and only if Ω is a round ball.

For the static metrics we have:

COROLLARY 2.6. *Let (M, g, v) be an asymptotically flat solution to problem* (1.7) *with ADM mass equal to $m > 0$. Then, for every $p \ge 2 - 1/(n-1)$, the inequality*

$$\left\|\frac{\partial v}{\partial \nu}\right\|_{L^p(\partial M)} \le \left(\frac{n-2}{2}\right)\sqrt{\left\|\frac{\mathrm{R}^{\partial M}}{(n-1)(n-2)}\right\|_{L^{p/2}(\partial M)}} \tag{2.9}$$

holds true, where $\mathrm{R}^{\partial M}$ is the scalar curvature of the metric induced by g on ∂M. Moreover, the equality holds if and only if (M, g, v) is isometric to a Schwarzschild solution with ADM mass equal to $m > 0$. Finally, letting $p \to +\infty$ in the previous inequality, one has that

$$\max_{\partial M}\left|\frac{\partial v}{\partial \nu}\right| \le \left(\frac{n-2}{2}\right)\sqrt{\max_{\partial M}\left|\frac{\mathrm{R}^{\partial M}}{(n-1)(n-2)}\right|}. \tag{2.10}$$

COROLLARY 2.7. *Let (M, g, v) be an asymptotically flat solution to problem* (1.7) *with ADM mass equal to $m > 0$. Then, for every $p \ge 2 - 1/(n-1)$, the inequality*

$$2m \le \frac{|\partial M|}{|\mathbb{S}^{n-1}|}\left(\fint_{\partial M}\left|\frac{\mathrm{R}^{\partial M}}{(n-1)(n-2)}\right|^{p/2} \mathrm{d}\sigma\right)^{1/p} \tag{2.11}$$

holds true, where $\mathrm{R}^{\partial M}$ is the scalar curvature of the metric induced by g on ∂M. Moreover, the equality holds if and only if (M, g, v) is isometric to a Schwarzschild *solution with ADM mass equal to $m > 0$. Finally, letting $p \to +\infty$ in the previous inequality, one has that*

$$2m \leq \frac{|\partial M|}{|\mathbb{S}^{n-1}|} \sqrt{\max_{\partial M} \left| \frac{\mathrm{R}^{\partial M}}{(n-1)(n-2)} \right|}. \tag{2.12}$$

Moreover, the equality is fulfilled if and only if (M, g, v) is isometric to a Schwarzschild solution with ADM mass equal to $m > 0$.

Remark 2. In both the inequalities (2.6) and (2.10) it would be interesting to see if the equality case can be characterized in terms of the rotational symmetry of the solution.

2.2. *Consequences of the global aspects of the monotonicity*

So far we have used the local feature of the monotonicity, namely the facts that $U_p' \leq 0$, $V_p' \leq 0$ and $V_p''(0) \leq 0$, to deduce a first group of corollaries of Theorems 1.1 and 1.2. To state further consequences, we now exploit the global feature of the monotonicity, comparing our quantities on different level sets of the function u or v. By keeping one of these level sets fixed and letting the other become larger and larger, for every $t \in [0, 1)$ and $p \geq 2 - 1/(n-1)$, we have that

$$U_p(t) \geq \lim_{\tau \to 1^-} U_p(\tau) = [\mathrm{Cap}(\Omega)]^p (n-2)^p |\mathbb{S}^{n-1}|$$

and

$$V_p(t) \geq \lim_{\tau \to 1^-} V_p(\tau) = m^p (n-2)^p |\mathbb{S}^{n-1}|.$$

Setting $t = 0$ and using (2.5) and (2.9) as well as the definitions of U_p and V_p, we get the following chains of sharp inequalities

$$|\mathbb{S}^{n-1}|^{\frac{1}{p}} [\mathrm{Cap}(\Omega)]^{1 - \frac{(p-1)(n-1)}{p(n-2)}} \leq \left\| \frac{\mathrm{D}u}{n-2} \right\|_{L^p(\partial\Omega)} \leq \left\| \frac{\mathrm{H}}{n-1} \right\|_{L^p(\partial\Omega)} \tag{2.13}$$

and

$$|\mathbb{S}^{n-1}|^{\frac{1}{p}} (2m)^{1 - \frac{(p-1)(n-1)}{p(n-2)}} \leq \left\| \frac{2\mathrm{D}v}{n-2} \right\|_{L^p(\partial M)} \leq \sqrt{\left\| \frac{\mathrm{R}^{\partial M}}{(n-1)(n-2)} \right\|_{L^{p/2}(\partial M)}} \tag{2.14}$$

where the equalities are fulfilled if and only if the solutions are rotationally symmetric. Setting $p = n - 1$ in the above inequalities, the factors involving the capacity of Ω and (twice) the mass of (M, g) become 1 and we can deduce some purely geometric consequences of our theory. In the case of the electrostatic potentials, we re-obtain the well known Willmore inequality, together with the corresponding rigidity statement (see [19], [9], and also [8, Theorem 3]).

COROLLARY 2.8 (Willmore inequality). *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 3$, be a bounded domain with smooth boundary. Then, the inequality*

$$|\mathbb{S}^{n-1}| \leq \int_{\partial\Omega} \left|\frac{\mathrm{H}}{n-1}\right|^{n-1} \mathrm{d}\sigma \tag{2.15}$$

holds true, where H *is the mean curvature of* $\partial\Omega$. *Moreover, the equality is fulfilled if and only if* Ω *is a round ball.*

In the case of static metrics, we obtain the following Willmore-type inequality.

COROLLARY 2.9 (Willmore-type inequality). *Let* (M, g, v) *be an asymptotically flat solution to problem* (1.7) *with ADM mass equal to* $m > 0$. *Then, for every* $p \geq 2 - 1/(n-1)$, *the inequality*

$$|\mathbb{S}^{n-1}| \leq \int_{\partial M} \left|\frac{\mathrm{R}^{\partial M}}{(n-1)(n-2)}\right|^{(n-1)/2} \mathrm{d}\sigma \tag{2.16}$$

holds true, where $\mathrm{R}^{\partial M}$ *is the scalar curvature of the metric induced by* g *on* ∂M. *Moreover, the equality holds if and only if* (M, g, v) *is isometric to a Schwarzschild solution with ADM mass equal to* $m > 0$.

2.3. *On the classification of static vacuum Einstein metrics*

We conclude this note with the description of further consequences of our analysis in the setting of problem (1.7). Using (1.10), one can rewrite the first inequality in (2.14) as

$$\left[\frac{⨍_{\partial M} |\mathrm{D}v| \,\mathrm{d}\sigma}{\left(⨍_{\partial M} |\mathrm{D}v|^p \,\mathrm{d}\sigma\right)^{1/p}}\right]^{\frac{p(n-2)}{(p-1)(n-1)}} \leq 2m\left(\frac{|\partial M|}{|\mathbb{S}^{n-1}|}\right)^{-\frac{n-2}{n-1}}.$$

On the other hand, $|\mathrm{D}v|$ is constant on the boundary of M and thus the left hand side is equal to 1. We have thus obtained the Riemannian Penrose Inequality for *static solution*

$$\frac{1}{2}\left(\frac{|\partial M|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \leq m \tag{2.17}$$

in every dimension. Such inequality is known to hold up to dimension $n = 7$, in the more general context of asymptotically flat manifolds with nonnegative scalar curvature and compact (outward minimizing) minimal boundary. For a comprehensive discussion about the general Riemannian Penrose Inequality and its generalizations up to dimension $n = 7$ we refer the reader to [5, 6, 13] and the references therein.

In our context it is also possible to obtain an interesting upper bound for m. To see this, it is sufficient to observe that inequality (2.4), restricted to $p = 2$ and coupled with Jensen inequality and with (1.10), yields

$$(2m)^2\Big(\frac{|\mathbb{S}^{n-1}|}{|\partial M|}\Big)(n-1)(n-2)|\mathbb{S}^{n-1}| \leq \int_{\partial M} \mathrm{R}^{\partial M}\, \mathrm{d}\sigma.$$

Up to some algebraic manipulations, we have obtained the Reverse Riemannian Penrose Inequality

$$m \leq \frac{1}{2}\Big(\frac{|\partial M|}{|\mathbb{S}^{n-1}|}\Big)^{\frac{n-2}{n-1}} \sqrt{\Big(\frac{|\partial M|}{|\mathbb{S}^{n-1}|}\Big)^{-\frac{n-3}{n-1}} \frac{\int_{\partial M} \mathrm{R}^{\partial M}\, \mathrm{d}\sigma}{(n-1)(n-2)|\mathbb{S}^{n-1}|}}. \tag{2.18}$$

Combining (2.17) with (2.18) gives the following theorem.

THEOREM 2.10 (Penrose Inequality and Reverse Penrose Inequality for static metrics). *Let (M, g, v) be an asymptotically flat solution to problem* (1.7) *with ADM mass equal to $m > 0$. Then, for every $n \geq 3$, the inequalities*

$$\frac{1}{2}\Big(\frac{|\partial M|}{|\mathbb{S}^{n-1}|}\Big)^{\frac{n-2}{n-1}} \leq m \leq \frac{1}{2}\Big(\frac{|\partial M|}{|\mathbb{S}^{n-1}|}\Big)^{\frac{n-2}{n-1}} \sqrt{\Big(\frac{|\partial M|}{|\mathbb{S}^{n-1}|}\Big)^{-\frac{n-3}{n-1}} \frac{\int_{\partial M} \mathrm{R}^{\partial M}\, \mathrm{d}\sigma}{(n-1)(n-2)|\mathbb{S}^{n-1}|}} \tag{2.19}$$

hold true. Moreover, the equality holds in either the first or in the second inequality in the above formula if and only if the static solution (M, g, v) is isometric to a Schwarzschild solution with ADM mass equal to $m > 0$.

To describe some immediate consequences of the above theorem, we observe that, in dimension $n = 3$, the Gauss–Bonnet Formula gives

$$\int_{\partial M} \mathrm{R}^{\partial M}\, \mathrm{d}\sigma = 4\pi\chi(\partial M) \leq 8\pi,$$

where $\chi(\partial M)$ is the Euler characteristic of ∂M. In particular, the term under square root in (2.19) is always bounded above by 1. Hence, the equality holds in (2.19) and we can recover the classical 3-dimensional Black Hole Uniqueness Theorem.

THEOREM 2.11 (Black Hole Uniqueness Theorem). *Let (M, g, v) be a 3-dimensional asymptotically flat solution to problem* (1.7) *with ADM mass equal to $m > 0$. Then, (M, g, v) is isometric to a Schwarzschild solution with ADM mass equal to $m > 0$.*

Now, going back to formula (2.19), it is important to notice that the term under the square root is scaling invariant. In fact, it can be rewritten in terms of the $(n-1)$-dimensional renormalized Einstein–Hilbert functional. We recall that for a compact $(n-1)$-dimensional manifold Σ, this functional is defined as

$$g \mapsto \mathscr{E}^{\Sigma}_{n-1}(g) = |\Sigma|_g^{-\frac{n-3}{n-1}} \int_{\Sigma} \mathrm{R}_g\, \mathrm{d}\sigma_g, \tag{2.20}$$

where $|\Sigma|_g$ represents the $(n-1)$-dimensional volume of Σ computed with respect to the metric g, whereas $\mathrm{d}\sigma_g$ and R_g are respectively the volume element and the scalar curvature of g. The minimizers of the *renormalized Einstein–Hilbert functional* over a given conformal class are constant scalar curvature metrics called Yamabe metrics. It follows from the celebrated works of Aubin and Schoen on the resolution of the Yamabe problem that for every compact $(n-1)$-dimensional manifold Σ, with $n \geq 4$, it holds

$$\sup\{\mathscr{E}_{n-1}^{\Sigma}(g) \,|\, g \text{ is a Yamabe metric on } \Sigma\} \leq \mathscr{E}_{n-1}^{\mathbb{S}^{n-1}}(g_{\mathbb{S}^{n-1}}).$$

In this setting, formula (2.19) can be rephrased as

$$\frac{1}{2}\left(\frac{|\partial M|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \leq m \leq \frac{1}{2}\left(\frac{|\partial M|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \sqrt{\frac{\mathscr{E}_{n-1}^{\partial M}(g_{\partial M})}{\mathscr{E}_{n-1}^{\mathbb{S}^{n-1}}(g_{\mathbb{S}^{n-1}})}}. \tag{2.21}$$

This gives the following theorem, which shows how the rotational symmetry of the static solution (M, g, v) can be detected from the knowledge of the intrinsic geometry of the boundary, in dimension $n \geq 4$.

THEOREM 2.12. *For every $n \geq 4$, let (M, g, v) be a n-dimensional asymptotically flat solution to problem* (1.7) *with ADM mass equal to $m > 0$. Then, we have*

$$\mathscr{E}_{n-1}^{\mathbb{S}^{n-1}}(g_{\mathbb{S}^{n-1}}) \leq \mathscr{E}_{n-1}^{\partial M}(g_{\partial M}), \tag{2.22}$$

where $g_{\partial M}$ is the metric induced by g on ∂M. Moreover, the equality holds if and only if (M, g, v) is isometric to a Schwarzschild solution with ADM mass equal to $m > 0$. In particular, if $g_{\partial M}$ is a Yamabe metric, then (M, g, v) is rotationally symmetric.

ADDED NOTE. A positive answer to the question raised in Remark 2 has been recently given in [4] for both inequalities (2.6) and (2.10).

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