Rend. Lincei Mat. Appl. 28 (2017), 103–117
DOI 10.4171/RLM/754

Calculus of Variations — *Optimal bounds for periodic mixtures of nearest-neighbour ferromagnetic interactions*, by Andrea Braides and Leonard Kreutz, communicated on November 11, 2016.

In memoriam Ennio De Giorgi

Abstract. — We give optimal bounds for the homogenization of mixtures of two types of ferromagnetic interactions. This is done by characterizing the possible Γ-limits of the corresponding energies in a discrete-to-continuum setting. We show that for nearest-neighbour systems this characterization can be provided by a description of the possible limit Wulff shapes in terms of the percentage of one of the two types of interactions.

Key words: Spin systems, homogenization, optimal design, Γ-convergence, conducting networks

Mathematics Subject Classification: 35B27, 74Q20, 82B20, 49J45, 74E30, 49Q20

1. INTRODUCTION

The homogenization of periodic ferromagnetic spin systems in a variational framework has been addressed by Caffarelli and de la Llave [15] using the notion of plane-like minimizers and by Braides and Piatnitsky [13] in a discrete-to-continuum setting by Γ-convergence (see also [3, 7]). In this paper we consider the problem of describing the overall properties of periodic mixtures of two types of nearest-neighbour interactions; i.e., of characterizing the homogenization of periodic Ising systems of the form

$$\sum_{ij} c_{ij}(u_i - u_j)^2 \tag{1}$$

where $u_i \in \{-1, +1\}$, $i \in \mathbb{Z}^2$, the sum runs over all nearest neighbours in a square lattice, and the "bonds" c_{ij} are periodic coefficients that may only take two positive values α and β with

$$\alpha < \beta. \tag{2}$$

A representation theorem in [13] shows that the variational properties of spin energies (1) are approximately described (for large number of interactions) by an interfacial energy

$$\int_{\partial^*\{u=1\}} \varphi(v)\, d\mathscr{H}^1 \tag{3}$$

defined on the "magnetization" parameter $u \in BV_{\rm loc}(\mathbb{R}^2; \{-1, +1\})$, which is a continuum counterpart of the spin variable. We give a precise description of all the *homogenized surface tension* φ that may be obtained in this way in terms of the *proportion* θ (or volume fraction) of β-bonds as follows. We show that, with fixed θ, all possible such φ are the (even positively homogeneous of degree one) convex functions such that

$$\alpha(|v_1| + |v_2|) \leq \varphi(v) \leq c_1|v_1| + c_2|v_2| \quad \text{for all } v \in S^1 \tag{4}$$

for some c_1 and c_2, where the coefficients c_1 and c_2 satisfy

$$c_1 \leq \beta, \quad c_2 \leq \beta, \quad c_1 + c_2 = 2(\theta\beta + (1-\theta)\alpha). \tag{5}$$

Note that since the volume fraction θ is rational, such bounds are understood as extended to all $\theta \in [0, 1]$ by approximation. These relations are a particular case of bounds obtained in [11] when also not-nearest neighbour are taken into account. When only nearest-neighbour interaction are considered as in this paper, a simplified proof using a homogenization formula on paths is possible, and a nice description of bounds in terms of the Wulff shapes of the continuum energies can be given.

1.1. *A localization principle*

We note that the characterization of bounds has an application far beyond the description of periodic Ising systems. Indeed, a general *localization principle* proved in [11] shows that the description of the φ above allows the analysis of the behaviour of arbitrary sequences (parameterized by $n \in \mathbb{N}$)

$$\sum_{ij} c^n_{ij}(u_i - u_j)^2 \tag{6}$$

without any periodicity assumption on c^n_{ij}. More precisely, in a discrete-to-continuum approach, we may define (up to subsequences) the *local volume fraction* $\theta(x)$ as the density of the weak*-limit of the measures

$$\frac{1}{4n^2} \sum_{\{(i,j) \in \mathscr{Z} : c^n_{ij} = \beta\}} \delta_{(i+j)/2n} \tag{7}$$

with respect to the Lebesgue measure. Note that the normalization factor is such that the weak*-limit is the constant θ times the Lebesgue measure (θ the volume fraction defined above) when $c^n_{ij} = c_{ij}$ independent of n with c_{ij} periodic. Then the localization principle states that all possible continuum counterparts of (6) are energies of the form

$$\int_{\partial^*\{u=1\}} \varphi(x, v)\, d\mathscr{H}^1 \tag{8}$$

defined on $BV_{\rm loc}(\mathbb{R}^2;\{-1,+1\})$, where $\varphi(x,\cdot)$ satisfies the bounds described above when $\theta=\theta(x)$ for almost every x.

1.2. *Description of the optimal geometry of bounds*

The discrete setting allows to give a (relatively) easy proof of the optimal bounds in a way similar to the treatment of mixtures of linearly elastic discrete structures [10]. The bounds obtained by sections and by averages in the elastic case have as counterpart *bounds by projection*, where the homogenized surface tension is estimated from below by considering the minimal value of the coefficient on each section, and *bounds by averaging*, where coefficients on a section are substituted with their average. The discrete setting allows to construct (almost-)optimal periodic geometries, which optimize one type or the other of bounds in each direction.

We briefly describe the 'extreme' geometries in Fig. 1 and Fig. 2, where α-connections are represented as dotted lines, β-connections are represented as solid lines, and the nodes with the value $+1$ or -1 as white circles or black circles,

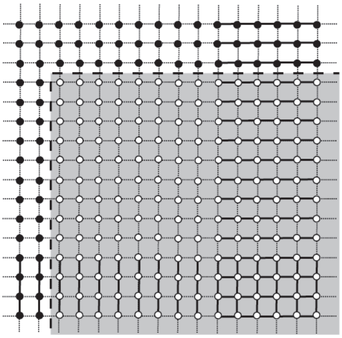

Figure 1. periodicity cell for a mixture giving the lower bound

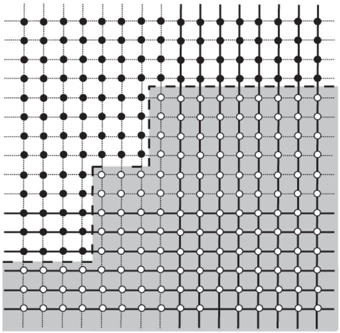

Figure 2. periodicity cell for a mixture giving an upper bound

respectively. In Fig. 1 there are pictured the periodicity cell of a mixture giving as a result the lower bound $\alpha(|v_1| + |v_2|)$ and an interface with minimal energy. Fig. 2 represents the periodicity cell of a mixture giving a upper bound of the form $c_1|v_1| + c_2|v_2|$. Note that the interface pictured in that figure crosses exactly a number of bonds proportional to the percentage θ_v of β-bonds in the horizontal direction.

It must be noted that, contrary to the elastic case, the bounds (i.e., the sets of possible φ) are increasing with θ, and in particular they always contain the minimal surface tension $\alpha(|v_1| + |v_2|)$, which can be achieved with an arbitrarily small amount of α-bonds.

1.3. Description of the optimal bounds in terms of Wulff shapes

We can picture the bounds in terms of their *Wulff shapes*; i.e., of the solutions A_φ centered in 0 to the problem

$$\max\left\{|A| : \int_{\partial A} \varphi(v)\, d\mathscr{H}^1(x) = 1\right\}.$$

If $\varphi(v) = c_1|v_1| + c_2|v_2|$ then such a Wulff shape is simply the rectangle centered in 0 with one vertex in $(1/(8c_2), 1/(8c_1))$. A general φ satisfying (4) and (5) corresponds to a convex symmetric set contained in the square of side length $1/(4\alpha)$ (which is the Wulff shape corresponding to $\alpha(|v_1| + |v_2|)$) and containing one of such rectangles for c_1 and c_2 satisfying (5). The envelope of the vertices of such rectangles lies in the curve (see Fig. 3).

$$\frac{1}{|x_1|} + \frac{1}{|x_2|} = 16(\theta\beta + (1-\theta)\alpha) \tag{9}$$

In terms of that envelope, we can describe the Wulff shapes of φ as follows:

- if $\theta \le 1/2$ then it is any symmetric convex set contained in the square of side length $1/(4\alpha)$ and intersecting the four portions of the set of points satisfying (9) contained in that square (see Fig. 4(a));

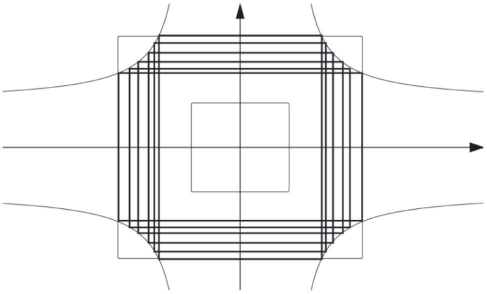

Figure 3. envelope of rectangular Wulff shapes

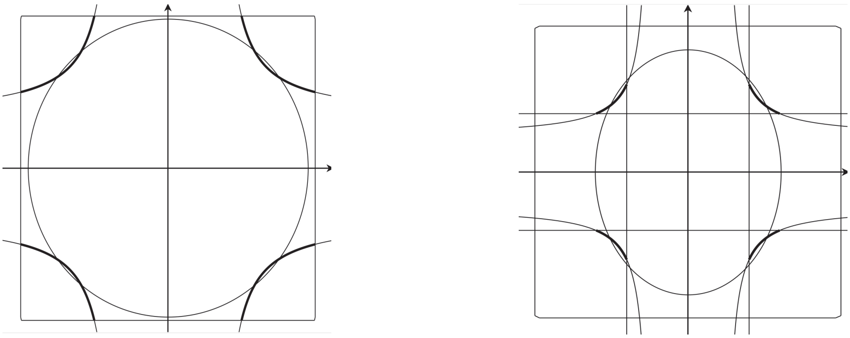

Figure 4. possible Wulff shapes with: (a) $\theta \le 1/2$ and (b) $\theta \ge 1/2$

- if $\theta \ge 1/2$ then it is any symmetric convex set contained in the square of side length $1/(4\alpha)$ and intersecting the four portions of the set of points satisfying (9) with $|x_1| \ge 1/(8\beta)$ and $|x_2| \ge 1/(8\beta)$ contained in that square (see Fig. 4(b)). This second condition is automatically satisfied if $\theta \le 1/2$.

1.4. Connection with continuum problems defined on Finsler metrics

The continuous counterpart of the problem of optimal bounds for (1) is the determination of optimal bounds for Finsler metrics obtained from the homogenization of periodic Riemannian metrics (see [1, 9, 8]) of the form

$$\int_a^b a\Big(\frac{u(t)}{\varepsilon}\Big)|u'|^2\,dt,$$

and $a(u)$ is a periodic function in $\mathbb{R}^2$ taking only the values α and β. This problem has been studied in [16], where it is shown that homogenized metrics satisfy

$$\alpha \le \varphi(v) \le (\theta\beta + (1-\theta)\alpha),$$

but the optimality of such bounds is not proved. The connection with energies (3) is that a 'dual' equivalent formulation in dimension two is obtained by considering the homogenization of periodic perimeter functionals of the form

$$\int_{\partial A} a\Big(\frac{x}{\varepsilon}\Big)\,d\mathscr{H}^1(x)$$

with the same type of a as above (see [4, 5]). The corresponding φ in this case can be interpreted as the homogenized surface tension of the homogenized perimeter functional.

2. SETTING OF THE PROBLEM

We consider a discrete system of nearest-neighbour interactions in dimension two with coefficients $c_{ij} = c_{ji} \ge 0$, $i, j \in \mathbb{Z}^2$. The corresponding ferromagnetic spin energy is

$$F(u) = \frac{1}{8} \sum_{(i,j) \in \mathscr{Z}} c_{ij}(u_i - u_j)^2, \tag{10}$$

where $u : \mathbb{Z}^2 \to \{-1, +1\}$, $u_i = u(i)$, and the sum runs over the set of *nearest neighbours* or *bonds* in $\mathbb{Z}^2$, which is denoted by

$$\mathscr{Z} = \{(i, j) \in \mathbb{Z}^2 \times \mathbb{Z}^2 : |i - j| = 1\}.$$

Such energies correspond to inhomogeneous surface energies on the continuum [2, 13].

Definition 1. Let $\{c_{ij}\}$ be indices as above with $\inf_{ij} c_{ij} > 0$ and periodic; i.e., such that there exists $T \in \mathbb{N}$ such that

$$c_{(i+e_1T)(j+e_1T)} = c_{(i+e_2T)(j+e_2T)} = c_{ij}.$$

Then, we define the *homogenized energy density* of $\{c_{ij}\}$ as the convex positively homogeneous function of degree one $\varphi : \mathbb{R}^2 \to [0, +\infty)$ such that for all $v \in S^1$ we have

$$\varphi(v) = \lim_{R \to +\infty} \inf \left\{ \frac{1}{R} \sum_{n=1}^{N} c_{i_n j_n} : i_N - i_0 = v^\perp R + o(R) \right\}. \tag{11}$$

The infimum is taken over all *paths of bonds*; i.e., pairs (i_n, j_n) such that the unit segment centred in $\frac{i_n + j_n}{2}$ and orthogonal to $i_n - j_n$ has an endpoint in common with the unit segment centred in $\frac{i_{n-1} + j_{n-1}}{2}$ and orthogonal to $i_{n-1} - j_{n-1}$. This is a good definition thanks to [13].

Remark 2. The definition above can be interpreted in terms of a passage from a discrete to a continuous description as follows. We consider the scaled energies

$$F_\varepsilon(u) = \frac{1}{8} \sum_{(i,j) \in \varepsilon\mathscr{Z}} \varepsilon c^\varepsilon_{ij}(u_i - u_j)^2,$$

where $u : \varepsilon\mathbb{Z}^2 \to \{-1, +1\}$, the factor 1/8 is a normalization factor, the sum runs over nearest neighbours in $\varepsilon\mathbb{Z}^2$, and

$$c^\varepsilon_{ij} = c_{\frac{i}{\varepsilon}, \frac{j}{\varepsilon}}.$$

Upon identifying u with its piecewise-constant interpolation, we can regard these energies as defined on $L^1_{\rm loc}(\mathbb{R}^2)$. Their Γ-limit in that space is infinite outside

$BV_{\rm loc}(\mathbb{R}^2, \{\pm 1\})$, where it has the form

$$F_\varphi(u) = \int_{\partial^*\{u=1\}} \varphi(\nu)\, d\mathscr{H}^1$$

with φ as above.

Periodic mixtures of two types of bonds. We will consider the case when

$$c_{ij} \in \{\alpha, \beta\} \quad \text{with } 0 < \alpha < \beta; \tag{12}$$

If we have such coefficients, we define the *volume fraction* of β-bonds as

$$\theta(\{c_{ij}\}) = \frac{1}{4T^2} \#\left\{(i,j) \in \mathscr{Z} : \frac{i+j}{2} \in [0,T)^2, c_{ij} = \beta\right\}. \tag{13}$$

Definition 3. Let $\theta \in [0,1]$. The set of *homogenized energy densities of mixtures of α and β bonds, with volume fraction θ* (of β bonds) is defined as

$$\begin{aligned}\mathbf{H}_{\alpha,\beta}(\theta) = \{\varphi : \mathbb{R}^2 \to [0,+\infty) : &\textit{ there exist } \theta_k \to \theta,\ \varphi_k \to \varphi \\ &\textit{and } \{c^k_{ij}\} \textit{ with } \theta(\{c^k_{ij}\}) = \theta_k \textit{ and } \varphi_k \\ &\textit{homogenized energy density of } \{c^k_{ij}\}\}.\end{aligned} \tag{14}$$

The following theorem completely characterizes the set $\mathbf{H}_{\alpha,\beta}(\theta)$.

Theorem 4 (optimal bounds). *The elements of the set $\mathbf{H}_{\alpha,\beta}(\theta)$ are all even convex positively homogeneous functions of degree one $\varphi : \mathbb{R}^2 \to [0,+\infty)$ such that*

$$\alpha(|x_1| + |x_2|) \le \varphi(x_1, x_2) \le c_1|x_1| + c_2|x_2| \tag{15}$$

for some $\alpha \le c_1, c_2 \le \beta$ such that

$$c_1 + c_2 = 2(\theta\beta + (1-\theta)\alpha). \tag{16}$$

Note that the lower bound for functions in $\mathbf{H}_{\alpha,\beta}(\theta)$ is independent of β. Note moreover that in the case $\theta = 1$ we have all functions satisfying the trivial bound

$$\alpha(|x_1| + |x_2|) \le \varphi(x_1, x_2) \le \beta(|x_1| + |x_2|). \tag{17}$$

This is due to the fact that in that case by considering $\theta_k \to 1$ we allow a vanishing volume fraction of α bonds, which is nevertheless sufficient to allow for all possible φ.

3. Optimality of bounds

We first give two bounds valid for every set of periodic coefficients $\{c_{ij}\}$.

Proposition 5 (bounds by projection). *Let φ be the homogenized energy density of $\{c_{ij}\}$; then we have*

$$\varphi(x) \geq c_1^p|x_1| + c_2^p|x_2|, \tag{18}$$

where

$$c_1^p = \frac{1}{T}\sum_{k=0}^{T-1} \min\{c_{ij} : i_2 = j_2 = k\} \tag{19}$$

and

$$c_2^p = \frac{1}{T}\sum_{k=0}^{T-1} \min\{c_{ij} : i_1 = j_1 = k\}. \tag{20}$$

Proof. The lower bound (18) immediately follows from the definition of φ, by subdividing the contributions of $c_{i_{n-1}i_n}$ in (11) into those with $(i_n)_2 = (i_{n-1})_2$ (or equivalently such that $i_n - i_{n-1} = \pm e_1$) and those with $(i_n)_1 = (i_{n-1})_1$ (or equivalently $i_n - i_{n-1} = \pm e_2$, and estimating

$$c_{i_{n-1}i_n} \geq \min\{c_{ij} : i_2 = j_2 = (i_n)_2\}$$

and

$$c_{i_{n-1}i_n} \geq \min\{c_{ij} : i_1 = j_1 = (i_n)_1\},$$

respectively, in the two cases. □

Proposition 6 (bounds by averaging). *Let φ be the homogenized energy density of $\{c_{ij}\}$; then we have*

$$\varphi(x) \leq c_1^a|x_1| + c_2^a|x_2|, \tag{21}$$

where c_1^a is the average over horizontal bonds

$$c_1^a = \frac{1}{T^2}\sum\left\{c_{ij} : \frac{i+j}{2} \in [0,T)^2, i_2 = j_2\right\} \tag{22}$$

and c_2^a is the average over vertical bonds

$$c_2^a = \frac{1}{T^2}\sum\left\{c_{ij} : \frac{i+j}{2} \in [0,T)^2, i_1 = j_1\right\}. \tag{23}$$

PROOF. The proof is obtained by construction of a suitable competitor $\{i_n, j_n\}$ for the characterization (11) of φ. To that end let $n_1, n_2 \in \{1, \ldots, T\}$ be such that

$$\frac{1}{T}\sum_{k=1}^{T} c_{(n_1-1,k),(n_1,k)} \le \frac{1}{T^2}\sum\left\{c_{ij} : \frac{i+j}{2} \in [0,T)^2, i_2 = j_2\right\}$$

and

$$\frac{1}{T}\sum_{k=1}^{T} c_{(k,n_2-1),(k,n_2)} \le \frac{1}{T^2}\sum\left\{c_{ij} : \frac{i+j}{2} \in [0,T)^2, i_1 = j_1\right\}.$$

Up to a translation, we may suppose that $n_1 = n_2 = 1$. It is not restrictive to suppose that $v_1 \ge 0$ and $v_2 \ge 0$. We define $i_0 = (\lfloor Rv_2 \rfloor, 0)$ and $i_N = (0, \lfloor Rv_1 \rfloor)$. It suffices then to take in Definition 3 the path of bonds $\{i_n, j_n\}$ obtained by concatenating the two paths of bonds defined by

$$i_n^1 = (\lfloor Rv_2 \rfloor - n, 0), \quad j_n^1 = (\lfloor Rv_2 \rfloor - n, 1), \quad n = 0, \ldots, \lfloor Rv_2 \rfloor - 1$$

and

$$i_n^2 = (0, n), \quad j_n^2 = (1, n), \quad n = 1, \ldots, \lfloor Rv_1 \rfloor.$$

We then have

$$\lim_{R\to+\infty} \frac{1}{R}\left(\sum_{n=1}^{\lfloor Rv_2 \rfloor} c_{(n,0)(n,1)} + \sum_{n=1}^{\lfloor Rv_1 \rfloor} c_{(0,n)(1,n)}\right)$$
$$= |v_2|\frac{1}{T}\sum_{n=1}^{T} c_{(n,0)(n,1)} + |v_1|\frac{1}{T}\sum_{n=1}^{T} c_{(0,n)(1,n)},$$

and the desired inequality. □

We now specialize the previous bound to mixtures of two types of bonds. Given $\{c_{ij}\}$ satisfying (12) we define the *volume fraction of horizontal β-bonds* as

$$\theta_h(\{c_{ij}\}) = \frac{1}{2T^2}\#\left\{(i,j) \in \mathscr{X} : \frac{i+j}{2} \in [0,T)^2,\ c_{ij} = \beta, i_2 = j_2\right\}. \tag{24}$$

and the *volume fraction of vertical β-bonds* as

$$\theta_v(\{c_{ij}\}) = \frac{1}{2T^2}\#\left\{(i,j) \in \mathscr{X} : \frac{i+j}{2} \in [0,T)^2,\ c_{ij} = \beta, i_1 = j_1\right\}. \tag{25}$$

Note that

$$\frac{\theta_h(\{c_{ij}\}) + \theta_v(\{c_{ij}\})}{2} = \theta(\{c_{ij}\}). \tag{26}$$

Proposition 7. *Let* $\{c_{ij}\}$ *satisfy* (12), *let* $\theta_h = \theta_h(\{c_{ij}\})$ *and* $\theta_v = \theta_v(\{c_{ij}\})$, *and let* φ *be the homogenized energy density of* $\{c_{ij}\}$. *Then*

$$\varphi(v) \leq (\theta_h\beta + (1-\theta_h)\alpha)|v_1| + (\theta_v\beta + (1-\theta_v)\alpha)|v_2| \tag{27}$$

Proof. It suffices to rewrite c_1^a and c_2^a given by the previous proposition using (24) and (25). □

The previous proposition, together with (26) and the trivial bound (17) gives the bounds in the statement of Theorem 4. We now prove their optimality. First we deal with a special case, from which the general result will be deduced by approximation.

Proposition 8. *Let*

$$\varphi(v) = c_1|v_1| + c_2|v_2|$$

with $\alpha \leq c_1, c_2 \leq \beta$ *and*

$$c_1 + c_2 \leq 2(\beta\theta + (1-\theta)\alpha) \tag{28}$$

for some $\theta \in (0,1)$. *Then* $\varphi \in \mathbf{H}_{\alpha,\beta}(\theta)$.

Proof. The case $\theta = 1$ is trivial. In the other cases, since the set of (c_1, c_2) as above coincides with the closure of its interior, by approximation it suffices to consider the case when indeed

$$\alpha < c_1, c_2 < \beta, \quad c_1 + c_2 < 2(\beta\theta + (1-\theta)\alpha). \tag{29}$$

In particular, we can find $\theta_1 \in (0,1)$ and $\theta_2 \in (0,1)$ such that $\theta_1 + \theta_2 = 2\theta$ and

$$c_1 < \beta\theta_1 + (1-\theta_1)\alpha, \quad c_2 < \beta\theta_2 + (1-\theta_2)\alpha. \tag{30}$$

We then write

$$c_1 = \beta t_1 + (1-t_1)\alpha, \quad c_2 = \beta t_2 + (1-t_2)\alpha. \tag{31}$$

for some $t_1 < \theta_1$ and $t_2 < \theta_2$.

We construct $\{c_{ij}\}$ with period $T \in \mathbb{N}$ and with

$$\theta_h(\{c_{ij}\}) = \theta_1, \quad \theta_v(\{c_{ij}\}) = \theta_2$$

by defining separately the horizontal and vertical bonds. Upon an approximation argument we may suppose that $N_i = t_iT \in \mathbb{N}$, and that $T^2\theta_i \in \mathbb{N}$ for $i = 1, 2$. We only describe the construction for the horizontal bonds. We define

$$c_{(j,n),(j+1,n)} = \begin{cases} \beta & \text{if } j = 0, \dots, T-1 \text{ and } n = 0, \dots, N_1 - 1 \\ \alpha & \text{if } j = 0 \text{ and } n = N_1, \dots, T-1 \end{cases}$$

and any choice of α and β for other indices i, j, only subject to the total constraint that $\theta_h(\{c_{ij}\}) = \theta_1$. With this choice of c_{ij} we have

$$\min\{c_{ij} : i_2 = j_2 = n\} = \begin{cases} \beta & \text{if } n = 0, \dots, N_1 - 1 \\ \alpha & \text{if } n = N_1, \dots, T - 1 \end{cases}$$

The analogous construction for vertical bonds gives

$$\min\{c_{ij} : i_1 = j_1 = n\} = \begin{cases} \beta & \text{if } n = 0, \dots, N_2 - 1 \\ \alpha & \text{if } n = N_2, \dots, T - 1 \end{cases}$$

Then, Proposition 5 gives that the homogenized energy density of $\{c_{ij}\}$ satisfies

$$\varphi(v) \geq c_1|v_1| + c_2|v_2|.$$

To give a lower bound we use the same construction of the proof of Proposition 6, after noticing that the vertical and horizontal paths with $i^1_n = (0, n)$, $j^1_n = (1, n)$ or $i^2_n = (n, 0)$, $j^2_n = (n, 1)$ are such that

$$\frac{1}{T}\sum_{n=1}^{T} c_{i^1_n, j^1_n} = c_1, \quad \frac{1}{T}\sum_{n=1}^{T} c_{i^2_n, j^2_n} = c_2.$$

In this way we obtain the estimate

$$\varphi(v) \leq c_1|v_1| + c_2|v_2|.$$

and hence the desired equality. □

Proof of Theorem 4. Thanks to Remarks 2 and 9 it suffices to approximate F_φ in the sense of Γ-convergence with functionals F_ε associated to c^ε_{ij}. Since all the functionals involved are equicoercive we can make some simplyfing assumptions on φ.

Step 1: We may suppose that

$$\begin{aligned} \alpha(|v_1| + |v_2|) < \varphi(v) &< (\beta\theta_1 + (1 - \theta_1)\alpha)|v_1| + (\beta\theta_2 + (1 - \theta_2)\alpha)|v_2| \\ &=: c_1|v_1| + c_2|v_2| \end{aligned} \tag{32}$$

for some $\theta_1, \theta_2 \in (0, 1)$ such that

$$\theta_1 + \theta_2 = 2\theta.$$

Moreover we can assume that φ is crystalline, i.e. the set $\{\varphi \leq 1\}$ is a convex polyhedron whose vertices correspond to rational directions and contain the directions e_1, e_2, i.e. there exists $\{e_1, e_2\} \subset \{v_k\}_{k=1}^N \subset S^1$, $v_j \neq v_k$, $j \neq k$ such that for all $k \in \{1, \dots, N\}$ there exists $\lambda_k \in \mathbb{R}$ such that $\lambda_k v_k \in \mathbb{Z}^2$, and we have

$$\varphi(v) = \sum_{k=1}^{N} c_k |\langle v, v_k \rangle|, \tag{33}$$

with $c_k \geq 0$.

Step 2: For every φ satisfying (32) and (33) there exists F_ε that approximates F_φ, where F_ε is of the form

$$F_\varepsilon(u) = \int_{\partial^*\{u=1\}} f\Big(\frac{x}{\varepsilon}, v\Big)\, \mathrm{d}\mathscr{H}^1 \tag{34}$$

where

$$f_\varepsilon(y, v) = \begin{cases} \varphi(v_k) & \text{if } y \in A_k,\ k = 1, \dots, N \\ c_1|v_1| + c_2|v_2| & \text{otherwise,} \end{cases}$$

with $A_k := \Pi_{v_k} + \mathbb{Z}^2$. In fact by [5] F_ε Γ-converge to F_φ as $\varepsilon \to 0$ with respect to the $L^1_{\mathrm{loc}}(\mathbb{R}^2)$-topology.

Step 3: Note that for every $k \in \{1, \dots, N\}$ we can write

$$\varphi(v_k) = c_1^k |(v_k)_1| + c_2^k |(v_k)_2| \tag{35}$$

for some $\alpha < c_i^k < c_i$, $i = 1, 2$. We can therefore consider equivalently

$$f(y, v) = \begin{cases} c_1^k |(v_k)_1| + c_2^k |(v_k)_2| & \text{if } y \in \Pi_{v_k} + \mathbb{Z}^d,\ k = 1, \dots, N \\ c_1|v_1| + c_2|v_2| & \text{otherwise.} \end{cases}$$

Every functional of the form (34) can be approximated by functionals of the form

$$F_{\delta,\varepsilon}(u) = \int_{\partial^*\{u=1\}} f_\delta\Big(\frac{x}{\varepsilon}, v_u(x)\Big)\, \mathrm{d}\mathscr{H}^d(x) \tag{36}$$

where for $\delta > 0$ $f_\delta : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, +\infty)$ is defined by

$$f_\delta(y, v) = \begin{cases} c_1^k |(v_k)_1| + c_2^k |(v_k)_2| & \begin{array}{l} \text{if } y \in A_{k,\delta},\ y \notin A_{j,\delta} \\ \text{for all } j \neq k,\ k = 1, \dots, N \end{array} \\ \alpha(|v_1| + |v_2|) & \begin{array}{l} \text{if } y \in A_{k,\delta} \cap A_{j,\delta} \\ \text{for some } j, k \in \{1, \dots, N\},\ j \neq k \end{array} \\ c_1|v_1| + c_2|v_2| & \text{otherwise,} \end{cases}$$

with $A_{j,\delta} = \{y \in \mathbb{R}^2 : \mathrm{dist}_\infty(y, \Pi_{v_j} + \mathbb{Z}^2) \leq \delta\}$.

Step 4: Every functional of the form (36) can be approximated by functionals of the form

$$F_{\eta,\delta,\varepsilon}(u) = \int_{\partial^*\{u=1\}} f_{\eta,\delta}\Big(\frac{x}{\varepsilon}, \nu_u(x)\Big)\, \mathrm{d}\mathscr{H}^1(x) \tag{37}$$

where for $\eta,\delta > 0$ $f_{\eta,\delta} : \mathbb{R}^2 \times \mathbb{R}^2 \to [0,+\infty)$ is defined by

$$f_{\eta,\delta,N}(y,v) = \begin{cases} c_1^k|(v_k)_1| + c_2^k|(v_k)_2| & \text{if } y \in A_{k,\delta},\ y \notin A_{j,\delta} \\ & \text{for all } j \neq k,\ k = 1,\dots,N \\ \alpha(|v_1| + |v_2|) & \text{if } y \in A_{k,\delta} \cap A_{j,\delta} \text{ for some} \\ & j,k \in \{1,\dots,N\},\ j \neq k \\ \beta(|v_1| + |v_2|) & \text{if } y \in A_{k,\delta+\eta} \backslash A_{k,\delta},\ y \notin A_{j,\delta} \\ & \text{for all } j \neq k,\ k = 1,\dots,N \\ c_1|v_1| + c_2|v_2| & \text{otherwise,} \end{cases}$$

Step 5: By localizing the construction in Proposition 8 we can find $c_{ij}^n = c^{n,\eta,\delta,\varepsilon}_{\frac{i}{n}\frac{j}{n}}$ n-periodic such that

$$F_{n,\eta,\delta,\varepsilon}(u) = \frac{1}{8} \sum_{(i,j)\in\frac{1}{n}\mathscr{Z}} \frac{1}{n} c_{ij}^n (u_i - u_j)^2,$$

and $F_{n,\eta,\delta,\varepsilon}$ Γ-converges to $F_{\eta,\delta,\varepsilon}$ with respect to the $L^1_{\mathrm{loc}}(\mathbb{R}^2)$ as $n \to \infty$ and $\theta(\{c^n_{i,j}\}) \to \theta$ as $\eta \to 0$. Using a diagonal argument we can find $c_{ij}^\varepsilon = c_{\frac{i}{\varepsilon}\frac{j}{\varepsilon}}$ $\frac{1}{\varepsilon}$-periodic such that

$$F_\varepsilon(u) = \sum_{(i,j)\in\varepsilon\mathscr{Z}} \varepsilon c^\varepsilon_{i,j}(u_i - u_j)^2$$

Γ-converges to

$$F_\varphi(u) = \int_{\partial^*\{u=1\}} \varphi(v)\, \mathrm{d}\mathscr{H}^1$$

as well as $\theta(\{c_{ij}^\varepsilon\}) \to \theta$ as $\varepsilon \to 0$. We can conclude using the following remark.

Remark 9. In order to prove that the homogenized energy densities φ_ε of c_{ij}^ε converge to φ if c_{ij}^ε $\frac{1}{\varepsilon}$-periodic and F_ε Γ-converges to F_φ, we extend our functionals 1-homogenously to $BV_{\mathrm{loc}}(\mathbb{R}^2)$ by

$$E_\varepsilon(u) = \frac{1}{4} \sum_{(i,j)\in\varepsilon\mathscr{Z}} \varepsilon c_{ij}^\varepsilon |u_i - u_j|$$

such that $E_\varepsilon(u) = F_\varepsilon(u)$ whenever $u \in BV_{\mathrm{loc}}(\mathbb{R}^2, \{-1,+1\})$. Using [12], Theorem 2.1, one can prove that the energy densities of the Γ-limits defined on

$E : BV_{\mathrm{loc}}(\mathbb{R}^2) \to [0, +\infty]$ and $F : BV_{\mathrm{loc}}(\mathbb{R}^2, \{-1, +1\}) \to [0, +\infty]$ agree. Furthermore by a convexity argument, we see that the homogenized energy densities φ_ε of the energies defined on $BV_{\mathrm{loc}}(\mathbb{R}^2)$ (and therefore the homogenized energy densities of the c^ε_{ij}) converge to the limit energy density φ. (See [11] for details)

By the application of this remark the approximation procedure is completed. □

Acknowledgments. The first author gratefully acknowledges the hospitality of the Mathematical Institute in Oxford and the financial support of the EPSRC Science and Innovation award to the Oxford Centre for Nonlinear PDE (EP/E035027/1).

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Received 25 August 2016,
and in revised form 7 October 2016.

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