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Functional Analysis — *Strongly nonlinear Gagliardo–Nirenberg inequality in Orlicz spaces and Boyd indices*, by Claudia Capone, Alberto Fiorenza and Agnieszka Kałamajska, communicated on November 11, 2016.

To the memory of Professor Ennio De Giorgi

Abstract. — Given a N-function A and a continuous function h satisfying certain assumptions, we derive the inequality

$$\int_{\mathbb{R}} A(|f'(x)|h(f(x)))\,dx \le C_1 \int_{\mathbb{R}} A\big(C_2 \sqrt[p]{|\mathcal{M}f''(x)\mathcal{T}_{h,p}(f,x)|}\cdot h(f(x))\big)\,dx,$$

with constants C_1, C_2 independent of f, where $f \ge 0$ belongs locally to the Sobolev space $W^{2,1}(\mathbb{R})$, f' has compact support, $p > 1$ is smaller than the lower Boyd index of A, $\mathcal{T}_{h,p}(\cdot)$ is certain nonlinear transform depending of h but not of A and $\mathcal{M}$ denotes the Hardy–Littlewood maximal function. Moreover, we show that when $h \equiv 1$, then $\mathcal{M}f''$ can be improved by f''. This inequality generalizes a previous result by the third author and Peszek, which was dealing with $p = 2$.

Key words: Gagliardo–Nirenberg inequalities, interpolation inequalities, capacities, isoperimetric inequalities

Mathematics Subject Classification (primary; secondary): 46E35; 26D10

1. Introduction

In [21] the third author, together with Peszek, obtained the inequality

$$\int_{\mathbb{R}} |f'(x)|^p h(f(x))\,dx \le (\sqrt{p-1})^p \int_{\mathbb{R}} \big(\sqrt{|f''(x)\mathcal{T}_h(f(x))|}\big)^p h(f(x))\,dx \tag{1.1}$$

where $p \ge 2$, $f \ge 0$ belongs to the Sobolev space $W^{2,1}_{\mathrm{loc}}(\mathbb{R})$ and f' has bounded support. The function h is given and $\mathcal{T}_h$ is a transform depending of h, but not of p. Such inequality generalizes the following variant of the classical Gagliardo–Niremberg inequality

$$\int_{\mathbb{R}} |f'(x)|^p\,dx \le (\sqrt{p-1})^p \int_{\mathbb{R}} |ff''|^{p/2}\,dx, \quad f \in C_0^\infty(\mathbb{R}),\ p \ge 2$$

(see [21, Lemma 1.1]), and, in view of the applications described therein, (1.1) has been proved in the rather general form. Another extension, dealing with the case

$h(x) = x^{\alpha}$ under the slightly less general assumptions, has been obtained in [17]. In both mentioned papers such inequalities were applied as a main tool to the asymptotic behavior and regularity of solutions of singular ODEs like, for example, to Emden–Fowler equations having the form $u''(x) = g(x)(u(x))^{\alpha}$ in (a,b), $g \in L^p(a,b)$. Another application is also in the study of the regularity of solutions to Cucker–Smale flocking model, see Peszek [28].

In [22] the Orlicz version of (1.1)

$$\int_{\mathbb{R}} A(|f'(x)|h(f(x)))\,dx \le \bar{C}_{A,2} \int_{\mathbb{R}} A\left(\sqrt{|f''(x)\mathcal{T}_{h,2}(f,x)|} \cdot h(f(x))\right) dx \tag{1.2}$$

has been obtained, where A stands for a general N-function which replaces the original power p. Here A has to satisfy the assumptions

$$d_A \frac{A(t)}{t} \le A'(t) \le D_A \frac{A(t)}{t}, \quad t > 0, \tag{1.3}$$

where $D_A \ge d_A > 2$ and

$$L := \max\left\{ \limsup_{\lambda \to 0} \frac{A(t)}{t^{D_A}}, \limsup_{\lambda \to \infty} \frac{A(t)}{t^{d_A}} \right\} < \infty.$$

Such inequality was applied in [22] to derive certain second order isoperimetric inequalities and capacitary estimates.

In this paper we improve the condition $D_A \ge d_A > 2$, this way weakening the assumption on A. In order to do this, we note that inequality (1.2) is invariant under certain equivalence classes of N-functions A. Taking this into account, as a key tool, we applied an idea by the second author and Krbec [8]. Here a characterization of Boyd indices, based again on the same invariance, has been obtained and revealed useful for questions related to variational integrals and extrapolation of integral operators.

As an example, let us consider the function

$$A(t) := \begin{cases} t^{7/3} & \text{if } 0 \le t < 1 \\ (2t-1)t^{1/3} & \text{if } 1 \le t < 2 \\ t^{7/3}/2 + t^{1/3} & \text{if } t \ge 2. \end{cases}$$

It does not satisfy (1.3) with any $d_A \ge 2$, because its lower Simonenko index, which intuitively is the highest possible d_A satisfying (1.3) (see (2.5) for the definition), is $5/3$. As it is smaller than 2, it cannot be considered in the previous statement (1.2), but it can be considered in the statement of our Theorem 3.1, because its lower Boyd index is $7/3$, greater than 2. Let us mention that the computation of the lower Boyd index can be done without using the artificial definition, but using the simple formula found in [9]. The heart of the matter is the following: the growth of $A(t)$ must be, in fact, according to the arguments in [22], at least like the power t^2, but the way (1.3) to "measure" the growth of A is not optimal and it can be refined, by means of the Boyd indices. In the case of the example above, the smaller growth of A in the interval $[1, 2[$ influences the Simonenko index but not the Boyd index. Indeed, according to [9], only the values in the neighborhoods of 0 and ∞ are relevant for the computation of the Boyd indices. The idea in [8], in this case, is that there exists another function A_1 which is equivalent to A (in the sense that A_1/A is bounded away from 0 and ∞) and satisfies (1.3) with $d_{A_1} > 2$. Therefore A_1 can replace A in (1.1) with a worst constant.

We stress that our generalization will lead to inequalities true not only for N-functions $A(t)$ essentially growing faster than t^2, but also for N-functions $A(t)$ essentially growing faster than t^p, with any fixed $p > 1$. As it is natural to expect, in this case the square root on the right hand side of (1.2) must be replaced by the p-root (see (3.1)). Moreover, $\mathcal{T}_{h,2}(\cdot)$ is then replaced by a more general transform $\mathcal{T}_{h,p}(\cdot)$ which is local when $p = 2$ and nonlocal in the other cases. This means that its value at the given point x, when $p \neq 2$, depends not only on the value of the involved function f at x, but it depends of the distribution of the values of f over all interval $(-\infty, x)$. It is described by a certain integration formula (see (2.1) and Remark 2.1) and, contrary to the case $p = 2$, it depends also on f'.

As an effect of this investigation, we obtain the inequality

$$(1.4) \qquad \int_{\mathbb{R}} A(|f'(x)|h(f(x)))\,dx \le \bar{C}_{A,p} \int_{\mathbb{R}} A\Big(\sqrt[p]{|\mathcal{M}f''(x)\mathcal{T}_{h,p}(f,x)|} \cdot h(f(x))\Big)\,dx,$$

with constant $\bar{C}_{A,p}$ independent of f, where f is nonnegative and belongs locally to the Sobolev space $W^{2,1}(\mathbb{R})$, f' has compact support, $p > 1$ is smaller than the lower Boyd index of A, and $\mathcal{M}g$ denotes the Hardy–Littlewood maximal function of g.

In the case $h \equiv 1$ the above inequality reduces to

$$(1.5) \qquad \int_{\mathbb{R}} A(|f'(x)|)\,dx \le C_{A,p} \int_{\mathbb{R}} A\Big(\sqrt[p]{|f''(x)\mathcal{T}_{h\equiv 1,p}(f,x)|}\Big)\,dx,$$

where $\mathcal{T}_{h\equiv 1,p}(f) = \int_{-\infty}^{x} |f'(y)|^{p-2} f'(y)\,dy = \Delta^{-1}(\Delta_p f)$, Δ^{-1} is the inverse of the classical Laplacian operator and $\Delta_p f = (|f'|^{p-2}f')'$ is the p-Laplacian, both acting in 1-d (see (2.1)). We remark that when $p \neq 2$ the nonlocality of such transform is clearly visible and that it depends only on f'.

For example, the choice of $h(t) = t^\alpha$ where $\alpha \ge 0$ and $A(t) = t^q$ where $1 < p < q < \infty$ implies the inequality

$$\int_{\mathbb{R}} |f'(y)|^q (f(y))^\alpha\,dy$$
$$\le C\Big(\int_{\mathbb{R}} (|f'(y)|(f(y))^\alpha)^{p-1}\,dy\Big)^{\frac{q}{p}} \int_{\mathbb{R}} (\mathcal{M}f''(y)(f(y))^\alpha)^{\frac{q}{p}}\,dy,$$

see Theorem 4.2, while the choice of $h \equiv 1$, $f := \int_{-\infty}^{x} |g(y)|\,dy$, where $g \in W^{1,1}(\mathbb{R})$ is compactly supported, implies the following variant of the Gagliardo–Nirenberg inequality

$$\int_{\mathbb{R}} A(|g(y)|)\,dy \le C_{A,p} \int_{\mathbb{R}} A\Big(\sqrt[p]{|g'(x)| \int_{-\infty}^{x} |g(y)|^{p-1}\,dy}\Big)\,dx,$$

which seems to be unknown also for A being of power type, see Theorem 4.3.

Let us mention that contrary to (1.5) and (1.2), in the generalization (1.4), we had to substitute f'' by its maximal function $\mathscr{M}f''$. This is not a surprise, because several other interpolation inequalities, having the pointwise character, were already known with the maximal function involved. We refer for instance to the following pioneering inequality due to Maz'ja and Kufner (see [27, formula (1.9)])

$$(f'(x))^2 \le 2f(x)\mathscr{M}f''(x), \quad f \in W^{2,1}_{\mathrm{loc}}(\mathbb{R}),\ f \ge 0,$$

which will be applied in the proof of our main result. For further generalizations of pointwise inequalities see also [15, 16, 25, 30, 31].

Inequality (1.1) has been already applied to mathematical models related to singular elliptic PDEs [17, 21, 28], while inequality (1.2) was considered to derive isoperimetric inequalities and capacitary estimates [22]. Therefore we believe that our new inequality is not only of purely theoretical interest but that it can contribute in the future to the isoperimetric inequalities and models in the similar way. For other Gagliardo–Nirenberg inequalities and their applications, see e.g. [5, 11, 20, 29].

2. Notation and preliminaries

2.1. *Preliminaries*

Compositions with Lipschitz functions. The following simple observation is a classical fact (see e.g. [26], [1, Appendix A]):

Lemma 2.1. *If $0 < R \le +\infty$, $f : [-R, R] \to [\alpha, \beta]$ is absolutely continuous and $L : [\alpha, \beta] \to \mathbb{R}$ is a Lipschitz function, then $(L \circ f)(x) := L(f(x))$ is absolutely continuous on $[-R, R]$.*

Some special transforms. Let $p \ge 1$ and set

$$\Phi_p(t) := \begin{cases} |t|^{p-2}t & \text{if } t \ne 0 \\ 0 & \text{if } t = 0 \end{cases}$$

Observe that $\Phi_2(t) = t$ and for general $p \ge 1$ the function Φ_p is odd, and on $(0, \infty)$ is continuous and locally Lipschitz. Moreover, $\Phi_p(t) = t^{p-1}$ when

$t \geq 0$, $\Phi_p(ab) = \Phi_p(a)\Phi_p(b)$, and $\Phi_p'(t) = (p-1)|t|^{p-2}$, $(|t|^p)' = p\Phi_p(t)$, when $t \neq 0$.

The following definition will be crucial for our considerations.

Definition 2.1. Let $p \geq 1$, let $h : (0,\infty) \to (0,\infty)$ be a continuous function such that $h \in L^1((0,r))$ for any finite r, and let $f \in W^{2,1}_{\mathrm{loc}}(\mathbb{R})$, be nonnegative and such that f' is compactly supported and $h(f(\cdot))^{p-1}\chi_{\{f(\cdot)>0\}} \in L^1_{\mathrm{loc}}(\mathbb{R})$. We define the following transforms of h and f:

$$\mathscr{G}_{h,p}(f,x) := \int_{-\infty}^{x} |f'(y)h(f(y))|^{p-1}\operatorname{sgn} f'(y)\chi_{\{f(y)\neq 0\}}\,dy \tag{2.1}$$

$$= \int_{-\infty}^{x} h(f(y))^{p-1}\Phi_p(f'(y))\chi_{\{f(y)\neq 0\}}\,dy$$

$$= \int_{-\infty}^{x} \Phi_p(h(f(y))f'(y))\chi_{\{f(y)\neq 0\}}\,dy, \tag{2.2}$$

$$\mathscr{T}_{h,p}(f,x) := \begin{cases} \frac{\mathscr{G}_{h,p}(f,x)}{h(f(x))^{p-1}} & \text{when } f(x) \neq 0 \\ 0 & \text{when } f(x) = 0 \end{cases}$$

where we use the convention that if $F \subseteq \mathbb{R}$ and function g is defined on F, by $g\chi_F$ we denote the extension of g by zero outside set F.

Remark 2.1. Let $p \geq 1$ and let $H : [0,\infty) \to \mathbb{R}$ be the locally absolutely continuous primitive of h such that $H(0) = 0$, i.e. $H(t) = \int_0^t h(s)\,ds$. The mapping $x \mapsto \mathscr{G}_{h,p}(f,x)$ is locally absolutely continuous and

$$\mathscr{G}_{h,2}(f,x) = H(f(x)),$$

$$\mathscr{T}_{h,2}(f,x) = (H/h)(f(x))\chi_{\{f(x)\neq 0\}}(x),$$

$$\mathscr{T}_{h\equiv 1,p}(f,x) = \mathscr{G}_{h\equiv 1,p}(f,x)\cdot\chi_{\{f(x)\neq 0\}}(x) = \int_{-\infty}^{x}\Phi_p(f'(y))\,dy\cdot\chi_{\{f(x)\neq 0\}}(x),$$

$$\frac{d}{dx}(\mathscr{G}_{h,p}(f,x)) = \Phi_p(h(f(x))f'(x)), \quad \text{when } p > 1.$$

Indeed, only the first statement requires some explanation. This follows because $(h(f(x))^{p-1}\Phi_p(f'(x))\chi_{\{f(x)\neq 0\}}$ is integrable (f' is continuous and compactly supported, so bounded, while $(h(f(x))^{p-1})\chi_{\{f(x)\neq 0\}}$ is integrable) and so its integral $\mathscr{G}_{h,p}(f,x)$ is locally absolutely continuous function. Let us also note that $f'(x) = 0$ whenever $f(x) = 0$ because f is nonnegative.

The assumptions on h in Definition 2.1 and in Remark 2.1 will be used in the statement of our main result. We introduce the following condition:

(h) $h : (0, \infty) \to (0, \infty)$ is of class $C^1(0, \infty)$, and there exists a constant $C_h > 0$ such that

$$|h'(t)|t \le C_h h(t), \quad t > 0. \tag{2.3}$$

Moreover, when h is unbounded nearby zero, we assume that it is strictly decreasing in some neighborhood of zero.

Easy verification shows that the above condition is satisfied for example by $h(t) = t^\alpha$, $\alpha \in \mathbb{R}$ or by $h(t) = t^\alpha(\ln(1+t))^\beta$ where $\alpha \in \mathbb{R}$, $\beta > 0$.

2.2. *N*-functions

By N-function we mean any function $A : [0, \infty) \to [0, \infty)$ which is continuous, strictly increasing, convex, such that $A(0) = 0$, $\lim_{t\to 0} A(t)/t = \lim_{t\to\infty} t/A(t) = 0$. For A, A_1 given N-functions, we will say that A_1 is equivalent to A ($A_1 \sim A$) if there exist constants $c_1, c_2 > 0$ such that

$$c_1 A(t) \le A_1(t) \le c_2 A(t), \quad t \ge 0.$$

The symbol A^* will denote the Legendre transform of A:

$$A^*(s) = \sup\{st - A(t) : t > 0\}, \quad s \ge 0.$$

It is known that A^* is also a N-function and the mapping $A \mapsto A^*$ defined in the class of N-functions is an involution (see e.g. [24, Theorem 4.3]).

In the sequel we will assume that A is a differentiable N-function satisfying the condition:

$$d_A \frac{A(t)}{t} \le A'(t) \le D_A \frac{A(t)}{t}, \quad t > 0, \tag{2.4}$$

where $D_A \ge d_A \ge 1$. Without loss of generality we may assume that d_A is the largest possible value and D_A is the smallest possible value such that (2.4) holds for A, by defining:

$$d_A := \inf_{t>0} \frac{tA'(t)}{A(t)}, \quad D_A := \sup_{t>0} \frac{tA'(t)}{A(t)}. \tag{2.5}$$

These numbers are called Simonenko lower and upper index of A, respectively [33].

Remark 2.2. We recall that the condition $d_A \frac{A(t)}{t} \le A'(t)$ is equivalent to the fact that $\frac{A(t)}{t^{d_A}}$ is nondecreasing, and, analogously, the condition $D_A \frac{A(t)}{t} \ge A'(t)$ is equivalent to the fact that $\frac{A(t)}{t^{D_A}}$ is nonincreasing. This latter inequality implies that A satisfies the so called Δ_2-condition:

$$A(2t) \le CA(t), \quad t \ge 0,$$

with some constant $C > 1$ independent of t. The condition $d_A > 1$ in (2.4) is equivalent to the Δ_2-condition for A^* (see e.g. [24, Theorems 4.1 and 4.3]).

We conclude this section recalling the notion of lower and upper Boyd index of A, denoted by i_A and I_A respectively (see Boyd [3], Gustavsson and Peetre [12] and [35]). Such indices can be defined by the following formulae ([8, Theorem 1.1]), based on the fact that lower and upper Simonenko index are not invariant under equivalence relation $\sim$, i.e. the condition $A_1 \sim A$ does not imply $d_{A_1} = d_A$ (see [8, Remark 3.5]):

$$i_A := \sup_{A_1 \sim A} d_{A_1} \quad I_A := \inf_{A_1 \sim A} D_{A_1}.$$

From these expressions it is clear that Boyd indices are invariants of relation $\sim$.

3. The main result

By $\mathscr{M}h$ we denote the Hardy–Littlewood maximal function of the locally integrable function $h : \mathbb{R} \to \mathbb{R}$, defined by

$$\mathscr{M}h(x) := \sup_{x \in I} \frac{1}{|I|} \int_I |h(y)|\, dy, \quad x \in \mathbb{R},$$

where I varies among all intervals containing x. Its most known and important property is the boundedness as operator from $L_p(\mathbb{R})$ to $L_p(\mathbb{R})$, $1 < p \le \infty$. Due to the several applications, e.g. in Harmonic Analysis and PDEs, this property has been intensively studied and extended in a variety of unweighted and weighted function spaces. For a quite recent picture, which starts from the first remarks on $\mathscr{M}$ as operator, see e.g. [6] and several references therein. For our purposes we need to recall that by the classical Lorentz–Shimogaki theorem, see e.g. [2, Theorem 5.17 p. 154], $\mathscr{M}$ is bounded on an Orlicz space $L_\Phi(\mathbb{R})$ if and only if $i_\Phi > 1$. If A is a N-function satisfying (2.4), and if we set $\Phi(t) = A(t^{1/p})$ where $1 < p < i_A$, then it is $i_\Phi = i_A/p > 1$ (see e.g. [8, Proposition 2.1]) and therefore, taking into account that A satisfies the Δ_2 condition, for any g locally integrable in $\mathbb{R}$ it is $\Phi(g) \in L_1(\mathbb{R})$ if and only if $\Phi(\mathscr{M}g) \in L_1(\mathbb{R})$. These considerations will be of help in the discussion about the finiteness of the right hand side of next (3.2), which are part of the following result.

Theorem 3.1. *Suppose that A satisfies (2.4) with $d_A > 1$ and let $1 < p < i_A$. Then we have*

i) *For every nonnegative $f \in W^{2,1}_{\mathrm{loc}}(\mathbb{R})$ such that f' is compactly supported*

$$\int_{\mathbb{R}} A(|f'(x)|)\, dx \le C_{A,p} \int_{\mathbb{R}} A\left(\sqrt[p]{|f''(x)\mathscr{T}_{h\equiv 1,p}(f,x)|}\right) dx. \tag{3.1}$$

ii) *Assume that the function* $h : (0, \infty) \to (0, \infty)$ *is as in Definition* 2.1 *and satisfies* (h). *Moreover, let* $f \in W^{2,1}_{\mathrm{loc}}(\mathbb{R})$ *be nonnegative and such that* f' *is compactly supported,* $h(f(\cdot))^{p-1} \in L^1_{\mathrm{loc}}(\mathbb{R})$, *and additionally that* $\int_{\mathbb{R}} A(\sqrt[p]{f''(x)|})\, dx < \infty$ *when* $h(\cdot)$ *is bounded in some neighborhood of zero and that*

$$\int_{\mathbb{R}} A(\sqrt[p]{\mathcal{M}f''(x)h(f(x))})\, dx < \infty$$

when $h(\cdot)$ *is unbounded and strictly decreasing near zero. Then we have*

$$\int_{\mathbb{R}\cap\{x:f(x)>0\}} A(|f'(x)|h(f(x)))\, dx \le \bar{C}_{A,p} \int_{\mathbb{R}\cap\{x:f(x)>0\}} A(\sqrt[p]{\mathcal{M}f''(x)|\mathcal{T}_{h,p}(f,x)|} \cdot h(f(x)))\, dx, \tag{3.2}$$

where $\mathcal{M}f''$ *is Hardy–Littlewood maximal function of* f''.

Constants $C_{A,p}, \bar{C}_{A,p} > 0$ *are independent on* f *and* $\mathcal{T}_{h,p}$ *is as in Definition* 2.1.

Remark 3.1. Inequality (3.1) dealing with $p = 2$ was obtained earlier in [18] in general dimension. Inequality (3.2) can be interpreted as certain extension of the pointwise inequality which will appear in (4.10).

Remark 3.2. Inequalities (3.1) and (3.2) may become trivial when, of course, their right hand sides are $+\infty$. In the notation fixed at the beginning of this Section, we remark that in the case when h is bounded near zero, right hand side in (3.2) is finite provided that $\Phi(f'') \in L_1(\mathbb{R})$. Indeed, under the assumptions of the theorem, there exists a bounded closed interval where f' is absolutely continuous, while f' is zero outside. Therefore f' is bounded, and so is f, $h(f)$ and $\mathcal{G}_{h,p}(f,\cdot)$, because we assume that h is bounded near zero. In that case we have

$$\begin{aligned} |\mathcal{T}_{h,p}(f,x)| &= \frac{|\mathcal{G}_{h,p}(f,x)|}{h(f(x))^{p-1}} \\ &\le \frac{1}{h(f(x))^{p-1}} \int_{-\infty}^{x} (h(f(y)))^{p-1}|f'(y)|^{p-1}\, dy, \end{aligned}$$

$$\begin{aligned} |\mathcal{T}_{h,p}(f,x)|^{\frac{1}{p}} h(f(x)) &= (h(f(x)))^{\frac{1}{p}} \Big(\int_{-\infty}^{x} (h(f(y)))^{p-1}|f'(y)|^{p-1}\, dy \Big)^{\frac{1}{p}} \\ &\le \sup\{h(t) : t \in (0, \|f\|_\infty)\} \Big(\int_{\mathbb{R}} |f'(y)|^{p-1}\, dy \Big)^{\frac{1}{p}} < \infty \end{aligned} \tag{3.3}$$

and the conclusion is the same as before. Hence, right hand side in (3.2) is finite, provided that $\Phi(\mathcal{M}f'') \in L_1(\mathbb{R})$, i.e. when $\Phi(f'') \in L_1(\mathbb{R})$.

Remark 3.3. The considerations at the beginning of this Section could be obtained equivalently applying the main result of [4] in the case $w \equiv 1$ or, alternatively, the main result of [10], recently generalized in [14].

4. Proof of the main result

We begin with few preliminary results.

Lemma 4.1. *Suppose that A is an N-function satisfying* (2.4) *and* $1 \le p < d_A$. *Then we have*

i) *The function*

$$A_{p-1}(t) := \begin{cases} \frac{A(|t|)}{\Phi_p(t)} & \text{when } t \neq 0 \\ 0 & \text{when } t = 0 \end{cases} \tag{4.1}$$

defined on $\mathbb{R}$, *is nondecreasing, locally Lipschitz and*

$$(A_{p-1})'(t) \le (D_A - p + 1)\frac{A(|t|)}{|t|^p}\chi_{t\neq 0}, \quad \text{when } t \in \mathbb{R}; \tag{4.2}$$

ii) *For all* $a, b > 0$ *we have*

$$\frac{A(a)}{a^p} b^p \le \left(1 - \frac{p}{D_A}\right) A(a) + \frac{p}{d_A} A(b). \tag{4.3}$$

Proof.

i) We observe that $A_{p-1}(\cdot)$ is odd and zero at 0. To show that it is locally Lipshitz, we first verify that the derivative of A_{p-1} is bounded when $t \in (0, t_0)$ for any $t_0 > 0$. We note that when $t > 0$ we have $A_{p-1}(t) = \frac{A(t)}{t^{d_A}} \cdot t^{d_A - p + 1}$, so that $A_{p-1}(t)$ is increasing by Remark 2.2. Therefore when $t > 0$

$$0 \le A'_{p-1}(t) = \frac{t^{p-1}A'(t) - (p-1)t^{p-2}A(t)}{t^{2p-2}} \le \frac{t^{p-1}}{t^{2p-2}}\left(D_A \frac{A(t)}{t} - (p-1)\frac{A(t)}{t}\right)$$
$$= (D_A - p + 1)\frac{A(t)}{t^p} = (D_A - p + 1)\frac{A(t)}{t^{d_A}} \cdot t^{d_A - p}.$$

Since the latter function is nondecreasing, it is bounded by $(D_A - p + 1)\frac{A(t_0)}{t_0^{d_A}} \cdot t_0^{d_A - p}$ and so (4.2) follows when $t \neq 0$. The estimate for $t = 0$ holds as well, because

$$\frac{A_{p-1}(t) - A_{p-1}(0)}{t} = \frac{A_{p-1}(t)}{t} = \frac{A(|t|)}{|t|^p} = \frac{A(|t|)}{|t|^{d_A}} \cdot |t|^{d_A - p} \xrightarrow{t \to 0} 0.$$

ii) We apply the Oppenheim inequality known since 1927 (see Theorem 158 in [13] or [32]):

$$\prod_{\nu} f_\nu(a_\nu) \le \sum_{\nu} \int_0^{a_\nu} \prod_{\mu \ne \nu} f_\mu(x)\, df_\nu(x), \tag{4.4}$$

whenever $\nu = 1, 2 \ldots, n$, $a_\nu \ge 0$, f_ν are nonnegative, continuous and strictly increasing functions and one of $f_\nu(0)$ equals 0. For this we consider at first $p < d_A$, so that the map $A(t)/t^p$ is strictly increasing and it can be naturally extended continuously to 0 at 0. Therefore we can apply (4.4) with $f_1(a) = \frac{A(a)}{a^p}$ and $f_2(b) = b^p$, to get

$$\begin{aligned}\frac{A(a)}{a^p} b^p &\le \int_0^a x^p d\Big(\frac{A(x)}{x^p}\Big) + \int_0^b \frac{A(x)}{x^p} d(x^p) = \int_0^a x^p \frac{A'(x)x^p - \frac{A(x)}{x} p x^p}{x^{2p}} dx \\ &\quad + \int_0^b \frac{A(x)}{x^p} p x^{p-1}\, dx = \int_0^a \Big(A'(x) - \frac{A(x)}{x} p\Big)\, dx + \int_0^b \frac{A(x)}{x} p\, dx \\ &\overset{(2.4)}{\le} \int_0^a \Big(A'(x) - \frac{p}{D_A} A'(x)\Big)\, dx + \int_0^b \frac{p}{d_A} A'(x)\, dx \\ &= \Big(1 - \frac{p}{D_A}\Big) A(a) + \frac{p}{d_A} A(b).\end{aligned}$$

Inequality for $p = d_A$ follows after letting $p \to d_A$. □

Remark 4.1. In the case $A(t) = t^p$ we could not apply Oppenheim inequality directly as then $D_A = d_A = p$ and $A(t)/t^p$ is not strictly increasing, but, on the other hand, the inequality is obvious because it reads

$$\frac{a^p}{a^p} b^p = \frac{A(a)}{a^p} b^p \le \Big(1 - \frac{p}{D_A}\Big) A(a) + \frac{p}{d_A} A(b) = A(b).$$

Remark 4.2. For $p = 1$ inequality (4.3) was known and applied earlier to obtain Hardy inequalities, some of them with best constants, see e.g. Lemma 4.2 in [19] or proof of Lemma 3.1 in [34].

Proposition 4.1 ([23]). *If A is a N-function satisfying* (2.4), *then*

$$\min(t^{d_A}, t^{D_A}) A(r) \le A(tr) \le \max(t^{d_A}, t^{D_A}) A(r), \quad r, t > 0.$$

Proof of Theorem 3.1. It suffices to show the proof under the assumption $1 < p < d_A$ as inequalities (3.1) and (3.2) stay invariant up to the constants if we substitute there $A_1 \sim A$ instead of A. Clearly, we can assume that right hand sides in the inequalities (3.1) and (3.2) are finite and that their left hand sides are strictly positive.

i) Set

$$I := \int_{\mathbb{R}} A(|f'(x)|)\, dx, \quad J := \int_{\mathbb{R}} A(\sqrt[p]{|f''(x)\mathscr{T}_{h\equiv 1,p}(f,x)|})\, dx.$$

Note that $I < \infty$. According to the formulae (4.1) and Remark 2.1, we have

$$I = \int_{\mathbb{R}} A_{p-1}(f'(x)) \cdot (\mathscr{G}_{h\equiv 1,p}(f,x))'\, dx.$$

By Lemma 4.1 the function $A_{p-1}(t)$ is locally Lipschitz. Moreover, f' belongs to $W^{1,1}(\mathbb{R})$ and is compactly supported and bounded. Therefore by Lemma 2.1 the function $A_{p-1}(|f'|)$ is absolutely continuous on $\mathbb{R}$ and compactly supported. This and the fact that $\mathscr{G}_{h\equiv 1,p}(f,x)$ belongs to $W^{1,1}_{\mathrm{loc}}(\mathbb{R})$ allows us to integrate by parts in the expression above, to get:

$$I = -\int_{\mathbb{R}} (A_{p-1}(f'(x)))' \cdot \mathscr{G}_{h\equiv 1,p}(f,x)\, dx. \tag{4.5}$$

Therefore, as $(A_{p-1}(f'(x)))' = 0$ a.e. on level set of f' and $\{x : f'(x) \neq 0\} \subseteq \{x : f(x) \neq 0\}$ for the nonnegative f, we obtain

$$\begin{aligned} I &= -\int_{\mathbb{R}\cap\{x: f'(x)\neq 0\}} (A_{p-1}(f'(x)))' \cdot \mathscr{G}_{h\equiv 1,p}(f,x)\, dx \\ &= -\int_{\mathbb{R}\cap\{x: f(x)\neq 0\}} (A_{p-1}(f'(x)))' \cdot \mathscr{G}_{h\equiv 1,p}(f,x)\, dx \\ &= -\int_{\mathbb{R}} (A_{p-1}(f'(x)))' \cdot \mathscr{T}_{h\equiv 1,p}(f,x)\, dx. \end{aligned}$$

Note that $A'_{p-1}(\cdot)$ is even and

$$0 \le A'_{p-1}(t) \overset{(4.2)}{\le} (D_A - p + 1)\frac{A(|t|)}{|t|^p}\chi_{t\neq 0},$$

$$(A_{p-1}(f'(x)))' = A'_{p-1}(f'(x))f''(x).$$

Consequently

$$\begin{aligned} I &\le \left| \int_{\mathbb{R}} (A_{p-1}(f'(x)))' \cdot \mathscr{T}_{h\equiv 1,p}(f,x)\, dx \right| \\ &\le (D_A - p + 1) \int_{\mathbb{R}\cap\{x: f'(x)\neq 0\}} \frac{A(|f'(x)|)}{|f'(x)|^p} |f''(x)\mathscr{T}_{h\equiv 1,p}(f,x)|\, dx. \end{aligned} \tag{4.6}$$

Let $a = (D_A - p + 1)$. We apply (4.3) to estimate

$$(4.7)\quad I \le a\delta \int_{\mathbb{R}\cap\{x: f'(x)\neq 0\}} \Big(\frac{A(|f'|)}{|f'|^p}\Big)\cdot\bigg(\bigg\{\frac{|f''\mathscr{T}_{h\equiv 1,p}(f,x)|}{\delta}\bigg\}^{1/p}\bigg)^p\,dx$$

$$\le a\Big(1-\frac{p}{D_A}\Big)\delta\int_{\mathbb{R}} A(|f'|)\,dx$$

$$+\,a\frac{p}{d_A}\delta\int_{\mathbb{R}\cap\{x: f'(x)\neq 0\}} A\bigg(\sqrt[p]{\frac{|f''\mathscr{T}_{h\equiv 1,p}(f,x)|}{\delta}}\bigg)\,dx,$$

where $\delta > 0$ will be chosen later. The first integral equals $a\big(1-\frac{p}{D_A}\big)\delta I$. To estimate the second one we note that according to Lemma 4.1: $A\big(\frac{t}{\sqrt[p]{\delta}}\big) \le \max\big\{\big(\frac{1}{\delta}\big)^{\frac{D_A}{p}}, \big(\frac{1}{\delta}\big)^{\frac{d_A}{p}}\big\}A(t)$. This implies:

$$I \le (D_A - p + 1)\Big(1-\frac{p}{D_A}\Big)\delta I$$

$$+\,(D_A - p + 1)\frac{p}{d_A}\delta\cdot\max\bigg\{\Big(\frac{1}{\delta}\Big)^{\frac{D_A}{p}}, \Big(\frac{1}{\delta}\Big)^{\frac{d_A}{p}}\bigg\}\int_{\mathbb{R}} A(\sqrt[p]{|f''\mathscr{T}_{h\equiv 1,p}(f,x)|})\,dx$$

$$= (D_A - p + 1)\Big(1-\frac{p}{D_A}\Big)\delta I$$

$$+\,(D_A - p + 1)\frac{p}{d_A}\delta\cdot\max\bigg\{\Big(\frac{1}{\delta}\Big)^{\frac{D_A}{p}}, \Big(\frac{1}{\delta}\Big)^{\frac{d_A}{p}}\bigg\}J.$$

We choose $\delta = \delta_0$ such that $(D_A - p + 1)\big(1-\frac{p}{D_A}\big)\delta_0 = \frac{1}{2}$, i.e. $\delta_0 = \frac{1}{2(D_A - p+1)\left(1-\frac{p}{D_A}\right)}$. Therefore setting

$$C_1(\delta_0) := (D_A - p + 1)\frac{p}{d_A}\delta_0\cdot\max\bigg\{\Big(\frac{1}{\delta_0}\Big)^{\frac{D_A}{p}}, \Big(\frac{1}{\delta_0}\Big)^{\frac{d_A}{p}}\bigg\},$$

we arrive at the inequality

$$I \le 2C_1(\delta_0)J,$$

which is what we wanted to prove.

ii) Let $f \in W^{2,1}_{\mathrm{loc}}(\mathbb{R})$, and denote $f_\varepsilon = f + \varepsilon$

$$I(\varepsilon) := \int_{\mathbb{R}} A(|f'|h(f_\varepsilon(x)))\,dx,$$

$$J(\varepsilon) := \int_{\mathbb{R}} A(\sqrt[p]{|\mathscr{M}f''(x)\mathscr{T}_{h,p}(f_\varepsilon,x)|}h(f_\varepsilon(x)))\,dx.$$

Then as before we have $I(\varepsilon) < \infty$ and

$$I(\varepsilon) = \int_{\mathbb{R}} (A_{p-1}(f_\varepsilon' h(f_\varepsilon)))\cdot\Phi_p(f_\varepsilon' h(f_\varepsilon))\,dx.$$

By Lemma 4.1 the function $A_{p-1}(t)$ is locally Lipschitz. Moreover, the function $f_\varepsilon' h(f_\varepsilon)$ belongs to $W^{1,1}(\mathbb{R})$ and is compactly supported and bounded. Indeed, the function $h(\cdot)$ is Lipschitz on every interval of the form $[\varepsilon, r)$ and f_ε is bounded, therefore by Lemma 2.1 we have $h(f_\varepsilon) \in W^{1,1}_{\mathrm{loc}}(\mathbb{R})$. As f_ε' belongs to $W^{1,1}(\mathbb{R})$, by the multiplicative properties of $W^{1,1}$ we have $f_\varepsilon' h(f_\varepsilon)$ belongs to $W^{1,1}(\mathbb{R})$ and it is compactly supported. We note that $\mathscr{G}_{h,p}(f_\varepsilon, x)$ is also locally absolutely continuous and by Remark 2.1 $(\mathscr{G}_{h,p}(f_\varepsilon, x))' = \Phi_p(f_\varepsilon' h(f_\varepsilon))$ in sense of distributions and almost everywhere. We integrate by parts in the expression above, and we get:

$$
\begin{aligned}
(4.8) \qquad I(\varepsilon) &= \int_{\mathbb{R}} (A_{p-1}(f_\varepsilon' h(f_\varepsilon))) \cdot (\mathscr{G}_{h,p}(f_\varepsilon, x))' \, dx \\
&= -\int_{\mathbb{R}} (A_{p-1}(f_\varepsilon' h(f_\varepsilon)))' \cdot \mathscr{G}_{h,p}(f_\varepsilon, x) \, dx.
\end{aligned}
$$

Setting $\lambda_{f_\varepsilon}(x) = f'(x) h(f_\varepsilon)(x)$, we have

$$
(A_{p-1}(\lambda_{f_\varepsilon}))' = A_{p-1}'(\lambda_{f_\varepsilon}) \{ f'' h(f_\varepsilon) + (f')^2 h'(f_\varepsilon) \}
$$

Therefore, by (4.2)

$$
\begin{aligned}
(4.9) \qquad I(\varepsilon) &\le \left| \int_{\mathbb{R} \cap \{\lambda_{f_\varepsilon} \neq 0\}} (A_{p-1}(f_\varepsilon' h(f_\varepsilon)))' \cdot \mathscr{G}_{h,p}(f_\varepsilon, x) \, dx \right| \\
&\le (D_A - p + 1) \int_{\mathbb{R}} \frac{A(|\lambda_{f_\varepsilon}|)}{|\lambda_{f_\varepsilon}|^p} |f'' h(f_\varepsilon) \mathscr{G}_{h,p}(f_\varepsilon, x)| \chi_{\{f'(x) \neq 0\}} \, dx \\
&\quad + (D_A - p + 1) \int_{\mathbb{R}} \frac{A(|\lambda_{f_\varepsilon}|)}{|\lambda_{f_\varepsilon}|^p} (f')^2 |h'(f_\varepsilon) \mathscr{G}_{h,p}(f_\varepsilon, x)| \chi_{\{f'(x) \neq 0\}} \, dx \\
&=: \mathrm{I}_1 + \mathrm{I}_2.
\end{aligned}
$$

We put $a = (D_A - p + 1)$ and apply (4.3) to get

$$
\begin{aligned}
\mathrm{I}_1 &= a\delta \int_{\mathbb{R} \cap \{x : f'(x) \neq 0\}} \left(\frac{A(|f'| h(f_\varepsilon))}{|f' h(f_\varepsilon)|^p} \right) \cdot \left(\left\{ \frac{|f'' h(f_\varepsilon) \mathscr{G}_{h,p}(f_\varepsilon, x)|}{\delta} \right\}^{1/p} \right)^p dx \\
&= a\delta \int_{\mathbb{R} \cap \{x : f'(x) \neq 0\}} \left(\frac{A(|f'| h(f_\varepsilon))}{|f' h(f_\varepsilon)|^p} \right) \cdot \left(\left\{ \frac{|f''(h(f_\varepsilon))^p \mathscr{T}_{h,p}(f_\varepsilon, x)|}{\delta} \right\}^{1/p} \right)^p dx \\
&\le a\left(1 - \frac{p}{D_A}\right) \delta \int_{\mathbb{R} \cap \{f'(x) \neq 0\}} A(|f'| h(f_\varepsilon)) \, dx \\
&\quad + a \frac{p}{d_A} \delta \int_{\mathbb{R} \cap \{f'(x) \neq 0\}} A\left(\sqrt[p]{\frac{|f'' \mathscr{T}_{h,p}(f_\varepsilon, x)|}{\delta}} h(f_\varepsilon) \right) dx,
\end{aligned}
$$

and as $\mathscr{M}h(x) \geq |h(x)|$ a.e., we obtain inequality

$$\mathrm{I}_1 \leq (D_A - p + 1)\Big(1 - \frac{p}{D_A}\Big)\delta I(\varepsilon) + (D_A - p + 1)\frac{p}{d_A}\delta \cdot \max\left\{\Big(\frac{1}{\delta}\Big)^{\frac{D_A}{p}}, \Big(\frac{1}{\delta}\Big)^{\frac{d_A}{p}}\right\} J(\varepsilon).$$

We are left with the estimates for expression I_2. Using the assumption (2.3), then the following pointwise inequality due to Maz'ya and Kufner ([27], Remark 1.7), valid for every nonnegative function $f \in W^{2,1}_{\mathrm{loc}}(\mathbb{R})$

$$(f'(x))^2 \leq 2f(x)\mathscr{M}f''(x), \quad x \in \mathbb{R}, \tag{4.10}$$

we compute that

$$\begin{aligned}\mathrm{I}_2 &= a\int_{\mathbb{R}\cap\{f'(x)\neq 0\}} \frac{A(|\lambda_{f_\varepsilon}|)}{|\lambda_{f_\varepsilon}|^p}(f_\varepsilon')^2\left|\frac{h'(f_\varepsilon)}{h(f_\varepsilon)}\frac{\mathscr{G}_{h,p}(f_\varepsilon,x)}{(h(f_\varepsilon))^{p-1}}\right|(h(f_\varepsilon))^p\,dx \\ &\leq aC_h\int_{\mathbb{R}\cap\{f'(x)\neq 0\}} \frac{A(|\lambda_{f_\varepsilon}|)}{|\lambda_{f_\varepsilon}|^p}\cdot\frac{(f_\varepsilon')^2}{f_\varepsilon}\cdot\{|\mathscr{T}_{h,p}(f_\varepsilon,x)|\}(h(f_\varepsilon))^p\,dx \\ &\leq 2aC_h\delta\int_{\mathbb{R}\cap\{f'(x)\neq 0\}} \frac{A(|\lambda_{f_\varepsilon}|)}{|\lambda_{f_\varepsilon}|^p}\cdot\frac{\mathscr{M}f''(x)\cdot\{|\mathscr{T}_{h,p}(f_\varepsilon,x)|\}|h(f_\varepsilon)|^p}{\delta}\,dx\end{aligned}$$

where $\delta > 0$ will be chosen later, and again by (4.3), we get

$$\begin{aligned}\mathrm{I}_2 &\leq 2aC_h\Big(1 - \frac{p}{D_A}\Big)\delta\int_{\mathbb{R}\cap\{f'(x)\neq 0\}} A(|\lambda_{f_\varepsilon}|)\,dx \\ &\quad + 2aC_h\frac{p}{d_A}\delta\int_{\mathbb{R}\cap\{f'(x)\neq 0\}} A\Big(\frac{\sqrt[p]{\mathscr{M}f''(x)|\mathscr{T}_{h,p}(f_\varepsilon,x)|}\,h(f_\varepsilon)}{\delta^{\frac{1}{p}}}\Big)\,dx \\ &\leq 2aC_h\Big(1 - \frac{p}{D_A}\Big)\delta I(\varepsilon) + 2aC_h\frac{p}{d_A}\delta\cdot\max\left\{\Big(\frac{1}{\delta}\Big)^{\frac{D_A}{p}}, \Big(\frac{1}{\delta}\Big)^{\frac{d_A}{p}}\right\}J(\varepsilon).\end{aligned}$$

In the end we get

$$I(\varepsilon) \leq a(\delta)\delta I(\varepsilon) + a(\delta)J(\varepsilon),$$

where

$$a(\delta) = (D_A - p + 1)\Big(1 - \frac{p}{D_A}\Big)(1 + 2C_h)\delta,$$

$$b(\delta) = (D_A - p + 1)\frac{p}{d_A}(1 + 2C_h)\delta\cdot\max\left\{\Big(\frac{1}{\delta}\Big)^{\frac{D_A}{p}}, \Big(\frac{1}{\delta}\Big)^{\frac{d_A}{p}}\right\}$$

Now it suffices to choose the sufficiently small δ, for example to have $a(\delta) = 1/2$, and rearrange to get (3.2) with f_ε instead of f.

We complete the arguments by letting $\varepsilon \to 0$. For this we have to explain the convergence

$$I(\varepsilon) \overset{\varepsilon \to 0}{\longrightarrow} I(0), \quad \text{and} \quad J(\varepsilon) \overset{\varepsilon \to 0}{\longrightarrow} J(0), \tag{4.11}$$

where we set $f_0 = f$. We consider two cases: a) when h is bounded near zero and b) when h is decreasing in some neighborhood of zero.

In case a) convergence follows from Lebesgue's Dominated Convergence Theorem. Indeed, in that case $|f'h(f_\varepsilon)|$ is bounded uniformly in ε and compactly supported and so is $|\mathcal{T}_{h,p}(f_\varepsilon, x)|^{1/p} h(f_\varepsilon(x))$ by similar considerations as in (3.3). Therefore

$$\begin{aligned} A\left(\sqrt[p]{|\mathcal{M}f''(x)\mathcal{T}_{h,p}(f_\varepsilon, x)|}\, h(f_\varepsilon(x))\right) &\le A\left(C\sqrt[p]{|\mathcal{M}f''(x)}\right) \overset{A \in \Delta_2}{\le} C_1 A\left(\sqrt[p]{|\mathcal{M}f''(x)}\right) \\ &= C_1 \Phi(|\mathcal{M}f''(x)|), \end{aligned}$$

where constants C_1, C do not depend on ε and we recall that property of $\Phi(\cdot) = A \circ |\cdot|^{1/p}$ were discussed at the beginning of this section. Last term above is integrable because by our assumption $\Phi(|f''(x)|)$ belongs to $L^1(\mathbb{R})$, which is equivalent to the fact that $\Phi(|\mathcal{M}f''(x)|)$ belongs to $L^1(\mathbb{R})$. In case b) we will find constant $D > 0$ such that

$$h(f_\varepsilon(x)) \le D h(f(x)).$$

Indeed, let us fix $\kappa > 0$ such that h is decreasing on $(0, \kappa]$ and let $\varepsilon < \kappa/2$. Then we have for every x such that $f(x) > 0$

$$\begin{aligned} h(f_\varepsilon(x)) &= h(f(x) + \varepsilon)\chi_{\{0 < f(x) \le \kappa/2\}} + h(f(x) + \varepsilon)\chi_{\{f(x) > \kappa/2\}} \\ &\le h(f(x))\chi_{\{0 < f(x) \le \kappa/2\}} + h(f(x))\frac{h(f(x) + \varepsilon)}{h(f(x))}\chi_{\{f(x) > \kappa/2\}} \\ &\le h(f(x))\chi_{\{0 < f(x) \le \kappa/2\}} + D h(f(x))\chi_{\{f(x) > \kappa/2\}} \\ &\le D h(f(x)), \end{aligned}$$

where

$$D := \frac{\sup\{h(t) : t \in [\kappa/2, \|f\|_\infty + \kappa/2]\}}{\inf\{h(t) : t \in [\kappa/2, \|f\|_\infty]\}}$$

We will verify that for every x

$$\sqrt[p]{|\mathcal{T}(f_\varepsilon, x)|}\, h(f_\varepsilon(x)) \le K \sqrt[p]{h(f(x))},$$

with constant K independent on ε. Indeed, this follows from sequence of estimations

$$\begin{aligned}
&\left(\left|\int_{-\infty}^{x} \Phi_p(h(f_\varepsilon(y))f'(y))\,dy\right| \cdot h(f_\varepsilon(x))\right)^{\frac{1}{p}} \\
&\le \left(\|f'\|_\infty^{p-1} \int_{f'\neq 0} h^{p-1}(f_\varepsilon(y))\,dy \cdot h(f_\varepsilon(x))\right)^{\frac{1}{p}} \\
&\le \|f'\|_\infty^{1-\frac{1}{p}}\left(\int_{f'\neq 0} D^{p-1}h^{p-1}(f(y))\,dy \cdot Dh(f(x))\right)^{\frac{1}{p}} \\
&\le \left\{D\|f'\|_\infty^{1-\frac{1}{p}}\left(\int_{f'\neq 0} h^{p-1}(f(y))\,dy\right)^{\frac{1}{p}}\right\} \cdot (h(f(x)))^{\frac{1}{p}} \\
&=: K\sqrt[p]{h(f(x))}.
\end{aligned}$$

Using this estimation and the Δ_2 condition for A we get

$$A\left(\sqrt[p]{|\mathcal{M}f''(x)\mathcal{T}_{h,p}(f_\varepsilon, x)|\, h(f_\varepsilon(x))}\right) \le EA\left(\sqrt[p]{|\mathcal{M}f''(x)h(f(x))}\right) \in L^1(\mathbb{R})$$

Therefore we can use Lebesgue's Dominated Convergence Theorem to justify properties from (4.11). □

The following statements follow from careful analysis of the proof of Theorem 3.1 and its modifications. Our first statement contributes to the analysis to the case $p = d_A$, which was not covered by Theorem 3.1.

THEOREM 4.1. *Let* $1 < p < \infty$. *Then*

i) *For every nonnegative* $f \in W^{2,1}_{\text{loc}}(\mathbb{R})$ *such that* f' *is compactly supported, we have*

$$\begin{aligned}
\int_{\mathbb{R}} |f'(x)|^p\,dx &\le \int_{\mathbb{R}} |f''(x)| \int_{-\infty}^{x} |f'(y)|^{p-1}\,dy\,dx \\
&\le \int_{\mathbb{R}} |f'(x)|^{p-1}\,dx \int_{\mathbb{R}} |f''(x)|\,dx;
\end{aligned}$$

ii) *Assume that the function* $h : (0,\infty) \to (0,\infty)$ *is as in Definition* 2.1 *and satisfies* (h). *Moreover, let* $f \in W^{2,1}_{\text{loc}}(\mathbb{R})$ *be nonnegative and such that* f' *is compactly supported,* $(h(f(\cdot)))^{p-1} \in L^1_{\text{loc}}(\mathbb{R})$, *and additionally that* $\int_{\mathbb{R}} \mathcal{M}f''(x)\,dx < \infty$, *when* $h(\cdot)$ *is bounded in some neighborhood of zero and that*

$$\int_{\mathbb{R}} \mathcal{M}f''(x)h(f(x))\,dx < \infty$$

when $h(\cdot)$ is unbounded and strictly increasing near zero. Then we have

$$\int_{\mathbb{R}\cap\{f(x)>0\}} |f'(x)|^p h(f(x))^p\,dx \le C\int_{\mathbb{R}\cap\{f(x)>0\}} \mathscr{M}f''(x)\cdot|\mathscr{T}_{h,p}(f,x)|h(f(x))^p\,dx,$$

where $C = 1 + 2C_h$.

Remark 4.3. Since from the assumptions of the theorem it follows that f'' is compactly supported, the condition $\int_{\mathbb{R}} \mathscr{M}f''(x)\,dx < \infty$ is equivalent to the fact that $\int_R |f''(x)|(\ln|f''(x)|)^+\,dx < \infty$, by celebrated Stein inequality.

Proof.

i) This follows immediately from (4.5) after we note that for $A(t) = t^p$ we have $A_{p-1}(t) = t$, so that $(A_{p-1}(f'))' = f''$.
ii) We note that (4.8) reads as

$$I(\varepsilon) = -\int_{\mathbb{R}} (f_\varepsilon' h(f_\varepsilon))'\cdot\mathscr{G}_{h,p}(f_\varepsilon,x)\,dx \quad \text{and}$$
$$(f_\varepsilon' h(f_\varepsilon))' = f''h(f_\varepsilon) + (f')^2 h'(f_\varepsilon).$$

This, the condition (h), the fact that for almost every x in $\{x : f'(x) = 0\}$ we have $f''(x) = 0$ and that $\{x : f'(x) \ne 0\} \subseteq \{x : f(x) > 0\}$ give

$$\begin{aligned} I(\varepsilon) &\le \int_{\mathbb{R}} |f''h(f_\varepsilon)|\,|\mathscr{G}_{h,p}(f_\varepsilon,x)|\,dx + C_h\int_{\mathbb{R}\cap\{f'(x)\ne 0\}} \frac{(f_\varepsilon')^2}{f_\varepsilon} h(f_\varepsilon)|\mathscr{G}_{h,p}(f_\varepsilon,x)|\,dx \\ &\overset{(4.10)}{\le} \int_{\mathbb{R}} |f''h(f_\varepsilon)|\,|\mathscr{G}_{h,p}(f_\varepsilon,x)|\,dx + 2C_h\int_{\mathbb{R}\cap\{f'(x)\ne 0\}} \mathscr{M}f''\cdot h(f_\varepsilon)|\mathscr{G}_{h,p}(f_\varepsilon,x)|\,dx \\ &\le C\int_{\mathbb{R}\cap\{f(x)>0\}} \mathscr{M}f''\cdot|\mathscr{T}_{h,p}(f_\varepsilon,x)|h(f_\varepsilon)^p\,dx, \end{aligned}$$

where $C = 1 + 2C_h$. When letting $\varepsilon \to 0$ and applying the same arguments as at the end of the proof of Theorem 3.1, we get

$$\int_{\mathbb{R}\cap\{f(x)>0\}} |f'(x)|^p h(f(x))^p\,dx \le C\int_{\mathbb{R}\cap\{f(x)>0\}} \mathscr{M}f''(x)\cdot|\mathscr{T}_{h,p}(f,x)|h(f(x))^p\,dx,$$

where $C = 1 + 2C_h$, under the suitable assumptions on f. □

Precise analysis of the case $A(t) = t^q$ in Theorem 3.1, together with the more precise estimates give the following result.

THEOREM 4.2. *For any $p, q \in \mathbb{R}$ where $1 < p < q < \infty$, we have*

i) *For every nonnegative $f \in W^{2,1}_{\mathrm{loc}}(\mathbb{R})$ such that f' is compactly supported, we have*

$$\int_{\mathbb{R}} |f'(x)|^q \, dx \le D(p,q) \int_{\mathbb{R}} |f''(x)|^{\frac{q}{p}} \Big(\int_{-\infty}^{x} |f'(y)|^{p-1} \, dy \Big)^{\frac{q}{p}} dx \tag{4.12}$$
$$\le D(p,q) \Big(\int_{\mathbb{R}} |f''(x)|^{\frac{q}{p}} dx \Big) \Big(\int_{\mathbb{R}} |f'(x)|^{p-1} \, dx \Big)^{\frac{q}{p}},$$

where $D(p,q) = (q - p + 1)^{\frac{q}{p}}$;

ii) *Let $\alpha \in \mathbb{R} \setminus \{0\}$, $f \in W^{2,1}_{\mathrm{loc}}(\mathbb{R})$ be nonnegative and such that f' is compactly supported, $(f(\cdot))^{\alpha(p-1)} \in L^1_{\mathrm{loc}}(\mathbb{R})$, and additionally that $\int_{\mathbb{R}} |f''(x)|^{\frac{q}{p}} dx < \infty$ when $\alpha > 0$ and that $\int_{\mathbb{R}} (\mathscr{M} f''(x)(f(x))^{\alpha})^{\frac{q}{p}} dx < \infty$ when $\alpha < 0$. Then*

$$\int_{\mathbb{R}} |f'(x)|^q (f(x))^{\alpha} \, dx \tag{4.13}$$
$$\le D(p,q,\alpha) \int_{\mathbb{R}} (\mathscr{M} f''(x))^{\frac{q}{p}} \Big(\int_{-\infty}^{x} |f'(y)|^{p-1} (f(y))^{\alpha(p-1)} \, dy \Big)^{\frac{q}{p}} f(x)^{\frac{\alpha q}{p}} dx$$
$$\le D(p,q,\alpha) \Big(\int_{\mathbb{R}} (|f'(y)|(f(y))^{\alpha})^{p-1} \, dy \Big)^{\frac{q}{p}} \int_{\mathbb{R}} (\mathscr{M} f''(y)(f(y))^{\alpha})^{\frac{q}{p}} dy,$$

where $D(p,q,\alpha) = [(q - p + 1)(1 + 2|\alpha|)]^{\frac{p}{q}}$.

PROOF.

i) We apply the identity (4.6) and we note that in our case $D_A = d_A = q$. This together with Hölder inequality gives

$$\int_{\mathbb{R}} |f'(x)|^q \, dx$$
$$\le (q - p + 1) \int_{\mathbb{R}} |f'(x)|^{q-p} \cdot (|f''(x)| \mathscr{T}_{h \equiv 1, p}(f, x)) \, dx$$
$$\le (q - p + 1) \Big(\int_{\mathbb{R}} |f'(x)|^q \, dx \Big)^{1 - \frac{p}{q}} \Big(\int_{\mathbb{R}} (\sqrt[p]{|f''(x)| \mathscr{T}_{h \equiv 1, p}(f, x)})^q \, dx \Big)^{\frac{p}{q}},$$

which implies (4.12).

ii) We consider $h(t) = t^\alpha$, where $\alpha \in \mathbb{R}\setminus\{0\}$ in part ii) of Theorem 3.1. Then

$$\mathscr{T}_{|t|^\alpha,p}(f,x) = \int_{-\infty}^{x} (f(y))^{\alpha(p-1)}\Phi_p(f'(y))\,dy,$$

$$\mathscr{G}_{|t|^\alpha,p}(f,x) = \frac{1}{(f(x))^{\alpha(p-1)}}\int_{-\infty}^{x} (f(y))^{\alpha(p-1)}\Phi_p(f'(y))\,dy \quad \text{for } f(x) > 0.$$

To obtain (4.13), we apply (4.9) and note that in our case $|h'(t)| = |\alpha|\frac{h(t)}{t}$. According to the notation in (4.9) this gives

$$\begin{aligned} I(\varepsilon) &= \int_r |f'(x)|^q (f_\varepsilon(x))^{\alpha q}\,dx \\ &\le (q-p+1)\bigg\{\int_{\mathbb{R}} |f'(f_\varepsilon)^\alpha|^{q-p}|f''(f_\varepsilon)^\alpha \cdot \mathscr{G}_{|\cdot|^\alpha,p}(f_\varepsilon,x)\,dx \\ &\qquad + |\alpha|\int_{\mathbb{R}} |f'(f_\varepsilon)^\alpha|^{q-p}|\frac{(f')^2}{f_\varepsilon}(f_\varepsilon)^\alpha \cdot \mathscr{G}_{|t|^\alpha,p}(f_\varepsilon,x)\,dx\bigg\}. \end{aligned}$$

Applying inequality $|f''| \le \mathscr{M}f''$ and (4.10), we get

$$I(\varepsilon) \le (q-p+1)(1+2|\alpha|)\int_{\mathbb{R}} (|f'(f_\varepsilon)^\alpha|)^{q-p} \cdot (|\mathscr{M}f''(f_\varepsilon)^\alpha \cdot \mathscr{G}_{|t|^\alpha,p}(f_\varepsilon,x))\,dx,$$

which together with Hölder inequality, and (4.11), implies (4.13). □

Remark 4.4. First inequality in (4.12) is the variant of mixed norm Gagliardo–Nirenberg type inequality and seems to us new.

Remark 4.5. It would be interesting to know whether constants in the inequalities (4.12) and (4.13) are optimal.

Our next statement does not involve second derivatives and is the strong variant of the Gagliardo–Nirenberg inequality.

Theorem 4.3. *Let us consider an arbitrary compactly supported function $g \in W^{1,1}(\mathbb{R})$. Then*

i) *For any $1 < p < q < \infty$ we have*

$$\int_{\mathbb{R}} |g(x)|^q\,dx \le (q-p+1)^{\frac{q}{p}}\int_{\mathbb{R}} |g'(x)|^{\frac{q}{p}}\Big(\int_{-\infty}^{x} |g(y)|^{p-1}\,dy\Big)^{\frac{q}{p}}\,dx. \tag{4.14}$$

ii) *When A satisfy* (2.4) *and $d_A > 1$, $1 < p < i_A$, then*

$$\int_{\mathbb{R}} A(|g(y)|)\,dy \le C_{A,p}\int_{\mathbb{R}} A\left(\sqrt[p]{|g'(x)|\int_{-\infty}^{x}|g(y)|^{p-1}\,dy}\right)dx.$$

where the constant $C_{A,p}$ does not depend on g.

Proof.

i) The function $f(x) := \int_{-\infty}^{x}|g(y)|\,dy$ is nonnegative and belongs to $W^{2,1}_{\mathrm{loc}}(\mathbb{R})$, $f'(x) = |g(x)|$ has bounded support and $|f''(x)| = |g'(x)|$ a.e.. Inequality (4.12) leads to (4.14).
ii) We use the same substitution as before to the statement i) in Theorem 3.1. □

Remark 4.6. Note that (4.14) involves the (nonlinear) nonlocal operator $\int_{-\infty}^{x}|g(y)|^{p-1}\,dy$ and that it implies the following inequality:

$$\left(\int_{\mathbb{R}}|g(x)|^q\,dx\right)^{\frac{1}{q}} \le (q-p+1)^{\frac{1}{p}}\left(\int_{\mathbb{R}}|g'(x)|^{\frac{q}{p}}\,dx\right)^{\frac{1}{q}}\left(\int_{\mathbb{R}}|g(x)|^{p-1}\,dx\right)^{\frac{1}{p}}.$$

The substitution of $r := \frac{q}{p} > 1$ and $s := p-1 > 1$ implies the Gagliardo–Nirenberg type inequality:

$$\|g\|_{L^q(\mathbb{R})} \le \tilde{D}(r,s)\|g'\|^{\alpha}_{L^r(\mathbb{R})}\|g\|^{1-\alpha}_{L^s(\mathbb{R})}, \tag{4.15}$$

where $\tilde{D}(r,s) = (s(r-1)+r)^{\frac{1}{s+1}}$, $q = r(s+1)$ and $\alpha = \frac{1}{s+1}$. Note that in particular $\frac{1}{q} = (\frac{1}{r}-1)\alpha + \frac{1}{s}(1-\alpha)$.

The general Gagliardo–Nirenberg type inequalities in dimension n have the form:

$$\|D^j u\|_{L^q(\mathbb{R}^n)} \le C\|D^m u\|^{\alpha}_{L^r(\mathbb{R}^n)}\|u\|^{1-\alpha}_{L^s(\mathbb{R}^n)},$$

where $\frac{1}{q} = \frac{j}{n} + \left(\frac{1}{r} - \frac{m}{n}\right)\alpha + \frac{1}{s}(1-\alpha)$, $u : \mathbb{R}^n \to \mathbb{R}$, $1 \le s, r \le \infty$, $j, m \in \mathbb{N}$, $\frac{j}{m} \le \alpha \le 1$. We deal with the case $j = 0$, $m = n = 1$.

Remark 4.7. Let us consider inequality (4.15) in the case $r = s = 2$, $q = 6$. According to our result it holds with constant $\tilde{D}(2,2) = 4^{1/3}$. This inequality is known to hold with optimal constant $C_{GN}(6) = \pi^{1/3}$. Indeed, for this we use results of Section 3.1. in [7] and the notation there, showing that the optimal constant $C_{GN}(6) = \left(\frac{C_1(6)}{c(6)}\right)^{\frac{16}{24}}$, where

$$C_1(6) = \left[\frac{8^{\frac{8}{16}}}{4^{\frac{4}{16}}4^{\frac{4}{16}}} \cdot I_2^{\frac{2\cdot 4}{16}}\right] = 2I_2^{\frac{1}{2}} = 2\sqrt{\pi}, \quad I_2 = \frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{8}{8}\right)} = \pi \quad \left(\text{as } \Gamma\left(\frac{1}{2}\right) = \pi\right),$$

$$c(6) = \left(\frac{8}{2}\right)^{\frac{2\cdot 4}{16}} = 2, \quad \text{so that}$$

$$C_{GN}(6) = \pi^{\frac{1}{3}}.$$

Hence, our constant $\tilde{D}(2,2)$ is not optimal in that inequality. On the other hand, our inequality (4.15) follows as a consequence of stronger inequality (4.14), so perhaps it explains the nonoptimality of the derived constant.

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