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Solid Mechanics — *New frontiers in Zanaboni's formulation of Saint-Venant's principle*, by R. J. Knops, communicated on January 13, 2017.

Dedicated to the memory of Professor Giuseppe Grioli.

Abstract. — A synoptic account is presented of a revised Zanaboni procedure for Saint-Venant's principle. Modifications required for conservative and dissipative coupled systems are illustrated by application to piezoelectricity and thermoelasticity.

Key words: Saint-Venant's principle, piezoelectricity, thermolasticity

Mathematics Subject Classification: 35J57, 74B05, 74F05, 74F15

1. Introduction

Saint-Venant's principle, first introduced in 1853 [30] and 1856 [31], states that for an elastic body in equilibrium subject to zero body force, the effect of self-equilibrated specified surface loads on an otherwise free boundary does not persist in the bulk of the body. The detailed spatial distribution of the loads affects the stress and other elastic fields only in the neighbourhood of the load surface. The principle, used as pragmatic guidance by structural engineers and others to simplify calculations, evolved from Saint-Venant's empirical experience.

The imprecise expression of the original statement compromises its use for computer aided and other calculations. Indeed, the history of the subject is characterised by attempts both to rigorously define the principle and to establish an equally rigorous mathematical proof of its validity. In any given problem, the measure adopted for the solution (i.e., the function space to which it belongs), geometry, and type of boundary data should be unambiguously defined since they are likely to affect decay rates. It is also legitimate to enquire whether the principle applies only to elastic materials. Observation suggests that appropriate versions are ubiquitous regardless of geometry and constitutive assumptions.

The principle's validity was early challenged by Boussinesq who appealed to known solutions to boundary value problems including those for the half-space. On the other hand, exact solutions especially for the semi-infinite solid and hollow cylinder and the half-strip, notably by Dougall, Synge, Gregory, Horvay, Stephen and co-workers, and in particular Grioli [6], confirm the principle and suggest that decay rates are exponential for cylinders and algebraic for cones. V. Mises and Sternberg extend the study of exact solutions using Green's functions. Their results show that the depth of penetration of the self-equilibrated boundary loads depends upon the size of the load region and the type of load.

That the principle is not restricted to bodies of particular geometry was recognised by Southwell and Goodier who used the strain energy of subregions as a measure of average behaviour. The notion of using the strain energy was significantly developed by Toupin [29] who constructed a differential inequality for volume measures of the strain energy on a cylinder and estimated corresponding decay rates. Further related developments are due to Fichera, Oleinik, and Weck. Differential inequalities were also used by Berdichevsky [1] to obtain estimated decay rates for regions whose cross-sections are perpendicular to a given direction and which may become unbounded with increasing distance along this direction. They are therefore more general than cylindrical regions. Sternberg and Knowles explain how the minimum strain energy principle can be used in Saint-Venant's problem to distinguish the solution in the interior of a cylinder that persists due to non-self-equilibrated resultant loads distributed over the ends. Numerous contributions, especially by Payne and co-workers, demonstrate how Saint-Venant's principle may be regarded as the decay component in a Phragmén–Lindelöf principle.

The comparative success of treatments based upon differential inequalities initiated by Toupin obscures an equally promising procedure introduced by Zanaboni in a series of Lincei publications [35, 36, 37]. Although the strain energy is again employed, the technique is not limited to regions of any special geometry. Unlike Southwell and Goodier, Zanaboni's method produces less vague conclusions. It fails, however, to establish meaningful decay estimates. A world war combined with somewhat outmoded arguments may have contributed to its subsequent neglect. The present aim, however, is to reassess Zanaboni's method and to provide a simplified derivation of a key component in his argument. This considerably broadens the range of accessible problems as illustrated by selected examples considered in later sections.

Surveys of the general literature devoted to Saint-Venant's principle including that cited above may be found in the comprehensive reviews by Maissonenuve [23]. Horgan and Knowles [12], Horgan [10, 11] and the concise account by Rionero [27].

Zanaboni's treatment of Saint-Venant's principle is outlined in Section 2 which also briefly comments on his mainly algebraic proof that involves convergence of monotone sequences. The geometry of regions, including an intuitive description of elongated regions, is contained in Section 3, while Sections 4 and 5 employ the simple examples of heat conduction and the dielectric to demonstrate essential points of the revised procedure. Section 6 illustrates how the procedure may be extended to coupled conservative systems by application to linear piezoelectricity. Although a slightly different type of monotone sequence is generated, standard properties may be adapted to discuss convergence. Section 7 considers the coupled dissipative system of linear thermostatics. Modifications required in the construction of the relevant fundamental inequality require a non-standard application of an embedding inequality derived in Section A. Nevertheless, the monotone sequence that occurs in piezoelectricity is recovered and in consequence the same convergence argument can be used. Section 8 concludes the paper with remarks on some open problems.

The same Cartesian system of rectangular coordinates is employed throughout. Scalar, vector and tensor quantities are not typographically distinguished, while the usual conventions are adopted of summation over repeated indices, and a subscript comma to indicate partial differentiation. Roman subscripts are in the numerical range $1, 2, 3$. Only ellipitc problems are studied and therefore a suitably smooth solution is always assumed to exist.

2. Zanaboni's procedure for Saint-Venant's principle

The emperical nature of Saint-Venant's original statement of the principle was reformulated by Zanaboni in a form which may be paraphrased as (cp, [35, 36, 37]):

> *The stored energy in those parts of an elongated linear elastic body in equilibrium that are increasinlgy remote from the load surface tends to zero at a rate independent of the self-equilibrated applied surface load and its distribution, but dependent upon the composition of the body and its geometry.*

This may be contrasted with a somewhat less precise version due to Southwell [33]:

> *The effect of a system of forces statically equivalent to zero force and zero couple (distributed over part of the surface) has negligible magnitude at distances which are large compared to the linear dimensions of that part.*

In order to prove his assertion, Zanaboni developed a procedure consisting of two main steps:

1. A fundamental inequality.
2. Convergence properties of monotone bounded sequences.

Zanaboni's derivation of decay estimates is less convincing than that devised, for example, by Berdichevsky [1], although there are certain common features. This aspect is beyond the intended scope of the present paper and is not discussed further.

Zanaboni applied his method to linear elastic bodies in equilirbrium in the absence of body force and subject to self-equilibrated loads distributed over a part (the load surface) of an otherwise traction-free smooth boundary. Within this context, his derivation of the fundamental inequality in part relies upon the minimum strain energy principle of linear elasticity, but as originally presented the argument is unfamiliar if not arcane. Limited elucidation is offered in books by Biezanno and Grammel [4] and by Fung [5], although that by Robinson [28] provides an alternative proof based upon Dirichlet's principle. Gurtin [7], Maisonneuve [23] and the survey by Horgan and Knowles [12] also refer to Zanaboni's contributions.

Here, we provide a transparent proof of Step 1 that involves a comparatively easy integration by parts supplemented as required by appropriate standard inequalities and embedding theorems. The new treatment first introduced in [19] for linear elasticity, is applied in Section 4 to heat conduction. The intrinsic simplicity of the argument facilitates its transposition to several other linear (coupled) conservative and "dissipative" elliptic systems. In addition to linear nonhomogeneous anisotropic compressible elasticity [19], incompressible isotropic linear elasticity, certain semi-linear elliptic equations and the p-Laplacian operator are among applications that can be successfully treated [14]. Non-linear elasticity is also considered in [19], having earlier been discussed by Locatelli [22, 21]. A variant of the present approach establishes the corresponding Zanaboni version of Saint-Venant's principle for selected theories of plastic behaviour [18]. The Zanaboni formulation, however, is not universally valid. Counterexamples are presented in [20] which also contains further illustrative examples. Other well-known counterexamples to the traditional Saint-Venant principle are due to Toupin [29] and Hoff [8]. Horgan [9] and Stephen [34] employ examples to show that very slow decay rates may render the principle inappropriate for composite materials. As already mentioned, determination of precise or estimated decay behaviour remains elusive by Zanaboni's procedure since it is difficult to ascribe precise meaning to terms such as "sufficiently remote". Consequently, Zanaboni's approach should presently be regarded as leading to a qualitative rather than a quantative description of spatial stabllity.

3. Geometry

Let $\Omega_n \subset \mathbb{R}^3$, $n = 1, 2, 3, \ldots$ denote members of an infinite sequence of successively embedded regions such that

$$\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \cdots \subset \Omega_n \subset \cdots. \tag{3.1}$$

where

$$\lim_{n\to\infty} |\Omega_n| \to \infty, \tag{3.2}$$

and $|\Omega_n|$ is the diameter of the region Ω_n. The sequence can include, for example, a family of increasing regions exterior to a bounded region, a family interior to a cone of semi-vertical angle $0 < \alpha \leq \pi/2$ or a family of thick plates. Present concern, however, is with a family of elongated regions, which may be roughly described as regions whose dimension in a specific (curvilinear) direction is significantly larger than in other directions. For simplicity, it is supposed that plane cross-sections oblique to the specified direction have area uniformly bounded below by a positive constant, and are simply-connected, a property inherited by each region Ω_n. Elongated regions can be contained in an semi-infinite helix or in entangled non-contiguous knotted curvilinear cylinders. Elongated regions that are multiply-connected or whose cross-sections collapse to zero require separate treatment. Exterior regions, however, appear amenable to the arguments developed here.

The surface $\partial\Omega_n$ of Ω_n, $n = 1, 2, \ldots$ is assumed to be smooth and to possess a unit outward vector normal whose Cartesian components are generically denoted by n_i. Other assumptions imposed on certain parts of the surface $\partial\Omega_n$ are stated where necessary. All surfaces $\partial\Omega_n$, $n = 1, 2, \ldots$ intersect in a common non-empty connected set Γ, known as the *load surface*, which satisfies

$$\emptyset \neq \Gamma \subset \partial\Omega_n \cap \partial\Omega_{n+1}, \quad n = 1, 2, 3, \ldots. \tag{3.3}$$

Furthermore, Σ_n designates that part of the surface $\partial\Omega_n$ that intersects the interior of Ω_{n+1}; that is

$$\Sigma_n = \partial\Omega_n \backslash (\partial\Omega_{n+1} \cap \partial\Omega_n). \tag{3.4}$$

4. Heat conduction

We illustrate our approach to the basic steps in Zanaboni's procedure by a boundary value problem for steady heat conduction. The example of the anisotropic dielectric considered in Section 5 further illustrates the technique. Another example is the incompressible fluid.

Each of the previously introduced regions Ω_n is occupied by the same inhomogeneous anisotropic heat conduction material whose symmetric non-negative heat conduction tensor κ has Cartesian components κ_{ij} that satisfy

$$\kappa_{ij}(x) = \kappa_{ji}(x), \quad x \in \Omega_n, \quad n = 1, 2, \ldots, \tag{4.1}$$

$$0 \leq \kappa_{ij}(x)\xi_i\xi_j, \quad x \in \Omega_n, \quad n = 1, 2, \ldots, \tag{4.2}$$

where $\xi \in \mathbb{R}^3$.

Let $\theta^{(n)}(x)$ denote the positive temperature in the region Ω_n. With respect to the same Cartesian set of rectangular coordinates, the system of Dirichlet boundary value problems we study in the absence of heat sources is given by

$$(\kappa_{ij}(x)\theta^{(n)}_{,i}(x))_{,j} = 0, \quad x \in \Omega_n, \tag{4.3}$$

$$\theta^{(n)}(x) = Q, \quad x \in \Gamma, \tag{4.4}$$

$$= 0, \quad x \in \partial\Omega_n \backslash \Gamma. \tag{4.5}$$

where the same scalar quantity Q is specified for each Ω_n and accordingly is independent of n.

Dirichlet data is adopted for simplicity.

Spatial behaviour appropriate to the system (4.3)–(4.5) is described in terms of the bilinear function defined by

$$V_{\Omega_n}(\phi, \psi) = \int_{\Omega_n} \kappa_{ij}\phi_{,i}\psi_{,j}\, dx, \tag{4.6}$$

where $\phi(x)$, $\psi(x)$ are differentiable scalar functions.

The thermal energy U_{Ω_n} in the region Ω_n is contained as a special case of V_{Ω_n} so that we have

$$U_{\Omega_n}(\theta^{(n)}) = V_{\Omega_n}(\theta^{(n)}, \theta^{(n)}). \tag{4.7}$$

In particular, it is assumed that

$$U_{\Omega_1}(\theta^{(1)}) < \infty. \tag{4.8}$$

As mentioned, the first step in the procedure is to establish a fundamentel inequality subseqeuently used to generate a convergent bounded monotone sequence. An integration by parts and appeal to (4.4) yields

$$\begin{aligned} V_{\Omega_n}(\theta^{(n)}, \theta^{(n+1)}) &= \int_{\Omega_n} (\kappa_{ij}\theta^{(n)}\theta^{(n+1)}_{,j})_{,i} - \int_{\Omega_n} \theta^{(n)}(\kappa_{ij}\theta^{(n+1)}_{,j})_{,i} \\ &= \int_\Gamma Q n_i \kappa_{ij} \theta^{(n+1)}_{,j}\, dS \qquad (4.9) \\ &= \int_\Gamma \theta^{(n+1)} n_i \kappa_{ij} \theta^{(n+1)}_{,j} \\ &= U_{\Omega_{n+1}}(\theta^{(n+1)}), \qquad (4.10) \end{aligned}$$

upon recalling the symmetry $\kappa_{ij} = \kappa_{ji}$.

On the other hand, an application of Schwarz's inequality and the arithmetic mean-geometric mean inequality (Young's inequality) leads to

$$\begin{aligned} V_{\Omega_n}(\theta^{(n)}, \theta^{(n+1)}) &\leq [U_{\Omega_n}(\theta^{(n)})U_{\Omega_n}(\theta^{(n+1)})]^{1/2} \\ &\leq \frac{1}{2}U_{\Omega_n}(\theta^{(n)}) + \frac{1}{2}U_{\Omega_n}(\theta^{(n+1)}). \end{aligned} \tag{4.11}$$

Elimination of the bilinear function $V_{\Omega_n}(\theta^{(n)}, \theta^{(n+1)})$ between (4.10) and (4.11) gives

$$U_{\Omega_{n+1}}(\theta^{(n+1)}) \leq \frac{1}{2}U_{\Omega_n}(\theta^{(n)}) + \frac{1}{2}U_{\Omega_n}(\theta^{(n+1)}),$$

which upon rearrangement becomes the required *fundamental inequality*:

$$U_{\Omega_{n+1}\setminus\Omega_n}(\theta^{(n+1)}) + U_{\Omega_{n+1}}(\theta^{(n+1)}) \leq U_{\Omega_n}(\theta^{(n)}). \tag{4.12}$$

For the moment, the non-negative first term on the left of (4.12) is discarded, which leads to the recursive generation of the bounded monotone non-increasing sequence given by

$$0 \leq U_{\Omega_{n+1}}(\theta^{(n+1)}) \leq U_{\Omega_n}(\theta^{(n)}) \leq \cdots U_{\Omega_1}(\theta^{(1)}) < \infty, \tag{4.13}$$

which is therefore convergent. Consequently, for $\varepsilon > 0$ there exists n_0 such that

$$U_{\Omega_n}(\theta^{(n)}) - U_{\Omega_m}(\theta^{(m)}) \leq \varepsilon, \quad n_0 \leq n < m. \tag{4.14}$$

Insertion of inequality (4.14) into (4.12) yields

$$\begin{aligned} U_{\Omega_{n+1}\backslash\Omega_n}(\theta^{(n+1)}) &\leq U_{\Omega_n}(\theta^{(n)}) - U_{\Omega_{n+1}}(\theta^{(n+1)}) \\ &\leq \varepsilon, \quad n_0 \leq n, \end{aligned} \tag{4.15}$$

which represents the mathematical expression of Zanaboni's version of Saint-Venant's principle.

For the particular problems under consideration, it is worth repeating that apart from a smooth surface, the regions Ω_n must satisfy condition (3.2). This is a vital restriction. The application of convergence properties requires in the limit an infinite number of regions Ω_n. But the condition

$$\lim_{n\to\infty} |\Omega_n| < \infty.$$

implies that in the limit $|\Omega_{n+1}\backslash\Omega_n|$ must tend to zero, which in turn implies a similar behaviour for the corresponding thermal energy and (4.15) becomes automatically satisfied. The property that Ω_n in the limit is an elongated region is essential to avoid a nugatory result.

Pointwise estimates of the solution $\theta^{(n)}$ may be obtained provided that the heat conduction tensor is convex. That is, for $n = 1, 2, \ldots$, we replace (4.2) by

$$\kappa_0\xi_i\xi_i \leq \kappa_{ij}\xi_i\xi_j, \quad x \in \Omega_n, \tag{4.16}$$

where κ_0 is a prescribed positive constant independent of n, and $\xi \in \mathbb{R}^3$.

In addition, a constraint is imposed on the regions $\Omega_{n+1}\backslash\Omega_n$. We stipulate that each point $x \in \Omega_{n+1}\backslash\Omega_n$ is at the centre of a sphere $S(x)$ completely contained in $\Omega_{n+1}\backslash\Omega_n$. Let $G(z, y)$ be the Green's function of second kind for $S(x)$ and equation (4.3). Then for $z, y \in S(x)$ we have

$$\begin{aligned} [\theta^{(n+1)}(z)]^2 &= \left[\int_{S(x)} \theta^{(n+1)}_{,i}(y)G_{,i}(z, y)\, dy\right]^2 \\ &\leq \int_{S(x)} \theta^{(n+1)}_{,i}(y)\theta^{(n+1)}_{,i}(y)\, dy \int_{S(x)} G_{,i}(z, y)G_{,i}(z, y)\, dy \\ &\leq C_1 U_{\Omega_{n+1}\backslash\Omega_n}(\theta^{(n+1)}), \end{aligned} \tag{4.17}$$

where C_1 is a computable positive constant. The estimate (4.15) then implies for $n_0 \leq n$ that

$$|\theta^{(n+1)}(x)| \leq \varepsilon, \quad x \in \Omega_{n+1}\backslash\Omega_n. \tag{4.18}$$

A similar conclusion that does not involve the explicit introduction of Green's functions may be established by methods developed in [25, 2].

Although the above proof is given for Dirichlet data, it may easily be extended to Neumann data subject to appropriate normalistions. It must be observed that it has not been possible so far to derive precise estimates of the rate at which the energy functions decay to zero with distance from the heated surface Γ. This is in contrast to treatments based upon differential inequalities for special geometries.

5. The anisotropic dielectric

In the absence of electric charge and other sources, Maxwell's equations imply tthat the electric vector field $E^{(n)}(x)$, $x \in \Omega_n$ is the gradient of a scalar function $\phi^{(n)}(x)$ such that

$$E_i^{(n)}(x) = \phi_{,i}^{(n)}(x), \quad x \in \Omega_n. \tag{5.1}$$

We also have that $E^{(n)}(x)$ is related to the electric displacement vector field $D^{(n)}(x)$ by

$$D_i^{(n)}(x) = \kappa_{ij}(x) E_j^{(n)}(x) = \kappa(x)_{ij} \phi_{,j}^{(n)}(x), \quad x \in \Omega_n, \tag{5.2}$$

where without confusion we let $\kappa(x)$ be the symmetric non-negative dielectric tensor satisfying (4.1) and (4.2).

In the absence of sources, the electric displacement is solenoidal so that

$$D_{i,i}^{(n)} = 0, \quad x \in \Omega_n. \tag{5.3}$$

We suppose that the electric displacement vector field is specified on $\partial\Omega^{(n)}$ and consequently the system of boundary value problems is given by

$$\begin{aligned} (\kappa_{ij}\phi_{,i}^{(n)})_{,j} &= 0, \quad x \in \Omega_n, \\ \frac{\partial \phi^{(n)}}{\partial n} &= Q, \quad x \in \Gamma, \\ &= 0, \quad x \in \partial\Omega_n \backslash \Gamma, \end{aligned}$$

where Q, assigned the new meaning of surface charge density, is prescribed independently of n, and the conormal derivative on $\partial\Omega_n$ is generically specified by

$$\frac{\partial \phi^{(n)}}{\partial n} = n_j \kappa_{ij} \phi_{,j}^{(n)}, \quad x \in \partial\Omega_n. \tag{5.4}$$

The boundary value problems to be treated are therefore similar to those encountered in Section 4. An analogous procedure leads to conclusions corresponding to (4.15). The argument, however, used to derive (4.18) cannot be followed to obtain a pointwise estimate for $E^{(n+1)}$ which requires an estimate for the gradient of $\phi^{(n+1)}$. Instead, we appeal to the mean value theorem for $\phi^{(n+1)}(x)$ given, for instance, by [3, eqn (3.2)]. In the notation of Section 4, and subject to the convexity condition (4.16) and the pointwise bound (4.17), we proceed as follows

$$\begin{aligned}\left[\frac{\partial\phi^{(n+1)}(x)}{\partial x_i}\right]^2 &= \left[\int_{S(x)}\phi^{(n+1)}(y)\frac{\partial F(x,y)}{\partial x_i}\,dy\right]^2\\ &\le \int_{S(x)}\phi^{(n+1)}\phi^{(n+1)}\,dy\int_{S(x)}\left(\frac{\partial F(x,y)}{\partial x_i}\right)^2 dy\\ &\le C_2\int_{\Omega_{n+1}\setminus\Omega_n}\kappa_{ij}\phi^{(n+1)}_{,j}\phi^{(n+1)}_{,j}\,dy,\end{aligned}$$

where $F(x, y)$ is the Levi function corresponding to our problem (see, for example [24]), and C_2 is a computable positive constant independent of n.

6. Piezoelectricity

The treatment of Sections 4 and 5 is extended to the coupled conservative system of linear piezoelectricity. Each of the embedded regions $\Omega_n \subset \Omega_{n+1}$, $n = 1, 2, \dots$ defined in Section 3 is composed of the same piezoelectric material in equilibrium under zero body-force and zero electric and other sources. The boundary conditions are homogeneous except on the common non-empty load surface Γ.

The electric field vector $E^{(n)}(x)$ in Ω_n satisfies the boundary value problem specified in Section 5 except that the dielectric tensor is now supposed to be convex in the sense of condition (4.16).

Components of the symmetric stress tensor and the elastic displacement vector in Ω_n are denoted by $\sigma^{(n)}_{ij}(x)$ and $u^{(n)}_i(x)$ respectively, while the corresponding linear symmetric strain tensor $e^{(n)}(x)$ has components

$$e^{(n)}_{ij}(x) = \frac{1}{2}(u^{(n)}_{i,j}(x) + u^{(n)}_{j,i}(x)), \quad x \in \Omega_n. \tag{6.1}$$

We consider the following system of boundary value problems derived from the equilibrium theory of linear piezoelectricity. The dielectric components are included for completeness.

$$\sigma^{(n)}_{ij} = c_{ijkl}e^{(n)}_{kl} + \mu_{ijk}\phi^{(n)}_{,k}, \quad D^{(n)}_i = \kappa_{ij}\phi^{(n)}_{,j}, \quad x \in \Omega_n, \tag{6.2}$$

$$\sigma^{(n)}_{ij,j} = 0, \quad D^{(n)}_{i,i} = (\kappa_{ij}\phi^{(n)}_{,j})_{,i} = 0, \quad x \in \Omega_n, \tag{6.3}$$

$$u^{(n)}_i = h_i, \quad x \in \Gamma_M, \tag{6.4}$$

$$= 0, \quad x \in \partial\Omega_n\setminus\Gamma_M, \tag{6.5}$$

$$\frac{\partial \phi^{(n)}}{\partial n} = Q, \quad x \in \Gamma_E, \tag{6.6}$$

$$= 0, \quad x \in \partial\Omega_n \backslash \Gamma_E, \tag{6.7}$$

where the conormal derivative in (6.6) and (6.7) is defined in (5.4), $h_i(x)$ and $Q(x)$ are functions prescribed independently of n, and $\emptyset \neq \Gamma_M \subset \Gamma$, $\emptyset \neq \Gamma_E \subset \Gamma$, or equivalently $\Gamma = \Gamma_M \cup \Gamma_E$, are the load surfaces. The symmetric nonhomogeneous elastic tensor $c(x)$ and the symmetric nonhomogeneous dielectric tensor $\kappa(x)$ are supposed convex (positive-definite) so that for all Ω_n, $n = 1, 2, \ldots$, we have

$$c_{ijkl} = c_{klij} = c_{jikl}, \quad \kappa_{ij} = \kappa_{ji}, \tag{6.8}$$

and

$$c_0 \xi_{ij} \xi_{ij} \leq c_{ijkl} \xi_{ij} \xi_{kl}, \quad \forall \xi_{ij} = \xi_{ji}, \tag{6.9}$$

$$\kappa_0 \xi_i \xi_i \leq \kappa_{ij} \xi_i \xi_j, \quad \xi_i \in \mathbb{R}^3, \tag{6.10}$$

where c_0 and κ_0 are prescribed positive constants.

Specification of the boundary value problems is completed by requiring the components of the dielectric permeability tensor to be symmetric in the first pair of indices, and bounded in the sense that

$$\mu^2 = \max_{\forall \Omega_n} \mu_{ijk} \mu_{ijk} < \infty. \tag{6.11}$$

Let $w(x)$ be a vector field with corresponding strain components denoted by

$$e_{ij}(w) = \frac{1}{2}(w_{i,j} + w_{j,i}),$$

and for scalar function $\psi(x)$ define the elastic and electric energies on Ω_n to be

$$W_{\Omega_n}(w) = \int_{\Omega_n} c_{ijkl} e_{ij}(w) e_{kl}(w)\, dx,$$

$$U_{\Omega_n}(\psi) = \int_{\Omega_n} \kappa_{ij} \psi_{,i} \psi_{,j}\, dx.$$

Moreover, we suppose that the respective energies are bounded on Ω_n; that is

$$W_{\Omega_n}(u^{(n)}) < \infty, \quad n = 1, 2, \ldots \tag{6.12}$$

$$U_{\Omega_n}(\phi^{(n)}) < \infty, \quad n = 1, 2, \ldots \tag{6.13}$$

The electric field is uncoupled from the elastic fields, and therefore the results of Section 5 indicate that the electric energy satisfies the inequality

$$U_{\Omega_{n+1} \backslash \Omega_n}(\phi^{(n+1)}) \leq U_{\Omega_n}(\phi^{(n)}) - U_{\Omega_{n+1}}(\phi^{(n+1)}).$$

Consequently, the term on the left vanishes for sufficiently large n in accordance with Zanaboni's version of Saint-Venant's principle.

The fundamental inequality appropriate to linear piezoelectricity is expressed in terms of a linear combination of the elastic and electrical energies given by

$$E_{\Omega_n}(w,\psi) = W_{\Omega_n}(w) + \Lambda U_{\Omega_n}(\psi), \tag{6.14}$$

where the positive constant Λ is chosen to satisfy

$$\Lambda = \gamma\delta^2, \quad \delta^2 = \frac{\mu^2}{c_0\kappa_0}, \tag{6.15}$$

and the positive constant γ satisfies

$$4\gamma > 1. \tag{6.16}$$

Derivation of the fundamental inequality also involves the following bilinear form

$$I_{\Omega_n}(u^{(n)},\phi^{(n)};u^{(n+1)},\phi^{(n+1)}) = \int_{\Omega_n} \sigma_{ij}^{(n+1)} e_{ij}^{(n)}\,dx + \Lambda\int_{\Omega_n} D_i^{(n+1)}\phi_{,i}^{(n)}\,dx. \tag{6.17}$$

Integration by parts and use of (6.2)–(6.7) leads to

$$\begin{aligned}
&I_{\Omega_n}(u^{(n)},\phi^{(n)};u^{(n+1)},\phi^{(n+1)})\\
&\quad= \int_{\Gamma_M} n_j\sigma_{ij}^{(n+1)}u_i^{(n)}\,dS + \Lambda\int_{\Gamma_E} n_j\kappa_{ij}\phi_{,i}^{(n)}\phi^{(n+1)}\,dS\\
&\quad= \int_{\Omega_{n+1}} \sigma_{ij}^{(n+1)}e_{ij}^{(n+1)}\,dx + \Lambda\int_{\Omega_{n+1}} \kappa_{ij}\phi_{,i}^{(n+1)}\phi_{,j}^{(n+1)}\,dx\\
&\quad= \int_{\Omega_{n+1}} [c_{ijkl}e_{ij}^{(n+1)}e_{kl}^{(n+1)} + \mu_{ijk}e_{ij}^{(n+1)}\phi_{,k}^{(n+1)}]\,dx\\
&\qquad+ \Lambda\int_{\Omega_{n+1}} \kappa_{ij}\phi_{,i}^{(n+1)}\phi_{,j}^{(n+1)}\,dx \qquad (6.18)\\
&\quad= I_{\Omega_{n+1}}(u^{(n+1)},\phi^{(n+1)};u^{(n+1)},\phi^{(n+1)}). \qquad (6.19)
\end{aligned}$$

A lower bound for the last expression is obtained by application of Schwarz's inequality to the second term on the right of (6.18). We have

$$\begin{aligned}
\pm\int_{\Omega_{n+1}} \mu_{ijk}e_{ij}^{(n+1)}\phi_{,k}^{(n+1)}\,dx &\le \left[\int_{\Omega_{n+1}} e_{ij}^{(n+1)}e_{ij}^{(n+1)}\,dx\int_{\Omega_{n+1}} \mu_{ijk}\mu_{ijk}\phi_{,p}^{(n+1)}\phi_{,p}^{(n+1)}\,dx\right]^{1/2}\\
&\le \left[\frac{\mu^2}{c_0\kappa_0}W_{\Omega_{n+1}}(u^{(n+1)})U_{\Omega_{n+1}}(\phi^{(n+1)})\right]^{1/2}.
\end{aligned}$$

In consequence, Young's inequality applied to (6.19) gives for arbitrary positive constant α_1 to be chosen,

$$I_{\Omega_n}(u^{(n)},\phi^{(n)};u^{(n+1)},\phi^{(n+1)}) \geq \Big(1-\frac{\alpha_1\delta}{2}\Big)W_{\Omega_{n+1}}(u^{(n+1)}) + \Lambda\Big(1-\frac{\delta}{2\alpha_1\Lambda}\Big)U_{\Omega_{n+1}}(\phi^{(n+1)}). \tag{6.20}$$

The next step in the derivation of the fundamental inequality is to construct an upper bound for the bilinear function (6.17). Standard inequalities give

$$\begin{aligned} I_{\Omega_n}&(u^{(n)},\phi^{(n)};u^{(n+1)},\phi^{(n+1)}) \\ &= \int_{\Omega_n}[c_{ijkl}e_{ij}^{(n)}e_{kl}^{(n+1)} + \mu_{ijk}e_{ij}^{(n)}\phi_{,k}^{(n+1)}]\,dx + \Lambda\int_{\Omega_n}\kappa_{ij}\phi_{,i}^{(n)}\phi_{,j}^{(n+1)}\,dx \\ &\leq \Big[\int_{\Omega_n}c_{ijkl}e_{ij}^{(n)}e_{kl}^{(n)}\,dx\int_{\Omega_n}c_{ijkl}e_{ij}^{(n+1)}e_{kl}^{(n+1)}\,dx\Big]^{1/2} \\ &\quad + \Big[\mu^2\int_{\Omega_n}e_{ij}^{(n)}e_{ij}^{(n)}\,dx\int_{\Omega_n}\phi_{,i}^{(n+1)}\phi_{,i}^{(n+1)}\,dx\Big]^{1/2} \\ &\quad + \Lambda\Big[\int_{\Omega_n}\kappa_{ij}\phi_{,i}^{(n)}\phi_{,j}^{(n)}\,dx\int_{\Omega_n}\kappa_{ij}\phi_{,i}^{(n+1)}\phi_{,j}^{(n+1)}\,dx\Big]^{1/2} \\ &\leq \frac{1}{2\alpha_2}W_{\Omega_n}(u^{(n+1)}) + \frac{1}{2}(\alpha_2+\alpha_3\delta)W_{\Omega_n}(u^{(n)}) \\ &\quad + \frac{\Lambda}{2}\Big(\frac{1}{\alpha_4}+\frac{\delta}{\alpha_3\Lambda}\Big)U_{\Omega_n}(\phi^{(n+1)}) + \frac{\alpha_4\Lambda}{2}U_{\Omega_n}(\phi^{(n)}), \end{aligned} \tag{6.21}$$

where α_2, α_3, α_4 are arbitrary positive constants to be determined.

Elimination of the bilinear function between (6.20) and (6.21) generates the inequality

$$\begin{aligned} \Big(1-\frac{\alpha_1\delta}{2}\Big)W_{\Omega_{n+1}}(u^{(n+1)}) &- \frac{1}{2\alpha_2}W_{\Omega_n}(u^{(n+1)}) + \Lambda\Big(1-\frac{\delta}{2\alpha_1\Lambda}\Big)U_{\Omega_{n+1}}(\phi^{(n+1)}) \\ &- \frac{\Lambda}{2}\Big(\frac{1}{\alpha_4}+\frac{\delta}{\alpha_3\Lambda}\Big)U_{\Omega_n}(\phi^{(n+1)}) \\ &\leq \frac{1}{2}(\alpha_2+\alpha_3\delta)W_{\Omega_n}(u^{(n)}) + \frac{\alpha_4\Lambda}{2}U_{\Omega_n}(\phi^{(n)}). \end{aligned} \tag{6.22}$$

We note (6.15), and set

$$\alpha_1\delta = \frac{(1+4\gamma)}{4\gamma},$$

$$\alpha_2 = \frac{4\gamma}{\varepsilon_1(4\gamma-1)},$$

$$\alpha_3\delta = \alpha_4 = \frac{2(1+4\gamma)}{\varepsilon_1(4\gamma-1)},$$

where

$$0 < \varepsilon_1 < 1.$$

This choice of α_1, α_2, α_3, and α_4 is selected for simplicity, and may not be optimal.

Insertion into (6.22), subsequent rearrangement, and introduction of Definition (6.14), leads to the fundamental inequality for linear piezoelectricity:

$$\begin{aligned} 2\varepsilon_1 W_{\Omega_{n+1}\setminus\Omega_n}(u^{(n+1)}) &+ 4\Lambda\varepsilon_1 \frac{(1+\gamma)}{(1+4\gamma)} U_{\Omega_{n+1}\setminus\Omega_n}(\phi^{(n+1)}) \\ &+ 2(1-\varepsilon_1)E_{\Omega_{n+1}}(u^{(n+1)}, \phi^{(n+1)}) \\ &\leq \frac{16\gamma(6\gamma+1)}{\varepsilon_1(4\gamma-1)^2} E_{\Omega_n}(u^{(n)}, \phi^{(n)}). \end{aligned} \tag{6.23}$$

Quantities $W_{\Omega_{n+1}\setminus\Omega_n}(u^{(n+1)})$ and $U_{\Omega_{n+1}\setminus\Omega_n}(\phi^{(n+1)})$ are positive-definite by assumption, and in accordance with the procedures of Sections 4 and 5, may be discarded from (6.23) to obtain the inequality

$$0 \leq E_{\Omega_{n+1}}(u^{(n+1)}, \phi^{(n+1)}) \leq qE_{\Omega_n}(u^{(n)}, \phi^{(n)}), \tag{6.24}$$

where

$$q = \frac{8\gamma(6\gamma+1)}{(1-\varepsilon_1)\varepsilon_1(4\gamma-1)^2}. \tag{6.25}$$

It is easy to show that $q > 1$. Inequality (6.24) recursively generates

$$\begin{aligned} E_{\Omega_{n+1}} &\leq qE_{\Omega_n} \\ &\leq q^2 E_{\Omega_{n-1}} \\ &\leq \vdots \\ &\leq q^{n+1-r_n} E_{\Omega_{r_n}} \qquad r_n = 1, 2, \ldots (n+1), \\ &\leq \vdots \\ &\leq q^n E_{\Omega_1}, \end{aligned}$$

in which the arguments of $E_{\Omega_n}(u^{(n)}, \phi^{(n)})$ are omitted for convenience.

We conclude that the terms

$$q^{n+1-r_n}E_{\Omega_{r_n}} \tag{6.26}$$

form a monotone decreasing bounded below sequence in r_n for each n. Our choice of arbitrary constants α_i, $i=1,2,3,4$, as mentioned, gives $q>1$ and consequently, terms in the sequence corresponding to large n and small r_n may become arbitrary large. On the other hand, assumptions (6.12) and (6.13) imply that all terms are bounded above and below for sufficiently large r_n irrespective of n. Accordingly, for $\varepsilon>0$ and $n\geq n_0$, where n_0 is sufficiently large, there exists a non-negative infinum E_n such that

$$\begin{aligned} E_n &\leq q^{n-r_n}E_{\Omega_{r_n+1}}(u^{(r_n+1)},\phi^{(r_n+1)}) \\ &\leq q^{n+1-r_n}E_{\Omega_{r_n}}(u^{(r_n)},\phi^{(r_n)}) \\ &\leq E_n+\varepsilon, \quad n_0\leq r_n\leq (n+1). \end{aligned} \tag{6.27}$$

Set $n_0\leq r_n=n$, $(n+1)$ to respectively obtain from (6.27) the bounds

$$\begin{aligned} E_n &\leq qE_{\Omega_n}(u^{(n)},\phi^{(n)})\leq E_n+\varepsilon, \\ E_n &\leq E_{\Omega_{n+1}}(u^{(n+1)},\phi^{(n+1)})\leq E_n+\varepsilon, \end{aligned}$$

which upon insertion into (6.23), and for convenience upon selecting $\varepsilon_1=1/2$ and $1<4\gamma<2$, yields for sufficiently large n

$$\begin{aligned} E_{\Omega_{n+1}\backslash\Omega_n}(u^{(n+1)},\phi^{(n+1)}) &\leq qE_{\Omega_n}(u^{(n)},\phi^{(n)})-E_{\Omega_{n+1}}(u^{(n+1)},\phi^{(n+1)}) \\ &\leq E_n+\varepsilon-E_n \\ &=\varepsilon. \end{aligned} \tag{6.28}$$

The last inequality immediately establishes the following Proposition that represents Zanaboni's version of Saint-Venant's principle for the coupled system of linear piezoelectricity on the elongated regions described in Section 3.

PROPOSITION 6.1. *Subject to the stated conditions and for sufficiently large n, we have*

$$\lim_{n\to\infty} E_{\Omega_{n+1}\backslash\Omega_n}(u^{(n+1)},\phi^{(n+1)})=0. \tag{6.29}$$

As a corollary, we deduce from the Proposition in conjunction with Definition (6.14), that besides the electric energy, the mechanical energy likewise tends to zero in regions sufficiently remote from the load surface Γ_M.

7. THERMOELASTOSTATICS

We study linear thermoelastostatics as an example of a coupled "dissipative" system in equilibrium. The main modification to the developements outlined in

Sections 4 and 6 is the replacement of Young's inequaltiy in the derivation of the fundamental inequality by a suitably adapted Poincaré embedding inequality. This is necessary in order to treat the thermal coupling term which combines the linear elastic strain with temperature and not with the temperature spatial gradient. The related embedding inequality is stated and its proof sketched in the Appendix.

Only the main components of the arguments are discussed. A complete account may be found in [13].

The geometry is that defined in Section 3. Each of the embedded regions $\Omega_n \subset \Omega_{n+1},\ n = 1, 2, \dots$ is respectively occupied by the same inhomogeneous linear anisotropic compressible thermoelastic material. Furthermore, each region Ω_n is in equilibrium under zero body-force and no heat sources. Zero Dirichlet boundary conditions are specified on each $\partial\Omega_n$ except on the common surface of intersection Γ which is subject to prescribed displacement on part $\emptyset \neq \Gamma_M \subset \Gamma$ and prescribed temperature on part $\emptyset \neq \Gamma_H \subset \Gamma$, where $\Gamma = \Gamma_M \cup \Gamma_H$. The corresponding traction boundary value problem with Neumann thermal boundary conditions may also be considered for suitably normalised solutions.

Without confusion, the same notation as before is largely retained, so that $\theta^{(n)}(x)$ denotes the positive temperature in Ω_n, while the corresponding displacement vector and linear strain tensor are given by $u^{(n)}$ and $e^{(n)}$ respectively.

The system of boundary value problems to be investigated is given by

$$(c_{ijkl}(x)e_{kl}^{(n)}(x) + \beta_{ij}(x)\theta^{(n)}(x))_{,j} = 0, \quad x \in \Omega_n, \tag{7.1}$$

$$(\kappa_{ij}(x)\theta_{,i}^{(n)})_{,j} = 0, \quad x \in \Omega_n, \tag{7.2}$$

$$u_i^{(n)}(x) = h_i(x), \quad x \in \Gamma_M, \quad \theta^{(n)}(x) = Q(x), \quad x \in \Gamma_H, \tag{7.3}$$

$$u_i^{(n)}(x) = 0 \quad x \in \partial\Omega_n \backslash \Gamma_M, \quad \theta^{(n)}(x) = 0, \quad x \in \partial\Omega_n \backslash \Gamma_H, \tag{7.4}$$

where the prescribed functions $h_i(x)$ and $Q(x)$ are independent of n. It is further supposed that the symmetric nonhomogeneous elasticity tensor $c(x)$ and symmetric heat conduction tensor $\kappa(x)$ are convex and therefore satisfy conditions corresponding to (6.9)–(6.10).

In addition, we require that the symmetric heat coupling tensor $\beta(x)$ is bounded in the sense that

$$\beta^2 = \max_{\forall\Omega_n} \beta_{ij}\beta_{ij} < \infty, \tag{7.5}$$

and that the mechanical and thermal energies in Ω_n are bounded for all n. Conseqeuntly, we assume that for $n = 1, 2, 3, \dots$ there holds

$$\int_{\Omega_n} c_{ijkl} e_{ij}^{(n)} e_{kl}^{(n)}\, dx < \infty, \tag{7.6}$$

$$\int_{\Omega_n} \kappa_{ij}\theta_{,i}^{(n)}\theta_{,j}^{(n)}\, dx < \infty. \tag{7.7}$$

In analogy with the treatment of Section 6, the appropriate fundamental inequality is derived in terms of the quadratic form

$$E_{\Omega_n}(u^{(n)}, \theta^{(n)}) = \int_{\Omega_n} c_{ijkl} e_{ij}^{(n)} e_{kl}^{(n)}\, dx + \Lambda \int_{\Omega_n} \kappa_{ij} \theta_{,i}^{(n)} \theta_{,j}^{(n)}\, dx, \tag{7.8}$$

where the positive constant Λ now satisfies

$$\Lambda = \gamma \bar{\delta}^2, \quad \bar{\delta}^2 = \frac{\beta^2 C}{4 c_0 \kappa_0}, \quad 1 < 4\gamma,$$

and C is the positive constant appearing in tbe generalised Poincaré inequality of Proposition A.1.

Derivation of the fundamental inequality again involves a bilinear form defined to be

$$I_{\Omega_n}(u, \theta; w, \phi) = \int_{\Omega_n} (c_{ijkl} e_{ij} e_{kl}(w) + \beta_{ij} e_{ij} \phi)\, dx + \Lambda \int_{\Omega_n} \kappa_{ij} \theta_{,i} \phi_{,j}\, dx, \tag{7.9}$$

where we recall that

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}),$$

$$e_{ij}(w) = \frac{1}{2}(w_{i,j} + w_{j,i}).$$

Integration by parts gives

$$I_{\Omega_n}(u^{(n)}, \theta^{(n)}; u^{(n+1)}, \theta^{(n+1)}) = I_{\Omega_{n+1}}(u^{(n+1)}, \theta^{(n+1)}; u^{(n+1)}, \theta^{(n+1)}). \tag{7.10}$$

The coupling term occurring in the integral on either side of (7.10) is bounded as follows. We consider only the left hand term. The right hand term is treated similarly. We have

$$\begin{aligned} &\pm \int_{\Omega_n} \beta_{ij} e_{ij}^{(n+1)} \theta^{(n)}\, dx \\ &\quad \le \left[\int_{\Omega_n} e_{ij}^{(n+1)} e_{ij}^{(n+1)}\, dx \int_{\Omega_n} \beta_{ij} \beta_{ij} \theta^{(n)} \theta^{(n)}\, dx \right]^{1/2} \\ &\quad \le \left(\frac{\beta^2 C}{c_0 \kappa_o} \right)^{1/2} \left[\int_{\Omega_n} c_{ijkl} e_{ij}^{(n+1)} e_{kl}^{(n+1)}\, dx \int_{\Omega_n} \kappa_{ij} \theta_{,i}^{(n)} \theta_{,j}^{(n)}\, dx \right]^{1/2}, \end{aligned} \tag{7.11}$$

where the embedding inequality (A.3) is employed.

Inequality (7.11) is next used to bound respectively from above and below the bilinear forms on the left and right of (7.10). Alternative manipulations to those of Section 6 lead to the relations

$$E_{\Omega_{n+1}}(u^{(n+1)}, \theta^{(n+1)}) \leq [(1-\varepsilon_2)E_{\Omega_n}(u^{(n+1)}, \theta^{(n+1)}) + \varepsilon_3 E_{\Omega_n}(u^{(n)}, \theta^{(n)})], \tag{7.12}$$

where the computable positive constants ε_2, ε_3 satisfy

$$0 < \varepsilon_2 < \frac{1}{2} < \varepsilon_3 < 1.$$

Rearrangement of (7.12) yields the required sequence of fundamental inequalities, namely

$$(1-\varepsilon_2)E_{\Omega_{n+1}\backslash\Omega_n}(u^{(n+1)}, \theta^{(n+1)}) \leq \varepsilon_3 E_{\Omega_n}(u^{(n)}, \theta^{(n)}) - \varepsilon_2 E_{\Omega_{n+1}}(u^{(n+1)}, \theta^{(n+1)}). \tag{7.13}$$

The quantities E_{Ω_n} are non-negative by assumption, and in consequence on discarding the term on the left of (7.13), we conclude that

$$0 \leq E_{\Omega_{n+1}}(u^{(n+1)}, \theta^{(n+1)}) \leq \frac{\varepsilon_3}{\varepsilon_2} E_{\Omega_n}(u^{(n)}, \theta^{(n)}), \tag{7.14}$$

and by recursive application that the terms

$$\left(\frac{\varepsilon_3}{\varepsilon_2}\right)^{n+1-r_n} E_{\Omega_{r_n}}(u^{(r_n)}, \theta^{(r_n)}), \quad r_n = 1, 2, \ldots (n+1), \tag{7.15}$$

form a monotone decreasing bounded sequence in r_n for each n. On setting $q = \varepsilon_3/\varepsilon_2 > 1$, expressions (7.13)–(7.15) become identical in type to (6.23)–(6.26). The same conclusions therefore are valid and corresponding to Proposition 6.1 may be embodied in the following final result which represents the thermoelastic version of Zanaboni's formulation of Saint-Venant's principle on elongated regions.

PROPOSITION 7.1. *Subject to the stated conditions and for sufficiently large n, we have for linear thermoelastostatics the limiting behaviour*

$$\lim_{n\to\infty} E_{\Omega_{n+1}\backslash\Omega_n}(u^{(n+1)}, \theta^{(n+1)}) = 0. \tag{7.16}$$

As in Section 6, expression (7.8) immediately implies that both the mechanical and thermal energies tend to zero in regions sufficiently remote from the load surface Γ.

8. Concluding remarks

It has frequently been commented that lack of appreciation and understanding, and even neglect, of historical contributions can be inimical to future progress.

Indeed, examination of old frontiers can afford exciting new horizons that open prospects of new frontiers.

An outstanding challenge to the approach described here is the derivation of decay estimates for elongated regions from the fundamental inequality. It is also of interest to extend the application of our procedure to non-elongated regions, such as cones treated in [17, 26, 15, 1] by means of differential inequalities, and to elongated regions composed of fibre reinforced and functionally graded materials. Other open problems concern the Navier-Stokes equations and stochastic processes. A preliminary incursion into time-dependent systems is undertaken in [16] which explores the example of transient heat conduction. As mentioned already, the innate simplicity of the approach suggests that these and other problems can be successfully treated.

A. Generalised Poincaré inequality

This Appendix is devoted to a statement and outline proof of an inequality employed in Section 7 for the derivation of the fundamental inequaltiy required in thermoelasticity to establish Saint-Venant's principle. The regions Ω_n to which the inequalities apply are defined in Section 3.

Proposition A.1. *Let the continuously differentiable scalar function $v(x)$ be defined on the region Ω_n and satisfy the boundary conditions*

$$v(x) \neq 0, \quad x \in \Gamma_H, \tag{A.1}$$

$$= 0, \quad x \in \partial\Omega_n \backslash \Gamma_H, \tag{A.2}$$

where $\emptyset \neq \Gamma_H \subset \partial\Omega_n$.

Then

$$\int_{\Omega_n} v^2 \, dx \leq C \int_{\Omega_n} v_{,i} v_{,i} \, dx, \tag{A.3}$$

where C is a computable positive constant independent of n.

Proof. The proof, described in detail in [13], depends upon the decomposition of Ω_n into disjoint regions Ω and Δ_n such that

$$\Omega_n = \Delta_n \cup \Omega, \quad \Delta_n \cap \Omega = \emptyset, \tag{A.4}$$

where $\emptyset \neq \Omega \subset \Omega_n$ is a fixed bounded region chosen independently of Ω_n. The surface of Ω is such that

$$\partial\Omega = \Sigma_1 \cup \Sigma_2, \tag{A.5}$$

where $\Gamma_H \subset \Sigma_2$, and Σ_1 is planar.

Embedding inequalities are separately derived for Δ_n and Ω. First, we apply the usual Poincaré inequality to each plane cross-section D_n of Δ_n chosen parallel to the plane cross-section Σ_1. The area $|D_n|$ of each cross-section, by hypothesis, is bounded and consequently the Faber-Krahn inequality and integration over Δ_n leads to

$$\int_{\Delta_n} v^2\,dx \le \frac{D}{j_0^2\pi}\int_{\Delta_n} v_{,i}v_{,i}\,dx, \tag{A.6}$$

where j_0 is the smallest positive zero of the Bessel function $J_0(.)$, and the constant D is given by

$$D = \max_{n}\ \max_{D_n\in\Delta_n} |D_n|. \tag{A.7}$$

The second component inequality is for the fixed region Ω and is derived in [2] (see also [32]). It states that there exists a computable positive constant C_Ω, dependent on Ω such that

$$\int_{\Omega} v^2\,dx \le C_\Omega \int_{\Omega} v_{,i}v_{,i}\,dx, \tag{A.8}$$

where $v(x)$ now satisfies the boundary conditions

$$\begin{aligned} v(x) &\neq 0, \quad x\in\Gamma_H\cup\Sigma_1\\ &= 0, \quad x\in\partial\Omega\backslash(\Gamma_H\cup\Sigma_1), \end{aligned}$$

Addition of inequalities (A.6) and (A.8) leads to (A.3) with

$$C = \max\Big(\frac{D}{j_0^2\pi}, C_\Omega\Big).$$

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