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Solid Mechanics — *Uniqueness and stability in triple porosity thermoelasticity*, by Brian Straughan, communicated on November 11, 2016.[1]

This paper is dedicated to the memory of Professor Giuseppe Grioli.

Abstract. — A general model for a triple porosity thermoelastic solid is presented in the linear anisotropic case. This allows for cross coupling of inertia coefficients, and cross coupling of interaction coefficients representing actions between pressures in the macro, meso and micro structures. Sufficient conditions are derived to demonstrate uniqueness and stability when the elastic coefficients are, in a precise sense, positive. Uniqueness is further demonstrated in the dynamical problem when the elastic coefficients are not sign-definite and possess only the major symmetry. An indication is given as to how one would proceed to obtain continuous dependence upon the inital data in the Hölder sense. The proof of uniqueness in the indefinite elasticity tensor case involves a logarithmic convexity method which proceeds by a novel choice of functional.

Key words: Thermoelasticity, triple porosity, stability, uniqueness

Mathematics Subject Classification: 74F99, 74F05, 74B99

1. Introduction

Triple porosity elastic materials are the subject of intense current research. Interest is driven mainly by the many engineering applications of this topic. A triply porous elastic body is a solid with pores on the macro scale, pores on a much smaller meso scale, and pores (also called fissures or cracks) at an altogether smaller scale known as micro pores. The notion of macro, meso and micro porosity may change depending on the application and sometimes these are known, respectively, as matrix porosity, cavern porosity, and fracture porosity, or matrix, fracture, and channel components. Among the many possible application areas for triple porosity elasticity we mention recovery of oil from an underground reservoir, see e.g. Bai et al. [4], Bai & Roegiers [5], Wang et al. [52], Ali et al. [2], Aguilera & Aguilera [1], Olusola et al. [33], Deng et al. [12]; recovering methane gas from an underground coal bed, see e.g. Zou et al. [58], Wei & Zhang [53]; involvement in fuel cell technology, see e.g. Yuan & Sundén [55]; provision of good drinking water from a carbonate aquifer, see e.g. Zuber & Motyka

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[59], Ghasemizadeh et al. [16]; design of membrane – based bioartifical livers, Dufresne et al. [13]; hydraulic fracturing (fracking) of underground rocks to recover natural gas from shale gas reservoirs, see e.g. Huang et al. [21], Zhao et al. [56], Kim & Moridis [25]; analysing interstitial fluid flow in bones or bone replacement/recovery technology, see e.g. Svanadze & Scalia [50, 51], Sakamoto & Matsumoto [40], Zhou et al. [57]. In fact, replacing long bones which are damaged (in human beings) is a major problem for surgery since the porosity can vary from 14% in the outer layer of the bone to 52% in the inner layer and multi – porosity theory may need to cater even for graded porosity materials, cf. Zhou et al. [57].

In this article we present and analyse a partial differential equation model for a triple porosity thermoelastic medium. The inclusion of thermal effects is very important since they can induce micro cracking in rocks such as granite, see e.g. Homand-Etienne & Houpert [20], David et al. [10], Siratovich et al. [44].

The model we present allows for cross coupling of inertia effects between the macro pressure, meso pressure, micro pressure, and the temperature field. We also allow cross interaction between macro and meso scales, meso and micro scales, and the often neglected macro and micro scales. As Svanadze [49] points out, Khalili [24] has demonstrated that neglecting coupling even in a double porosity elastic system can lead to loss of important information and the deformation cannot be simulated correctly.

Throughout the paper we employ standard indicial notation together with the Einstein summation convention. For example, a subscript $,a$ denotes $\partial/\partial x_a$, and a superposed dot denotes $\partial/\partial t$, where t is time and x_a, $a = 1, 2, 3$, is a spatial point. Let $u_i(\mathbf{x}, t)$ denote the elastic displacement, $p(\mathbf{x}, t)$ denote the pressure in the macro pores, $q(\mathbf{x}, t)$ denote the pressure in the meso pores, $s(\mathbf{x}, t)$ denote the pressure in the micro pores, and let $\theta(\mathbf{x}, t)$ denote the temperature in the body. Then, generalizing the approach in Svanadze [49], the linear anisotropic equations for a triple porosity elastic material are written as

$$
\begin{aligned}
&\rho\ddot{u}_i = (a_{ijkh}u_{k,h})_{,j} - (\beta_{ij}p)_{,j} - (\gamma_{ij}q)_{,j} - (\omega_{ij}s)_{,j} - (a_{ij}\theta)_{,j} + \rho f_i,\\
&\alpha\dot{p} + \alpha_1\dot{q} + \alpha_2\dot{s} + \alpha_3\dot{\theta} = (k_{ij}p_{,j})_{,i} - \beta_{ij}\dot{u}_{i,j} - \gamma(p-q) - \omega(p-s),\\
&\alpha_1\dot{p} + \beta\dot{q} + \beta_1\dot{s} + \beta_2\dot{\theta} = (m_{ij}q_{,j})_{,i} - \gamma_{ij}\dot{u}_{i,j} + \gamma(p-q) - \xi(q-s),\\
&\alpha_2\dot{p} + \beta_1\dot{q} + \varepsilon\dot{s} + \varepsilon_1\dot{\theta} = (\ell_{ij}s_{,j})_{,i} - \omega_{ij}\dot{u}_{i,j} + \xi(q-s) + \omega(p-s),\\
&\alpha_3\dot{p} + \beta_2\dot{q} + \varepsilon_1\dot{s} + a\dot{\theta} = (r_{ij}\theta_{,j})_{,i} - a_{ij}\dot{u}_{i,j} + \rho r.
\end{aligned}
\tag{1}
$$

In these equations $\rho = \rho(\mathbf{x}) > 0$ is the density, $a_{ijkh}(\mathbf{x})$ are the elastic coefficients, β_{ij}, γ_{ij}, ω_{ij} and a_{ij} are coupling coefficients and they are all symmetric tensors. Likewise k_{ij}, m_{ij}, ℓ_{ij} and r_{ij} are all symmetric tensors. The inertia coefficients $\alpha, \alpha_1, \ldots, a$ and the interaction coefficients γ, ω, ξ all depend on $\mathbf{x}$ and further information is provided in section 2 of this article. The elastic coefficients are required to be symmetric in the sense that

$$
a_{ijkh} = a_{khij} = a_{jikh}. \tag{2}
$$

The term $f_i(\mathbf{x}, t)$ represents a prescribed body force whereas $r(\mathbf{x}, t)$ is a prescribed heat source. The effect of a heat source may well be important in triple porosity thermoelasticity as it could contribute to inducing internal thermal stresses.

We point out that Svanadze [49] presents an analogous system to (1) but for a *double* porosity body which is *isotropic*. He does, however, include temperature and cross inertia effects, see also Scarpetta et al. [43], Scarpetta & Svanadze [42]. In addition we observe that cross interaction effects are included in tridisperse materials where internal fluid flow is considered in a rigid skeleton by Kuznetsov & Nield [29], Nield & Kuznetsov [32], see also Straughan [48], pp. 203, 204.

In the next section we establish uniqueness for a solution to a boundary initial value problem for equations (1) using an energy technique. This is followed by a section where we establish continuous dependence on the initial data, continuous dependence on the body force, and continuous dependence on the heat source. In the following section we strongly relax the conditions on the elastic tensor and demonstrate uniqueness by a logarithmic convexity method. It is worth noting that Straughan [47] employs a logarithmic convexity method to establish continuous dependence and uniqueness for a solution to a model for double porosity elasticity. However, the model adopted by Straughan [47] does not have temperature nor cross coupling inertia terms.

2. Non-negative elastic coefficients

Throughout the paper we let Ω denote a bounded domain in $\mathbb{R}^3$ with boundary Γ sufficiently smooth to allow application of the divergence theorem. We shall now derive sufficient conditions to establish uniqueness and stability of a solution to the basic equations (1). We suppose that (u_i, p, q, s, θ) is a solution to equations (1) in the domain $\Omega \times (0, T)$ where $T < \infty$ is a fixed time. This solution is required to satisfy the boundary conditions

$$\begin{aligned} &u_i(\mathbf{x}, t) = u_i^B(\mathbf{x}, t), \quad p(\mathbf{x}, t) = p^B(\mathbf{x}, t), \quad q(\mathbf{x}, t) = q^B(\mathbf{x}, t), \\ &s(\mathbf{x}, t) = s^B(\mathbf{x}, t), \quad \theta(\mathbf{x}, t) = \theta^B(\mathbf{x}, t), \quad \mathbf{x} \in \Gamma \times (0, T), \end{aligned} \tag{3}$$

where u_i^B, p^B, q^B, s^B, and θ^B are prescribed. In addition, we require the solution (u_i, p, q, s, θ) to satisfy the initial conditions

$$\begin{aligned} &u_i(\mathbf{x}, 0) = v_i(\mathbf{x}), \quad \dot{u}_i(\mathbf{x}, 0) = w_i(\mathbf{x}), \quad p(\mathbf{x}, 0) = P(\mathbf{x}), \\ &q(\mathbf{x}, 0) = Q(\mathbf{x}), \quad s(\mathbf{x}, 0) = S(\mathbf{x}), \quad \theta(\mathbf{x}, 0) = \Theta(\mathbf{x}), \quad \mathbf{x} \in \Omega, \end{aligned} \tag{4}$$

where the functions v_i, w_i, P, Q, S and Θ are prescribed. Denote the boundary initial value problem for equations (1) together with (3) and (4) by $\mathscr{P}$.

We suppose that the coefficients k_{ij}, m_{ij}, ℓ_{ij} and r_{ij} are functions of $\mathbf{x}$ with

$$k_{ij}\xi_i\xi_j \geq 0, \quad m_{ij}\xi_i\xi_j \geq 0, \quad \ell_{ij}\xi_i\xi_j \geq 0, \quad r_{ij}\xi_i\xi_j \geq 0, \tag{5}$$

for all ξ_i. In addition, the coupling coefficients β_{ij}, γ_{ij}, ω_{ij} and a_{ij} are functions of $\mathbf{x}$ while the interaction coefficients γ, ω and ξ are positive functions of $\mathbf{x}$.

Likewise, the inertia coefficients α, β, ε, a, α_1, α_2, α_3, β_1, β_2 and ε_1 are functions of the spatial variable $\mathbf{x}$. These coefficients are such that $\alpha > 0$, $\beta > 0$, $\varepsilon > 0$ and $a > 0$ together with the requirement that the symmetric matrix

$$A = \begin{pmatrix} \alpha & \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \beta & \beta_1 & \beta_2 \\ \alpha_2 & \beta_1 & \varepsilon & \varepsilon_1 \\ \alpha_3 & \beta_2 & \varepsilon_1 & a \end{pmatrix}$$

is positive – definite. Thus, there exist positive constants $k_1, \ldots, k_4$ such that

$$\mathbf{y}^T A \mathbf{y} \geq k_1^2 y_1^2 + k_2 y_2^2 + k_3 y_3^2 + k_4 y_4^2. \tag{6}$$

In this section we shall also suppose that the elastic coefficients a_{ijkh} are functions of $\mathbf{x}$ and

$$a_{ijkh}\xi_{ij}\xi_{kh} \geq a_0 \xi_{ij}\xi_{ij} \tag{7}$$

for all ξ_{ij}, where $a_0 \geq 0$ to establish uniqueness and $a_0 > 0$ when we are answering the stability question.

To establish uniqueness for a solution to $\mathscr{P}$ we let the functions $(u_i^1, p^1, q^1, s^1, \theta^1)$ and $(u_i^2, p^2, q^2, s^2, \theta^2)$ be two solutions to $\mathscr{P}$ for the same boundary data functions h_i^B, p^B, q^B, s^B, θ^B, for the same initial data functions v_i, w_i, P, Q, S, Θ, and for the same body force f_i and for the same heat source r. Define the difference variables u_i, p, q, s and θ by the relations

$$\begin{aligned} &u_i = u_i^1 - u_i^2, \quad p = p^1 - p^1, \quad q = q^1 - q^2, \\ &s = s^1 - s^2, \quad \theta = \theta^1 - \theta^2. \end{aligned} \tag{8}$$

Then by subtraction, one sees from equations (1), (3) and (4), that the solution (u_i, p, q, s, θ) satisfies the boundary initial value problem

$$\begin{aligned} &\rho \ddot{u}_i = (a_{ijkh}u_{k,h})_{,j} - (\beta_{ij}p)_{,j} - (\gamma_{ij}q)_{,j} - (\omega_{ij}s)_{,j} - (a_{ij}\theta)_{,j}, \\ &\alpha\dot{p} + \alpha_1\dot{q} + \alpha_2\dot{s} + \alpha_3\dot{\theta} = (k_{ij}p_{,j})_{,i} - \gamma(p-q) - \omega(p-s) - \beta_{ij}\dot{u}_{i,j}, \\ &\alpha_1\dot{p} + \beta\dot{q} + \beta_1\dot{s} + \beta_2\dot{\theta} = (m_{ij}q_{,j})_{,i} + \gamma(p-q) - \xi(q-s) - \gamma_{ij}\dot{u}_{i,j}, \\ &\alpha_2\dot{p} + \beta_1\dot{q} + \varepsilon\dot{s} + \varepsilon_1\dot{\theta} = (\ell_{ij}s_{,j})_{,i} + \omega(p-s) + \xi(q-s) - \omega_{ij}\dot{u}_{i,j}, \\ &\alpha_3\dot{p} + \beta_2\dot{q} + \varepsilon_1\dot{s} + a\dot{\theta} = (r_{ij}\theta_{,j})_{,i} - a_{ij}\dot{u}_{i,j}, \end{aligned} \tag{9}$$

where now the boundary conditions are

$$\begin{aligned} &u_i(\mathbf{x}, t) = 0, \quad p(\mathbf{x}, t) = 0, \quad q(\mathbf{x}, t) = 0, \\ &s(\mathbf{x}, t) = 0, \quad \theta(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \Gamma, \end{aligned} \tag{10}$$

and the initial conditions are

$$
\begin{aligned}
&u_i(\mathbf{x},0)=0, \quad \dot{u}_i(\mathbf{x},0)=0, \quad p(\mathbf{x},0)=0,\\
&q(\mathbf{x},0)=0, \quad s(\mathbf{x},0)=0, \quad \theta(\mathbf{x},0)=0, \quad \mathbf{x}\in\Omega.
\end{aligned} \tag{11}
$$

To establish uniqueness of a solution to $\mathscr{P}$ under the conditions (5) and (7) we multiply equation $(9)_1$ by $\dot{u}_i$ and integrate over Ω. Then we multiply equations $(9)_2$–$(9)_5$, respectively, by p, q, s and θ and integrate each over Ω. We integrate by parts, use the boundary conditions, and add the results of each together. Let us denote by $\langle\cdot\rangle$ integration over Ω, e.g. $\langle f\rangle = \int_\Omega f\,dx$. Then we define an energy function $E(t)$ by

$$
\begin{aligned}
E(t) = {}& \frac{1}{2}\langle\rho\dot{u}_i\dot{u}_i\rangle + \frac{1}{2}\langle a_{ijkh}u_{i,j}u_{k,h}\rangle + \frac{1}{2}\langle\alpha p^2\rangle + \frac{1}{2}\langle\beta q^2\rangle\\
&+ \frac{1}{2}\langle\varepsilon s^2\rangle + \frac{1}{2}\langle a\theta^2\rangle + \langle\alpha_1 pq\rangle + \langle\alpha_2 sp\rangle\\
&+ \langle\alpha_3\theta p\rangle + \langle\beta_1 qs\rangle + \langle\beta_2 q\theta\rangle + \langle\varepsilon_1\theta s\rangle.
\end{aligned} \tag{12}
$$

By proceeding in the manner outlined above we may now show that $E(t)$ satisfies the energy equation

$$
\begin{aligned}
E(t) &+ \int_0^t \langle k_{ij}p_{,i}p_{,j}\rangle\,da + \int_0^t \langle m_{ij}q_{,i}q_{,j}\rangle\,da\\
&+ \int_0^t \langle \ell_{ij}s_{,i}s_{,j}\rangle\,da + \int_0^t \langle r_{ij}\theta_{,i}\theta_{,j}\rangle\,da\\
&+ \int_0^t \langle\gamma(p-q)^2\rangle\,da + \int_0^t \langle\omega(p-s)^2\rangle\,da\\
&+ \int_0^t \langle\xi(s-q)^2\rangle\,da = E(0),
\end{aligned} \tag{13}
$$

where da denotes integration with respect to time.

For uniqueness $E(0)=0$ and then from (13) we deduce by employing conditions (5) and (6) that

$$
p\equiv 0, \quad q\equiv 0, \quad s\equiv 0, \quad \theta\equiv 0, \quad \text{and} \quad \dot{u}_i\equiv 0,
$$

and so $u_i\equiv 0$ in $\Omega\times(0,T)$. Thus, uniqueness of a solution to $\mathscr{P}$ follows.

3. Stability

3.1. Continuous dependence upon the initial data

We commence with an analysis of stability in the sense of continuous dependence upon the initial data. Thus, suppose there are two solutions $(u_i^1, p^1, q^1, s^1, \theta^1)$ and

$(u_i^2, p^2, q^2, s^2, \theta^2)$ which satisfy equations (1) for the same body force f_i and for the same heat source r. These solutions also satisfy the boundary conditions (3) for the same functions u_i^B, p^B, q^B, s^B and θ^B. However, they are subject to different initial conditions in that $(u_i^1, p^1, q^1, s^1, \theta^1)$ is required to satisfy the initial conditions

$$\begin{aligned} &u_i^1(\mathbf{x},0) = v_i^1(\mathbf{x}), \quad \dot{u}_i^1(\mathbf{x},0) = w_i^1(\mathbf{x}), \quad p^1(\mathbf{x},0) = P^1(\mathbf{x}),\\ &q^1(\mathbf{x},0) = Q^1(\mathbf{x}), \quad s^1(\mathbf{x},0) = S^1(\mathbf{x}), \quad \theta^1(\mathbf{x},0) = \Theta^1(\mathbf{x}), \end{aligned} \tag{14}$$

for $\mathbf{x} \in \Omega$, whereas $(u_i^2, p^2, q^2, s^2, \theta^2)$ satisfies another set of initial conditions

$$\begin{aligned} &u_i^2(\mathbf{x},0) = v_i^2(\mathbf{x}), \quad \dot{u}_i^2(\mathbf{x},0) = w_i^2(\mathbf{x}), \quad p^2(\mathbf{x},0) = P^2(\mathbf{x}),\\ &q^2(\mathbf{x},0) = Q^2(\mathbf{x}), \quad s^2(\mathbf{x},0) = S^2(\mathbf{x}), \quad \theta^2(\mathbf{x},0) = \Theta^2(\mathbf{x}), \end{aligned} \tag{15}$$

for $\mathbf{x} \in \Omega$. In equations (14) and (15) the functions $v_i^1(\mathbf{x}), \dots, \Theta^2(\mathbf{x})$ are prescribed.

One again forms the difference solution (u_i, p, q, s, θ) as in (8) and one may verify that this solution satisfies the differential equations (9) and the boundary conditions (10). Let us now define the data functions v_i, w_i, P, Q, S and Θ by

$$\begin{aligned} &v_i(\mathbf{x}) = v^1(\mathbf{x}) - v_i^2(\mathbf{x}), \quad w_i(\mathbf{x}) = w_i^1(\mathbf{x}) - w_i^2(\mathbf{x}),\\ &P(\mathbf{x}) = P^1(\mathbf{x}) - P^2(\mathbf{x}), \quad Q(\mathbf{x}) = Q^1(\mathbf{x}) - Q^2(\mathbf{x}),\\ &S(\mathbf{x}) = S^1(\mathbf{x}) - S^2(\mathbf{x}), \quad \Theta(\mathbf{x}) = \Theta^1(\mathbf{x}) - \Theta^2(\mathbf{x}), \quad \mathbf{x} \in \Omega. \end{aligned} \tag{16}$$

Then, instead of satisfying the initial conditions (9) one sees that the difference solution (u_i, p, q, s, θ) is subject to the initial conditions

$$\begin{aligned} &u_i(\mathbf{x},0) = v_i(\mathbf{x}), \quad \dot{u}_i(\mathbf{x},0) = w_i(\mathbf{x}), \quad p(\mathbf{x},0) = P(\mathbf{x}),\\ &q(\mathbf{x},0) = Q(\mathbf{x}), \quad s(\mathbf{x},0) = S(\mathbf{x}), \quad \theta(\mathbf{x},0) = \Theta(\mathbf{x}), \quad \mathbf{x} \in \Omega. \end{aligned} \tag{17}$$

Thus, the difference solution (u_i, p, q, s, θ) in this section satisfies the boundary initial value problem $\mathscr{P}_1$ comprising equations (9), (10) and (17).

By proceeding as in the analysis leading to equation (13) one finds that this equation again holds, but now $E(0) > 0$. One may employ (5) and the fact that γ, ω and ξ are positive to then deduce from the analogous equation to (13) that

$$E(t) \le E(0) \tag{18}$$

where $E(t)$ is as given in (12) and $E(0) > 0$ is the constant given by

$$\begin{aligned} E(0) = &\frac{1}{2}\langle \rho w_i w_i \rangle + \frac{1}{2}\langle a_{ijkh} v_{i,j} v_{k,h} \rangle + \frac{1}{2}\langle \alpha P^2 \rangle + \frac{1}{2}\langle \beta Q^2 \rangle\\ &+ \frac{1}{2}\langle \varepsilon S^2 \rangle + \frac{1}{2}\langle a\Theta^2 \rangle + \langle \alpha_1 PQ \rangle + \langle \alpha_2 SP \rangle\\ &+ \langle \alpha_3 \Theta P \rangle + \langle \beta_1 QS \rangle + \langle \beta_2 Q\Theta \rangle + \langle \varepsilon_1 \Theta S \rangle. \end{aligned} \tag{19}$$

We next employ inequality (6) in inequality (18) to find

$$\frac{1}{2}\langle\rho\dot{u}_i\dot{u}_i\rangle+\frac{1}{2}\langle a_{ijkh}u_{i,j}u_{k,h}\rangle+k_1\|p\|^2+k_2\|q\|^2+k_3\|s\|^2+k_4\|\theta\|^2\le E(0), \tag{20}$$

where $\|\cdot\|$ is the norm on $L^2(\Omega)$. We then employ inequality (7) for a constant $a_0>0$ to obtain from inequality (20)

$$\frac{1}{2}\langle\rho\dot{u}_i\dot{u}_i\rangle+\frac{a_0}{2}\|\nabla\mathbf{u}\|^2+k_1\|p\|^2+k_2\|q\|^2+k_3\|s\|^2+k_4\|\theta\|^2\le E(0). \tag{21}$$

Inequality (21) establishes continuous dependence on the initial data for all $t>0$ in the L^2 measure for $\dot{\mathbf{u}}$, p, q, s, θ and in $H_0^1(\Omega)$ for $\mathbf{u}$.

3.2. *Continuous dependence upon the body force*

In this section we wish to consider continuous dependence of the solution to equations (1) upon changes in the body force, f_i, itself. This class of stability problem, where one effectively considers a change to the model itself, is known as structural stability. In fact, many writers believe structural stability is as important a concept as continuous dependence on the initial data. Certainly, structural stability features prominently in the classic text of Hirsch & Smale [19]. In the field of linear elastodynamics structural stability was analysed in depth by Knops & Payne [26] with significant improvements given by Knops & Payne [28]. It is also worth observing that structural stability occupies the whole of chapter 2 of the book by Straughan [45].

To investigate continuous dependence on the body force for a solution to equations (1) we let $(u_i^1,p^1,q^1,s^1,\theta^1)$ be a solution to equations (1) for a body force $f_i^1(\mathbf{x},t)$ and we let $(u_i^2,p^2,q^2,s^2,\theta^2)$ be a solution to equations (1) for a body force $f_i^2(\mathbf{x},t)$. We suppose that both solutions are subject to the same heat source $r(\mathbf{x},t)$ and both solutions satisfy the boundary conditions (3) and the initial conditions (4) for the *same* functions u_i^B, p^B, q^B, s^B, θ^B and v_i, w_i, P, Q, S and Θ.

We now define the difference body force f_i by

$$f_i(\mathbf{x},t)=f_i^1(\mathbf{x},t)-f_i^2(\mathbf{x},t). \tag{22}$$

By subtraction we see that the difference solution defined by (8) satisfies the boundary initial value problem

$$\begin{aligned}
&\rho\ddot{u}_i=(a_{ijkh}u_{k,h})_{,j}-(\beta_{ij}p)_{,j}-(\gamma_{ij}q)_{,j}-(\omega_{ij}s)_{,j}-(a_{ij}\theta)_{,j}+\rho f_i,\\
&\alpha\dot{p}+\alpha_1\dot{q}+\alpha_2\dot{s}+\alpha_3\dot{\theta}=(k_{ij}p_{,j})_{,i}-\gamma(p-q)-\omega(p-s)-\beta_{ij}\dot{u}_{i,j},\\
&\alpha_1\dot{p}+\beta\dot{q}+\beta_1\dot{s}+\beta_2\dot{\theta}=(m_{ij}q_{,j})_{,i}+\gamma(p-q)-\xi(q-s)-\gamma_{ij}\dot{u}_{i,j},\\
&\alpha_2\dot{p}+\beta_1\dot{q}+\varepsilon\dot{s}+\varepsilon_1\dot{\theta}=(\ell_{ij}s_{,j})_{,i}+\omega(p-s)+\xi(q-s)-\omega_{ij}\dot{u}_{i,j},\\
&\alpha_3\dot{p}+\beta_2\dot{q}+\varepsilon_1\dot{s}+a\dot{\theta}=(r_{ij}\theta_{,j})_{,i}-a_{ij}\dot{u}_{i,j},
\end{aligned} \tag{23}$$

together with

$$u_i(\mathbf{x},t)=0,\quad p(\mathbf{x},t)=0,\quad q(\mathbf{x},t)=0,$$
$$s(\mathbf{x},t)=0,\quad \theta(\mathbf{x},t)=0,\quad \mathbf{x}\in\Gamma, \tag{24}$$

and

$$u_i(\mathbf{x},0)=0,\quad \dot{u}_i(\mathbf{x},0)=0,\quad p(\mathbf{x},0)=0,$$
$$q(\mathbf{x},0)=0,\quad s(\mathbf{x},0)=0,\quad \theta(\mathbf{x},0)=0,\quad \mathbf{x}\in\Omega. \tag{25}$$

To analyse continuous dependence upon the body force we multiply equation $(23)_1$ by $\dot{u}_i$ and integrate over Ω. We then multiply equations $(23)_2$–$(23)_5$ each, respectively, by p, q, s and θ and integrate over Ω. After use of the boundary conditions (24) we add the results to find that

$$\frac{dE}{dt}+\langle k_{ij}p_{,i}p_{,j}\rangle+\langle m_{ij}q_{,i}q_{,j}\rangle+\langle \ell_{ij}s_{,i}s_{,j}\rangle+\langle r_{ij}\theta_{,i}\theta_{,j}\rangle$$
$$+\langle\gamma(p-q)^2\rangle+\langle\omega(p-s)^2\rangle+\langle\xi(s-q)^2\rangle=\langle\rho f_i\dot{u}_i\rangle, \tag{26}$$

where $E(t)$ is as defined in (12). To progress from this we employ the arithmetic–geometric mean inequality on the term on the right of (26) in the form

$$\langle\rho f_i\dot{u}_i\rangle\le\frac{1}{2}\langle\rho\dot{u}_i\dot{u}_i\rangle+\frac{1}{2}\langle\rho f_if_i\rangle. \tag{27}$$

We appeal to conditions (5) and discard terms on the left of (26) and then after using (27) in (26) we multiply by the integrating factor e^{-t} and integrate in t to obtain after using (6) the inequality

$$\frac{1}{2}\langle\rho\dot{u}_i\dot{u}_i\rangle+\frac{1}{2}\langle a_{ijkh}u_{i,j}u_{k,h}\rangle+\frac{1}{2}k_1\|p\|^2+\frac{1}{2}k_2\|q\|^2+\frac{1}{2}k_3\|s\|^2+\frac{1}{2}k_4\|\theta\|^2$$
$$\le\int_0^t e^{(t-a)}\langle\rho f_i(\mathbf{x},a)f_i(\mathbf{x},a)\rangle. \tag{28}$$

Finally, we employ (7) with $a_0>0$ to obtain

$$\frac{1}{2}\langle\rho\dot{u}_i\dot{u}_i\rangle+\frac{1}{2}a_0\|\nabla\mathbf{u}\|^2+\frac{1}{2}k_1\|p\|^2+\frac{1}{2}k_2\|q\|^2+\frac{1}{2}k_3\|s\|^2+\frac{1}{2}k_4\|\theta\|^2$$
$$\le\int_0^t e^{(t-a)}\langle\rho f_i(\mathbf{x},a)f_i(\mathbf{x},a)\rangle. \tag{29}$$

Inequality (29) is the required estimate and establishes continuous dependence of a solution to equations (1) upon the body force f_i.

3.3. Continuous dependence upon the heat source

In this section we analyse another problem of structural stability, namely, continuous dependence of a solution to equations (1) upon changes in the heat source $r(\mathbf{x}, t)$.

The procedure develops as in section 3.2 in that we let $(u_i^1, p^1, q^1, s^1, \theta^1)$ and $(u_i^2, p^2, q^2, s^2, \theta^2)$ be two solutions to equations (1) but now they satisfy the *same* body force, f_i, but different heat sources r^1 and r^2, respectively. The two solutions satisfy the same boundary and initial conditions. Thus, the difference solution defined by (8) again satisfies the boundary conditions (24) and initial conditions (25). However, the difference solution satisfies the partial differential equations

$$
\begin{aligned}
&\rho\ddot{u}_i = (a_{ijkh}u_{k,h})_{,j} - (\beta_{ij}p)_{,j} - (\gamma_{ij}q)_{,j} - (\omega_{ij}s)_{,j} - (a_{ij}\theta)_{,j},\\
&\alpha\dot{p} + \alpha_1\dot{q} + \alpha_2\dot{s} + \alpha_3\dot{\theta} = (k_{ij}p_{,j})_{,i} - \gamma(p-q) - \omega(p-s) - \beta_{ij}\dot{u}_{i,j},\\
&\alpha_1\dot{p} + \beta\dot{q} + \beta_1\dot{s} + \beta_2\dot{\theta} = (m_{ij}q_{,j})_{,i} + \gamma(p-q) - \xi(q-s) - \gamma_{ij}\dot{u}_{i,j},\\
&\alpha_2\dot{p} + \beta_1\dot{q} + \varepsilon\dot{s} + \varepsilon_1\dot{\theta} = (\ell_{ij}s_{,j})_{,i} + \omega(p-s) + \xi(q-s) - \omega_{ij}\dot{u}_{i,j},\\
&\alpha_3\dot{p} + \beta_2\dot{q} + \varepsilon_1\dot{s} + a\dot{\theta} = (r_{ij}\theta_{,j})_{,i} - a_{ij}\dot{u}_{i,j} + \rho r.
\end{aligned}
\tag{30}
$$

We commence as in section 3.2 and multiply each of $(30)_1$–$(30)_5$ by $\dot{u}_i$, p, q, s and θ, respectively, and integrate over Ω. This procedure now leads to the equation

$$
\begin{aligned}
\frac{dE}{dt} &+ \langle k_{ij}p_{,i}p_{,j}\rangle + \langle m_{ij}q_{,i}q_{,j}\rangle + \langle \ell_{ij}s_{,i}s_{,j}\rangle + \langle r_{ij}\theta_{,i}\theta_{,j}\rangle\\
&+ \langle \gamma(p-q)^2\rangle + \langle \omega(p-s)^2\rangle + \langle \xi(s-q)^2\rangle = \langle \rho\theta r\rangle.
\end{aligned}
\tag{31}
$$

We again require k_{ij}, m_{ij} and ℓ_{ij} to satisfy (5) but now we request that r_{ij} satisfy the stronger (but usually physically acceptable) condition that

$$
r_{ij}\xi_i\xi_j \geq r_0\xi_i\xi_i, \tag{32}
$$

for all ξ, and for a constant $r_0 > 0$. In this case we employ inequality (32) on the appropriate term in (31) and then we employ the arithmetic–geometric mean inequality to find

$$
\frac{dE}{dt} + r_0\|\nabla\theta\|^2 \leq \frac{1}{2b}\langle \theta^2\rangle + \frac{b}{2}\langle \rho^2 r^2\rangle, \tag{33}
$$

for a constant $b > 0$ at our disposal. We next employ Poincaré's inequality in the form $\|\nabla\theta\|^2 \geq \lambda_1\|\theta\|^2$ where $\lambda_1 > 0$ is the first eigenvalue in the membrane problem for Ω. We pick $b = 1/2r_0\lambda_1$ and then from (33) we derive

$$
\frac{dE}{dt} \leq \frac{b}{2}\langle \rho^2 r^2\rangle. \tag{34}
$$

Upon integration in time we utilize inequality (6) to find

$$\langle \rho \dot{u}_i \dot{u}_i \rangle + \langle a_{ijkh} u_{i,j} u_{k,h} \rangle + k_1 \|p\|^2 + k_2 \|q\|^2 + k_3 \|s\|^2 + k_4 \|\theta\|^2 \leq b \int_0^t \langle \rho^2 r^2 \rangle \, da. \tag{35}$$

Finally we employ (7) with $a_0 > 0$ to derive the estimate

$$\langle \rho \dot{u}_i \dot{u}_i \rangle + a_0 \|\nabla \mathbf{u}\|^2 + k_1 \|p\|^2 + k_2 \|q\|^2 + k_3 \|s\|^2 + k_4 \|\theta\|^2 \leq \frac{1}{2 r_0 \lambda_1} \int_0^t \langle \rho^2 r^2 \rangle \, da. \tag{36}$$

The estimate (36) is an *a priori* inequality which demonstrates continuous dependence of a solution to the triple porosity thermoelastic equations (1) upon changes in the heat source $r(\mathbf{x}, t)$.

4. Indefinite elastic coefficients

In this section we shall not require any definiteness conditions on the elasticity tensor a_{ijkh} and we suppose only symmetry as in (2).

Many modern, often man made, materials may well have elastic coefficients which are not sign-definite. For example, auxetic materials, Greaves et al. [17], Sanami et al. [41]; graded systems, Jou et al. [23]; chiral bodies, Lakes [30], Ha et al. [18], Iesan & Quintanilla [22]; some composites, Miller et al. [31], Greaves et al. [17]; may possess properties not normally associated with typical elastic behaviour. To amplify this we mention briefly isotropic linear elastodynamics where the equations are, see e.g. Knops & Payne [27], p. 16,

$$\rho \ddot{u}_i = \mu \Delta u_i + (\lambda + \mu) u_{j,ij}. \tag{37}$$

In these equations λ and μ are the Lamé and shear moduli, respectively, and they are connected to Poisson's ratio σ, and Young's modulus E, by, see Knops & Payne [27], p. 10,

$$\lambda = \frac{2\mu\sigma}{1 - 2\sigma}, \qquad \mu = \frac{E}{2(1+\sigma)}. \tag{38}$$

Many classical works on elasticity have required positive-definiteness of the elastic coefficients and for an isotropic elastic body in three dimensions this means λ and μ must satisfy, Knops & Payne [27], p. 19,

$$3\lambda + 2\mu > 0, \qquad \mu > 0, \tag{39}$$

which is equivalent to

$$-1 < \sigma < \frac{1}{2}, \quad \mu > 0. \tag{40}$$

A weaker condition is often employed, namely that of strong ellipticity and this requires, Knops & Payne [27], p. 21, Chirita & Ghiba [9],

$$\mu(\lambda + 2\mu) > 0, \tag{41}$$

which is equivalent to

$$-\infty < \sigma < \frac{1}{2}, \quad 1 < \sigma < \infty, \quad \mu \neq 0. \tag{42}$$

While Poisson's ratio is allowed to be negative by the criteria (40) and (42), it has often been argued in the past that it should always be positive. With the advent of many new materials it is now a known fact that Poisson's ratio may be negative, see Xinchun & Lakes [54], and other non-classical effects are often reported from experiments, cf. Greaves et al. [17], Ha et al. [18]. In view of the range of new effects being discovered in elasticity we believe it is desirable to analyse the case of indefinite elastic coefficients, especially in the context of a body with a triple porosity structure. Another very strong motivation for requiring indefiniteness of the elastic tensor is that when the equations are linearized about a state of nonlinear deformation then the elastic coefficients contain the effect of pre-stress and then it is not at all obvious why the elasticity tensor should be positive. This point is succinctly made on page 47 of the book by Straughan [46], cf. also Flavin & Green [15] and Flavin [14].

We again let $(u_i^1, p^1, q^1, s^1, \theta^1)$ and $(u_i^2, p^2, q^2, s^2, \theta^2)$ be solutions to $\mathscr{P}$ for the same body force, boundary conditions and initial conditions as in section 2 and define the difference solution (u_i, p, q, s, θ) as in (8). Thus, the difference solution (u_i, p, q, s, θ) satisfies equations (9), (10) and (11). We again require conditions (5) and (6). However, we relax the positivity condition (7) and require of a_{ijkh} only the symmetry condition (2).

To establish uniqueness we commence by introducing the functions $\eta(\mathbf{x}, t)$, $\zeta(\mathbf{x}, t)$, $\phi(\mathbf{x}, t)$ and $\psi(\mathbf{x}, t)$ by

$$\begin{aligned} \eta(\mathbf{x}, t) &= \int_0^t p(\mathbf{x}, a)\, da, \quad \zeta(\mathbf{x}, t) = \int_0^t q(\mathbf{x}, a)\, da, \\ \phi(\mathbf{x}, t) &= \int_0^t s(\mathbf{x}, a)\, da, \quad \psi(\mathbf{x}, t) = \int_0^t \theta(\mathbf{x}, a)\, da, \end{aligned} \tag{43}$$

where da denotes integration in time. We next introduce the functional $F(t)$ by

$$F(t) = \langle \rho u_i u_i \rangle + \int_0^t \langle k_{ij}\eta_{,i}\eta_{,j} \rangle\, da + \int_0^t \langle m_{ij}\zeta_{,i}\zeta_{,j} \rangle\, da + \int_0^t \langle \ell_{ij}\phi_{,i}\phi_{,j} \rangle\, da + \int_0^t \langle r_{ij}\psi_{,i}\psi_{,j} \rangle\, da + \int_0^t \langle \gamma(\eta-\zeta)^2 \rangle\, da + \int_0^t \langle \xi(\phi-\zeta)^2 \rangle\, da + \int_0^t \langle \omega(\eta-\phi)^2 \rangle\, da. \tag{44}$$

For the uniqueness case the initial data are zero and then one may differentiate F in (44) to see that

$$F'(t) = 2\langle \rho u_i \dot{u}_i \rangle + 2\int_0^t \langle k_{ij}\eta_{,i}p_{,j} \rangle\, da + 2\int_0^t \langle m_{ij}\zeta_{,i}q_{,j} \rangle\, da + 2\int_0^t \langle \ell_{ij}\phi_{,i}s_{,j} \rangle\, da + 2\int_0^t \langle r_{ij}\psi_{,i}\theta_{,j} \rangle\, da + 2\int_0^t \langle \gamma(\eta-\zeta)(p-q) \rangle\, da + 2\int_0^t \langle \xi(\phi-\zeta)(s-q) \rangle\, da + 2\int_0^t \langle \omega(\eta-\phi)(p-s) \rangle\, da. \tag{45}$$

After a further differentiation one obtains,

$$F''(t) = 2\langle \rho u_i \ddot{u}_i \rangle + 2\langle \rho \dot{u}_i \dot{u}_i \rangle + 2\langle k_{ij}\eta_{,i}p_{,j} \rangle + 2\langle m_{ij}\zeta_{,i}q_{,j} \rangle + 2\langle \ell_{ij}\phi_{,i}s_{,j} \rangle + 2\langle r_{ij}\psi_{,i}\theta_{,j} \rangle + 2\langle \gamma(\eta-\zeta)(p-q) \rangle + 2\langle \xi(\phi-\zeta)(s-q) \rangle + 2\langle \omega(\eta-\phi)(p-s) \rangle. \tag{46}$$

To progress we need to write equations $(9)_2$–$(9)_5$ in terms of η, ζ, ϕ and ψ and so we integrate these equations in time to derive the equations

$$\begin{aligned}
\alpha p + \alpha_1 q + \alpha_2 s + \alpha_3 \theta &= (k_{ij}\eta_{,j})_{,i} - \gamma(\eta-\zeta) - \omega(\eta-\phi) - \beta_{ij}u_{i,j},\\
\alpha_1 p + \beta q + \beta_1 s + \beta_2 \theta &= (m_{ij}\zeta_{,j})_{,i} + \gamma(\eta-\zeta) - \xi(\zeta-\phi) - \gamma_{ij}u_{i,j},\\
\alpha_2 p + \beta_1 q + \varepsilon s + \varepsilon_1 \theta &= (\ell_{ij}\phi_{,j})_{,i} + \omega(\eta-\phi) + \xi(\zeta-\phi) - \omega_{ij}u_{i,j},\\
\alpha_3 p + \beta_2 q + \varepsilon_1 s + a\theta &= (r_{ij}\psi_{,j})_{,i} - a_{ij}u_{i,j}.
\end{aligned} \tag{47}$$

We now multiply equation $(47)_1$ by p, equation $(47)_2$ by q, equation $(47)_3$ by s, equation $(47)_4$ by θ, and integrate over Ω. We multiply equation $(9)_1$ by u_i and integrate over Ω. The next step is to form the sum of the results and in this manner after some integration by parts and use of the boundary conditions we obtain the relation

$$
\begin{aligned}
(48)\qquad & \langle \alpha p^2\rangle + \langle \beta q^2\rangle + \langle \varepsilon s^2\rangle + \langle a\theta^2\rangle + 2\langle \alpha_1 pq\rangle + 2\langle \alpha_2 sp\rangle + 2\langle \alpha_3 \theta p\rangle \\
& + 2\langle \beta_1 sq\rangle + 2\langle \beta_2 q\theta\rangle + 2\langle \varepsilon_1 \theta s\rangle + \langle a_{ijkh} u_{i,j} u_{k,h}\rangle \\
& = -\langle \rho u_i \ddot{u}_i\rangle - \langle r_{ij}\psi_{,j}\theta_{,i}\rangle - \langle \ell_{ij}\phi_{,j} s_{,i}\rangle - \langle m_{ij}\zeta_{,j} q_{,i}\rangle - \langle k_{ij}\eta_{,j} p_{,i}\rangle \\
& - \langle \xi(\phi-\zeta)(s-q)\rangle - \langle \omega(\eta-\phi)(p-s)\rangle - \langle \gamma(p-q)(\eta-\zeta)\rangle.
\end{aligned}
$$

The next step is to substitute the expression on the right hand side of (48) into equation (46) for F'' and rewrite this function as

$$
\begin{aligned}
(49)\qquad F'' = & 4\langle \rho \dot{u}_i \dot{u}_i\rangle - 2\langle \alpha p^2\rangle - 2\langle \beta q^2\rangle - 2\langle \varepsilon s^2\rangle - 2\langle a\theta^2\rangle \\
& - 4\langle \alpha_1 pq\rangle - 4\langle \alpha_2 sp\rangle - 4\langle \alpha_3 \theta p\rangle - 4\langle \beta_1 sq\rangle \\
& - 4\langle \beta_2 q\theta\rangle - 4\langle \varepsilon_1 \theta s\rangle - 2\langle a_{ijkh} u_{i,j} u_{k,h}\rangle - 2\langle \rho \dot{u}_i \dot{u}_i\rangle.
\end{aligned}
$$

The energy equation (13) holds with $E(0) = 0$ and then we substitute from this equation into equation (49) to further rewrite F'' as

$$
\begin{aligned}
(50)\qquad F'' = & 4\langle \rho \dot{u}_i \dot{u}_i\rangle + 4\int_0^t \langle k_{ij} p_{,i} p_{,j}\rangle\, da \\
& + 4\int_0^t \langle m_{ij} q_{,i} q_{,j}\rangle\, da + 4\int_0^t \langle \ell_{ij} s_{,i} s_{,j}\rangle\, da \\
& + 4\int_0^t \langle r_{ij}\theta_{,i}\theta_{,j}\rangle\, da + 4\int_0^t \langle \gamma(p-q)^2\rangle\, da \\
& + 4\int_0^t \langle \omega(p-s)^2\rangle\, da + 4\int_0^t \langle \xi(s-q)^2\rangle\, da.
\end{aligned}
$$

We now form the combination $FF'' - (F')^2$ from equations (50), (45) and (44) and then we appeal to the Cauchy–Schwarz inequality to deduce that

$$
(51)\qquad FF'' - (F')^2 \geq 0.
$$

Thus, on the interval $(0, T)$, $\log F(t)$ is a convex function of t. From inequality (51) one may show $F(t) \equiv 0$ on $(0, T)$, see e.g. Ames & Straughan [3], p. 17. Actually, one has to take care with the uniqueness proof since $F(0) = 0$ and consequently $\log F(0)$ is not defined. One has to employ a contradiction argument and assume $F \neq 0$ on an interval, (ε, T) say, with $\varepsilon > 0$. Then from (51) one shows

$$
\log F(t) \leq \log F(\varepsilon)\Big(\frac{T-t}{T-\varepsilon}\Big) + \log F(T)\Big(\frac{t-\varepsilon}{T-\varepsilon}\Big)
$$

for $t \in (\varepsilon, T)$ and a continuity argument allowing $\varepsilon \to 0$ establishes $F \equiv 0$. Alternatively, one may define $\mathscr{F}(t) = F(t) + \varepsilon$ where F is defined by (44) and demonstrate $\mathscr{F}$ satisfies an inequality like (51). Again, a continuity argument allowing $\varepsilon \to 0$ furnishes a proof that $F \equiv 0$. Thus, from the definition of $F(t)$ in (44) one deduces $u_i(\mathbf{x}, t) \equiv 0$ in $\Omega \times (0, T)$. Now with $u_i \equiv 0$ the energy identity (13) together with the definition of $E(t)$ in (12) allows one to also deduce

$$p \equiv 0, \quad q \equiv 0, \quad s \equiv 0, \quad \text{and} \quad \theta \equiv 0.$$

Thus, uniqueness of a solution to $\mathscr{P}$ follows with only the symmetry condition (2) on the elastic coefficients..

One may take this logarithmic convexity analysis further to establish a stability result with only symmetry imposed upon the elastic coefficients a_{ijkh}, in the sense that one may demonstrate Hölder continuous dependence upon the initial data on compact intervals of $(0, T)$. The proof is somewhat tricky and one has to extend the definition of η, ζ, ϕ and ψ by adding contributions to account for non-zero initial data. One has also to strengthen conditions (5) and instead request that

$$\begin{aligned} k_{ij}\xi_i\xi_j \geq k_0\xi_i\xi_i \quad & m_{ij}\xi_i\xi_j \geq m_0\xi_i\xi_i, \\ \ell_{ij}\xi_i\xi_j \geq \ell_0\xi_i\xi_i, \quad & r_{ij}\xi_i\xi_j \geq r_0\xi_i\xi_i, \end{aligned} \tag{52}$$

for all ξ_i, and for constants $k_0 > 0$, $m_0 > 0$, $\ell_0 > 0$, $r_0 > 0$.

Details are similar to those in Straughan [47], equations (23)–(25). It will be necessary to consider two separate cases, namely when $E(0) \leq 0$ and when $E(0) > 0$, as analysed by Straughan [47], and the functional chosen for each case is different. However, the proof may be completed following the method in Straughan [47], *mutatis mutandis*.

5. Conclusions

We have presented a system of partial differential equations to describe the evolutionary behaviour of a triply porous elastic body in a non-isothermal setting. Due to the numerous applications a theory of triple porosity has, many of which are outlined in the introduction to this article, we believe the development of such a system of equations will prove beneficial. In particular, the inclusion of temperature effects is important in real life since thermal stresses can induce cracking behaviour, as pointed out in the introduction. A particular feature of the model is to allow cross inertia effects and cross interaction effects between the macro, meso and micro porosity structure of the elastic matrix.

We have commenced an analysis of the mathematical properties of the system of partial differential equations for a triply porous thermoelastic body and have demonstrated uniqueness and stability in the sense of continuous dependence upon the initial data in a precise mathematical manner. We have also begun an analysis of structural stability on the model itself and we have established continuous dependence upon changes in the body force and also upon the heat source. The conditions imposed upon the elastic coefficients are important and we have considered the case where these are positive, in a precise sense. However, we have also considered the case which is likely to be important for many new materials where the elasticity tensor is sign indefinite. To accommodate the cross effects and indefiniteness has necessitated the introduction of a novel technique.

It is worth mentioning that thermal convection in a multi-porosity body has also been attracting much attention, see e.g. chapter 13 of the book by Straughan [48]. The models in that case have been restricted to the situation where the solid skeleton is rigid and are usually referred to as bidisperse or tridisperse media. However, the effect of a moving elastic skeleton upon thermal convection would be a highly interesting topic. This would be particularly so in practical scenarios where often there are additionally many different salts present. For a clear fluid or a single porosity case thermal convection with many salts is a topic of much recent research, see e.g. Rionero [34–39], Capone & De Luca [7, 6, 8], Deepika & Narayana [11], and the references therein.

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