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Mechanics — *Upper and lower estimates of multicomponent convection instability threshold via auxiliary Bénard problems*, by Salvatore Rionero, communicated on December 16, 2016.

This paper is dedicated to the memory of Professor Giuseppe Grioli.

Abstract. — The onset of convection in a m-component Navier–Stokes fluid mixture, filling a horizontal layer L heated from below and salted partly from above and partly from below, $\forall m \in \mathbb{N}$, is investigated. The difficulties of handling this nonlinear problem, grow drastically with m and only under special circumstances the instability threshold can be given in algebraic simple closed form. Then the problem of finding, in simple algebraic closed forms, suitable lower and upper estimates of the instability threshold arises. In the present paper, for any type of boundary (rigid-rigid, one rigid-one free, free-free), it is shown that: 1) a linearization principle in the energy norm holds; 2) upper and lower estimates of the instability threshold can be obtained via auxiliary classical Bénard problems. The estimates obtained appear to be of interest not only for theoreticians but also for experimentalists investigating natural phenomena and/or industrial processes related to the onset of convection.

Key words: Multi-component mixtures, convection, instability threshold, upper and lower estimates, global stability, Bénard auxiliary systems

Mathematics Subject Classification: 76D05, 76RXX, 35B35

1. Introduction

The heat and mass transfer by convection in horizontal layers attracts the attention of many scientists, theoreticians and experimentalists. The books and the papers devoted – in various physical circumstances – to the onset of convection, constitute a very extensive literature (see [1]–[32], and the references therein). This is because convection occurs in many natural phenomena (of atmosphere, climate, meteorology, sea water, geothermically heated lakes, Earth's core, air and water pollution, subterranean flows, . . .), industrial processes and technological problems (thermal engineering, solar cells, insulation of walls, crystal growth in vapor transport process, . . .). In absence of chemical species (salts) dissolved in the fluid, the convection phenomenon is called *Bénard Problem* because of his laboratory experiments (1900). It is named *binary (or double), ternary, quaternary, . . . , m-component convection according to the number of salts* $(1, 2, 3, \dots, m)$, *dissolved in the fluid*. Although the subject of double diffusive convection is still a very active area of research ([25]–[28]), its more difficult counterpart involving more than two components, has been increasingly attracting attention (see [10]–[12], [29]–[32] and the references therein). This is because in natural phenomena and in industrial processes, the fluid mixtures at stake generally have various salts dissolved in.

This paper is concerned with the onset of convection in a horizontal layer L filled by a Navier–Stokes m-component fluid mixture – heated from below and salted from below by $r \geq 0$ and from above by $m - r \geq 0$ salts – $\forall m \in \mathbb{N}$. The difficulties in handling this nonlinear problem grow rapidly with m. Denoting by P_r, P_α, R, R_α, $(\alpha = 1, 2, \dots, m)$, the fluid and salts Prandtl and Rayleigh numbers and by R_C the critical value of R^2 such that convection occurs if and only if $R^2 \geq R_C$, R_C is given in simple algebraic closed form only for $m < 2$. In fact the determination of R_C, already in the case $m = 2$, is not easy and becomes very difficult for $m > 2$ (see [12], pp. 386–387). As far as we know, for $m > 2$, in the existing literature, only numerical evaluations are provided. In the free-free case lower bounds of R_C have been found in [12] by looking for symmetries and skew-symmetries hidden in the system equations governing the perturbations to the thermal conduction. In the present paper our target is to obtain – via a new strategy – estimates of R_C, in simple algebraic closed form, for any type of boundary (rigid-rigid, one rigid-one free, free-free). Precisely our aim, first of all, is to show that $\forall m \in \mathbb{N}$, an Energy Linearization Principle holds and that an estimate of the thermal critical Rayleigh number $R_C(m)$ of a m-component fluid mixture can be given by the analogous number of a suitable auxiliary $(m-1)$-component virtual fluid mixture and hence, by successive applications to obtain that a lower bound of $R_C(m)$ can be given by the critical Rayleigh number of an auxiliary virtual Bénard problem. Analogously, an upper bound of $R_C(m)$ can be given by introducing a second auxiliary virtual Bénard problem. Denoting by $F(m)$ the fluid mixture at hand and by $E(m)$ and $\hat{E}(m)$, the L^2-energy of the perturbations to the thermal conduction, evaluated respectively along their non-linear and linear evolution equations, we obtain the following basic properties.

Property 1 (Linearization Principle). *For any $m \in \mathbb{N}$, the decay of $\hat{E}(m)$ at $t = 0$, for any admissible initial data, implies the decay of $E(m)$ at any instant.*

Property 2. *There exists an auxiliary virtual fluid mixture F_1 with $m - 1$ components, which energy $\hat{E}_1(m-1)$ is lower than $\hat{E}(m)$ and such that*

$$R_C(F_1) \leq R_C. \tag{1.1}$$

Property 3. *For any $m \in \mathbb{N}$,*

$$R^2 < \sum_{\alpha=1}^{r} \frac{R_\alpha^2}{P_\alpha} - \sum_{\alpha=r+1}^{m} \frac{R_\alpha^2}{P_\alpha} + \frac{\gamma}{P_*}, \tag{1.2}$$

with

$$P_* = \max(1, P_1, P_2, \dots, P_m) \tag{1.3}$$

and γ critical Rayleigh number of the classical Bénard problem associated to the boundary conditions given by

$$\gamma = \begin{cases} 1707.762, & \text{rigid-rigid boundary,} \\ 1100.65, & \text{one rigid-one free boundary,} \\ \dfrac{27}{4}\pi^4, & \text{free-free boundary,} \end{cases} \tag{1.4}$$

guarantees the nonlinear global asymptotic stability of the thermal conduction (absence of convection).

PROPERTY 4. *For any $m \in \mathbb{N}$,*

$$R^2 > \sum_{\alpha=1}^{r} \frac{R_\alpha^2}{P_\alpha} - \sum_{\alpha=r+1}^{m} \frac{R_\alpha^2}{P_\alpha} + \frac{\gamma}{\bar{P}_*}, \tag{1.5}$$

with

$$\bar{P}_* = \min(1, P_1, \ldots, P_m), \tag{1.6}$$

guarantees the instability of thermal conduction (onset of convection).

2. PRELIMINARIES

Let $\mathtt{L}$ be a horizontal layer of depth d filled by a Navier–Stokes fluid mixture in which m chemical species ("salts") S_α ($\alpha = 1, 2, \ldots, m$), are dissolved in and let $Oxyz$ be an orthogonal frame of reference with fundamental unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ ($\mathbf{k}$ pointing vertically upwards). We suppose that $\mathtt{L}$ is uniformly heated from below and salted, from below by $r \geq 0$ and from above by $m - r \geq 0$ salts. The equations governing the fluid motion, in the Boussinesq approximation, are [30]:

$$\begin{cases} \rho_0(\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v}) = -\nabla p + \rho_0 \nu \Delta \mathbf{v} \\ \qquad -\rho_0 \left[1 - A(T - T_0) + \displaystyle\sum_{\alpha=1}^{m} A_\alpha (C_\alpha - \hat{C}_\alpha) \right] g\mathbf{k}, \\ \nabla \cdot \mathbf{v} = 0, \\ T_t + \mathbf{v}\cdot\nabla T = k\Delta T, \\ C_{\alpha t} + \mathbf{v}\cdot\nabla C_\alpha = k_\alpha \Delta C_\alpha, \quad \alpha = 1, 2, \ldots, m, \end{cases} \tag{2.1}$$

with

$$\rho_0 = \text{constant density}, \quad \mathbf{x} = (x, y, z), \quad \mathbf{v} = \text{fluid velocity},$$
$$T = \text{temperature}, \quad C_\alpha = \text{salt } S_\alpha \text{ concentration},$$
$$p = \text{pressure}, \quad T_0 = \text{reference temperature}, \quad \mathbf{g} = -g\mathbf{k} = \text{gravity},$$

$\hat{C}_\alpha =$ reference salt S_α concentration, $\quad \nu =$ kinematic viscosity,
$A =$ thermal expansion coefficient, $\quad A_\alpha =$ salt S_α expansion coefficient,
$k =$ thermal diffusivity, $\quad k_\alpha =$ salt S_α diffusivity, $\quad \alpha = 1, 2, \ldots, m.$

To (2.1) we append the boundary conditions

$$
(2.2) \qquad \begin{cases} T(x, y, 0, t) = T_l, \quad T(x, y, d, t) = T_u, \quad T_l > T_u \\ C_\alpha(x, y, 0, t) = C_{\alpha_l}, \quad C_\alpha(x, y, d, t) = C_{\alpha_u}, \quad \alpha = 1, 2, \ldots, m \\ \mathbf{v} \cdot \mathbf{k} = 0, \quad \text{on } z = 0, d, \end{cases}
$$

and the stress-free boundary conditions. The boundary value problem (2.1)–(2.2) admits the thermal conduction solution $\overline{m}_0 = (\bar{p}, \bar{\mathbf{v}}, \overline{T}, \overline{C}_1, \ldots, \overline{C}_m)$ given by

$$
(2.3) \qquad \begin{cases} \bar{\mathbf{v}} = \mathbf{0}, \quad \overline{T} = T_l - \dfrac{\delta T}{d} z, \quad \overline{C}_\alpha = C_{\alpha_l} - \dfrac{\delta C_\alpha}{d} z, \\ \delta T = T_l - T_u, \quad \delta C_\alpha = C_{\alpha_l} - C_{\alpha_u}, \quad \alpha = 1, 2, \ldots, m \\ \bar{p}(z) = \bar{p}_0 - \rho_0 g z \left[1 - A(T_l - T_0) + \displaystyle\sum_{\alpha=1}^m A_\alpha (C_{\alpha l} - \hat{C}_\alpha) \right] \\ \quad - \dfrac{\rho_0 g z^2}{2d} \left[A \delta T - \displaystyle\sum_{\alpha=1}^m A_\alpha \delta C_\alpha \right], \quad \bar{p}_0 = \text{const.} > 0. \end{cases}
$$

Setting

$$
(2.4) \qquad p = \bar{p} + \pi, \quad \mathbf{v} = \bar{\mathbf{v}} + \mathbf{u}, \quad T = \overline{T} + \theta, \quad C_\alpha = \overline{C}_\alpha + \Phi_\alpha,
$$

introducing the non dimensional scalings

$$
(2.5) \qquad \begin{cases} t = t^* \dfrac{d^2}{k}, \quad \mathbf{u} = \mathbf{u}^* \dfrac{\nu}{d}, \quad \pi = \pi^* \dfrac{\nu^2 \rho_0}{d^2}, \quad \mathbf{x} = \mathbf{x}^* d, \quad \theta = \theta^* T^\#, \\ \Phi_\alpha = \Phi_\alpha^* \Phi_\alpha^\#, \quad T^\# = \left(\dfrac{\nu^3 |\delta T|}{A g k d^3} \right)^{\frac{1}{2}}, \quad \Phi_\alpha^\# = \left(\dfrac{\nu^3 |\delta C_\alpha|}{A_\alpha g k_\alpha d^3} \right)^{\frac{1}{2}}, \end{cases}
$$

(2.1) (omitting the asterisks) reduces to

$$
(2.6) \qquad \begin{cases} P_r^{-1} \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \pi + \Delta \mathbf{u} + \left(R\theta - \displaystyle\sum_{\alpha=1}^m R_\alpha \Phi_\alpha \right) \mathbf{k}, \\ \nabla \cdot \mathbf{u} = 0, \\ \theta_t + P_r \mathbf{u} \cdot \nabla \theta = R w + \Delta \theta, \\ P_\alpha (\Phi_{\alpha t} + P_r \mathbf{u} \cdot \nabla \Phi_\alpha) = H_\alpha R_\alpha w + \Delta \Phi_\alpha, \quad \alpha = 1, 2, \ldots, m \end{cases}
$$

with

$$(2.7)\quad \begin{cases} R^2 = \dfrac{Agd^3|\delta T|}{\nu k} \text{ thermal Rayleigh number,} \\ R_\alpha^2 = \dfrac{A_\alpha g d^3 |\delta C_\alpha| P_\alpha}{\nu k} \text{ salt } S_\alpha \text{ Rayleigh number,} \quad H_\alpha = \mathrm{sgn}(\delta C_\alpha), \\ P_r = \dfrac{\nu}{k} \text{ fluid Prandtl number,} \quad P_\alpha = \dfrac{k}{k_\alpha} \text{ salt } S_\alpha \text{ Prandtl number.} \end{cases}$$

To (2.6) the boundary conditions of the free-free or rigid-rigid or rigid-free or free-rigid case have to be appended. Setting $\mathbf{u} = (u, v, w)$, the boundary conditions to be taken are [1]

$$(2.8)\quad \begin{cases} u = v = w = \dfrac{\partial w}{\partial z} = \theta = \Phi_\alpha = 0, \quad \text{on rigid boundary;} \\ \dfrac{\partial u}{\partial z} = \dfrac{\partial v}{\partial z} = w = \dfrac{\partial^2 w}{\partial z^2} = \theta = \Phi_\alpha = 0, \quad \text{on stress-free boundary.} \end{cases}$$

We assume that:

i) the perturbations $(\nabla\pi, u, v, w, \theta, \Phi_1, \dots, \Phi_m)$ are periodic in the x and y directions, respectively of periods $2\pi/a_x$, $2\pi/a_y$;

ii) $\Omega = [0, 2\pi/a_x] \times [0, 2\pi/a_y] \times [0, 1]$ is the periodicity cell;

iii) u, v, w, θ, $\Phi_1, \dots, \Phi_m$ are such that together with all their first derivatives and second spatial derivatives are square integrable in Ω, $\forall t \in \mathbb{R}^+$ and can be expanded in a Fourier series uniformly convergent in Ω via complete orthogonal sequences $\{\varphi_n\}$, $\{\bar\varphi_n\}$, $\{\bar{\bar\varphi}_n\}$ – according to the boundary case at stark – with eigenvalues

$$(2.9)\qquad \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots$$

Let us denote by $\mathscr{A}(\Omega)$ the set of functions Ψ such that:

1) $\Psi : (\mathbf{x}, t) \in \Omega \times \mathbb{R}^+ \to \Psi(\mathbf{x}, t) \in \mathbb{R}$, $\Psi \in W^{2,2}(\Omega)$, $\forall t \in \mathbb{R}^+$, Ψ is periodic in the x and y directions of period $\dfrac{2\pi}{a_x}$, $\dfrac{2\pi}{a_y}$ respectively;

2) Ψ, together with all the first derivatives and second spatial derivatives, can be expanded in a Fourier series absolutely uniformly convergent in Ω, $\forall t \in \mathbb{R}^+$;

3) $(\Psi)_{z=0} = (\Psi)_{z=1} = 0$

and by $\mathscr{B}(\Omega)$ the set of eigenfunctions $\bar\varphi$ verifying 1)–2) and

4) $\left[\dfrac{\partial\bar\varphi}{\partial z}\right]_{z=0} = \left[\dfrac{\partial\bar\varphi}{\partial z}\right]_{z=1} = 0.$

Since the sequence $\{\sin n\pi z\}_{n\in\mathbb{N}}$, is a complete orthogonal system for $L^2(0,1)$ under the boundary conditions $[\Psi]_{z=0} = [\Psi]_{z=1} = 0$, by virtue of periodicity, it turns out that $\forall\Psi \in \mathscr{A}(\Omega)$, there exists a sequence $\{\tilde{\Psi}_n(x, y, t)\}$ such that

$$(2.10)\qquad \begin{cases} \Psi = \displaystyle\sum_{n=1}^{\infty} \tilde{\Psi}_n \sin n\pi z, \quad \frac{\partial\Psi}{\partial t} = \sum_{n=1}^{\infty} \frac{\partial\tilde{\Psi}_n}{\partial t} \sin n\pi z, \\ \Delta_1\Psi = -a^2\Psi, \quad \Delta\Psi = -a^2\Psi + \dfrac{\partial^2\Psi}{\partial z^2}, \\ \Delta_1 = \partial^2/\partial x^2 + \partial^2/\partial y^2, \quad a^2 = a_x^2 + a_y^2 \end{cases}$$

the series appearing in (2.10) being absolutely uniformly convergent in Ω. Analogously, since the sequence $\{\cos n\pi z\}_{n\in\mathbb{N}}$ is a complete orthogonal system for $L^2(0,1)$ under the boundary conditions $\left[\dfrac{\partial\bar{\varphi}}{\partial z}\right]_{z=0} = \left[\dfrac{\partial\bar{\varphi}}{\partial z}\right]_{z=1} = 0$, by virtue of periodicity, it turns out that $\forall\bar{\varphi} \in \mathscr{B}(\Omega)$, there exists a sequence $\{\tilde{\varphi}_n(x, y, t)\}$

$$(2.11)\qquad \begin{cases} \bar{\varphi} = \displaystyle\sum_{n=1}^{\infty} \bar{\varphi}_n = \sum_{n=1}^{\infty} \tilde{\varphi}_n \cos n\pi z, \quad \frac{\partial\bar{\varphi}}{\partial t} = \sum_{n=1}^{\infty} \frac{\partial\tilde{\varphi}_n}{\partial t} \cos n\pi z, \\ \Delta_1\bar{\varphi} = -a^2\bar{\varphi}, \quad \Delta\bar{\varphi} = -\displaystyle\sum_{n=1}^{\infty} \xi_n\bar{\varphi}_n \cos n\pi z. \end{cases}$$

We end by remarking that:

i) Setting

$$(2.12)\qquad \zeta = (\nabla \times \mathbf{u}) \cdot \mathbf{k} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},$$

the horizontal components of $\mathbf{u}$ are given by

$$(2.13)\qquad u = \frac{1}{a^2}\left(\frac{\partial^2 w}{\partial x\partial z} + \frac{\partial\zeta}{\partial y}\right), \quad v = \frac{1}{a^2}\left(\frac{\partial^2 w}{\partial y\partial z} - \frac{\partial\zeta}{\partial x}\right)$$

and – in view of $\mathbf{u} = \sum_{n=1}^{\infty} \mathbf{u}_n$, $\zeta_n = \dfrac{\partial v_n}{\partial x} - \dfrac{\partial u_n}{\partial y}$ – it follows that

$$(2.14)\qquad \begin{cases} u_n = \dfrac{1}{a^2}\left(\dfrac{\partial^2 w_n}{\partial x\partial z} + \dfrac{\partial\zeta_n}{\partial y}\right), \quad v_n = \dfrac{1}{a^2}\left(\dfrac{\partial^2 w_n}{\partial y\partial z} - \dfrac{\partial\zeta_n}{\partial x}\right), \\ \nabla \cdot \mathbf{u}_n = \left(\dfrac{1}{a^2}\Delta_1 w_n + w_n\right)_z = 0 \end{cases}$$

ii) on the free boundary $w, \theta, \Phi_\alpha \in \mathscr{A}(\Omega)$ and $u, v \in \mathscr{B}(\Omega)$;

iii) on the rigid boundary

$$w = \frac{\partial w}{\partial z} = 0 \tag{2.15}$$

requires a different basis $\{\bar{\bar{\varphi}}_n\}$ given in [1].

3. Linearization principle

Let us introduce the *linear system* ($\alpha = 1, 2, \ldots, m$)

$$\begin{cases} P_r^{-1} \dfrac{\partial \hat{\mathbf{u}}}{\partial t} = \Delta \hat{\mathbf{u}} + \Big(R\hat{\theta} - \displaystyle\sum_{\alpha=1}^{m} R_\alpha \hat{\Phi}_\alpha \Big) \mathbf{k}, \\ \nabla \cdot \hat{\mathbf{u}} = 0, \\ \dfrac{\partial \hat{\theta}}{\partial t} = R\hat{w} + \Delta\hat{\theta}, \\ P_\alpha \dfrac{\partial \hat{\Phi}_\alpha}{\partial t} = H_\alpha R_\alpha \hat{w} + \Delta \hat{\Phi}_\alpha, \quad \alpha = 1, 2, \ldots, m \end{cases} \tag{3.1}$$

under the initial boundary conditions

$$\begin{cases} (\hat{\mathbf{u}})_{(t=0)} = \hat{\mathbf{u}}^{(0)}, \quad (\hat{\theta})_{(t=0)} = \hat{\theta}^{(0)}, \quad (\hat{\Phi}_\alpha)_{(t=0)} = \hat{\Phi}_\alpha^{(0)}, \\ \hat{u}_z = \hat{v}_z = \hat{w} = \hat{\theta} = \hat{\Phi}_\alpha = 0, \quad \alpha = 1, 2, \ldots, m, \quad \text{on stress-free boundary}, \\ \hat{u} = \hat{v} = \hat{w} = \dfrac{\partial \hat{w}}{\partial z} = \hat{\theta} = \hat{\Phi}_\alpha = 0, \quad \alpha = 1, 2, \ldots, m, \quad \text{on rigid boundary}. \end{cases} \tag{3.2}$$

We denote by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ the $L^2(\Omega)$-norm and the scalar product respectively and introduce the energy

$$E = \frac{1}{2}\left[P_r^{-1} \|\mathbf{u}\|^2 + \|\theta\|^2 + \sum_{\alpha=1}^{m} \|\Phi_\alpha\|^2 \right], \tag{3.3}$$

of (2.6), (2.8) solutions and the energy

$$\hat{E} = \frac{1}{2}\left[P_r^{-1} \|\hat{\mathbf{u}}\|^2 + \|\hat{\theta}\|^2 + \sum_{\alpha=1}^{m} \|\hat{\Phi}_\alpha\|^2 \right], \tag{3.4}$$

of (3.1)–(3.2) solutions with

$$\|\mathbf{u}\|^2 = \|u\|^2 + \|v\|^2 + \|w\|^2, \quad \|\hat{\mathbf{u}}\|^2 = \|\hat{u}\|^2 + \|\hat{v}\|^2 + \|\hat{w}\|^2. \tag{3.5}$$

Denoting by Q, $\bar{Q}$ the quadratic forms

$$Q = \begin{cases} -\Big(\|\nabla \mathbf{u}\|^2 + \|\nabla\theta\|^2 + \sum_{\alpha=1}^{m} \frac{1}{P_\alpha}\|\nabla\Phi_\alpha\|^2\Big) \\ \quad + \Big\langle 2R\theta - \sum_{\alpha=1}^{m}\Big(1 - \frac{H_\alpha}{P_\alpha}\Big)R_\alpha\Phi_\alpha, w\Big\rangle, \end{cases} \tag{3.6}$$

$$\hat{Q} = \begin{cases} -\Big(\|\nabla \hat{\mathbf{u}}\|^2 + \|\nabla\hat{\theta}\|^2 + \sum_{\alpha=1}^{m} \frac{1}{P_\alpha}\|\nabla\hat{\Phi}_\alpha\|^2\Big) \\ \quad + \Big\langle 2R\hat{\theta} - \sum_{\alpha=1}^{m}\Big(1 - \frac{H_\alpha}{P_\alpha}\Big)R_\alpha\hat{\Phi}_\alpha, \hat{w}\Big\rangle, \end{cases} \tag{3.7}$$

it follows that the time derivatives of E and $\hat{E}$ respectively along (2.6), (2.8) and (3.1)–(3.2) are given by

$$\frac{dE}{dt} = Q, \quad \frac{d\hat{E}}{dt} = \hat{Q}. \tag{3.8}$$

Then the following linearization principle holds.

Theorem 3.1. *Let*

$$\Big(\frac{d\hat{E}}{dt}\Big)_{(t=0)} < 0, \tag{3.9}$$

for arbitrary (*admissible*) *initial data. Then*

$$\Big(\frac{dE}{dt}\Big) < 0, \quad \forall t \geq 0. \tag{3.10}$$

Proof. Let $\tau \in \mathbb{R}^+$. Then at $t = \tau$ it follows that

$$\begin{aligned} \Big(\frac{dE}{dt}\Big)_{(t=\tau)} = Q_{(t=\tau)} &= -\Big(\|\nabla \mathbf{u}\|^2 + \|\nabla\theta\|^2 + \sum_{\alpha=1}^{m} \frac{1}{P_\alpha}\|\nabla\Phi_\alpha\|^2\Big)_{(t=\tau)} \\ &\quad + \Big\langle 2R\theta(\tau) - \sum_{\alpha=1}^{m}\Big(1 - \frac{H_\alpha}{P_\alpha}\Big)R_\alpha\Phi_\alpha(\tau), w(\tau)\Big\rangle. \end{aligned} \tag{3.11}$$

Then, on choosing

$$(\hat{\mathbf{u}})_{(t=0)} = \mathbf{u}(\tau), \quad (\hat{\theta})_{(t=0)} = \theta(\tau), \quad (\hat{\Phi}_\alpha)_{(t=0)} = \Phi_\alpha(\tau), \tag{3.12}$$

it follows that (in view of (3.9) for any initial data)

$$Q_{(t=\tau)} = \hat{Q}_{(t=0)} \tag{3.13}$$

and hence (3.10) immediately follows.

REMARK 3.1. We remark that:

i) denoting by $\hat{\mathbf{u}}_n$, $\hat{\theta}_n$, $\hat{\Phi}_{\alpha n}$ the Fourier components of $\hat{\mathbf{u}}$, $\hat{\theta}$, $\hat{\Phi}_\alpha$ and denoting by

$$\hat{E}_n = \frac{1}{2}\left[P_r^{-1}\|\hat{\mathbf{u}}_n\|^2 + \|\hat{\theta}_n\|^2 + \sum_{\alpha=1}^{m} P_\alpha \|\hat{\Phi}_{\alpha n}\|^2\right], \tag{3.14}$$

the $L^2(\Omega)$-energy of $\hat{\mathbf{u}}_n$, $\hat{\theta}_n$, $\hat{\Phi}_{\alpha n}$, it follows that

$$\begin{cases} \hat{E} = \displaystyle\sum_{n=1}^{\infty} \hat{E}_n, \\ \dfrac{d\hat{E}_n}{dt} = \hat{Q}_n = -\Big(\|\nabla\hat{\mathbf{u}}_n\|^2 + \|\nabla\hat{\theta}_n\|^2 + \displaystyle\sum_{\alpha=1}^{m} \frac{1}{P_\alpha}\|\nabla\hat{\Phi}_{\alpha n}\|^2\Big) \\ \qquad + \Big\langle 2R\hat{\theta}_n - \displaystyle\sum_{\alpha=1}^{m}\Big(1 - \frac{H_\alpha}{P_\alpha}\Big) R_\alpha \hat{\Phi}_{\alpha n}, \hat{w}_n \Big\rangle \end{cases} \tag{3.15}$$

and hence

$$\Big(\frac{d\hat{E}_n}{dt}\Big)_{(t=0)} < 0, \tag{3.16}$$

for any initial data $\forall n \in \mathbb{N}$, implies (3.10);

ii) $\hat{\mathbf{u}}_n$, $\hat{\theta}_n$, $\hat{\Phi}_{\alpha n}$ are governed – in view of the linearity of (3.1) – by

$$\begin{cases} P_r^{-1}\dfrac{\partial \hat{\mathbf{u}}_n}{\partial t} = \Delta\hat{\mathbf{u}}_n + \Big(R\hat{\theta}_n - \displaystyle\sum_{\alpha=1}^{m} R_\alpha \hat{\Phi}_{\alpha n}\Big)\mathbf{k}, \\ \nabla\cdot\hat{\mathbf{u}}_n = 0, \\ \dfrac{\partial\hat{\theta}_n}{\partial t} = R\hat{w}_n + \Delta\hat{\theta}_n, \\ P_\alpha \dfrac{\partial\hat{\Phi}_{\alpha n}}{\partial t} = H_\alpha R_\alpha \hat{w}_n + \Delta\hat{\Phi}_{\alpha n}, \quad \alpha = 1, 2, \ldots, m \end{cases} \tag{3.17}$$

with

$$\begin{cases} \dfrac{\partial\hat{u}_n}{\partial z} = \dfrac{\partial\hat{v}_n}{\partial z} = \hat{w}_n = \hat{\theta}_n = \hat{\Phi}_{\alpha n} = 0, \quad \text{on stress-free boundary}, \\ \hat{u}_n = \hat{v}_n = \hat{w}_n = \dfrac{\partial\hat{w}_n}{\partial z} = \hat{\theta}_n = \hat{\Phi}_{\alpha n} = 0, \quad \text{on rigid boundary}. \end{cases} \tag{3.18}$$

iii) $\hat{u}_n$, $\hat{v}_n$ in view of (2.10) are of type

$$\Psi_n = \tilde{\Psi}_n(x, y, t)\varphi_n'(z) \tag{3.19}$$

and it follows that (Poincarè inequality)

$$\langle \Psi_n, \Delta\Psi_n \rangle < -\pi^2 \|\Psi_n\|^2; \tag{3.20}$$

iv) $(3.17)_1$ implies

$$P_r^{-1}\frac{\partial \hat{u}_n}{\partial t} = \Delta\hat{u}_n, \quad P_r^{-1}\frac{\partial \hat{v}_n}{\partial t} = \Delta\hat{v}_n, \tag{3.21}$$

then, by virtue of $(3.20)_3$, one obtains

$$\frac{1}{2}P_r^{-1}(\|\hat{u}_n\|^2 + \|\hat{v}_n\|^2) < -\pi^2(\|\hat{u}_n\|^2 + \|\hat{v}_n\|^2) < 0; \tag{3.22}$$

v) setting

$$\tilde{\mathscr{E}}_n = \frac{1}{2}\left[P_r^{-1}\|\tilde{\mathbf{u}}_n\|^2 + \|\tilde{\theta}_n\|^2 + \sum_{\alpha=1}^{m} P_\alpha \|\tilde{\Phi}_{\alpha n}\|^2\right], \tag{3.23}$$

then

$$\left(\frac{d\tilde{\mathscr{E}}_n}{dt}\right)_{(t=0)} < 0, \tag{3.24}$$

$\forall n \in \mathbb{N}$ and for any initial data, implies (3.9) for any initial data;

vi) equations governing $\hat{w}_n$, $\hat{\theta}_n$, $\hat{\Phi}_{\alpha n}$ can be reduced to

$$\begin{cases} P_r^{-1}\dfrac{\partial \Delta\hat{w}_n}{\partial t} = \Delta_1\left(R\hat{\theta}_n - \displaystyle\sum_{\alpha=1}^{m} R_\alpha \hat{\Phi}_{\alpha n}\right) + \Delta\Delta\hat{w}_n, \\ \dfrac{\partial \hat{\theta}_n}{\partial t} = R\hat{w}_n + \Delta\hat{\theta}_n, \\ P_\alpha \dfrac{\partial \hat{\Phi}_{\alpha n}}{\partial t} = H_\alpha R_\alpha \hat{w}_n + \Delta\hat{\Phi}_{\alpha n} \end{cases} \tag{3.25}$$

where $(3.25)_1$ is the third component of the double curl of $(3.17)_1$.

Remark 3.2. In view of theorem 3.1, one is led to find conditions, necessary and sufficient, for guaranteeing (3.9). As matter of fact, these conditions require that the eigenvalues of (3.1)–(3.2) have, $\forall n \in \mathbb{N}$, negative real part.

Then the solutions of the system are given, $\forall n \in \mathbb{N}$, by functions of type $f_n = \exp[(a_n + ib_n)t]g_n(x, y, z)$ and of type $F_n = t^q[\exp(a_n + ib_n)t]g_n(x, y)$ with $a_n < 0$ and $q = 1, 2, \ldots$, according to $a_n + ib_n$ is a double, triple,... root of the spectral equation. Denoting by $\bar{f}_n$ and $\bar{F}_n$ the complex conjugate of f_n and F_n, it follows that

$$
(3.26) \qquad \begin{cases}
\dfrac{1}{2}\dfrac{d}{dt}\langle f_n, \bar{f}_n\rangle = \dfrac{1}{2}\dfrac{d}{dt}e^{2a_n t}\|g_n\|^2 = a_n e^{2a_n t}\|g_n\|^2 < 0, \\
\left[\dfrac{1}{2}\dfrac{d}{dt}\langle f_n, \bar{f}_n\rangle\right]_{(t=0)} = a_n\|g_n\|^2, \\
\left[\dfrac{1}{2}\dfrac{d}{dt}\langle F_n, \bar{F}_n\rangle\right]_{(t=0)} = [(qt^{2q-1} + a_n t^{2q})e^{2a_n t}]_{(t=0)}\|g_n\| = 0.
\end{cases}
$$

Hence (3.9) is verified for any initial data. Therefore, the necessary and sufficient conditions guaranteeing that the eigenvalues of (3.1) have, $\forall n \in \mathbb{N}$, negative real part – in view of theorem 3.1 – are necessary and sufficient for guaranteeing that Q, along the solutions of (2.6), is negative definite. Vice-versa, conditions guaranteeing that Q is negative definite are only sufficient to guarantee that (3.1)-eigenvalues have, $\forall n \in \mathbb{N}$, negative real part. These conditions can be too restrictive and not able to put in evidence the stabilizing effect of skew-symmetric terms appearing in (2.6), but giving a zero contribution to Q_n. This happens, for instance, when $\mathtt{L}$ rotates uniformly about z. Then in $(2.6)_1$ appears the term $\mathscr{T}\mathbf{u} \times \mathbf{k}$, with $\mathscr{T}$ (Taylor number) positive constant and, in view of $\mathbf{u} \cdot \mathbf{u} \times \mathbf{k} = 0$, does not give any contribution to Q. Further this happens also in MHD convection [13].

4. Basic lemmas for lower estimates of the instability threshold

Setting

$$
(4.1) \qquad \hat{\theta}^* = R\hat{\theta}, \quad \hat{\mathbf{\Phi}}^*_\alpha = R_\alpha\hat{\mathbf{\Phi}}_\alpha,
$$

(3.25) in view of (3.20) omitting the stars, becomes

$$
(4.2) \qquad \begin{cases}
P_r^{-1}\dfrac{\partial \Delta\hat{w}_n}{\partial t} = \Delta_1\left(\hat{\theta}_n - \displaystyle\sum_{\alpha=1}^{m}\hat{\Phi}_{\alpha n}\right) + \Delta\Delta\hat{w}_n, \\
\dfrac{\partial\hat{\theta}_n}{\partial t} = R^2\hat{w}_n + \Delta\hat{\theta}_n, \\
P_\alpha\dfrac{\partial\hat{\Phi}_{\alpha n}}{\partial t} = R_\alpha^2\hat{w}_n + \Delta\hat{\Phi}_{\alpha n}, \quad \alpha = 1, 2, \ldots, r, \\
P_\alpha\dfrac{\partial\hat{\Phi}_{\alpha n}}{\partial t} = -R_\alpha^2\hat{w}_n + \Delta\hat{\Phi}_{\alpha n}, \quad \alpha = r+1, \ldots, m,
\end{cases}
$$

with

$$\begin{cases} \hat{w}_n = \hat{\theta}_n = \hat{\Phi}_{\alpha n} = 0, \quad \forall n \in \mathbb{N}, \text{ on stress-free boundary,} \\ \hat{w}_n = \dfrac{\partial \hat{w}_n}{\partial z} = \hat{\theta}_n = \hat{\Phi}_{\alpha n} = 0, \quad \forall n \in \mathbb{N}, \text{ on rigid boundary.} \end{cases} \tag{4.3}$$

The following basic Lemmas hold.

Lemma 4.1. *Let*

$$P_{ij} = \max(P_i, P_j), \quad i \neq j \in \{1, 2, \dots, n\} \tag{4.4}$$

and let us associate to

$$\begin{cases} \dfrac{\partial \hat{\Phi}_{in}}{\partial t} = \dfrac{H_i R_i^2}{P_i} \hat{w}_n + \dfrac{1}{P_i} \Delta \hat{\Phi}_{in}, \\ \dfrac{\partial \hat{\Phi}_{jn}}{\partial t} = \dfrac{H_j R_j^2}{P_j} \hat{w}_n + \dfrac{1}{P_j} \Delta \hat{\Phi}_{jn}, \end{cases} \tag{4.5}$$

under the boundary conditions (4.3)*, the auxiliary system*

$$\begin{cases} \dfrac{\partial \bar{\Phi}_{in}}{\partial t} = \dfrac{H_i R_i^2}{P_i} \overline{w}_n + \dfrac{1}{P_{ij}} \Delta \bar{\Phi}_{in}, \\ \dfrac{\partial \bar{\Phi}_{jn}}{\partial t} = \dfrac{H_j R_j^2}{P_j} \overline{w}_n + \dfrac{1}{P_{ij}} \Delta \bar{\Phi}_{jn}, \end{cases} \tag{4.6}$$

under the boundary conditions

$$\begin{cases} \overline{w}_n = \bar{\theta}_n = \bar{\Phi}_{in} = \bar{\Phi}_{jn} 0, \quad \forall n \in \mathbb{N}, \textit{ on stress-free boundary,} \\ \overline{w}_n = \dfrac{\partial \overline{w}_n}{\partial z} = \bar{\theta}_n = \bar{\Phi}_{in} = \bar{\Phi}_{jn} 0, \quad \forall n \in \mathbb{N}, \textit{ on rigid boundary.} \end{cases} \tag{4.7}$$

Then

$$\left[\frac{d}{dt} (\|\bar{\Phi}_{in}\|^2 + \|\bar{\Phi}_{jn}\|^2) \right]_{(t=0)} < 0, \tag{4.8}$$

for any initial data, implies

$$\left[\frac{d}{dt} (\|\hat{\Phi}_{in}\|^2 + \|\hat{\Phi}_{jn}\|^2) \right]_{(t=0)} < 0, \tag{4.9}$$

for any initial data.

PROOF. It easily follows that

$$
(4.10)\quad \begin{cases}
\left[\dfrac{d}{dt}(\|\bar{\Phi}_{in}\|^2 + \|\bar{\Phi}_{jn}\|^2)\right]_{(t=0)} \\
\quad = 2\left[\left\langle \bar{w}_n^{(0)}, \dfrac{H_i R_i^2}{P_i}\bar{\Phi}_{in}^{(0)} + \dfrac{H_j R_j^2}{P_j}\bar{\Phi}_{jn}^{(0)} \right\rangle - \dfrac{1}{P_{ij}}(\|\nabla\bar{\Phi}_{in}^{(0)}\|^2 + \|\nabla\bar{\Phi}_{jn}^{(0)}\|^2)\right], \\
\left[\dfrac{d}{dt}(\|\hat{\Phi}_{in}\|^2 + \|\hat{\Phi}_{jn}\|^2)\right]_{(t=0)} \\
\quad = 2\left[\left\langle \hat{w}_n^{(0)}, \dfrac{H_i R_i^2}{P_i}\hat{\Phi}_{in}^{(0)} + \dfrac{H_j R_j^2}{P_j}\hat{\Phi}_{jn}^{(0)} \right\rangle - \dfrac{1}{P_i}\|\nabla\hat{\Phi}_{in}^{(0)}\|^2 - \dfrac{1}{P_j}\|\nabla\hat{\Phi}_{jn}^{(0)}\|^2\right],
\end{cases}
$$

with

$$
(4.11)\qquad f^{(0)} = (f)_{(t=0)}, \quad \forall f \in (\bar{w}, \hat{w}, \bar{\Phi}_i, \hat{\Phi}_i, \bar{\Phi}_j, \hat{\Phi}_j).
$$

On choosing

$$
(4.12)\qquad \bar{w}_n^{(0)} = \hat{w}_n^{(0)}, \quad \bar{\Phi}_{in}^{(0)} = \hat{\Phi}_{in}^{(0)}, \quad \bar{\Phi}_{jn}^{(0)} = \hat{\Phi}_{jn}^{(0)},
$$

in view of (4.10) and (4.4), (4.9) immediately follows.

LEMMA 4.2. *Let*

$$
(4.13)\qquad \left[\frac{d}{dt}(\|\bar{\Phi}_{in} + \bar{\Phi}_{jn}\|^2)\right]_{(t=0)} < 0,
$$

for any initial data. Then

$$
(4.14)\qquad \frac{d}{dt}(\|\bar{\Phi}_{in}\|^2 + \|\bar{\Phi}_{jn}\|^2) < 0,
$$

for any initial data.

PROOF. Setting

$$
(4.15)\qquad \bar{U}_{ijn} = \bar{\Phi}_{in} + \bar{\Phi}_{jn}, \quad \bar{V}_{ijn} = \frac{H_j R_j^2}{P_j}\bar{\Phi}_{in} - \frac{H_i R_i^2}{P_i}\bar{\Phi}_{jn},
$$

(4.6) can be substituted by the equivalent system

$$
(4.16)\qquad \begin{cases}
\dfrac{\partial \bar{U}_{ijn}}{\partial t} = \left(\dfrac{H_i R_i^2}{P_i} + \dfrac{H_j R_j^2}{P_j}\right)\bar{w}_n + \dfrac{1}{P_{ij}}\Delta\bar{U}_{ijn}, \\
\dfrac{\partial \bar{V}_{ijn}}{\partial t} = \dfrac{1}{P_{ij}}\Delta\bar{V}_{ijn},
\end{cases}
$$

under the zero boundary conditions for $z = 0, 1$ implied by (4.7). In view of $(4.16)_2$, $\overline{V}$ appears to be an independent field and since

$$\frac{d}{dt}\|\overline{V}_{ijn}\|^2 = -\frac{2}{P_{ij}}\|\nabla\overline{V}_{ijn}\|^2, \tag{4.17}$$

(4.13) guarantees that the initial energy of (4.16) is negative for any initial data. Then the energy of (4.6) decays if decays the energy of $(4.16)_1$, i.e. of

$$P_{ij}\frac{\partial}{\partial t}(\bar{\Phi}_{in} + \bar{\Phi}_{jn}) = R_{ij}\overline{w}_n + \Delta(\bar{\Phi}_{in} + \bar{\Phi}_{jn}), \tag{4.18}$$

with

$$R_{ij} = P_{ij}\Big(\frac{H_i R_i^2}{P_i} + \frac{H_j R_j^2}{P_j}\Big). \tag{4.19}$$

5. Proof of (1.1)

Starting from (4.2), by successive applications of Lemma 4.2, a "*sequence of stability auxiliary systems*" can be obtained such that the stability of one system of this sequence – implied by the request that all the eigenvalues have negative real part – in view of (3.26), guarantees that, for any initial data, the L^2-energy decreases and tends to zero exponentially.

We now apply the procedure of Lemma 4.2 to the salts salting L from below. Let $r > 1$

$$P_{12} = \max(P_1, P_2), \quad \Psi = \Phi_1 + \Phi_2 \tag{5.1}$$

and associate to (4.2) the auxiliary system

$$\begin{cases} P_r^{-1}\dfrac{\partial\Delta\overline{w}_n}{\partial t} = \Delta\Delta\overline{w}_n + \Delta_1\Big(\bar{\theta}_n - \bar{\Psi}_{1n} - \displaystyle\sum_{\alpha=3}^{m}\bar{\Phi}_{\alpha n}\Big), \\ \dfrac{\partial\bar{\theta}_n}{\partial t} = R^2\overline{w}_n + \Delta\bar{\theta}_n, \\ P_\alpha\dfrac{\partial\bar{\Phi}_{\alpha n}}{\partial t} = R_\alpha^2\overline{w}_n + \Delta\bar{\Phi}_{\alpha n}, \quad \alpha = 1, 2, \ldots, r \\ P_\alpha\dfrac{\partial\bar{\Phi}_{\alpha n}}{\partial t} = -R_\alpha^2\overline{w}_n + \Delta\bar{\Phi}_{\alpha n}, \quad \alpha = r+1, \ldots, m \\ \dfrac{\partial\bar{\Psi}_{1n}}{\partial t} = \Big(\dfrac{R_1^2}{P_1} + \dfrac{R_2^2}{P_2}\Big)\overline{w}_n + \dfrac{1}{P_{12}}\Delta\bar{\Psi}_{1n}, \end{cases} \tag{5.2}$$

under the boundary conditions

$$(5.3)\qquad \begin{cases} \bar{w}_n = \bar{\theta}_n = \bar{\Psi}_{1n} = \bar{\Phi}_{\alpha n} = 0, & \text{on stress-free boundary;} \\ \dfrac{\partial \bar{w}_n}{\partial z} = \bar{w}_n = \bar{\theta}_n = \bar{\Psi}_{1n} = \bar{\Phi}_{\alpha n} = 0, & \text{on rigid boundary.} \end{cases}$$

In view of lemmas 4.1–4.2 and remark 3.2, the stability of the null solution of (5.2) – guaranteed by the negativity of the real part of the eigenvalues – implies the stability of the null solution of (4.2). System (5.2)–(5.3) governs the onset of convection in L filled by a fluid mixture with $m-1$ components and salted from below by $r-1$ salts. Applying to (5.2) the procedure of lemma 4.2, one obtains a fluid mixture with $m-2$ components with L salted from below by $r-2$ salts. Therefore – by successive applications of the procedure of lemma 4.2 – one arrives to

$$(5.4)\qquad \begin{cases} P_r^{-1} \dfrac{\partial \Delta \bar{w}_n}{\partial t} = \Delta\Delta \bar{w}_n + \Delta_1 \Big(\bar{\theta}_n - \bar{Z}_{1n} - \displaystyle\sum_{\alpha=r+1}^{m} \bar{\Phi}_{\alpha n} \Big), \\ \dfrac{\partial \bar{\theta}_n}{\partial t} = R^2 \bar{w}_n + \Delta \bar{\theta}_n, \\ \dfrac{\partial \bar{Z}_{1n}}{\partial t} = \mathscr{R}_1^2 \bar{w}_n + \dfrac{\Delta \bar{Z}_{1n}}{\mathscr{P}_1}, \\ \dfrac{\partial \bar{\Phi}_{\alpha n}}{\partial t} = -\dfrac{R_\alpha^2}{P_\alpha} \bar{w}_n + \dfrac{\Delta \bar{\Phi}_{\alpha n}}{P_\alpha}, \quad \alpha = r+1, \dots, m \end{cases}$$

with

$$(5.5)\qquad \mathscr{P}_1 = \max(P_1, P_2, \dots, P_r), \quad \mathscr{R}_1^2 = \sum_{\alpha=1}^{r} \frac{R_\alpha^2}{P_\alpha}, \quad Z_1 = \sum_{\alpha=1}^{r} \Phi_\alpha$$

under the boundary conditions implied by (5.3). Starting from (5.4), with the same procedure applied successively to the salts salting L from above, one arrives to

$$(5.6)\qquad \begin{cases} P_r^{-1} \dfrac{\partial \Delta \bar{w}_n}{\partial t} = \Delta\Delta \bar{w}_n + \Delta_1 (\bar{\theta}_n - \bar{Z}_{1n} - \bar{Z}_{2n}), \\ \dfrac{\partial \bar{\theta}_n}{\partial t} = R^2 \bar{w}_n + \Delta \bar{\theta}_n, \\ \dfrac{\partial \bar{Z}_{1n}}{\partial t} = \mathscr{R}_1^2 \bar{w}_n + \dfrac{\Delta \bar{Z}_{1n}}{\mathscr{P}_1}, \\ \dfrac{\partial \bar{Z}_{2n}}{\partial t} = -\mathscr{R}_2^2 \bar{w}_n + \dfrac{\Delta \bar{Z}_{2n}}{\mathscr{P}_2}, \end{cases}$$

with

$$(5.7)\qquad \mathscr{P}_2 = \max(P_{r+1}, \dots, P_m), \quad \mathscr{R}_2^2 = \sum_{\alpha=r+1}^{m} \frac{R_\alpha^2}{P_\alpha}, \quad Z_2 = \sum_{\alpha=r+1}^{m} \Phi_\alpha$$

and successively to

$$
\left\{
\begin{aligned}
& P_r^{-1} \frac{\partial \Delta \overline{w}_n}{\partial t} = \Delta\Delta \overline{w}_n + \Delta_1(\bar{\theta}_n - \overline{\Psi}_n), \\
& \frac{\partial \bar{\theta}_n}{\partial t} = R^2 \overline{w}_n + \Delta \bar{\theta}_n, \\
& \frac{\partial \overline{\Psi}_n}{\partial t} = (\mathscr{R}_1^2 - \mathscr{R}_2^2)\overline{w}_n + \frac{\Delta \overline{\Psi}_n}{\mathscr{P}},
\end{aligned}
\right.
\tag{5.8}
$$

with

$$
\mathscr{P} = \max(P_1, \ldots, P_m), \quad \Psi = \sum_{\alpha=1}^{m} \Phi_\alpha,
\tag{5.9}
$$

under the boundary conditions implied by (5.3).

6. Proof of (1.2) via an auxiliary virtual Bénard problem

To (5.8) we associate the system

$$
\left\{
\begin{aligned}
& P_r^{-1} \frac{\partial \Delta \overline{w}_n}{\partial t} = \Delta\Delta \overline{w}_n + \Delta_1(\bar{\theta}_n - \overline{\Psi}_n), \\
& \frac{\partial \bar{\theta}_n}{\partial t} = R^2 \overline{w}_n + \frac{1}{P_*}\Delta \bar{\theta}_n, \\
& \frac{\partial \overline{\Psi}_n}{\partial t} = (\mathscr{R}_1^2 - \mathscr{R}_2^2)\overline{w}_n + \frac{1}{P_*}\Delta \overline{\Psi}_n,
\end{aligned}
\right.
\tag{6.1}
$$

which is equivalent to

$$
\left\{
\begin{aligned}
& P_r^{-1} \frac{\partial \Delta \overline{w}_n}{\partial t} = \Delta\Delta \overline{w}_n + \Delta_1(\bar{\theta}_n - \overline{\Psi}_n), \\
& \frac{\partial}{\partial t}(\bar{\theta}_n - \overline{\Psi}_n) = (R^2 - \mathscr{R}_1^2 - \mathscr{R}_2^2)\overline{w}_n + \frac{\Delta}{P_*}(\bar{\theta}_n - \overline{\Psi}_n), \\
& \frac{\partial}{\partial t}[(\mathscr{R}_1^2 - \mathscr{R}_2^2)\bar{\theta}_n - R^2 \overline{\Psi}_n] = \frac{\Delta}{P_*}[(\mathscr{R}_1^2 - \mathscr{R}_2^2)\bar{\theta}_n - R^2 \overline{\Psi}_n].
\end{aligned}
\right.
\tag{6.2}
$$

Setting

$$
\left\{
\begin{aligned}
& \theta^* = \theta - \sum_{\alpha=1}^{m} \Phi_\alpha = \theta - \Psi, \quad \Psi^* = (\mathscr{R}_1^2 - \mathscr{R}_2^2)\theta - R^2\Psi, \\
& \tau = \frac{t}{P_*}, \quad P_r^* = P_r P_*,
\end{aligned}
\right.
\tag{6.3}
$$

one is reduced to investigate the stability of the null solution of

$$\text{(6.4)} \qquad \begin{cases} \dfrac{1}{P_r^*}\dfrac{\partial \Delta w_n}{\partial \tau} = \Delta\Delta w_n + \Delta_1 \theta_n^*, \\ \dfrac{\partial \theta_n^*}{\partial \tau} = P^*(R^2 - \mathscr{R}_1^2 + \mathscr{R}_2^2) w_n + \Delta \theta_n^*, \end{cases}$$

under the boundary conditions

$$\text{(6.5)} \qquad \begin{cases} w_n = \theta_n^* = 0, \quad \text{on stress-free boundary}, \\ \dfrac{\partial w_n}{\partial z} = w_n = \theta_n^*, \quad \text{on rigid boundary} \end{cases}$$

System (6.4)–(6.5) is an auxiliary Bénard problem since (6.4)–(6.5) governs the onset of convection in $\mathtt{L}$, filled by a virtual fluid $\mathscr{F}$ having P^* as Prandtl number, heated from below or above according to $(R^2 - \mathscr{R}_1^2 + \mathscr{R}_2^2)$ is positive or negative. In the case $R^2 - \mathscr{R}_1^2 + \mathscr{R}_2^2 > 0$, introducing the auxiliary thermal Rayleigh number given by $R_*^2 = P^*(R^2 - \mathscr{R}_1^2 + \mathscr{R}_2^2)$ the critical values of R_*^2, as shown in [1], is given by $P_*(R^2 - \mathscr{R}_1^2 + \mathscr{R}_2^2) = \gamma$, hence (1.3) immediately follows. We remark that (1.3) continues to hold also for $R^2 - \mathscr{R}_1^2 + \mathscr{R}_2^2 \leq 0$. For the sake of simplicity, we give the proof in the free-free case.

In that case one has

$$\text{(6.6)} \qquad \begin{cases} \varphi_n = \sin n\pi z, \quad w_n = \tilde{w}_n \sin n\pi z, \quad \theta_n^* = \tilde{\theta}_n^* \sin n\pi z, \\ \Delta = -\xi_n, \quad \Delta\Delta = \xi_n^2 \end{cases}$$

and (6.4) becomes

$$\text{(6.7)} \qquad \begin{cases} \dfrac{\partial \tilde{w}_n}{\partial \tau} = -P_r^* \xi_n \tilde{w}_n + \dfrac{a^2}{\xi_n} P_r^* \tilde{\theta}_n^*, \\ \dfrac{\partial \tilde{\theta}_n^*}{\partial \tau} = P_*(R^2 - \mathscr{R}_1^2 + \mathscr{R}_2^2)\tilde{w}_n - \xi_n \tilde{\theta}_n^*. \end{cases}$$

The null solution of (6.7) is stable if and only if

$$\text{(6.8)} \qquad \xi_n^2 - P_* \frac{a^2}{\xi_n}(R^2 - \mathscr{R}_1^2 + \mathscr{R}_2^2) > 0$$

and, in view of

$$\text{(6.9)} \qquad \min_{(n,a^2)\in \mathbb{N}\times\mathbb{R}^+} \frac{(a^2 + n^2\pi^2)^3}{a^2} = \left[\frac{(a^2 + n^2\pi^2)^3}{a^2}\right]_{(a^2=\pi^2/2)}^{(n=1)} = \frac{27}{4}\pi^4,$$

(1.3) immediately follows for $\mathscr{R}_1^2 \leq \mathscr{R}_2^2$.

7. A better lower estimate of R_C via a binary auxiliary virtual convection problem

Condition (1.3) is sufficient for inhibiting the onset of convection since

$$R^2 < \sum_{\alpha=1}^{r} \frac{R_\alpha^2}{P_\alpha} - \sum_{\alpha=r+1}^{m} \frac{R_\alpha^2}{P_\alpha} + \frac{\gamma}{P_*} \Rightarrow R^2 < R_C. \tag{7.1}$$

Better estimates of R_C can be obtained via (5.8)–(5.9) and (5.6)–(5.7). We here confine ourselves to provide a bigger lower estimate of R_C via the binary virtual auxiliary convection system (5.8)–(5.9), in the free-free case for $\mathscr{R}_1^2 < \mathscr{R}_2^2$. Setting

$$\mathscr{L}_n = \begin{pmatrix} -P_r\xi_n & \dfrac{a^2}{\xi_n}P_r & -\dfrac{a^2}{\xi_n}P_r \\ R^2 & -\xi_n & 0 \\ \mathscr{R}_1^2 - \mathscr{R}_2^2 & 0 & -\dfrac{\xi_n}{\mathscr{P}} \end{pmatrix} \tag{7.2}$$

(5.8)–(5.9) becomes

$$\frac{\partial}{\partial t}\begin{pmatrix} \overline{w}_n \\ \bar{\theta}_n \\ \bar{\Psi}_n \end{pmatrix} = \mathscr{L}_n \begin{pmatrix} \overline{w}_n \\ \bar{\theta}_n \\ \bar{\Psi}_n \end{pmatrix}. \tag{7.3}$$

The eigenvalues of $\mathscr{L}_n$ are the roots of

$$\lambda^3 - \mathtt{I}_{1n}\lambda^2 + \mathtt{I}_{2n}\lambda - \mathtt{I}_{3n} = 0, \tag{7.4}$$

with $\mathtt{I}_{ns}$, $(s = 1, 2, 3)$, characteristic (invariants) values of (7.2) given by (see [38]–[39] and [12], pp. 386–387)

$$\begin{cases} \mathtt{I}_{1n} = \displaystyle\sum_{s=1}^{3} \lambda_{ns} = -\Big(1 + P_r + \frac{1}{\mathscr{P}}\Big)\xi_n, \\ \mathtt{I}_{2n} = \displaystyle\sum_{s \neq p}^{1-3} \lambda_{ns}\lambda_{np} = \begin{vmatrix} -P_r\xi_n & P_r\eta_n \\ R^2 & -\xi_n \end{vmatrix} + \begin{vmatrix} -\xi_n & 0 \\ 0 & -\dfrac{\xi_n}{\mathscr{P}} \end{vmatrix} \\ \qquad + \begin{vmatrix} -P_r\xi_n & -P_r\eta_n \\ \mathscr{R}_1^2 - \mathscr{R}_2^2 & -\dfrac{\xi_n}{\mathscr{P}} \end{vmatrix} = P_r\eta_n\left[\Big(1 + \dfrac{1+P_r}{\mathscr{P}P_r}\Big)\dfrac{\xi_n^2}{\eta_n} + (\mathscr{R}_1^2 - \mathscr{R}_2^2) - R^2\right], \\ \mathtt{I}_{3n} = \det \mathscr{L}_n = \displaystyle\prod_{s=1}^{3} \lambda_{ns} = a^2P_r\left[R^2 - (\mathscr{R}_1^2 - \mathscr{R}_2^2) - \frac{\xi_n^2}{\mathscr{P}\eta_n}\right], \end{cases} \tag{7.5}$$

with $\eta_n = \dfrac{a^2}{\xi_n}$. Since for $\mathscr{R}_1^2 < \mathscr{R}_2^2$, (7.2) is symmetrizable, the eigenvalues are real and instability occurs via the steady state associated to the lowest value of R^2 such that

$$\det \mathscr{L}_n = a^2 P_r \left[R^2 - (\mathscr{R}_1^2 - \mathscr{R}_2^2) - \frac{\xi_n^3}{\mathscr{P} a^2} \right] = 0, \tag{7.6}$$

i.e. at

$$R^2 = \mathscr{R}_1^2 - \mathscr{R}_2^2 + \frac{27}{4} \frac{\pi^4}{\mathscr{P}}. \tag{7.7}$$

Since

$$\mathscr{P} < P_* \Rightarrow \mathscr{R}_1^2 - \mathscr{R}_2^2 + \frac{27}{4} \pi^4 \frac{1}{P_*} < \mathscr{R}_1^2 - \mathscr{R}_2^2 + \frac{27}{4} \pi^4 \frac{1}{\mathscr{P}},$$

(7.7) for $\mathscr{P} < P_*$ guarantees stability also for

$$\mathscr{R}_1^2 - \mathscr{R}_2^2 + \frac{27}{4} \pi^4 \frac{1}{P_*} < R^2 < \mathscr{R}_1^2 - \mathscr{R}_2^2 + \frac{27}{4} \pi^4 \frac{1}{\mathscr{P}}.$$

8. Proof of property 4

Let (1.5) holds. Then in any ball of the L^2-phase space there exists a path along which the energy $\hat{E}$ increases and tends to infinity exponentially. This instability result is implied by the following Lemma.

LEMMA 8.1. *Let*

$$P_{ij}^* = \min(P_i, P_j), \quad i \neq j \in \{1, 2, \ldots, n\} \tag{8.1}$$

and let us associate to

$$\begin{cases} \dfrac{\partial \hat{\Phi}_{in}}{\partial t} = \dfrac{H_i R_i^2}{P_i} \hat{w}_n + \dfrac{1}{P_i} \Delta \hat{\Phi}_{in}, \\[2ex] \dfrac{\partial \hat{\Phi}_{jn}}{\partial t} = \dfrac{H_j R_j^2}{P_j} \hat{w}_n + \dfrac{1}{P_j} \Delta \hat{\Phi}_{jn}, \end{cases} \tag{8.2}$$

under the boundary conditions (4.3), *the auxiliary system*

$$\begin{cases} \dfrac{\partial \Phi_{in}^*}{\partial t} = \dfrac{H_i R_i^2}{P_i} w_n^* + \dfrac{1}{P_{ij}^*} \Delta \Phi_{in}^*, \\[2ex] \dfrac{\partial \Phi_{jn}^*}{\partial t} = \dfrac{H_j R_j^2}{P_j} w_n^* + \dfrac{1}{P_{ij}^*} \Delta \Phi_{jn}^*, \end{cases} \tag{8.3}$$

under the boundary conditions

$$
(8.4)\qquad \begin{cases} w_n^* = \theta_n^* = \Phi_{in}^* = \Phi_{jn}^* = 0, \quad \forall n \in \mathbb{N}, \textit{ on stress-free boundary},\\ w_n^* = \dfrac{\partial w_n^*}{\partial z} = \theta_n^* = \Phi_{in}^* = \Phi_{jn}^* = 0, \quad \forall n \in \mathbb{N}, \textit{ on rigid boundary}. \end{cases}
$$

Then

$$
(8.5)\qquad \frac{d}{dt}(\|\Phi_{in}^*\|^2 + \|\Phi_{jn}^*\|^2) > 0,
$$

implies

$$
(8.6)\qquad \frac{d}{dt}(\|\hat{\Phi}_{in}^*\|^2 + \|\hat{\Phi}_{jn}\|^2) > 0,
$$

on the same data.

PROOF. It easily follows that

$$
(8.7)\qquad \begin{cases} \left[\dfrac{d}{dt}(\|\Phi_{in}^*\|^2 + \|\Phi_{jn}^*\|^2)\right]_{(t=\tau)} \\ \quad = 2\left[\left\langle w_n^*, \dfrac{H_iR_i^2}{P_i}\Phi_{in}^* + \dfrac{H_jR_j^2}{P_j}\Phi_{jn}^*\right\rangle - \dfrac{1}{P_{ij}^*}(\|\nabla\Phi_{in}^*\|^2 + \|\nabla\Phi_{jn}^*\|^2)\right]_{(t=\tau)}, \\ \left[\dfrac{d}{dt}(\|\hat{\Phi}_{in}\|^2 + \|\hat{\Phi}_{jn}\|^2)\right]_{(t=\tau)} \\ \quad = 2\left[\left\langle \hat{w}_n, \dfrac{H_iR_i^2}{P_i}\hat{\Phi}_{in} + \dfrac{H_jR_j^2}{P_j}\hat{\Phi}_{jn}\right\rangle - \dfrac{1}{P_i}\|\nabla\hat{\Phi}_{in}\|^2 - \dfrac{1}{P_j}\|\nabla\hat{\Phi}_{jn}\|^2\right]_{(t=\tau)}. \end{cases}
$$

Therefore

$$
(8.8)\qquad \begin{aligned} &\hat{w}_n(\tau) = w_n^*(\tau), \quad \hat{\Phi}_{in}(\tau) = \Phi_{in}^*(\tau),\\ &(\nabla\hat{\Phi}_{in})_{(t=\tau)} = (\nabla\Phi_{in}^*)_{(t=\tau)}, \quad \forall i \in (1, 2, \ldots, n), \end{aligned}
$$

in view of (8.1), (8.4) implies (8.6).

Starting from (4.2), by successive applications of Lemma 8.1, a sequence of instability auxiliary systems can be obtained. The instability of one system of this sequence – implied by the existence of an eigenvalue with positive real part – in view of (3.26) with $a_n > 0$, guarantees the existence of a path, in any ball centered at the origin of the L^2-phase space, along which the L^2-energy increases exponentially and in view of Lemma 8.1 – in any ball (centered at the origin) each previous system has a path along which its L^2-energy increases exponentially. In fact, by successive applications of Lemma 8.1, following – mutatis mutandis – the procedure of section 5, one obtains the auxiliary system (analogous to (5.8))

$$\begin{cases} P_r^{-1}\dfrac{\partial \Delta w_n^*}{\partial t} = \Delta\Delta w_n^* + \Delta_1(\theta_n^* - \Psi_n^*), \\ \dfrac{\partial \theta_n^*}{\partial t} = R^2 w_n^* + \Delta\theta_n^*, \\ \dfrac{\partial \Psi_n^*}{\partial t} = (\mathscr{R}_1^2 - \mathscr{R}_2^2) w_n^* + \dfrac{\Delta \Psi_n^*}{\mathscr{P}^*}, \end{cases} \tag{8.9}$$

with

$$\mathscr{P}^* = \min(P_1, \ldots, P_m), \quad \Psi^* = \sum_{\alpha=1}^{n} \Phi_\alpha^*. \tag{8.10}$$

To (8.9) we associate the system

$$\begin{cases} P_r^{-1}\dfrac{\partial \Delta w_n^*}{\partial t} = \Delta\Delta w_n^* + \Delta_1(\theta_n^* - \Psi_n^*), \\ \dfrac{\partial \theta_n^*}{\partial t} = R^2 w_n^* + \dfrac{1}{\bar{P}_*}\Delta\theta_n^*, \\ \dfrac{\partial \Psi_n^*}{\partial t} = (\mathscr{R}_1^2 - \mathscr{R}_2^2) w_n^* + \dfrac{1}{\bar{P}_*}\Delta \Psi_n^*, \end{cases} \tag{8.11}$$

which is equivalent to

$$\begin{cases} P_r^{-1}\dfrac{\partial \Delta w_n^*}{\partial t} = \Delta\Delta w_n^* + \Delta_1(\theta_n^* - \Psi_n^*), \\ \dfrac{\partial}{\partial t}(\theta_n^* - \Psi_n^*) = (R^2 - \mathscr{R}_1^2 - \mathscr{R}_2^2) w_n^* + \dfrac{\Delta}{\bar{P}_*}(\theta_n^* - \Psi_n^*), \\ \dfrac{\partial}{\partial t}[(\mathscr{R}_1^2 - \mathscr{R}_2^2)\theta_n^* - R^2\Psi_n^*] = \dfrac{\Delta}{\bar{P}_*}[(\mathscr{R}_1^2 - \mathscr{R}_2^2)\theta_n^* - R^2\Psi_n^*], \end{cases} \tag{8.12}$$

via the (6.3), with $\bar{P}_*$ at the place of P_* and $\bar{P}_r^* = P_r\bar{P}_*$, one is reduced to investigate for the instability of the null solution of

$$\begin{cases} \dfrac{1}{\bar{P}_r^*}\dfrac{\partial \Delta w_n}{\partial t} = \Delta\Delta w_n + \Delta_1\theta_n^*, \\ \dfrac{\partial \theta_n^*}{\partial t} = \bar{P}_*(R^2 - \mathscr{R}_1^2 - \mathscr{R}_2^2) w_n + \Delta\theta_n^*, \end{cases} \tag{8.13}$$

under the boundary conditions (6.5). System ((8.13), (6.5)) is an auxiliary virtual Bénard problem governing the onset of convection in L filled by a virtual fluid $\mathscr{F}$, having $\bar{P}_*$ as Prandtl number, heated from below or from above according to $(R^2 - \mathscr{R}_1^2 + \mathscr{R}_2^2)$ is positive or negative. Then, via remarks analogous to the which ones of Sect. 6, one easily obtains the instability condition (1.4).

9. Discussion and comments

1) The onset of convection in a m-component, $\forall m \in \mathbb{N}$, Navier–Stokes fluid mixture, filling a horizontal layer L-heated from below and salted partly from above and partly from below – is investigated;
2) Since the difficulties of providing, in closed simple form, the instability threshold R_C grow drastically with m, the problem of finding simple useful estimates of R_C to be used not only by theoreticians but also for experimentalists, arises;
3) A linearization principle in the L^2-energy norm is obtained;
4) For any $m \in \mathbb{N}$, it is shown that exist two virtual auxiliary fluid mixtures with $m - 1$ components which instability thresholds give, respectively, a lower and an upper estimate of R_C;
5) Exist two virtual auxiliary Bénard problems which instability thresholds give, respectively, a lower and an upper estimate of R_C such that, according to (1.2)–(1.4), one has

$$\sum_{\alpha=1}^{r} \frac{R_\alpha^2}{P_\alpha} - \sum_{\alpha=r+1}^{m} \frac{R_\alpha^2}{P_\alpha} + \frac{\gamma}{P_*} < R_C < \sum_{\alpha=1}^{r} \frac{R_\alpha^2}{P_\alpha} - \sum_{\alpha=r+1}^{m} \frac{R_\alpha^2}{P_\alpha} + \frac{\gamma}{\bar{P}_*}, \tag{9.1}$$

and hence

$$R^2 < \sum_{\alpha=1}^{r} \frac{R_\alpha^2}{P_\alpha} - \sum_{\alpha=r+1}^{m} \frac{R_\alpha^2}{P_\alpha} + \frac{\gamma}{P_*}, \quad \text{inhibits convection,} \tag{9.2}$$

while

$$R^2 > \sum_{\alpha=1}^{r} \frac{R_\alpha^2}{P_\alpha} - \sum_{\alpha=r+1}^{m} \frac{R_\alpha^2}{P_\alpha} + \frac{\gamma}{\bar{P}_*}, \quad \text{guarantees convection;} \tag{9.3}$$

6) For

$$\sum_{\alpha=r+1}^{m} \frac{R_\alpha^2}{P_\alpha} > \sum_{\alpha=1}^{r} \frac{R_\alpha^2}{P_\alpha} + \frac{\gamma}{\bar{P}_*}, \tag{9.4}$$

(9.3) is verified $\forall R^2$ (i.e. irrespective of the temperature gradient) and guarantees the onset of "cold convection" [40];
7) Looking for symmetries and skew-symmetries hidden in (3.25) – as done in [12] in the free-free case – one can obtain, only for particular values of the

Prandtl numbers, conditions guaranteeing stability which generally does not appear to be more convenient of (9.2);

8) As far as we know, (9.1)–(9.4) are new in the existing literature;
9) (9.1)–(9.4) appear to be of interest not only for theoreticians but also for the experimentalists investigating natural phenomena and/or industrial processes related to the onset of convection.

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Salvatore Rionero
Department of Mathematics and Applications "R. Caccioppoli"
University of Naples Federico II
Complesso Universitario Monte S. Angelo
Via Cinzia
80126 Naples, Italy
and
Accademia Nazionale dei Lincei
Via della Lungara 10
00165 Rome, Italy
rionero@unina.it