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Mechanics — *On the equations governing nonlinear symmetric Kirchhoff's elastic rods*, by Edvige Pucci and Giuseppe Saccomandi, communicated on November 11, 2016.

This paper is dedicated to the memory of Professor Giuseppe Grioli.

Abstract. — A systematic study of the equations governing the nonlinear symmetric Kirchhoff elastic rods is proposed. Symmetric rods are characterized by the conservation of the contact torque along the tangent of the centre line of the rod. This additional conservation law enables the formal reduction to quadratures of the governing equations and a systematic study of general (or Lancret's) helical solutions. This improves previous analyses where only solutions relative to the case when the centre line is a circular helix have been investigated. Moroever, we consider the general helical solution for inhomogeneous symmetric rods and, conclusively, we study a special solution where the contact torque along the tangent of the centre line is zero. This solution is valid also for asymmetric Kirchhoff rods.

Key words: Rods mechanics, Lancret's helices, Kirchhoff's Kinetic Analogy

Mathematics Subject Classification: 74K10

1. Introduction and basic equations

In recent times there has been a renewed interest in the mechanical theory of rods because of the possible use of this theory in the framework of the modeling of biological structures. An example is the *mystical obsession for helical structures* [7] arising in DNA, collagen, bacterial flagella, etc..

From a historical perspective the earlier work by Euler on rod theory has been first extended by Kirchhoff [21], Clebsch [9] and then by Cosserat's brothers [10], [24, 32]. The modern age starts with the contributions by Ericksen and Truesdell [15], but the true impetus is more recent and it is linked to the names, among others, of Antman [2], Coleman [30], Goriely [27], Maddocks [25], Thompson and Van der Heijden [12]. A comprehensive treatment of the subject can be found in the monographs [3] and [32].

From a geometrical and kinematical point of view the basic idea is to model the rod as a continuum of planar sections with a material curve passing trough the centroid of each section. From the dynamical point of view there are several theories of rods. Here, we allow arbitrary bending and torsion but no axial extension or shear.

The equilibrium deformation of the rod is therefore completely determined considering the *centre line* $\boldsymbol{r}(s)$ parametrized by its arc length $s \in [0, L]$, where L is the length of the rod plus the *twist angle* $\varphi = \varphi(s)$ [24].

To determine $\boldsymbol{r}(s)$ we introduce a smooth unit vector $\boldsymbol{d}_1$ orthogonal to $\boldsymbol{r}$ at s. Since the rod is inexestensible, s is the arc length in any configuration, and $\boldsymbol{r}'(s) := d\boldsymbol{r}/ds$ is the unit tangent to the centre line at s. Prime denotes differentiation with respect to the arc length.

Considering $\boldsymbol{d}_3 = \boldsymbol{r}'(s)$ we introduce the right-handed orthonormal basis of the *directors* defined by $\{\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3\}$ where $\boldsymbol{d}_2 := \boldsymbol{d}_3 \times \boldsymbol{d}_1$. The *twist vector* is denoted $\boldsymbol{u}(s) = k_1\boldsymbol{d}_1 + k_2\boldsymbol{d}_2 + k_3\boldsymbol{d}_3$ and

$$\boldsymbol{d}_i' = \boldsymbol{u} \times \boldsymbol{d}_i, \quad i = 1, 2, 3. \tag{1.1}$$

The components of the twist vector can be expressed as functions of the twist angle $\varphi(s)$ between the vector $\boldsymbol{d}_1$ and the normal to the curve, the Frenet curvature k, and the torsion τ:

$$(k_1, k_2, k_3) = (k \sin\varphi, k\cos\varphi, \tau + \varphi'). \tag{1.2}$$

The triple $\{\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3\}$ can be related to a fixed orthonormal system $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ via the usual Euler angles (Love, 2013): θ the inclination angle between $\boldsymbol{d}_3$ and $\boldsymbol{e}_3$, ψ the azimuthal angle between $\boldsymbol{e}_2$ and $\boldsymbol{d}_3 \times \boldsymbol{e}_3$, ϕ the rotation angle between $\boldsymbol{d}_3 \times \boldsymbol{e}_3$ and $\boldsymbol{d}_2$. The components of $\{\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3\}$ with respect $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ can be found in [12] or in any textbook of classical mechanics.

The balance of the forces and the moments acting on each cross section leads to the following set of equilibrium equations

$$\frac{d}{ds}\boldsymbol{g} = \boldsymbol{0}, \tag{1.3}$$

and

$$\frac{d}{ds}\boldsymbol{m} + \boldsymbol{d}_3 \times \boldsymbol{g} = \boldsymbol{0}, \tag{1.4}$$

where $\boldsymbol{g}(s)$ is the contact force, $\boldsymbol{m}(s)$ the contact moment. We are assuming that the external loads are only couples and forces applied at either ends of the rod.

Considering a hyperelastic rod, we assume the existence of a strain-energy density function $W = W(k_1, k_2, k_3, s)$ such that the components m_i $(i = 1, 2, 3)$ of $\boldsymbol{m}$ with respect to $\{\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3\}$ are given by

$$m_1 = \frac{\partial W}{\partial k_1}, \quad m_2 = \frac{\partial W}{\partial k_2}, \quad m_3 = \frac{\partial W}{\partial k_3}.$$

In so doing we close the system of equations (1.3) and (1.4).

An explicit and important example of strain energy is the case of a homogeneous linear straight elastic rod

$$W = \frac{1}{2} E I_1 k_1^2 + \frac{1}{2} E I_2 k_2^2 + \frac{1}{2} \mu J k_3^2, \tag{1.5}$$

where the products EI_1, EI_2 are the principal bending stiffnesses of the rod and μJ is the torsional stiffness. (E is Young's modulus and μ the shear modulus, I_1, I_2 and J the principal moments of inertia of the cross-section.).

When (1.5) is enforced, the equations (1.3) and (1.4) are identical to the equations describing the motion of a heavy rigid body with one point fixed (the spinning top). This identification is known as the *Kirchhoff's Kinetic Analogy* [25, 27].

This analogy is quite powerful when we have a symmetric rod ($I_1 = I_2 = J/2$). This is because we have a complete analogy of the governing equations with the *Lagrange top* and it is possible to reduce (1.3) and (1.4) to an equivalent oscillator. The details of this reduction can be found in [27] and [12].

The aim of the present note is to study a generalization of the Kirchhoff's Kinetic Analogy that allows to solve the equations for symmetric rods with a general nonlinear strain-energy. Moreover, we wish to generalize the *Kirchhoff's problem* to consider Lancret's helices [4]. We remember that a space curve is a helix if the lines tangent to the curve make a constant angle with a fixed direction in space (the helical axis). Using Lancret's theorem [4] a necessary and sufficient condition for a space curve to be a helix is the constancy of the ratio curvature over torsion. For such reason general helices are also denoted Lancret's helices.

Kirchhoff showed, by using a semi-inverse method, that an initially straight prismatic rod, characterized by the constitutive equation (1.5) and a symmetric cross-section, admits helical solutions [24]. According to Kirchhoff a *helical deformation* is a deformation with constant curvature and torsion, i.e.: a circular helix, a circle, a straight line. The Kirchhoff problem has been considered for a general strain-energy $W = W(k_1, k_2, k_3)$ in [14] and solved for a general nonlinear symmetric theory with axial extension and cross section shear by Antman [2]. Preliminary results in this direction are contained in [33].

For a homogeneous and hyperelastic, but otherwise arbitrary, nonlinear rod subject to appropriate end loadings all the equilibria whose center lines form circular helices has been completely solved in [8]. The complete semi-inverse classification, initiated by Kirchhoff, of all infinite helical equilibria of inextensible, unshearable and uniform rods with elastic energies that are a general quadratic function of the flexures and twist is proposed in [7]. Where the case of *non-uniform* circular helices is considered. A curve is non-uniform if the derivative of the twist angle φ is not constant.

In the framework of Kirchhoff's Kinetic Analogy the paper by [7] is the corresponding of [17] paper about regular precessions for the spinning top. Indeed, in this analogy a regular precession is a circular helix (in a regular precession both the angular and the precessional speeds are constants see the remark at the end of this Section). In [18] a class of *non regular* precessional motions are determined.

These motions, in the above mentioned analogy, may correspond to Lancret's helices. While regular precession are completely equivalent to circular helices, non regular precession can be more general than helices. In the framework of rod theory the Lancret's helices have been studied for non homogeneous rods with a quadratic strain energy in [16].

A variational formulation of equations (1.3) and (1.4) can be found in [29]. Lancret helices in the framework of a variational theory of curves have been considered in various papers [5, 6, 31, 26]. In these papers it is possible to find only very special solutions to the general equations.

The plan of the paper is the following. In Section 2 we write down the equations of motion. It is fundamental for our computations to write the equations in the intrinsic frame. In Section 3 we restrict our attention to the symmetric case and we determine the integrability of the equations of motion by following the same approach of the integrable case of Lagrange for the heavy top. This simple fact seems to have been unnoticed in then literature. In Section 4 we consider the case of null tension and we find that all the possible solutions are at most Lancret's helices. In Section 5 we determine via a semi-inverse method all those solutions that are Lancret's helices for the nonlinear symmetric rod with tension. In Section 6 we show how our results can be easily extended to inhomogeneous rods. Section 7 is devoted to a special case that in rigid body mechanics is related to precessional motions but in rods mechanics provides only deformed centre lines with no torsion. We devote the last section to concluding remarks.

Throughout the paper we discuss the analogies with the celebrated results by Grioli in stereodynamics [17, 18, 19] and some of the solutions there considered. In [17] a class of regular precessional motions are found for the heavy top. Then in [18] the problem of the *non* regular precessional motions is considered. Grioli points out that this is the next step for finding special solutions for the heavy top, but he observes that solving the problem via the Euler angles may not be easy. For this reason Grioli introduced the idea of *generalized* precessions. In this case the starting point is an ansatz on the angular momentum. In so doing Grioli is able to find a special solution which is indeed a non regular precession.

In a subsequent paper [19] Grioli uses an intrinsic version of the Euler dynamical equations to show that the resulting equations are simplified and that a special solution can be found. We point out that Grioli was interested to a rigid body with general structure. For the gyroscope intrinsic equations have been used previously by [28].

Here, by using Grioli's approach we are able to solve in a very general and direct way the problem of the Lancret's helices for the symmetric nonlinear Kirchhoff rod and to push the Kirchhoff's Kinetic Analogy (in a *weak* direction) for a general strain-energy $W = W(k_1, k_2, k_3; s)$. This is because in the intrinsic frame $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$

$$\boldsymbol{d}_1 = \cos\varphi\boldsymbol{n} + \sin\varphi\boldsymbol{b}, \quad \boldsymbol{d}_2 = -\sin\varphi\boldsymbol{n} + \cos\varphi\boldsymbol{b},$$

and the twist vector is given as $\boldsymbol{u} = k\boldsymbol{b} + (\tau + \dot{\varphi})\boldsymbol{t}$. For an helix it must exists a fixed unit vector $\boldsymbol{a}$ such that $\boldsymbol{t} \cdot \boldsymbol{a} = \cos\alpha_0$, where α_0 is constant, i.e. $\boldsymbol{a} = \cos\alpha_0\boldsymbol{t} +$

$\sin\alpha_0 \boldsymbol{b}$. By derivation of this last expression with respect the arc length s we obtain $\mathbf{0} = (k\cos\alpha_0 - \tau\sin\alpha_0)\boldsymbol{n}$ and therefore it must be (Lancret's theorem)

$$\tau/k = \cot\alpha_0 := \omega.$$

On the other hand, it is easy to show that for an helix when $\alpha_0 \neq 0$

$$\boldsymbol{u} = \frac{k}{\sin\alpha_0}\boldsymbol{a} + \dot{\varphi}\boldsymbol{t}.$$

This formula is the analogue of the kinematical formula which characterizes precessional motions[1] in rigid body mechanics. This is the reason because Grioli's results are interesting in the framework of rods mechanics.

2. Equilibrium equations

For the equilibrium equation (1.3) we get $\boldsymbol{g} = \boldsymbol{g}_0$ where $\boldsymbol{g}_0$ is a constant vector. Choosing the fixed orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ such that $\boldsymbol{g} \parallel \boldsymbol{e}_3$ we have that $\boldsymbol{g}(s) = T\boldsymbol{e}_3$ (T is constant). Therefore, the basic equations are rewritten as

$$\frac{d}{ds}\boldsymbol{g} = 0, \quad \frac{d}{ds}\boldsymbol{m} + T\boldsymbol{d}_3 \times \boldsymbol{e}_3 = \mathbf{0}. \tag{2.1}$$

If we consider a quadratic uniform strain-energy these equations are straightforwardly equivalent to the spinning top equations.

Let us write down the equations (2.1) for a general strain-energy density:

$$\begin{aligned} &g_1' - (g_2k_3 - g_3k_2) = 0, \quad g_2' - (g_3k_1 - g_1k_3) = 0, \\ &g_3' - (g_1k_2 - g_2k_1) = 0; \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} &\left(\frac{\partial W}{\partial k_1}\right)' - \left(k_3\frac{\partial W}{\partial k_2} - k_2\frac{\partial W}{\partial k_3}\right) = g_2; \\ &\left(\frac{\partial W}{\partial k_2}\right)' - \left(k_1\frac{\partial W}{\partial k_3} - k_3\frac{\partial W}{\partial k_1}\right) = -g_1; \\ &\left(\frac{\partial W}{\partial k_3}\right)' - \left(k_2\frac{\partial W}{\partial k_1} - k_1\frac{\partial W}{\partial k_2}\right) = 0. \end{aligned} \tag{2.3}$$

Because

$$\boldsymbol{d}_1 = k^{-1}(k_2\boldsymbol{n} + k_1\boldsymbol{b}), \quad \boldsymbol{d}_2 = k^{-1}(-k_1\boldsymbol{n} + k_2\boldsymbol{b}),$$

[1] Clearly for $\alpha_0 = \pi/2$ we have a degeneration of the helix to a planar curve.

we have

$$\boldsymbol{m} = k^{-1}\Big(\frac{\partial W}{\partial k_1}k_2 - \frac{\partial W}{\partial k_2}k_1\Big)\boldsymbol{n} + k^{-1}\Big(\frac{\partial W}{\partial k_1}k_1 + \frac{\partial W}{\partial k_2}k_2\Big)\boldsymbol{b} + \frac{\partial W}{\partial k_3}\boldsymbol{t}. \tag{2.4}$$

A constitutive equation in the form $W = W(k_1, k_2, k_3, s)$ admits an equivalent form $\hat{W} = \hat{W}(k, \varphi, \tau + \varphi', s)$.

Symmetric rods are characterized by the a reduced dependence i.e. $W = W(k_1^2 + k_2^2, k_3, s)$ or $\hat{W} = \hat{W}(k, \tau + \varphi', s)$.

Moving from (k_1, k_2, k_3) to (k, τ, φ) using (2.4) we obtain

$$m_{\boldsymbol{n}} = k^{-1}\frac{\partial \hat{W}}{\partial \varphi}, \quad m_{\boldsymbol{b}} = \frac{\partial \hat{W}}{\partial k}, \quad m_{\boldsymbol{t}} = \frac{\partial \hat{W}}{\partial \tau}\Big(\equiv \frac{\partial \hat{W}}{\partial k_3}\Big). \tag{2.5}$$

Therefore $(2.1)_2$, in the intrinsic frame, reads

$$\begin{aligned} &m_{\boldsymbol{t}}' - k m_{\boldsymbol{n}} = 0 \to \Big(\frac{\partial \hat{W}}{\partial \tau}\Big)' - \frac{\partial \hat{W}}{\partial \varphi} = 0,\\ &m_{\boldsymbol{n}}' - \tau m_{\boldsymbol{b}} + k m_{\boldsymbol{t}} = g_{\boldsymbol{b}} \to \Big(k^{-1}\frac{\partial \hat{W}}{\partial \varphi}\Big)' - \tau\frac{\partial \hat{W}}{\partial k} + k\frac{\partial \hat{W}}{\partial \tau} = g_{\boldsymbol{b}},\\ &m_{\boldsymbol{b}}' + \tau m_{\boldsymbol{n}} = -g_{\boldsymbol{n}} \to \Big(\frac{\partial \hat{W}}{\partial k}\Big)' + \frac{\tau}{k}\frac{\partial \hat{W}}{\partial \varphi} = -g_{\boldsymbol{n}}, \end{aligned} \tag{2.6}$$

to which we add

$$g_{\boldsymbol{t}}' - k g_{\boldsymbol{n}} = 0, \quad g_{\boldsymbol{n}}' - \tau g_{\boldsymbol{b}} + k g_{\boldsymbol{t}} = 0, \quad g_{\boldsymbol{b}}' + \tau g_{\boldsymbol{n}} = 0. \tag{2.7}$$

If from $(2.6)_2$, $(2.6)_3$ and $(2.7)_2$ we deduce $g_{\boldsymbol{t}}$, $g_{\boldsymbol{n}}$, $g_{\boldsymbol{b}}$ and we substitute these relations in $(2.7)_1$ and $(2.7)_3$ considering $g_{\boldsymbol{t}}^2 + g_{\boldsymbol{n}}^2 + g_{\boldsymbol{b}}^2 = T^2$, we recover the same equations obtained by a variational method in [29].

3. Integrability of the homogeneous symmetric rod

For the system (2.1) it is possible to write down the following set of first integrals:

- conservation of the torque about the $\boldsymbol{e}_3$ axis:

$$\boldsymbol{m} \cdot \boldsymbol{e}_3 = C_1; \tag{3.1}$$

- conservation of the total energy density

$$\boldsymbol{m} \cdot \boldsymbol{k} - W + T\boldsymbol{d}_3 \cdot \boldsymbol{e}_3 = C_2. \tag{3.2}$$

To integrate the equations (2.2) and (2.3) we need an additional first integral. Both for the Lagrange's top and for Kirchhoff's Kinetic Analogy when the rods are symmetric, the third integral is supplied by the conservation of the torque about the body axis $\boldsymbol{d}_3$ i.e.

$$\boldsymbol{m} \cdot \boldsymbol{d}_3 = C_3. \tag{3.3}$$

We notice that the first integral (3.3) is valid if and only if the strain-energy is such that

$$W = W(k_1^2 + k_2^2, k_3). \tag{3.4}$$

Conservation laws for the theory of rods are considered in [25]. Clearly the C_i, $(i = 1, 2, 3)$, are arbitrary constants and, apologizing for the abuse of notation, we consider the strain-energy as a function of the curvature and k_3: $W = W(k, k_3)$.

Let us show that the three first integrals (3.1), (3.2) and (3.3) allow us to integrate our starting equations when (3.4) is assumed. Considering the Euler angles:

$$k_1 = \theta' \sin\phi - \psi' \sin\theta \cos\phi, \quad k_2 = \theta' \cos\phi + \psi' \sin\theta \sin\phi, \tag{3.5}$$

and

$$k_3 = \phi' + \psi' \cos\theta. \tag{3.6}$$

Clearly,

$$k^2 = k_1^2 + k_2^2 = \theta'^2 + \psi'^2 \sin^2\theta.$$

The first integrals (3.1), (3.2) and (3.3) are rewritten as

$$\begin{aligned}
&\frac{1}{k}\frac{\partial W}{\partial k}\psi' \sin^2\theta + C_3 \cos\theta = C_1, \\
&\frac{\partial W}{\partial k}k + C_3(\phi' + \psi' \cos\theta) - W + T\cos\theta = C_2, \\
&\frac{\partial W}{\partial k_3} = C_3.
\end{aligned} \tag{3.7}$$

Under suitable hypotheses of monotony and coerciveness [2], it is possible to use these equations to eliminate ψ' and ϕ' and to obtain a single differential equation in terms of the angle $\theta = \theta(s)$. This is the generalization of the approach adopted by (Van der Heijden and Thompson, 2000) for the classical quadratic energy where the reduced equilibrium equation can be written as an equivalent nonlinear oscillator. We point out that this result is neither contained in [3] and [32] and to the best of our knowledge seems to be original.

It is easy to check that for

$$W(k, k_3) = G_1(k)k_3 + G_2(k),$$

or

$$W(k, k_3) = Ak^2 + G_3(k_3),$$

where A is a constant and G_i $(i = 1, 2, 3)$ are arbitrary functions, it is possible to reduce the system to an equivalent nonlinear oscillator as in the classical quadratic case.

Clearly, in general, the necessity to solve higher grade algebraic or trascendental equations for ϕ' and ψ' gives no chance to reduce the problem to a single ordinary differential equation.

On the other hand, it is clear that, by derivation of the three first integrals with respect to s, we obtain a system of three differential equations of the second order. By making use of a numerical method it is therefore possible to solve any boundary-value problem, although the situation is not so simple as for an initial-value problem and this is an important difference with the heavy top.

An alternative to this situation is to look for special classes of simple solutions and this is our next step, starting from the case of the equations with null tension.

4. Null tension

When $\boldsymbol{g} \equiv \mathbf{0}$ the equations (2.3) are an integrable system of differential equations. Indeed they are a highly nonlinear version of the Poinsot system for the gyroscope

$$\begin{aligned} &\left(\frac{\partial W}{\partial k_1}\right)' - \left(k_3 \frac{\partial W}{\partial k_2} - k_2 \frac{\partial W}{\partial k_3}\right) = 0; \\ &\left(\frac{\partial W}{\partial k_2}\right)' - \left(k_1 \frac{\partial W}{\partial k_3} - k_3 \frac{\partial W}{\partial k_1}\right) = 0; \\ &\frac{\partial W}{\partial k_3} = C_3. \end{aligned} \tag{4.1}$$

It is convenient to work in the intrinsic frame where, considering $\partial W/\partial \varphi \equiv 0$, equations (2.6) read

$$\begin{aligned} &\left(\frac{\partial W}{\partial k_3}\right)' = 0, \quad \left(\frac{\partial W}{\partial k}\right)' = 0, \\ &-\tau \frac{\partial W}{\partial k} + k \frac{\partial W}{\partial k_3} = 0. \end{aligned} \tag{4.2}$$

Therefore for any solution it must result

$$\frac{\partial W}{\partial k_3} = C_3, \quad \frac{\partial W}{\partial k} = C_2, \quad -\tau C_2 + k C_3 = 0. \tag{4.3}$$

From (4.3) the determinant $\mathscr{D}$ of the 3×3 Jacobian

$$\begin{pmatrix} W_{kk} & W_{kk_3} & 0 \\ W_{kk_3} & W_{k_3k_3} & 0 \\ C_3 & 0 & -C_2 \end{pmatrix},$$

is

$$\mathscr{D} \equiv -C_2(W_{kk}W_{k_3k_3} - W^2_{kk_3}).$$

If $\mathscr{D} \neq 0$ from the third equation in (4.3) it must be $\tau = \omega k$ where $\omega = C_3/C_2$. Once we have fixed a strain energy, the first two equations in (4.3)

$$\left.\frac{\partial W}{\partial k_3}\right|_{\tau=\omega k} = C_3, \quad \left.\frac{\partial W}{\partial k}\right|_{\tau=\omega k} = C_2, \tag{4.4}$$

are a system for the curvature k and the derivative of the twist φ'. This means that only solutions that are uniform circular helices, as we have conjectured, are possible.

If $\mathscr{D} = 0$ and the Jacobian is of rank $= 1$ it must be or $C_2 = 0$ or $W(k, k_3)$ a linear function

$$W = C_3k_3 + C_2k. \tag{4.5}$$

If $C_2 = 0$ then $C_3 = 0$ and we are considering a mathematical situation of no mechanical meaning because all the moments are null.

On the other hand if the strain-energy is given by (4.5) all the solutions are Lancret's helices. These helices can be uniform or non uniform i.e. we can choose arbitrarly both k and φ'. Clearly the strain-energy (4.5) is not truly meaningful from a mechanical point of view because the components of the associated moment are constant.

If $\mathscr{D} = 0$ and the Jacobian is of rank $= 2$ the two equations in (4.4) are functionally dependent and it must result

$$W_{k_3k_3}W_{kk} - (W_{k_3k})^2 = 0. \tag{4.6}$$

Equation (4.6) is the classical Monge–Ampere equation. For this equation we know the general solution in parametric form

$$W = tk + F(t)k_3 + G(t), \quad k + \frac{dF}{dt}k_3 + \frac{dG}{dt} = 0,$$

where F, G are arbitrary functions.

We point out that strain-energies wich are solutions of the Monge–Ampere equation (4.6) could be still associated only with circular helices. This is the case of

$$W(k, k_3) = C_3k_3 + H(k).$$

Here $H(k)$ is an arbitrary function. The difference with the general case ($\mathscr{D} \neq 0$) is that now the circular helix may be non uniform.

On the other hand, two explicit examples of solutions of the Monge–Ampere equation associated with Lancret's helices are

$$W(k, k_3) = C_2 k + H(k_3), \tag{4.7}$$

where $H(k_3)$ is an arbitrary function, and

$$W(k, k_3) = \int H(\zeta)\, d\zeta, \quad \zeta = Ak + Bk_3, \tag{4.8}$$

where H is an arbitrary function of ζ and A, B are arbitrary constants.

In the case of (4.8) the two first integrals in (4.4) are written as

$$H(\zeta)B = C_3, \quad H(\zeta)A = C_2,$$

i.e. $AC_3 - BC_2 = 0$. Given an helix such that $\omega = B/A$ it must be

$$AB\varphi' = AH^{-1}\left(\frac{C_3}{B}\right) - (A^2 + B^2)k.$$

This means that for the strain-energy (4.8) all the solutions are *non uniform* Lancret's helices or uniform circular helices.

In [2] it is possible to find a proof that circular helices are solutions of the equations (4.2), but here we show that all solutions are helices and we characterize all the strain-energy density functions such that these helices are not circular.

5. Lancret's helices with tension

Consider the equations (2.6) when the strain energy does not depend on φ and therefore $m_{\boldsymbol{n}} = 0$. The balance equations for the angular moment reduce to

$$m'_{\boldsymbol{t}} = 0, \quad g_{\boldsymbol{b}} = -\tau m_{\boldsymbol{b}} + k m_{\boldsymbol{t}}, \quad g_{\boldsymbol{n}} = -m'_{\boldsymbol{b}}. \tag{5.1}$$

Therefore we have the conservation law

$$m_{\boldsymbol{t}} = C_3 \tag{5.2}$$

and introducing $(5.1)_{2,3}$ in (2.7) (balance of linear momentum) after some manipulation we obtain two differential equations

$$\left[(kC_3 - \tau m_{\boldsymbol{b}})\frac{\tau}{k}\right]' + \left(\frac{m''_{\boldsymbol{b}}}{k}\right)' + k m'_{\boldsymbol{b}} = 0, \tag{5.3}$$

and

$$(kC_3 - \tau m_{\boldsymbol{b}})' - \tau m'_{\boldsymbol{b}} = 0. \tag{5.4}$$

If we restrict our attention to Lancret's helices we set $\tau = \omega k$ with ω constant. In this case it is possible to reduce equations (5.3) and (5.4) to quadrature. Introducing the new dependent variable

$$\zeta = C_3 - \omega m_{\boldsymbol{b}}, \tag{5.5}$$

the equations (5.3) and (5.4) are rewritten as

$$\omega^2(k\zeta)' - \left(\frac{\zeta''}{k}\right)' - k\zeta' = 0, \tag{5.6}$$

and

$$(k\zeta)' + k\zeta' = 0 \tag{5.7}$$

The (5.7) is integrated as

$$\zeta = \frac{C_1}{\sqrt{k}}, \tag{5.8}$$

and, using this, the (5.6) is reduced to the third order equation

$$2k\mathscr{E}' - 7k'\mathscr{E} = 0, \tag{5.9}$$

where

$$\mathscr{E} \equiv 4k^4(\omega^2 + 1) + 2kk'' - 3k'^2.$$

Equation (5.9) restricts in a severe way the curvatures that are admissible. Therefore only very special Lancret's helices are possible.

By considering $\mathscr{E} = 0$ the special family of solutions of equation (5.9)

$$k(s) = \frac{\alpha}{\alpha^2(1+\omega^2) + (s+\beta)^2}, \tag{5.10}$$

can be found. Here α, β are integration constants.

By considering $k(s)$ given by (5.10) the parametric equations of the centre line [1] are

$$\begin{aligned} \boldsymbol{r}(s) = {} & \frac{1}{\sqrt{1+\omega^2}} \int \cos\left[\arctan\left(\frac{s+\beta}{\gamma}\right)\right] ds\boldsymbol{e}_1 \\ & + \frac{1}{\sqrt{1+\omega^2}} \int \sin\left[\beta \arctan\left(\frac{s+\beta}{\gamma}\right)\right] ds\boldsymbol{e}_2 \\ & + \omega s\boldsymbol{e}_3, \end{aligned} \tag{5.11}$$

where we have set $\gamma = \alpha\sqrt{1+\omega^2}$.

Computing the integrals in (5.11)

$$\int \cos\left[\arctan\left(\frac{s+\beta}{\gamma}\right)\right] ds = \int \frac{\gamma}{\sqrt{\gamma^2 + (s+\beta)^2}} ds = \gamma \sinh^{-1}\left(\frac{s+\beta}{\gamma}\right),$$

$$\int \sin\left[\arctan\left(\frac{s+\beta}{\gamma}\right)\right] ds = \int \frac{1+\beta}{\sqrt{\gamma^2 + (s+\beta)^2}} ds = \sqrt{(s+\beta)^2 + \gamma^2},$$

we obtain

$$x(s) = \alpha \sinh^{-1}\left(\frac{s+\beta}{\gamma}\right), \quad y(s) = \alpha\sqrt{1 + \left(\frac{s+\beta}{\gamma}\right)^2}, \quad z(s) = \omega s.$$

These curves *climb* on an open surface:

$$y = \pm\gamma \cosh\left(\frac{x}{\alpha}\right).$$

In figure 1 we plot the axis of the rod, if φ' is not constant it is important to remember that the planar section *wraps* the plotted curve in a non uniform way.

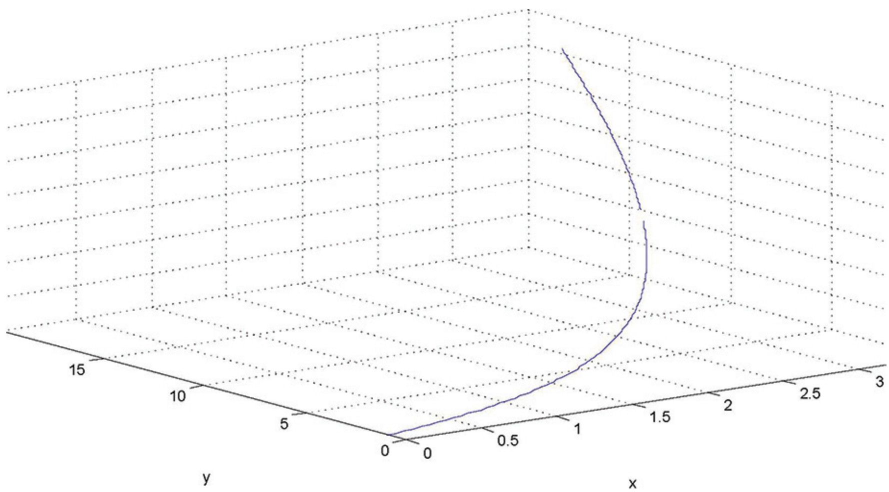

Figure 1. Example of centre line for $\omega = 0.35$, $\alpha = 1$, $\beta = 0$

We point out that given a strain-energy density it is not ensured that the curvatures solving (5.9) determines solutions that are admissible. This is because we have to ensure that the equations (5.2) and (5.5) are satisfied. Several comments are possible at this stage:

- *Circular helices*. Circular helices are a special case of Lancret's helices. If k is constant from (5.7) ζ is constant and (5.6) is satisfied, and the balance equations are reduced to

$$\omega \frac{\partial W}{\partial k}\bigg|_{k=k_0,\, \tau=\omega k_0} = C_3 - \zeta, \quad \frac{\partial W}{\partial k_3}\bigg|_{k=k_0,\, \tau=\omega k_0} = C_3.$$

  Solving these equations we obtain uniform circular helices. The difference of such solutions and the ones in the previous Section is that in this case the tension is not null. The case of null tension is obtained when $\zeta = 0$. These solutions are contained in [2].
- *Linear energies*. Energies that are linear in the curvature and the torsion are of limited interested because they imply a constant set of moments. Nothwithstanding, these energies have been considered in great detail for example in [5, 6]. Lancret's helices cannot be sustained in this case. This means that linear strain energies can sustain Lancret's helices only in the case of null tension.
- *Classical quadratic energies*. In the case of the constitutive equation (1.5) when $I_1 = I_2$ by a direct computation we obtain from (5.4) that the curvature must be constant. This means that the only helical solutions are circular helices.
- *A full nonlinear example*. A nonlinear strain-energy that can support Lancret's helices is given by

$$W = A\sqrt{k} + Bk + \int H(k_3),$$

  where A and B are constants and H is an arbitrary function. Indeed, in this case we have that the equation (5.2) is satisfied considering the twist such that

$$\tau + \varphi' = H^{-1}(C_3) \rightarrow \varphi' = -\omega k + H^{-1}(C_3),$$

  whereas (5.5) gives

$$C_1 = \frac{A\omega}{2}, \quad C_3 = \omega B.$$

  Clearly this is only an example, but this is sufficient to point out how special are those constitutive equations that can support Lancret's helices.

6. Lancret's helices for inhomogeneous symmetric rods

Let us consider a non-homogeneous strain-energy

$$W = W(k, k_3, s). \tag{6.1}$$

In this case we have that the system is no more integrable, but the computation we have done for the case of null tension and for the Lancret's helices is still valid. We point out that in [11] the authors have shown that the inhomogeneous non symmetric and non twisted rod is chaotic also in the null tension case.

This section is motivated by [16], where only quadratic constitutive equations are considered. Here we propose a more compact and exhaustive analysis than the one proposed by [16]. Our results are valid for any symmetric strain-energy density and for producing simple examples it is possible to confine attention to the special case

$$W = E\frac{1}{2}\frac{J(s)}{2}(k-\hat{k})^2 + \frac{1}{2}\mu J(s)(k_3-\hat{k}_3)^2 \tag{6.2}$$

where $J(s) = \pi R^4(s)/4$ depends on the *variable* radius of the circular cross section of the rod and $\hat{k}$ and $\hat{k}_3$ are related to the intrinsic curvature and torsion of the rod.

In case of null tension, once again the balance equations can be reduced to (4.2). Therefore all the solutions are Lancret's helices determined by (4.4). Introducing (6.2) in the relationship (4.4) we obtain

$$k(s) = \frac{2C_2}{EJ(s)} - \hat{k}, \quad \varphi'(s) = \frac{EC_3 - 2\omega\mu C_2}{\mu EJ(s)} + \omega\hat{k} + \hat{k}_3.$$

To simplify the algebra let $\hat{k} = \hat{k}_3 = 0$ and let us choose a linear law for the inertia $J(s) = \eta_0(1+\eta s)$, i.e. a sublinear growth for the radius of the rod. Now

$$k(s) = \frac{k_0}{1+\eta s}, \quad \varphi(s) = \varphi_0 \log(1+\eta s),$$

where $k_0 = 2C_2/E\eta_0$ and $\varphi_0 = (EC_3 - 2\omega\mu C_2)/E\mu\eta_0\eta$.

In so doing we find the free standing solution for a constitutive equation with a simple inhomogeneity given by

$$\begin{aligned}
x(s) &= \frac{k_0\sqrt{1+\omega^2}(1+\eta s)\sin\Omega(s) + \eta[(1+\eta s)\cos\Omega(s) - 1]}{D},\\
y(s) &= \frac{\eta(1+\eta s)\sin\Omega(s) - k_0\sqrt{1+\omega^2}[(1+\eta s)\cos\Omega(s) - 1]}{D},\\
z(s) &= \omega s.
\end{aligned}$$

Here $\Omega(s) = k_0\sqrt{1+\omega^2}\log(1+\eta s)/\eta$ and $D = \sqrt{1+\omega^2}(k_0^2(1+\omega^2)+\eta^2)$.

The surface associated with this solution is the conical surface

$$\sqrt{1+\omega^2}[(Dx+\eta)^2 + (Dy - k_0\sqrt{1+\omega^2})^2] - D(\omega+\eta z)^2 = 0.$$

Conical helices in rods with a variable radius are common in Nature. It is sufficient to think to any filamentary growth in Biology to have evidence of a variability of the radius of the filament. We were not able to find measurements to confirm if the variation of radius of such organism is compatible with our findings.

7. Beyond the symmetric case

The possibility to extend our analysis to the asymmetric case is not simple at all. It is well known that also in the quadratic energy case the asymmetric equations are integrable only in very special cases. The problem of the circular helical solution (the Kirchhoff problem for uniform and non uniform helices) for a general quadratic strain-energy with a reference configuration which is not straight has been solved in [7]. In this paper a very special example of strain-energy such that a non uniform circular helix is admissible is provided.

Here we want to investigate how the idea of *generalized* precession introduced in [18] can be helpful. The Grioli solutions are determined using an ansatz not on the angular velocity but on the momentum. The idea of Grioli is to consider that unit vector of the angular momentum is fixed in the space with a time dependent magnitude. This ansatz produces some solutions for the asymmetric rigid body that are non regular precession and such that $\boldsymbol{m} \cdot \boldsymbol{d}_3 = 0$. Therefore, these solutions are german to the case that we have investigated in the previous sections. It is worth noting out that here we have not the conservation of the torque along $\boldsymbol{d}_3$ (an opportunity that implies a symmetric rigid body or a symmetric rod) but an *invariant relation* generated from this conservation law. For information about the concept of invariant relation we refer to [23].

In our settings the Grioli solutions are obtained when we impose

$$\boldsymbol{m} = \kappa(s)\boldsymbol{\chi}, \tag{7.1}$$

where $\boldsymbol{\chi}$ is an unit vector fixed in the space. Clearly if we assume (7.1), because $d\boldsymbol{\chi}/ds = 0$, it must be that

$$\kappa'(s)\boldsymbol{\chi} + T\boldsymbol{d}_3 \times \boldsymbol{e}_3 = \boldsymbol{0}, \quad \to \boldsymbol{\chi} \cdot \boldsymbol{e}_3 = 0, \quad \boldsymbol{\chi} \cdot \boldsymbol{d}_3 = 0$$

and this is exactly the same situation we have in the Hess special integrability case of an heavy rigid body (Levi–Civita and Amaldi, 1923).

Being $\boldsymbol{\chi} \cdot \boldsymbol{d}_3 \equiv \chi_3 = 0$ we deduce from the fact that $\boldsymbol{\chi}$ is a constant vector

$$\chi_1' - k_3\chi_2 = 0, \quad \chi_2' + k_3\chi_1 = 0, \quad k_2\chi_1 - k_1\chi_2 = 0. \tag{7.2}$$

From $(7.2)_3$ being $\boldsymbol{\chi}$ an unit vector

$$\chi_1 = \pm\sin\varphi, \quad \chi_2 = \pm\cos\varphi,$$

and from $(7.2)_{1,2}$

$$k_3 = \varphi'.$$

Therefore it must be $\tau = 0$ and the centre line of the rod is a planar curve, but because φ is not constant any straight line on the mantle of the rod is deformed to a spatial curve.

Being $\boldsymbol{\chi} \cdot \boldsymbol{e}_3 \equiv \chi_1 g_1 + \chi_2 g_2 = 0$, it must be

$$g_1 = -h(s)\cos\varphi, \quad g_2 = h(s)\sin\varphi$$

and from the balance of angular momentum we have that $h = \kappa'$ and from the balance of linear momentum we obtain the definition of

$$g_3 = \kappa''/k,$$

and a compatibility equation

$$\left(\frac{\kappa''}{k}\right)' + k\kappa' = 0. \tag{7.3}$$

Clearly a first integral for equation (7.3) is just

$$\kappa'^2 + (\kappa''/k)^2 = T^2.$$

Given a strain-energy the ansatz (7.1) implies that the system composed by the equations (7.3) and

$$\frac{\partial W}{\partial k_1} = \pm\kappa\sin\varphi, \quad \frac{\partial W}{\partial k_2} = \pm\kappa\cos\varphi, \quad \frac{\partial W}{\partial k_3} = 0, \tag{7.4}$$

must be satisfied in the unknowns $\kappa = \kappa(s)$, $k = k(s)$ and $\varphi = \varphi(s)$. The compatibility of this system imposes severe restrictions to the admissible strain-energies.

Let us consider the quadratic strain-energy

$$W(k_1, k_2, k_3) = \frac{1}{2}(Ak_1^2 + Bk_2^2 + Ck_3^2) - A'k_2k_3. \tag{7.5}$$

Now it must be

$$Ak_1 = \pm\kappa\sin\varphi, \quad Bk_2 - A'k_3 = \pm\kappa\cos\varphi, \quad Ck_3 - A'k_2 = 0,$$

i.e.

$$\kappa = \pm Ak, \quad (B - A)C - A'^2 = 0, \quad \varphi' = \frac{AA'k\cos\varphi}{BC - A'^2}.$$

If the curvature solves

$$k'' + \frac{1}{2}k^3 = Ek,$$

($E = T^2/A^2$) we have a four parameters family of Grioli's solution. Similar solutions have been considered in rod mechanics in [22].

To obtain some example of nonlinear strain-energy is convenient to rewrite the equations (7.4) in the intrinsic frame. In so doing we obtain

$$\frac{\partial W}{\partial k} = \kappa, \quad \frac{\partial W}{\partial \varphi} = 0, \quad \frac{\partial W}{\partial k_3} = 0. \tag{7.6}$$

For example it is sufficient to consider the strain-energy in the form $W = G(k) + H(\varphi, k_3)$, G, H arbitrary function of their argument and to use the first equation in (7.6) to determine κ as a function of the curvature and to choose the function H such that the remaining equations in (7.6) are dependent i.e. they reduce to only one relation from which is formally possible to determine φ.

8. Concluding remarks

This note is devoted to symmetric Kirchhoff's rods in the fully nonlinear case. Despite the fact that we have been able to produce a simple and direct proof of the general formal integrability of this theory, our main goal was to find special solutions that have helical shape. Indeed, in complete analogy with Poinsot's motion for a gyroscope, we prove that helical curves (circular helices or Lancret's helices) are the only general solutions to the no tension case. Then we are able to characterize all Lancret's helices that can be sustained in a symmetric nonlinear Kirchhoff's rod when the tension is not null. These helices are very special curves and they can be sustained only by very particular classes of materials. These results are related to the findings contained in [2] and clearly discussed in the beautiful paper [8], but in these papers only circular helices and sufficient conditions are considered.

Our results about helices are also valid for inhomogeneous constitutive equations. In this framework, we provide more complete and general results than the ones provided in [16].

Finally we investigate if the idea of generalized precessions introduced by Grioli to obtain special solutions for the rigid body can work in the framework of nonlinear and non symmetric Kirchhoff rods. We succeeded in generating some of these solutions but in rod mechanics they are planar and non uniform.

Our results provide a clear mechanical ground to some recent results by [26, 31] and [5, 6]. These results are obtained using a variational formalism. As shown in [29] the variational approach is equivalent to the one considered here. On the other hand it must be pointed out that in previous studies other authors restrict themselves only to the special case of uniform center-lines. Therefore, we provide not only a clear and direct mechanical interpretation but also a significant generalization of their results.

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