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Mathematical Analysis — *Ancient solutions of semilinear heat equations on Riemannian manifolds*, by Daniele Castorina and Carlo Mantegazza, communicated on November 11, 2016.

This paper is dedicated to the memory of Professor Ennio De Giorgi.

Abstract. — We study the qualitative properties of ancient solutions of linear and semilinear heat equations in a Riemannian manifold, with particular attention to positivity and constancy in space.

Key words: Ancient solution, Liouville theorem, semilinear parabolic equation

Mathematics Subject Classification: 35K05, 58J35

1. Introduction

We will discuss some properties of solutions of linear and semilinear heat equations in $\mathbb{R}^n$ or in a Riemannian manifold. We will mainly consider the heat equation or the semilinear parabolic equation $u_t = \Delta u + u^2$, but most of the results can be extended to solutions of $u_t = \Delta u + f(u)$, where $f : \mathbb{R} \to \mathbb{R}$ will be a smooth, nonnegative and even real function, monotone increasing on $\mathbb{R}^+$ and decreasing on $\mathbb{R}^-$, with $f(s) = 0$ if and only if $s = 0$.

Definition 1.1. We call a solution of $u_t = \Delta u + f(u)$

- *ancient* if it is defined in $D \times (-\infty, T)$ for some $T \in \mathbb{R}$,
- *immortal* if it is defined in $D \times (T, +\infty)$ for some $T \in \mathbb{R}$,
- *eternal* if it is defined in $D \times \mathbb{R}$,

where D is some connected domain of a manifold.

We call a solution u *trivial* if it is constant in space, that is, $u(x, t) = u(t)$ and solves the ODE $u' = f(u)$. We say that u is simply *constant* if it is constant in space and time.

Notice that positive ancient (trivial) solutions always exist (the problem reduces to an ODE) and the same for negative immortal ones, while eternal solutions are more difficult to exist.

By means of a priori gradient, energy or entropy estimates, we will prove some Liouville type results (i.e. triviality in space variables) for ancient solutions of linear and semilinear heat equations on Riemannian manifolds with nonnegative

Ricci tensor (or simply bounded from below), under "mild" growth conditions. We underline that we do not assume positivity of the solutions, but we deduce it as a consequence of the boundedness from below of the Ricci tensor (see Section 2). There is a quite large literature on this topic, for a rather complete account we refer the interested reader to the paper of Souplet and Zhang [11], which was also an inspiration for our analysis of the semilinear case in the last section. Other interesting recent developments for the semilinear heat equation can be found in [7], as well as in [9] which gives important improvements of known results both for the scalar and system cases. Let us point out that, as a consequence of our positivity Theorem 2.4, it is possible to improve results such as Theorem 1 in [9] or Corollary 1.6 in [7].

2. Positivity

We start with an easy example of which kind of results we are going to discuss in this section.

Proposition 2.1. *Let (M, g) be a compact Riemannian manifold without boundary and u an ancient solution of the equation $u_t = \Delta u + u^2$ in $M \times (-\infty, T)$, for some $T \in \mathbb{R}$, which is uniformly bounded below, then either $u \equiv 0$ or $u > 0$ everywhere.*

Proof. We define $x_t \in M$ as the point such that $u(x_t, t) = \min_{x \in M} u(x, t)$ and we set $v(t) = u(x_t, t)$, then, by maximum principle or, more precisely, by Hamilton's trick (see [4] or [6, Lemma 2.1.3] for details), at almost every $t \in (-\infty, T)$ (precisely where $v'(t)$ exists – notice that v is locally Lipschitz) there holds $v'(t) \geq v^2(t)$.

Hence, v is nondecreasing and there exists $\lim_{t \to -\infty} v(t) = m > -\infty$, by the assumption on the uniform lower bound. Assume that $m \neq 0$, then $v'(t) \geq m^2/4 > 0$, almost everywhere for t small enough, but then, by integration, this implies $m = -\infty$, a contradiction. Thus, $m = 0$ and $u \geq 0$ everywhere. By strong maximum principle, actually $u > 0$ everywhere, otherwise $u \equiv 0$, and we are done. □

Corollary 2.2. *Let u be an eternal solution of $u_t = \Delta u + u^2$ in $M \times \mathbb{R}$ which is uniformly bounded below, then $u \equiv 0$.*

Proof. By the previous proposition, if $u \not\equiv 0$, the u is positive everywhere and (with same notation) $v(t_0) \geq \delta > 0$, for some $t_0, \delta \in \mathbb{R}$. Then, by integrating the differential inequality $v'(t) \geq v^2(t)$, we see that $v(t)$ goes to $+\infty$ in finite time, hence, the same holds also for u, against the hypothesis that it is eternal. □

We now deal with the general case, we follow the technical line of [2, Proposition 2.1]. Let (M, g) an n-dimensional, complete Riemannian manifold without boundary and u an ancient solution of $u_t = \Delta u + u^2$ in $M \times (-\infty, T)$. Notice that M can be noncompact and we are not asking any bound on u.

LEMMA 2.3. *Let the Ricci tensor of* (M, g) *bounded below by* $-K$, *with* $K \geq 0$. *Let* $u : M \times [0, T) \to \mathbb{R}$ *be a solution of the equation* $u_t = \Delta u + u^2$. *For any* $0 < \delta < 1$, *there is a constant* $C_\delta > 0$ *such that, if* $u \geq -L$, *for some positive* $L \in \mathbb{R}$, *in the ball* $B_{Ar_0}(x_0)$ *at* $t = 0$, *with*

$$A \geq 2 + 2(n-1)T/r_0^2 + 2(n-1)T\sqrt{K}/r_0,$$

then,

$$u(x,t) \geq \min\left\{-\frac{1}{(1-\delta)t + 1/L}, -\frac{C_\delta}{A^2 r_0^2}\right\},$$

for every $x \in B_{Ar_0/4}(x_0)$ *and* $t \in [0, T)$.

PROOF. By the *Laplacian comparison* theorem (see [8, Chapter 9, Section 3.3] and also [10]), if $\mathrm{Ric} \geq -K$, with $K \geq 0$, we have

$$-\Delta d(x,x_0) \geq -\frac{n-1}{d(x,x_0)} - (n-1)\sqrt{K} \geq -\frac{n-1}{r_0} - (n-1)\sqrt{K} \tag{2.1}$$

whenever $d(x, x_0) \geq r_0$, in the sense of support functions.

We consider the function $w(x,t) = u(x,t)\psi(x,t)$ with

$$\psi(x,t) = \varphi\left(\frac{d(x_0,x) + \left(\frac{n-1}{r_0} + (n-1)\sqrt{K}\right)t}{Ar_0}\right),$$

where φ is a fixed smooth, nonnegative and nonincreasing function such that $\varphi = 1$ on $(-\infty, \frac{3}{4}]$, and $\varphi = 0$ on $[1, +\infty)$.

Then,

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right)w &= \varphi\left(\frac{\partial}{\partial t} - \Delta\right)u + u\left(\frac{\partial}{\partial t} - \Delta\right)\psi - 2\nabla\psi\nabla u \\
&= \varphi u^2 + \varphi' \frac{u}{Ar_0}\left[\frac{n-1}{r_0} + (n-1)\sqrt{K}\right] - \Delta\psi - 2\nabla\psi\nabla u \\
&= \varphi u^2 + \varphi' \frac{u}{Ar_0}\left[-\Delta d(x_0,x) + \frac{n-1}{r_0} + (n-1)\sqrt{K}\right] \\
&\quad - \varphi'' \frac{u}{A^2 r_0^2} - 2\nabla\psi\nabla u,
\end{aligned} \tag{2.2}$$

at smooth points of distance function (notice that in the last passage we used the fact that $|\nabla d| = 1$).

Let $w_{\min}(t) = \min_M w(\cdot, t)$ be achieved at some point $x_t \in M$. If $u_{\min}(t) < 0$, also $w_{\min}(t) < 0$, hence $\psi(x_t, t) > 0$ and $\varphi' u \geq 0$. Hence, the factor in front of the second term in the right hand side of the above formula is nonnegative. Moreover, if $x_t \in B_{r_0}(x_0)$, we have

$$\begin{aligned}\frac{d(x_0,x)+\left(\frac{n-1}{r_0}+(n-1)\sqrt{K}\right)t}{Ar_0} &\leq \frac{r_0+\left(\frac{n-1}{r_0}+(n-1)\sqrt{K}\right)T}{Ar_0}\\ &\leq \frac{r_0+\frac{(n-1)T}{r_0}+(n-1)\sqrt{K}T}{2r_0+2(n-1)T/r_0+2(n-1)\sqrt{K}T}\\ &\leq 1/2\end{aligned}$$

hence, by the choice of φ, it is easy to see that $\nabla\psi(x_t,t)=\Delta\psi(x_t,t)=0$. It follows, by the second line in computation (2.2), that in such case $w'_{\min}(t)\geq \varphi u^2(x_t,t)=w^2_{\min}(t)$, at almost every time $t\in[0,T)$.

If instead $d(x_t,x_0)\geq r_0$, estimate (2.1) holds and at (x_t,t) we have, by the second line in computation (2.2),

$$\begin{aligned}\frac{\partial}{\partial t}w &\geq \varphi u^2-\varphi''\frac{u}{A^2r_0^2}-2\nabla\psi\nabla u=\varphi u^2-\varphi''\frac{u}{A^2r_0^2}-2u\frac{|\nabla\psi|^2}{\psi}\\ &=\varphi u^2+\frac{u}{A^2r_0^2}\left(\frac{2[\varphi']^2}{\varphi}-\varphi''\right),\end{aligned}$$

in the sense of support function, since $0=\nabla w=\psi\nabla u+u\nabla\psi$ at (x_t,t), by minimality, at almost every time $t\in[0,T)$.

Then, by maximum principle or, more precisely, by Hamilton's trick (see [4] or [6, Lemma 2.1.3]), for any $\delta\in(0,1)$, we have

$$\begin{aligned}\frac{d}{dt}w_{\min} &\geq \varphi u^2+\frac{u}{A^2r_0^2}\left(\frac{2[\varphi']^2}{\varphi}-\varphi''\right)\\ &\geq \varphi u^2-\frac{\delta}{2}\varphi u^2-\frac{1}{2\delta A^4r_0^4\varphi}\left(\frac{2[\varphi']^2}{\varphi}-\varphi''\right)^2\\ &\geq \frac{w^2_{\min}}{\varphi}(1-\delta/2)-\frac{C^2}{2\delta A^4r_0^4}\\ &\geq (1-\delta)w^2_{\min}+\frac{\delta}{2}\left(w^2_{\min}-\frac{C^2_\delta}{A^4r_0^4}\right),\end{aligned}$$

where we used Peter–Paul inequality, the estimate $\left|\frac{2[\varphi']^2}{\varphi}-\varphi''\right|\leq C\sqrt{\varphi}$ and that $1/\varphi\geq 1$.

Resuming, at almost every time $t\in[0,T)$ such that $w_{\min}(t)<0$ either $w'_{\min}(t)\geq w^2_{\min}(t)$ or the inequality (2.3) holds. Then, by integration of these differential inequalities, we conclude

$$w_{\min}(t)\geq \min\left\{-\frac{1}{(1-\delta)t+1/L},-\frac{C_\delta}{A^2r_0^2}\right\},$$

which implies

$$u(x,t) \geq \min\left\{-\frac{1}{(1-\delta)t+1/L}, -\frac{C_\delta}{A^2 r_0^2}\right\}$$

for every $x \in B_{Ar_0/4}(x_0)$ and $t \in [0,T)$. □

THEOREM 2.4. *Let the Ricci tensor of (M,g) be uniformly bounded below. If $u : M \times (-\infty, T) \to \mathbb{R}$ is an ancient solution of the equation $u_t = \Delta u + u^2$, then either $u \equiv 0$ or $u > 0$ everywhere*

PROOF. We only need to show that $u \geq 0$ everywhere, then the conclusion will follow by the strong maximum principle.

Since the estimate in the previous lemma is invariant by translation in time, for every $m \in \mathbb{N}$, we can consider the interval $[-m, T)$ and conclude that

$$u(x,t) \geq \min\left\{-\frac{1}{(1-\delta)(t+m)+1/L}, -\frac{C_\delta}{A^2 r_0^2}\right\}$$

for every $x \in B_{Ar_0/4}(x_0)$ and $t \in [-m, T)$, with $-L \leq \inf_{B_{Ar_0}(x_0)} u$ and

$$A \geq 2 + 2(n-1)T/r_0^2 + 2(n-1)(T+m)\sqrt{K}/r_0.$$

In particular, for every $t \in [-m+1, T)$ and $x \in B_{Ar_0/4}(x_0)$, sending L to $+\infty$, we have

$$u(x,t) \geq \min\left\{-\frac{1}{(1-\delta)(t+m)}, -\frac{C_\delta}{A^2 r_0^2}\right\},$$

sending now $m \in \mathbb{N}$ to $+\infty$, we have that for every $t \in (-\infty, T)$ and $x \in B_{Ar_0/4}(x_0)$,

$$u(x,t) \geq -C_\delta / A^2 r_0^2.$$

Sending finally also $A \to +\infty$, we conclude that $u \geq 0$ everywhere. □

REMARK 2.5. Notice that Theorem 2.4 does not hold for the standard *linear* heat equation, the (positive) nonlinearity plays a key role here.

REMARK 2.6. In the noncompact situation, the conclusion of Corollary 2.2 does not necessarily hold. Consider $M = \mathbb{R}^n$ and u given by a "Talenti's function" (an extremal of Sobolev inequalities, see [12] and also [1]),

$$u(x) = \frac{n(n-2)}{(1+|x|^2)^{\frac{n-2}{2}}}.$$

In the special case $n = 6$, we have

$$u(x) = \frac{24}{(1 + |x|^2)^2}$$

which, by a straightforward computation, satisfies $\Delta u + u^2 = 0$ in $\mathbb{R}^6$, in particular, u is a nonzero eternal solution for the semilinear heat equation $u_t = \Delta u + u^2$.

Remark 2.7. This theorem, in the special case $M = \mathbb{R}^n$, implies that the positivity hypothesis in Theorem 1 of [9] and Corollary 1.6 in [7] (dealing with ancient/eternal solutions of $u_t = \Delta u + u^p$), for $p = 2$, can be actually removed.

3. Triviality I – Heat equation

3.1. Gradient estimates.

Proposition 3.1. *Let (M, g) be a compact Riemannian manifold without boundary and u an ancient solution of the heat equation $u_t = \Delta u$ in $M \times (-\infty, T)$, for some $T \in \mathbb{R}$, which is uniformly bounded, then u is trivial (that is, $|\nabla u| \equiv 0$), hence, constant.*

Proof. We first compute the evolution equation for the gradient squared of u.

$$\frac{d}{dt}|\nabla u|^2 = 2\nabla u \nabla u_t = 2\nabla u \nabla \Delta u = 2\nabla u \Delta \nabla u = \Delta|\nabla u|^2 - 2|D^2 u|.$$

Hence,

$$\begin{aligned} \frac{d}{dt}[u^2 + 2(t - t_0)|\nabla u|^2] &= \Delta u^2 + 2(t - t_0)[\Delta|\nabla u|^2 - 2|D^2 u|^2] \\ &\leq \Delta[u^2 + 2(t - t_0)|\nabla u|^2]. \end{aligned}$$

Then, setting $v = u^2 + 2(t - t_0)|\nabla u|^2$, by maximum principle, we have $v'_{\max} \leq 0$, almost everywhere. We conclude that $v(x, t) \leq v_{\max}(t_0)$, that is

$$u^2(x, t) + 2(t - t_0)|\nabla u(x, t)|^2 \leq u^2_{\max}(\cdot, t_0) \leq C < +\infty,$$

for every $t_0 \in (-\infty, T)$ and $(x, t) \in M \times (t_0, T)$. It follows

$$|\nabla u(x, t)|^2 \leq \frac{C}{t - t_0},$$

sending $t_0 \to -\infty$, we get $\nabla u(x, t) = 0$ for every $(x, t) \in M \times (-\infty, T)$, that is, u is trivial. □

COROLLARY 3.2. *Let u be an eternal solution of the heat equation $u_t = \Delta u$ in $M \times \mathbb{R}$ on a compact manifold (M, g), which is uniformly bounded, then u is constant.*

REMARK 3.3. Notice that boundedness is necessary, the function $u(\theta, t) = e^{-t} \sin\theta$ is an ancient (actually eternal), nontrivial, unbounded (above and below) solution of the heat equation on $\mathbb{S}^1 \times \mathbb{R}$.

3.2. Energy estimates. Let $u : M \times (-\infty, T)$ be an ancient solution of the heat equation $u_t = \Delta u$ in a compact Riemannian manifold (M, g) (without boundary). Taking $t_0 \in (-\infty, T)$ and setting $u_0 = u(\cdot, t_0)$, supposing for a moment that M is flat, by differentiating and integrating by parts, we have

$$\frac{d}{dt}\int_M u\,dx = \int_M u_t\,dx = \int_M \Delta u\,dx = 0,$$

hence, $H(u) = \int_M u\,dx = \int_M u_0\,dx = H(u_0)$ (*heat conservation*), for every $t \in (t_0, T)$.

Then, for any $m \in \mathbb{N}$, we have

$$\begin{aligned}\frac{d}{dt}\int_M u^m\,dx &= m\int_M u^{m-1}u_t\,dx = m\int_M u^{m-1}\Delta u\,dx\\ &= -m(m-1)\int_M u^{m-2}|\nabla u|^2\,dx,\end{aligned}$$

hence, if $u \geq 0$, $E_m(u) = \int_M u^m\,dx$ is nonincreasing in time.

We have

$$\begin{aligned}\frac{d}{dt}\int_M |\nabla u|^2\,dx &= 2\int_M \nabla u\nabla u_t\,dx = 2\int_M \nabla u\nabla\Delta u\,dx\\ &= 2\int_M \nabla u\Delta\nabla u\,dx = -2\int_M |D^2u|^2\,dx.\end{aligned} \tag{3.1}$$

Finally, we consider the functional

$$W(u) = \int_M u^2 + 2|\nabla u|^2(t - t_0)\,dx.$$

Notice that $W = E_2 - (t - t_0)E_2'$, hence $W' = -(t - t_0)E_2''$ and

$$\frac{d}{dt}W(u) = -4(t - t_0)\int_M |D^2u|^2\,dx \leq 0,$$

which implies that $W(u)$ is a monotone nonincreasing function and $E_2(u)$ is a convex function for $t \in (t_0, T)$.

It clearly follows,

$$W(u(\cdot,t)) \leq W(u_0) = \int_M u_0^2 \, dx,$$

for every $t \in (t_0, T)$ and we conclude

$$\int_M |\nabla u|^2 \, dx \leq \frac{\|u(\cdot,t_0)\|^2_{L^2(M)}}{t - t_0},$$

for every $t \in (t_0, T)$.

Hence, assuming that the L^2 norm of u is uniformly bounded in time, taking the limit as $t_0 \to -\infty$, we get that u is necessarily constant in space ($|\nabla u| \equiv 0$ everywhere).

All these computations (and arguments) can be performed analogously if (M, g) is a compact Riemannian manifold without boundary, but passing from the third to the fourth term in equation (3.1) we interchanged space derivatives and this produces an extra "error" term, due to the curvature of (M, g), given by

$$-2 \int_M (t - t_0) \operatorname{Ric}(\nabla u, \nabla u) \, dx,$$

hence getting

$$\frac{d}{dt} W(u) = -4(t - t_0) \int_M |D^2 u|^2 + \operatorname{Ric}(\nabla u, \nabla u) \, dx. \tag{3.2}$$

Notice that the formulas $H' = 0$ and $E_2' = -2 \int_M |\nabla u|^2 \, dx \leq 0$ are not affected.

Clearly, the quantity on the right hand side of formula (3.2) is non positive if $\operatorname{Ric} \geq 0$. In such case we can conclude as before that $W(u)$ is a monotone nonincreasing function and $E_2(u)$ is a convex function for $t \in (t_0, T)$, moreover, the function u is trivial.

PROPOSITION 3.4. *Let (M, g) be a compact Riemannian manifold without boundary and* $\operatorname{Ric} \geq 0$. *If u an ancient solution of the heat equation $u_t = \Delta u$ in $M \times (-\infty, T)$ with uniformly bounded (in time) L^2 norm (in space), then u is trivial, hence constant.*

We remark that the results of this section can be extended to noncompact manifolds, if all the integrals are finite and integrations by parts are justified.

3.3. Entropy estimates. Let $u : M \times (-\infty, T)$ be an ancient *positive* solution of the heat equation $u_t = \Delta u$ in a compact Riemannian manifold (M, g) without boundary. Taking $t_0 \in (-\infty, T)$ and setting $u_0 = u(\cdot, t_0)$, supposing for a moment that M is flat, by differentiating and integrating by parts, we have

$$\frac{d}{dt}\int_M u\log u\,dx = \int_M u_t(\log u+1)\,dx = \int_M \Delta u(\log u+1)\,dx$$
$$= -\int_M \frac{|\nabla u|^2}{u}\,dx = -\int_M u|\nabla \log u|^2\,dx \le 0,$$

hence, $E(u) = \int_M u\log u\,dx \le \int_M u_0\log u_0\,dx = E(u_0)$ (*entropy dissipation*), for every $t \in (t_0, T)$.

Then, we consider the functional

$$F(u) = \int_M u(\log u - 1) + \frac{|\nabla u|^2}{u}(t-t_0)\,dx.$$

Notice that $F = E - H - (t-t_0)E'$, hence $F' = -(t-t_0)E''$.

$$\frac{d}{dt}F(u) = \int_M u_t(\log u - 1) + u_t + \frac{|\nabla u|^2}{u} + (t-t_0)\left[\frac{2\nabla u\nabla u_t}{u} - \frac{|\nabla u|^2}{u^2}u_t\right]dx \tag{3.3}$$
$$= \int_M \Delta u\log u + \frac{|\nabla u|^2}{u} + (t-t_0)\left[\frac{2\nabla u\nabla\Delta u}{u} - \frac{|\nabla u|^2}{u^2}\Delta u\right]dx$$
$$= \int_M (t-t_0)\left[-2\frac{\nabla u|\nabla u|^2\nabla u}{u^3} + \frac{\nabla u\nabla|\nabla u|^2}{u^2}\right.$$
$$\left. - 2\frac{|D^2u|^2}{u} - \frac{\nabla u D^2u\nabla u}{u^2}\Delta u\right]dx$$
$$= \int_M (t-t_0)\left[-2\frac{|\nabla u|^4}{u^3} - 2\frac{|D^2u|^2}{u} + 4\frac{\nabla u D^2u\nabla u}{u^2}\Delta u\right]dx$$
$$= -2(t-t_0)\int_M u|D^2\log u|^2\,dx$$
$$\le 0,$$

which implies that $F(u)$ is a monotone nonincreasing function and $E(u)$ is a convex function for $t \in (t_0, T)$, by the previous remark.

It clearly follows,

$$F(u(\cdot,t)) \le F(u_0) = \int_M u_0(\log u_0 - 1)\,dx,$$

for every $t \in (t_0, T)$.

By heat conservation and

$$-\frac{\mathrm{Vol}(M,g)}{e} \leq \int_M u \log u \, dx \leq \int_M u_0 \log u_0 \, dx \leq C(u_0) \leq C \,\mathrm{Vol}(M,g),$$

we conclude

$$\int_M \frac{|\nabla u|^2}{u} dx \leq \frac{E(u_0) + \mathrm{Vol}(M,g)/e}{t - t_0},$$

for every $t \in (t_0, T)$. Hence, taking the limit as $t_0 \to -\infty$, we get that u is necessarily constant in space ($|\nabla u| \equiv 0$ everywhere).

Hence, assuming that the entropy of u is uniformly bounded in time, taking the limit as $t_0 \to -\infty$, we get that u is trivial.

The same argument/computation can be performed analogously if (M, g) is a compact Riemannian manifold without boundary, but passing from the second to the third line in computation (3.3) we interchanged spatial derivatives and this produces an extra "error" term, due to the curvature of (M, g), given by

$$-\int_M (t - t_0) \frac{2\,\mathrm{Ric}(\nabla u, \nabla u)}{u} dx,$$

hence getting

$$\begin{aligned} \frac{d}{dt} F(u) &= -2(t - t_0) \int_M u \left[|D^2 \log u|^2 + \mathrm{Ric}\left(\frac{\nabla u}{u}, \frac{\nabla u}{u} \right) \right] dx \\ &= -2(t - t_0) \int_M u [|D^2 \log u|^2 + \mathrm{Ric}(\nabla \log u, \nabla \log u)] \, dx. \end{aligned} \tag{3.4}$$

Notice that the formulas $dH(u)/dt = 0$ and $dE(u)/dt = -\int_M u |\nabla \log u|^2 \, dx \leq 0$ are not affected.

Clearly, the quantity on the right hand side of formula (3.4) is nonpositive if $\mathrm{Ric} \geq 0$. In such case we can conclude as before that $F(u)$ is a monotone nonincreasing function and the entropy $E(u)$ is a convex function for $t \in (t_0, T)$, moreover, by the same argument above, the function u is constant in space, for every $t \in (-\infty, T)$.

Proposition 3.5. *Let (M, g) be a compact Riemannian manifold without boundary and* $\mathrm{Ric} \geq 0$. *If u an ancient solution of the heat equation $u_t = \Delta u$ in $M \times (-\infty, T)$ with uniformly bounded entropy, then u is trivial, hence constant.*

As before, all the results of this section can be extended to noncompact manifolds, if all the integrals are finite and integration by parts are justified.

4. Triviality II – Semilinear case

4.1. Gradient estimates. We will now prove a gradient estimate for positive solutions to the semilinear heat equation $u_t = \Delta u + u^2$ on manifolds with nonnegative Ricci tensor, following the line of Souplet and Zhang in [11], who showed an analogous (actually slightly stronger) result for the heat equation. We will then apply this estimate to study the triviality of ancient solutions of $u_t = \Delta u + u^2$ with a mild growth condition at minus infinity.

Lemma 4.1. *Let (M, g) be a Riemannian manifold such that $\mathrm{Ric}(M, g) \geq kg \geq 0$, for some $k \in \mathbb{R}$. Let u be a positive solution to the semilinear heat equation $u_t = \Delta u + u^2$ in $Q_{R,T} = B(x_0, R) \times [T_0 - T, T_0]$, with $B(x_0, R)$ the geodesic ball centered at $x_0 \in M$ of radius R. Assume that $u \leq D$ in $Q_{R,T}$. Then, there exists $C = C_n > 0$ such that on $Q_{R/2,T/2}$ there holds*

$$\frac{|\nabla u(x,t)|}{u(x,t)} \leq C\Big(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{(2-k)_+}\Big)\Big(1 + \log\frac{D}{u(x,t)}\Big) \tag{4.1}$$

Proof. Since estimate (4.1) is invariant under the rescaling $u \to u/D$, without loss of generality we can suppose that $0 < u \leq 1$. Let us define

$$f = \log u, \qquad w = \frac{|\nabla f|^2}{(1-f)^2}$$

Thanks to the semilinear heat equation we easily see that

$$f_t = \Delta f + |\nabla f|^2 + e^f,$$

which allows us to derive an equation for w. We have, in a orthonormal basis,

$$\begin{aligned}
w_t &= \frac{2\nabla f \nabla f_t}{(1-f)^2} + \frac{2|\nabla f|^2 f_t}{(1-f)^3} \\
&= \frac{2\nabla f \nabla(\Delta f + |\nabla f|^2 + e^f)}{(1-f)^2} + \frac{2|\nabla f|^2(\Delta f + |\nabla f|^2 + e^f)}{(1-f)^3} \\
&= \frac{2\nabla f \nabla(\Delta f + |\nabla f|^2)}{(1-f)^2} + \frac{2|\nabla f|^2(\Delta f + |\nabla f|^2)}{(1-f)^3} + \frac{(2-f)e^f}{1-f}\,\frac{2|\nabla f|^2}{(1-f)^2} \\
&= \frac{2f_{jii}f_j + 4f_i f_j f_{ij}}{(1-f)^2} + \frac{2f_i^2 f_{jj} + 2|\nabla f|^4}{(1-f)^3} + \frac{(2-f)e^f}{1-f}\,\frac{2|\nabla f|^2}{(1-f)^2} \\
&= \frac{2f_{iij}f_j - 2\,\mathrm{Ric}_{ij} f_i f_j + 4f_i f_j f_{ij}}{(1-f)^2} + \frac{2f_i^2 f_{jj} + 2|\nabla f|^4}{(1-f)^3} + \frac{(2-f)e^f}{1-f}\,\frac{2|\nabla f|^2}{(1-f)^2},
\end{aligned}$$

where we interchanged derivatives (hence, there is an "extra" error term given by the Ricci tensor), passing from the fourth to the fifth line and we use the usual convention of summing on repeated indexes.

Now,

$$\nabla_j w = \nabla_j\Big(\frac{f_i^2}{(1-f)^2}\Big) = \frac{2f_i f_{ji}}{(1-f)^2} + \frac{2f_i^2 f_j}{(1-f)^3} \tag{4.2}$$

and

$$\Delta w = \frac{2f_{ij}^2}{(1-f)^2} + \frac{2f_i f_{jji}}{(1-f)^2} + \frac{8f_i f_{ij} f_j}{(1-f)^3} + \frac{2f_i^2 f_{jj}}{(1-f)^3} + \frac{6f_i^2 f_j^2}{(1-f)^4}.$$

Hence, we get

$$\begin{aligned}
w_t - \Delta w &= \frac{2f_j f_{iij} - 2\,\mathrm{Ric}_{ij}\, f_i f_j + 4f_i f_{ij} f_j}{(1-f)^2} + \frac{2f_i^2 f_{jj} + 2|\nabla f|^4}{(1-f)^3} + \frac{(2-f)e^f}{1-f}\,\frac{2|\nabla f|^2}{(1-f)^2}\\
&\quad - \frac{2f_{ij}^2}{(1-f)^2} - \frac{2f_i f_{jji}}{(1-f)^2} - \frac{8f_i f_{ij} f_j}{(1-f)^3} - \frac{2f_i^2 f_{jj}}{(1-f)^3} - \frac{6f_i^2 f_j^2}{(1-f)^4}\\
&= \frac{4f_j f_{ij} f_j - 2\,\mathrm{Ric}_{ij}\, f_i f_j}{(1-f)^2} + \frac{2|\nabla f|^4}{(1-f)^3} + \frac{(2-f)e^f}{1-f}\,\frac{2|\nabla f|^2}{(1-f)^2}\\
&\quad - \frac{2f_{ij}^2}{(1-f)^2} - \frac{8f_i f_{ij} f_j}{(1-f)^3} - \frac{6f_i^2 f_j^2}{(1-f)^4}.
\end{aligned}$$

As, by hypothesis, $\mathrm{Ric}_{ij}\, f_i f_j \geq k|\nabla f|^2$ and $f \leq 0$, we have

$$\frac{(2-f)e^f}{1-f} = \Big(1 + \frac{1}{1-f}\Big)e^f \leq 2,$$

hence,

$$\begin{aligned}
w_t - \Delta w \leq{}& \frac{4f_i f_{ij} f_j}{(1-f)^2} + \frac{2|\nabla f|^4}{(1-f)^3} + \frac{2(2-k)|\nabla f|^2}{(1-f)^2}\\
&- \frac{2f_{ij}^2}{(1-f)^2} - \frac{8f_i f_{ij} f_j}{(1-f)^3} - \frac{6|\nabla f|^4}{(1-f)^4}.
\end{aligned}$$

Notice that by (4.2), there holds

$$\langle \nabla f \,|\, \nabla w\rangle = \frac{2f_i f_{ij} f_j}{(1-f)^2} + \frac{2|\nabla f|^4}{(1-f)^3},$$

hence, substituting, we get

$$
\begin{aligned}
(4.3) \qquad w_t - \Delta w \leq {} & 2\langle \nabla f \,|\, \nabla w\rangle - \frac{2|\nabla f|^4}{(1-f)^3} + \frac{2(2-k)|\nabla f|^2}{(1-f)^2} \\
& - \frac{2f_{ij}^2}{(1-f)^2} - \frac{8f_i f_{ij} f_j}{(1-f)^3} - \frac{6|\nabla f|^4}{(1-f)^4} \\
= {} & 2\langle \nabla f \,|\, \nabla w\rangle - \frac{2\langle \nabla f|\nabla w\rangle}{1-f} + \frac{2(2-k)|\nabla f|^2}{(1-f)^2} - \frac{2|\nabla f|^4}{(1-f)^3} \\
& - \frac{2f_{ij}^2}{(1-f)^2} - \frac{4f_i f_{ij} f_j}{(1-f)^3} - \frac{2|\nabla f|^4}{(1-f)^4} \\
= {} & -\frac{2f}{1-f}\langle \nabla f \,|\, \nabla w\rangle + \frac{2(2-k)|\nabla f|^2}{(1-f)^2} \\
& - \frac{2|\nabla f|^4}{(1-f)^3} - \frac{2}{(1-f)^2}\left(f_{ij} + \frac{f_i f_j}{1-f}\right)^2 \\
\leq {} & -\frac{2f}{1-f}\langle \nabla f \,|\, \nabla w\rangle + 2(2-k)w - 2(1-f)w^2.
\end{aligned}
$$

We introduce the following cut–off functions (of Li–Yau): let ψ to be a smooth cut–off function supported in $Q_{R,T}$ with the following properties:

(1) $\psi(x,t) = \varphi(d^M(x,x_0),t) \in [0,1]$ with $\varphi(r,t) \equiv 1$ if $r \leq R/2$ and $T_0 - T/4 \leq t \leq T_0$,
(2) φ is nonincreasing in the space variable r,
(3) $|\nabla\psi|/\psi^a = |\partial_r\varphi|/\varphi^a \leq C_a/R$ and $|\partial_r^2\varphi|/\varphi^a \leq C_a/R^2$, when $0 < a < 1$,
(4) $|\partial_t\psi|/\psi^{1/2} \leq C/T$,

for some constants C, C_a.

Then, by inequality (4.3) with a straightforward calculation, setting $b = -\frac{2f}{1-f}\nabla f$ one has

$$
\begin{aligned}
& \Delta(\psi w) + \langle b \,|\, \nabla(\psi w)\rangle - 2\left\langle \frac{\nabla\psi}{\psi} \,\middle|\, \nabla(\psi w)\right\rangle - (\psi w)_t \\
& \qquad \geq 2\psi(1-f)w^2 + \langle b \,|\, \nabla\psi\rangle w - 2\frac{|\nabla\psi|^2}{\psi}w + w\Delta\psi - \psi_t w + 2(k-2)w\psi.
\end{aligned}
$$

Suppose that the positive maximum of ψw is reached at some point $(x_1, t_1) \in Q_{R,T}$, which cannot be on the boundary where $\psi = 0$. By Li–Yau [5], we can assume that x_1 is not in the cut-locus of M. In fact, in general the distance is only Lipschitz on the cut-locus of M and the maximum principle would have to be intended in a weak sense. However, thanks to an argument of Calabi we can assume without loss of generality that ψw is smooth when applying the maximum principle, see for instance [8]. Then, at such point there holds $\Delta(\psi w) \leq 0$, $(\psi w)_t \geq 0$ and $\nabla(\psi w) = 0$, hence

$$2\psi(1-f)w^2(x_1,t_1) \leq -\left[\langle b \,|\, \nabla\psi\rangle w - 2\frac{|\nabla\psi|^2}{\psi}w + (\Delta\psi)w - \psi_t w + 2(k-2)w\psi\right](x_1,t_1). \tag{4.4}$$

We now estimate each term on the right-hand side. For the first term we have,

$$\begin{aligned}|\langle b \,|\, \nabla\psi\rangle w| &\leq \frac{2|f|}{1-f}|\nabla f|\,|\nabla\psi|w \\ &\leq 2w^{3/2}|f|\,|\nabla\psi| \\ &= 2[(1-f)\psi w^2]^{3/4}\frac{|f|\,|\nabla\psi|}{[(1-f)\psi]^{3/4}} \\ &\leq (1-f)\psi w^2 + C\frac{(f|\nabla\psi|)^4}{[(1-f)\psi]^3} \\ &\leq (1-f)\psi w^2 + C\frac{f^4}{R^4(1-f)^3},\end{aligned} \tag{4.5}$$

by the properties of the function ψ.

For the second term,

$$\frac{|\nabla\psi|^2}{\psi}w = \psi^{1/2}\frac{|\nabla\psi|^2}{\psi^{3/2}}w \leq \frac{1}{8}\psi w^2 + C\Big(\frac{|\nabla\psi|^2}{\psi^{3/2}}\Big)^2 \leq \frac{1}{8}\psi w^2 + \frac{C}{R^4}. \tag{4.6}$$

Thanks to the assumption on the nonnegative Ricci curvature, by the *Laplacian comparison* theorem (see [8, Chapter 9, Section 3.3] or [10]), one has

$$\begin{aligned}-(\Delta\psi)w &\leq -\Big(\partial_r^2\varphi + \frac{n-1}{r}\partial_r\varphi\Big)w \\ &\leq \Big(|\partial_r^2\varphi| + 2(n-1)\frac{|\partial_r\varphi|}{R}\Big)w \\ &\leq \varphi^{1/2}w\Big(\frac{|\partial_r^2\varphi|}{\varphi^{1/2}} + 2(n-1)\frac{|\partial_r\varphi|}{R\varphi^{1/2}}\Big) \\ &\leq \frac{1}{8}\varphi w^2 + C\left(\left[\frac{|\partial_r^2\varphi|}{\varphi^{1/2}}\right]^2 + \left[\frac{|\partial_r\varphi|}{R\varphi^{1/2}}\right]^2\right) \\ &\leq \frac{1}{8}\psi w^2 + \frac{C}{R^4},\end{aligned} \tag{4.7}$$

by the properties of the functions φ and ψ (n here is the dimension of the manifold M).

Now we estimate $|\psi_t|w$ as

$$|\psi_t|w = \psi^{1/2}\frac{|\psi_t|}{\psi^{1/2}}w \le \frac{1}{8}\psi w^2 + C\Big(\frac{|\psi_t|}{\psi^{1/2}}\Big)^2 \le \frac{1}{8}\psi w^2 + \frac{C}{T^2}, \tag{4.8}$$

again by the properties of ψ.

Finally, we deal with the last term,

$$2(2-k)w\psi \le 2(2-k)_+ w\psi \le \frac{1}{8}\psi w^2 + C(2-k)_+^2 \tag{4.9}$$

Substituting estimates (4.5), (4.6), (4.7), (4.8), (4.9) in the right-hand side of inequality (4.4), we deduce

$$2(1-f)\psi w^2 \le (1-f)\psi w^2 + C\frac{f^4}{R^4(1-f)^3} + \frac{1}{2}\psi w^2 + \frac{C}{R^4} + \frac{C}{T^2} + C(2-k)_+^2.$$

Recalling that $f \le 0$, it follows

$$\psi w^2(x_1, t_1) \le C\frac{f^4}{R^4(1-f)^4} + \frac{1}{2}\psi w^2(x_1, t_1) + \frac{C}{R^4} + \frac{C}{T^2} + C(2-k)_+^2$$

and, since $f^4/(1-f)^4 \le 1$, we conclude that

$$\begin{aligned}\psi^2(x,t)w^2(x,t) &\le \psi^2(x_1,t_1)w^2(x_1,t_1) \le \psi(x_1,t_1)w^2(x_1,t_1)\\ &\le \frac{C}{R^4} + \frac{C}{T^2} + C(2-k)_+^2,\end{aligned}$$

for all $(x,t) \in Q_{R,T}$.

As $\psi > 0$ in $Q_{R/2,T/2}$ and $w = |\nabla f|^2/(1-f)^2$, we finally have

$$\frac{|\nabla f|}{(1-f)} \le \frac{C}{R} + \frac{C}{\sqrt{T}} + C\sqrt{(2-k)_+}$$

for every $(x,t) \in Q_{R/2,T/2}$. Since $f = \log(u/D)$, we are done. □

Remark 4.2. Notice that if $k > 0$, then the manifold is compact, by Bonnet–Myers theorem (see [3]).

Corollary 4.3. *Let (M, g) be a compact Riemannian manifold such that* $\mathrm{Ric}(M,g) \ge kg \ge 0$, *for some $k \in \mathbb{R}$. Let u be a positive solution to the semilinear* *heat equation* $u_t = \Delta u + u^2$ *in* $M \times [T_0 - T, T_0]$. *Assume that* $u \leq D$, *then, there exists* $C = C_n > 0$ *such that on* $M \times [T_0 - T/2, T_0]$ *there holds*

$$\frac{|\nabla u(x,t)|}{u(x,t)} \leq C\Big(\frac{1}{\sqrt{T}} + \sqrt{(2-k)_+}\Big)\Big(1 + \log\frac{D}{u(x,t)}\Big) \tag{4.10}$$

Proof. The proof is the same, we simply consider analogous functions ψ which are constant in space. □

We can now prove the following triviality result.

Theorem 4.4. *Let* (M, g) *be a compact Riemannian manifold such that* $\mathrm{Ric}(M, g) \geq 2g$. *Let* u *be an ancient solution to the semilinear heat equation such that*

$$\log u(x,t) = o(\sqrt{|t|}), \quad \text{as } t \to -\infty \tag{4.11}$$

Then u *is trivial.*

Proof. By Theorem 2.4 we know that, if u is nontrivial, then u is necessarily positive. Hence, under the above hypothesis, we have that estimate (4.10) on $Q_{R/2,T/2}$ reads as

$$\frac{|\nabla u(x,t)|}{u(x,t)} \leq \frac{C}{\sqrt{T}}\Big(1 + \log\frac{D}{u(x,t)}\Big), \tag{4.12}$$

with $D = \max_{M\times[T_0-T,T_0]} u(x,t)$. Notice that thanks to the compactness of M and the hypothesis (4.11), we clearly have

$$\log D = o(\sqrt{|T|}), \quad \text{as } T \to -\infty$$

Now, if we fix (x_0, t_0) is space-time, using inequality (4.12) on the cube $B(x_0, R) \times [t_0 - R^2, t_0]$ and the growth hypothesis (4.11) we finally deduce

$$\frac{|\nabla u(x,t)|}{u(x,t)} \leq \frac{C}{R}(1 + o(R)). \tag{4.13}$$

Letting $R \to \infty$ in inequality (4.13) it follows that $|\nabla u(x_0, t_0)| = 0$ and being (x_0, t_0) arbitrary, u is necessarily constant. □

Remark 4.5. Let us point out that the positive curvature bound in Theorem 4.4 is necessary. A counterexample in the flat case $M = \mathbb{R}^n$ is given again by the "Talenti's function"

$$u(x) = \frac{24}{(1 + |x|^2)^2}$$

which satisfies $\Delta u + u^2 = 0$ in $\mathbb{R}^6$, hence, it is a nonconstant stationary solution (eternal) for the semilinear heat equation, contradicting Theorem 4.4 in the flat case.

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