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History of Mathematics — *Analytical mechanics and Levi-Civita's parallel transport*, by Giuseppe Iurato, communicated on December 16, 2016.[1]

Abstract. — This historical note is a detailed, technical reconstruction of the crucial role played by Lagrange's principle of virtual work in working out the fundamental notion of parallel transport on a Riemannian manifold described in the original Levi-Civita's argument.

Key words: Riemannian manifold, Levi-Civita's parallel transport, virtual work principle, linear connection

Mathematics Subject Classification: 01-08, 01A60, 01A85, 53-03, 53B05, 53B20, 70-03, 70F20

1. Introduction

The role of the principle of virtual works of analytical mechanics, in formulating Levi-Civita's parallel transport of Riemannian geometry, has already been emphasized in [12], to which the reader is referred for a more comprehensive historical contextualization. In [12] the subject is covered mostly at the descriptive level; this note deepens instead a more particular aspect of the matter, namely a technical reconstruction of Levi-Civita's original argument to generate this important idea of differential geometry, followed by its reformulation in today's language to enlighten its links to the modern formalism.

Tullio Levi-Civita wrote in the introduction to [14] that his initial motivation was the simplification of the calculation methods for the curvature of a generic Riemannian manifold, involving Riemann's symbols. To this end, a preliminary geometric examination of the question enabled him to work out an idea of parallelism on a Riemannian manifold. This idea turned out to be necessary to compute the manifold curvature by the methods available at the time, basically centered on vectorial circuitations along suitable infinitesimal closed contours (among which are the so-called *geodesic parallelogrammoids*), lying on the given manifold.

In pursuing this program for the computation of the curvature (which will lead to the so-called *Levi-Civita's geometric characterization of Riemannian curvature*[2], Levi-Civita devotes the first fourteen sections of his memoir – i.e., most of the whole paper – to introduce and explain the new concept of parallelism

[1] Presented by S. Graffi.
[2] Cf. [23], Ch. 8, pp. 316–317.

upon an arbitrary Riemannian manifold V_n with dimension $n \geq 2$, embedded in some ordinary Euclidean space $\mathbb{R}^N$, denoted by S_N. After that, he applies this geometric idea to simplify the computation of the Riemannian curvature. His main aim is to clarify the pioneering Riemann ideas on the curvature of metric manifolds, sketched in the 1854 *Habilitationschrift*. He succeeds in this task by means of an analogical-conceptual transcription of the initial purely geometric question into a suitable analytical mechanics framework.

Levi-Civita devotes the last two sections (17 and 18) of [14], just to make explicit what he deemed implicitly present in the original Riemann memoir, and accomplishes this task through suitable geometrical tools, primarily his notion of parallelism. Actually in these sections of [14], Levi-Civita uses his notion of parallelism on a generic Riemannian manifold to clarify and determine more easily the covariant behaviour of Riemann's symbols as well as the curvature of a Riemannian manifold with a generic metric. This is pursued by means of that usual method based on infinitesimal geodesic parallelogrammoids and commutation properties of the related first-order infinitesimal displacement operators (see later).

The main purpose of the present note is to clarify, by going through the technical formal steps of Levi-Civita's construction of the notion of parallelism, that analytical mechanics had a primary, central role in the development of this idea. Its influence is twofold: first on the intuition and then in guiding the analytical treatment of the question, which enabled him to obtain the local differential equations of parallelism. In what follows, we reconstruct and explain (adding the necessary details) the crucial points of the original Levi-Civita's procedure, which involves concepts and methods of analytical mechanics and, at the same time, of differential geometry, and conclude with a short reformulation of the main concept in todays' language.

2. A brief recall on the principle of virtual works

In this section, the principle of virtual works is recalled; we refer to [12] for a relevant historical discussion. To stay as close as possible to Levi-Civita's original formulation, also for the sake of a methodological coherence, here his famous treatise on rational mechanics [16] is followed, almost *verbatim*.[3]

The fundamental equations of dynamics for a general system of $N \geq 1$ material points of mass m_i, subjected to preassigned *active forces* $\vec{F}_i$ and *constraint reactions* $\vec{R}_i$, are written as

$$(1) \qquad (\vec{F}_i - m_i\vec{a}_i) + \vec{R}_i = \vec{0}, \quad i = 1, 2, \ldots, N,$$

where $-m_i\vec{a}_i$, $i = 1, \ldots, N$, are the *inertial forces*. These equations are the formal statement of the so-called *D'Alembert's principle*, which formally reduces any dynamical problem to a static one if the active forces $\vec{F}_i$, are replaced by the

[3] As already mentioned, the object of this paper is a detailed technical reconstruction of the original Levi-Civita's argument in [14]. Therefore, all also his related works are textually followed.

effective forces[4] $\vec{F}_i - m_i\vec{a}_i$, $i = 1, \ldots, N$. This principle, which may be stated in many equivalent forms, is a key principle of analytical mechanics, together the principle of virtual works.

The *principle of virtual works*, in turn, states that the total work of the reactions $\vec{R}_i$ is, in case of smooth constraints and virtual displacements compatible with them, non-negative for any irreversible virtual displacement, and zero for any reversible virtual displacement.[5] If the smooth constraints are bilateral, (expressed, for example, by r equalities which define a smooth manifold of codimension r), all compatible virtual displacements are reversible, and the virtual work of constraint reactions is zero. Hence, the principle of virtual works reads

$$\delta\Phi = \sum_i \vec{R}_i \cdot \delta\vec{P}_i = 0, \tag{2}$$

where $\delta\vec{P}_i$ is the (non-zero) first-order virtual displacement of the point of application of $\vec{R}_i$. Relation (2) is also said to be the *symbolic equation of statics*.[6]

From (1) and (2), it follows *Lagrange's principle of virtual work* which states a finite system of material points is in equilibrium when the active forces $\vec{F}_i$ acting on the system satisfy[7]

$$\delta L = \sum_i \vec{F}_i \cdot \delta\vec{P}_i = 0, \tag{3}$$

where $\delta\vec{P}_i$ is the (non-zero) first-order infinitesimal displacement of the application point of $\vec{F}_i$. Thus, if a system is at equilibrium, the virtual work of all active forces $\vec{F}_i$ will vanish for any virtual displacement. Relation (3) is also said to be the *symbolic equation of dynamics*.[8]

If the constraints are holonomic, and expressed as equalities in the intrinsic parameters of the system[9], then the vanishing of the virtual work of the constraint reactions assumes a well-known geometrical meaning. If, in particular, the material point is constrained to lie upon a smooth surface (or a smooth curve), then the reaction is perpendicular to the surface (curve), while every (non-zero) first-order virtual displacement lies on the tangent plane (tangent line) of the surface (curve). In this case, the constraint reaction makes no work.[10]

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[4] Cf. [16], Vol. I, Ch. XV, Sec. 1, No. 2; Vol. II, Part I, Ch. V, Sec. 3, Nos. 18–20.

[5] Cf. [16], Vol. I, Ch. XV; Vol. II, Part I, Ch. V, Sect. 3, Nos. 18–21; [18], Vol. I, Ch. XIV, Sect. 2, Nos. 4–8; Vol. II, Ch. V, Sect. 3, Nos. 17–19; [1], Vol. II, Ch. V, Sect. 1, No. 4; [2], Ch. I, Sects. 1–2; [9], Vol. 1, Ch. XIII, Sect. 4.

[6] Also said to be *D'Alembert–Lagrange principle* as reformulated by Lagrange (cf. [3], Ch. IV), or *general equation of virtual work* (cf. [4], Vol. I, Ch. XV, Sect. 318; [13], Vol. I; [24]). See also the references quoted in the previous footnote.

[7] Cf. [10], Ch. 1, Sect. 1.4, Eqs. (1.43)–(1.45); [25], Part I, Ch. 6, p. 210; Part III, Ch. 12, p. 441.

[8] Cf. [16], Vol. II, Part I, Ch. V, Sect. 3, No. 20.

[9] Cf. [16], Vol. I, Ch. VI, Sects. 1 and 3.

[10] Cf. [16], Vol. I, Ch. XV, Sec. 1, No. 3-a).

This latter instance will be emblematic in the following historical inquiry, when it will be discussed how and why the symbolic equation of dynamics (3) is so crucial in developing Levi-Civita's notion of parallel transport on a Riemannian manifold. This will bring back into consideration, again following Levi-Civita, a conceptual mechanical analogy concerning this example.

3. Levi-Civita's parallel transport: a technical reconstruction

As already mentioned, the present investigation basically consists in a detailed formal analysis of the original sources related to the question. Therefore, in this section, I strictly follow the original paper of Levi-Civita, i.e., [14], and other related works by Levi-Civita himself. Starting from the historical considerations of [12], my intention is, here, to reconstruct technically the pathway followed by Levi-Civita in reaching his notion of parallelism, through a detailed explicitation of those points in which he uses arguments of analytical mechanics.

In [14], Sect. 1, Levi-Civita begins with the consideration of two arbitrary directions $\vec{\alpha}$, $\vec{\alpha}'$ standing out from two infinitesimally close points P, P' of a generic Riemannian manifold V_n, embedded in a N-dimensional Euclidean space S_N (i.e., $\mathbb{R}^N$) of suitable dimension N. Thinking V_n immersed into S_N, we may consider $\vec{\alpha}$ and $\vec{\alpha}'$ in S_N; the Euclidean geometry condition of parallelism means that the two directions $\vec{\alpha}$, $\vec{\alpha}'$ are parallel if and only if

$$\text{angle}(\widehat{\vec{\alpha}, \vec{f}}) = \text{angle}(\widehat{\vec{\alpha}', \vec{f}}) \tag{4}$$

for any arbitrarily fixed direction $\vec{f}$ issued from both P and P', according to the equipollence relation in S_N.

Then Levi-Civita underlines that this parallelism condition, in V_n, a priori depends on the path joining P and P' on V_n; the independence of the path holds only in ordinary Euclidean spaces (which are flat). Now, condition (4) has to be specified. This will be done by analyzing the geometric behavior of $\vec{\alpha}$ issued from P, assumed to remain parallel (in S_N) according to (4), when P moves towards P' along a generic curve of V_n with endpoints P and P'. This (synthetic) geometrical view generates the (analytical) notion of parallelism in V_n, if interpreted in terms of analytical mechanics.

Levi-Civita considers[11] indeed a generic metric on an arbitrary finite-dimensional manifold V_n

$$ds^2 = \sum_{i,j=1}^{n} a_{ik}\, dx_i\, dx_j. \tag{5}$$

Then he embeds V_n in a Euclidean space S_N with dimension $N(\leq n(n+1)/2)$ large enough. Hence V_n may be described by the system of equations[12]

$$y_\nu = y_\nu(x_1, \ldots, x_n), \quad \nu = 1, 2, \ldots, N, \tag{6}$$

[11] Cf. [7], Ch. XXV.
[12] Cf. [14], Eq. (1), p. 4.

where the y_ν are (Cartesian) coordinates in S_N, while the x_k are intrinsic (equivalently, Lagrangian – see below) coordinates on V_n.

Now remark that the system (6) may be thought of as the configuration space of a constrained mechanical system with n degrees of freedom subjected to N smooth holonomic bilateral constraints, which identifies a differentiable manifold structure of dimension n. This is the central point of the mechanical interpretation of Levi-Civita's parallelism notion, the unique possible within this formal framework: the shift from a point on V_n to another infinitesimally close is performed through (6), in corresponding conceptual analogy with a holonomic mechanical system[13] of material points with unitary mass, whose lagrangian coordinates are $(x_1, \ldots, x_n)$ and whose kinetic energy T has by (5) the form $2T\,dt^2 = \sum_{i,j} a_{ij}\,dx_i\,dx_j$.

For the sake of simplicity, Levi-Civita identifies any unit vector in S_N, i.e., a direction in S_N, with a unit vector $\vec{\alpha}$ having direction cosines α_ν, $\nu = 1, \ldots, N$, and similarly any auxiliary direction of S_N is identified by the unit vector $\vec{f}$, with direction cosines f_ν, $\nu = 1, 2, \ldots, N$; both vectors are supposed to stand out or issue from an arbitrarily fixed point P of V_n, and immersed into S_N. Therefore, the direction cosines of both unit vectors $\vec{\alpha}$ and $\vec{f}$ are computed with respect to S_N. All that is possible because V_n is embedded in the ambient space S_N, so that each direction belonging to V_n also belongs to S_N.

The point P may be thought of as a unit-mass material point performing a motion along an arbitrary smooth curve $\mathscr{C}$ lying on V_n, parameterized by the curvilinear (or natural) abscissa s as in (5), so that $\alpha_\nu = \alpha_\nu(s)$, $\nu = 1, 2, \ldots, N$. Let $x_i = x_i(s)$, $i = 1, 2, \ldots, n$ be the intrinsic parametric equations of $\mathscr{C}$. Then $\mathscr{C}$, thought as embedded in S_N via (6), is represented also by the parametric equations $y_\nu = y_\nu(s)$, $\nu = 1, 2, \ldots, N$. Indeed $x_i = x_i(s)$, $i = 1, 2, \ldots, n$ yields

$$y_\nu = y_\nu(x_1(s), \ldots, x_n(s)), \quad \nu = 1, 2, \ldots, N. \tag{7}$$

Clearly in the analog constrained system considered above, $\mathscr{C}$ is a trajectory in the manifold of the admissible configurations V_n, parameterized by time t according to the parametric equations $x_\nu = x_\nu(t)$, $\nu = 1, 2, \ldots, N$, with $t \in \mathbb{R}^+$.

To find a generic unit direction issuing from an arbitrary point P of $\mathscr{C}$, Levi-Civita differentiates[14] the parametric representation (7) with respect to the natural abscissa s

$$y'_\nu = \sum_{i=1}^{n} \frac{\partial y_\nu}{\partial x_i} x'_i, \quad \nu = 1, 2, \ldots, N, \tag{8}$$

and obtains the direction cosines with respect to S_N (i.e., y'_ν), while x'_i, $i = 1, \ldots, n$ are the direction cosines of the same unit direction but with respect to V_n.

Then he considers, at some point P of $\mathscr{C}$, an arbitrarily fixed direction $\vec{\alpha}$ of V_n through P, whose direction cosines are $\xi^{(i)}$, $i = 1, 2, \ldots, n$ with respect to V_n,

[13] Cf. [16], Vol. II, Part I, Ch. V, Sect. 9, No. 63; Vol. II, Part II, Ch. XI, Sect. 4, No. 15.
[14] Cf. [14], (4), p. 5.

and α_ν, $\nu = 1, 2, \ldots, N$ with respect to S_N; by the identification $y'_\nu \to \alpha_\nu$ (in S_N), $x'_i \to \xi^{(l)}$ (in V_n), from (8) it also follows[15]

$$\alpha_\nu = \sum_{l=1}^{n} \frac{\partial y_\nu}{\partial x_l} \xi^{(l)}, \quad \nu = 1, 2, \ldots, N. \tag{9}$$

α_ν, $\nu = 1, \ldots, n$ is thus a linear form on $\xi^{(l)}$, i.e., on the direction cosines of $\vec{\alpha}$ with respect to V_n.

When P moves along $\mathscr{C}$, animated its motion on V_n, ordinary parallelism in S_N implies the equality of the angle between $\vec{\alpha}$ and an auxiliary direction $\vec{f}$ arbitrarily fixed in S_N, according to the Euclidean condition (4) (of synthetic geometry). Now, starting from this stance, Levi-Civita gradually introduces an intrinsic notion of parallelism in V_n, considering two nearby points P and P' of $\mathscr{C}$, lying on V_n, with P moving towards P' along $\mathscr{C}$, never leaving V_n. Therefore, considering the arbitrarily fixed direction $\vec{f}$ of S_N, with direction cosines f_ν (in S_N), the cosine of the angle between $\vec{f}$ and $\vec{\alpha}$ in S_N, is given by

$$\cos(\widehat{\vec{f}, \vec{\alpha}}) = \sum_{\nu=1}^{N} \alpha_\nu f_\nu. \tag{10}$$

Then Levi-Civita considers an infinitesimal variation ds of the natural abscissa s on V_n along $\mathscr{C}$. This implies that the cosine (10) undergoes the following first-order variation[16]

$$d[\cos(\widehat{\vec{f}, \vec{\alpha}})] = ds \sum_{\nu=1}^{N} \alpha'_\nu(s) f_\nu. \tag{11}$$

Now, the ordinary parallelism in S_N between the two directions $\vec{\alpha}(s)$ (in P) and $\vec{\alpha}(s + ds)$ (in P'), as expressed by (4), would require (11) to vanish when $\vec{f}$ varies arbitrarily in S_N, implying α_ν to be constant or uniform, and vice versa. Levi-Civita introduces the key argument which leads to the notion of parallelism on V_n exactly at this point. Since, indeed, his main purpose is the computation of the curvature of the Riemannian manifold V_n, his approach to the problem is first an intuitive one, namely he works out initially a geometrical intuition, and then

[15] Cf. [14], Eqs. (7), p. 6.

[16] As for this first-order variation, the arbitrariness with which $\vec{f}$ may be fixed entails that such an auxiliary direction – defined, as an ordinary vector, according to the equipollence relation in S_N – may be considered as independent of s, at least locally in $P \in V_n$ (i.e., in $T_P(V_n)$ – see later). Instead, the direction $\vec{\alpha}$ a priori depends on s as it varies with the motion of P along $\mathscr{C}$ on V_n, even in a neighborhood of P. The equipollence relation as settled in S_N, locally restricted to a neighborhood of $P \in V_n$ and defined at varying of all the curves $\mathscr{C}(\subseteq V_n)$ passing by P, will led to the individuation (and, later, to the modern definition) of the so-called *tangent space* $T_P(V_n)$ to V_n in P. In [14], Levi-Civita considers $\vec{f}$ as belonging only to $T_P(V_n)$, and not to the whole of S_N as in (4), to get his notion of parallelism on V_n.

carries on its analytical formulation within the framework of absolute differential calculus.

The usual way to determine the curvature of a Riemannian manifold V_n consists in computing the circuitation of a given vector around a suitable infinitesimal closed path entirely lying on the given manifold, usually a "parallelogrammoid" whose sides are first-order infinitesimal geodetic traits, drawn around an arbitrary point P of V_n. By definition, this vector is rotated, all around the circuit, in a parallel way to itself. After a complete circuitation, once the departure point is recovered, the deviation angle between initial and final directions in the coinciding final and initial points, provide a (first-order) computation of the curvature of V_n.

A preliminary notion of parallelism on a generic Riemannian manifold is thus needed to determine its curvature; it is exactly to this end that Levi-Civita, making appeal to analytical mechanics, gives a first geometrical sketch to the solution of this formal problem. In this geometrical view of the question, Levi-Civita considers only circuitations of a generic applied vector (as, for instance, $\vec{\alpha}$) whose application point P never leaves V_n.

This circuitation therefore takes place in a neighborhood of P on V_n. In Levi-Civita's geometrical intuition, its main purpose should be, so to speak, "to feel the shape of V_n", i.e., its distortion, bending, deformation, and so on. An intuitive way to accomplish this end, is to warrant the circuitating vector $\vec{\alpha}$ to stay always related, during the circuitation (which is a composition of sequential shifts along infinitesimal traits of smooth curves on V_n usually, geodesic curves), with some intrinsic geometric entity characterizing V_n – i.e., $T_P(V_n)$. Therefore the above considerations should *a fortiori* hold along each of these infinitesimal curve traits.

A Riemannian manifold is characterized by the property to be locally isomorphic to an Euclidean space S_N; hence the tangent space of V_n at some point P, denoted[17] $T_P(V_n)$, is just that geometric entity characterizing the local differentiable structure underlying any Riemannian manifold. Therefore, the Euclidean condition (11) will have an intrinsic meaning related only to V_n when the variability of $\vec{f}$ is restricted from the whole of S_N to $T_P(V_n)$, thus guaranteeing the above required reference of $\vec{\alpha}$ to V_n along its motion on V_n. This intrinsic geometrical restriction, in particular, should hold for (11).

At the same time, the restriction $\vec{f} \in T_P(V_n)$ also guarantees that the infinitesimal motion of $\vec{\alpha}$ along $\mathscr{C}$ "smooths out" V_n, thereby "feeling" its local curvature. This is the key geometric intuition of Levi-Civita in order to constructively define a notion of parallelism on a generic Riemannian manifold. The analytic formulation of this geometric idea represents the precise point where Levi-Civita introduces analytical mechanics into play. More precisely, through the principle of virtual works in its deepest geometric aspect.

[17] Here, we use a modern notation for the tangent vector space to a Riemannian manifold, not yet used by Levi-Civita in his 1917 memoir, in which he simply speaks of "a lying of S_N tangent in P to V_n" ([14], p. 2).

Hence Levi-Civita claims that, to this end, the directions $\vec{f}$ must be exactly those compatible with the constraints[18] (6), if one assumes valid the formal analogy which considers P as a (unit-mass) material point subjected to the smooth constraints (6). Accordingly, $\vec{f}$ must lie on $T_P(V_n)$, that is to say, $\vec{f}$ must be correlated, in this mechanical analogy, with (non-zero) first-order displacements compatible with the constraints (6). Under these conditions, if we wish to define a vectorial displacement (of $\vec{\alpha}$) intrinsically related to (or correlated with) V_n, then we should require that, while P moves along $\mathscr{C}$ on V_n, the direction $\vec{\alpha}$ must be transported, along $\mathscr{C}(\subseteq V_n)$, always in such a way that $\vec{f} \in T_P(V_n)$, that is to say, compatibly with the smooth constraints (6).

At this point, still inside the above geometric framework provided by analytical mechanics, Levi-Civita, in looking at the formal aspect of (11), glimpses a kind of "physical work" in S_N made by some "active forces"[19], whose Cartesian components formally correspond to $\alpha'_\nu(s)$, as applied to the (unit-mass) material point P moving along $\mathscr{C}$ with respect to the smooth constraints (6) which identify V_n as a holonomic mechanical system. Hence, in this mechanical analogy, Levi-Civita sees in (11) the formal expression of the principle of virtual works according to (3), if we replace $f_\nu \in T_P(V_n)$ with some quantities, say δy_ν, proportional to (non-zero) first-order virtual displacements compatible with smooth constraints (6). This replacement is formally allowed as these latter displacements δy_ν, by definition, lie on $T_P(V_n)$.

Therefore, with the identification[20] $\vec{F} \to \vec{\alpha}$, $\delta\vec{P} \to \vec{f}$, assumed in (3), hence with the further replacement of $\vec{f} = (f_\nu) \in T_P(V_n)$ (in S_N) with δy_ν (in V_n), the assumption $\delta L = 0$ entails that the Euclidean parallelism condition on $V_n(\hookrightarrow S_N)$, expressed by the vanishing of (11), reduces to

$$\sum_{\nu=1}^{N} \alpha'_\nu(s)\delta y_\nu = 0, \tag{12}$$

for any (non-zero) variation δy_ν, that is, "for any admissible first-order displacement compatible with the constraints" (6), as Levi-Civita writes textually in [14], p. 7. Under suitable mechanical interpretation of the $\alpha'_\nu(s)$, for instance considering them as a kind of generic mechanical action in S_N generated by some potential $\alpha_\nu(s)$, (12) is a formulation of the virtual work principle in S_N related to the smooth bilateral holonomic system defined by (6), hence to analytical mechanics on a Riemannian manifold.[21]

[18] Cf. [14], p. 7.

[19] Cf. [17], p. 120.

[20] We cannot consider the correspondence $\vec{R} \to \vec{\alpha}$ instead of $\vec{F} \to \vec{\alpha}$, because, a priori, not always $\vec{\alpha}$ is normal to V_n as required by reactions to smooth constraints. Therefore, in applying the principle of virtual works in this pattern analogy of Levi-Civita, we should take into account the symbolic equation of dynamics (as involving active forces $\vec{F}$), rather than the symbolic equation of statics (as involving reactions $\vec{R}$).

[21] Cf. [27], Ch. V; [22], Ch. V; [26], Ch. II; [19], Ch. VI, Sect. I, Nos. 87–89, 92; [3], Ch. IV; [5], Part II, Ch. V, Sect. 6; [11], Ch. 3, Sect. 2, No. 2.6.; [20], Ch. 15; [8], Ch. 1, Sects. 1.9–12, Ch. 4; [25], Part III, Ch. 12.

Nevertheless, the condition (12) is still related to S_N as in it still figures the dimension N of S_N, not the intrinsic dimension n of V_n, and the Cartesian coordinates y_ν of S_N, not the intrinsic ones x_k of V_n. Levi-Civita solved also this formal problem as follows. To get an intrinsic form, from (6), one first has[22]

$$\delta y_\nu = \sum_{k=1}^{n} \frac{\partial y_\nu}{\partial x_k} \delta x_k, \quad \nu = 1, 2, \ldots, N, \tag{13}$$

where δx_k, $k = 1, \ldots, n$, is an arbitrary (non-zero) first-order virtual displacement on $T_P(V_n)$ (with respect to V_n). Hence (12) reduces to[23]

$$\sum_{\nu=1}^{N} \alpha'_\nu(s) \frac{\partial y_\nu}{\partial x_k} = 0, \quad k = 1, 2, \ldots, n. \tag{14}$$

These N conditions, in the intrinsic variables (or Lagrangian coordinates) x_k, express the parallelism of the direction $\vec{\alpha}$ moving along $\mathscr{C}$ on V_n. To obtain then a fully intrinsic relation in V_n, it is necessary to involve only the parameters of V_n, eliminating any reference to S_N.

To this end, one replaces the direction cosines $\alpha_\nu(s)$ (in S_N) with their expression given by (9), which involve only the intrinsic direction cosines $\xi^{(i)}(s)$ (in V_n). Omitting the algebraic details, one finally deduces[24]

$$\frac{d\xi^{(i)}}{ds} + \sum_{j,l=1}^{n} \Gamma^i_{jl} \frac{dx_j}{ds} \xi^{(l)} = 0, \quad i = 1, 2, \ldots, n, \tag{15}$$

where Γ^i_{jl} are the Christoffel symbols of the second kind in the intrinsic coordinates x_k (of V_n), defined as follows[25]

$$\Gamma^i_{jl} = \sum_{k=1}^{n} a^{ik} \Big(\frac{\partial a_{kl}}{\partial x_j} + \frac{\partial a_{jk}}{\partial x_l} - \frac{\partial a_{jl}}{\partial x_k} \Big), \quad i, j, l = 1, 2, \ldots, n. \tag{16}$$

$\|a^{ik}\|$ is the coefficient matrix of the reciprocal form of (5). The (15) are the so-called (intrinsic) *Levi-Civita's equations of parallelism* on a Riemannian manifold V_n, equipped with a generic metric of the type (5).

They are first-order ordinary differential equations for the direction cosines $\xi^{(i)}$ of the arbitrary direction $\vec{\alpha}$, standing out from P, which is transported, along a curve $\mathscr{C}$ on V_n, up to the infinitesimal nearby point P', where a parallel direction $\vec{\alpha}'$ (to $\vec{\alpha}$) issues, with new direction cosines $\xi^{(i)} + d\xi^{(i)}$ such that

[22] Cf. [14], unnumbered equation before eqs. (8), p. 7.
[23] Cf. [14], Eq. (8), p. 7.
[24] Cf. [14], Eq. (I_a), p. 8.
[25] Cf. [6], Ch. II.

(by (15))

$$d\xi^{(i)} + \sum_{j,l=1}^{n} \Gamma^i_{jl}\, dx_j \xi^{(l)} = 0 \quad i = 1, 2, \ldots, n. \tag{17}$$

The equations (15), identify a (regular) linear system of ordinary differential equations for $\xi^{(i)}$, $i = 1, \ldots, n$. The standard existence and uniqueness theorems make possible to determine a (unique) direction parallel to every other preassigned. This is the starting point for each possible notion of *connection* of differential geometry; its coefficients are Γ^i_{jl} of (17).

This last point can be reformulated in modern notations as follows.

If the smooth curve $\mathscr{C}$ has parametric equations $x : [0, 1] \to V_n$, then Levi-Civita's *local* parallel transport along $\mathscr{C}$, expressed by the differential forms (17), establishes an isomorphism between the tangent spaces $T_{x(t)}(V_n)$, $t \in [0, 1]$ (if $P \in \mathscr{C}$ is identified by $x(t)$), of the tangent bundle $T(V_n) \doteq \bigcup_{P \in V_n} T_P(V_n)$ (with disjoint union), placed at infinitesimal nearby points of V_n. Levi-Civita's *global* parallel transport along $\mathscr{C}$, as expressed by the ODE system (15), is the isomorphism, denoted $\nabla_{\mathscr{C}}$, defined by

$$\nabla_{\mathscr{C}} : T_{x(0)}(V_n) \to T_{x(1)}(V_n). \tag{18}$$

Hence (15) yields, from the preassigned initial direction $(\xi^{(1)}(0), \ldots, \xi^{(n)}(0))$, the final direction $(\xi^{(1)}(1), \ldots, \xi^{(n)}(1))$ as the solution to (15). This solution exists and is unique by the well-known theorems of existence and uniqueness for the regular system of first-order ordinary differential equations (15). Therefore, one says that the vector $\xi^{(0)} \in T_{x(0)}(V_n)$ is *parallel* (in the sense of Levi-Civita) to the vector $\xi^{(t)} \doteq \nabla_{\mathscr{C}}(\xi^{(0)}) \in T_{x(t)}(V_n)$, along $\mathscr{C}(\subseteq V_n)$, for every $t \in]0, 1]$ arbitrarily fixed. Upon variation of $\mathscr{C}$ in the set of all possible smooth curves $\mathscr{C}$ of V_n, $\nabla_{\mathscr{C}}$ identifies a *linear connection* on V_n, denoted ∇, which generalizes the usual notion of directional derivative of ordinary Euclidean space to general Riemannian manifolds.[26]

Thus, the formal deduction of the intrinsic conditions (15) (or (17)), characterizing Levi-Civita's notion of parallel transport of the generic direction $\vec{\alpha}$ along an arbitrary curve[27] $\mathscr{C}(\subseteq V_n)$ as a function of its directional parameters $\xi_1, \ldots, \xi_n$ with respect to V_n, basically relies on the symbolic equation of dynamics (3). This yields the parallelism conditions (14), whence (15). We can say that the power of Levi-Civita's discovery is to put into relation infinitesimally nearby points of a Riemannian manifold V_n by means of a linear isomorphism (i.e., $\nabla_{\mathscr{C}}$) 'connecting' the related (linear) tangent spaces at V_n. This construction, deeply rooted in analytical mechanics, and never considered before, became therefore a milestone of differential geometry.

[26] Cf. [20], Part I, Ch. 1, Sect. 1.1.

[27] Levi-Civita also consideres (in [14], Sect. 7) the particular case where $\mathscr{C}$ is a geodetic curve of V_n, but this has not been neither the general case considered in [14] in deducing the main equations (17) nor the initial motivation of his 1917 memoir.

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