Rend. Lincei Mat. Appl. 28 (2017), 277–291
DOI 10.4171/RLM/762

Number Theory — *Integral points on the complement of the branch locus of projections from hypersurfaces*, by ANDREA CIAPPI, communicated on November 11, 2016.[1]

ABSTRACT. — We study the integral points on $\mathbb{P}_n \backslash D$, where D is the branch locus of a projection from a hypersurface in $\mathbb{P}_{n+1}$ to a hyperplane $H \simeq \mathbb{P}_n$. We extend to the general case a result by Zannier (whose approach we follow) and we also obtain a sharper bound that yields, in some cases, the finiteness of integral points. The results presented are effective and the proofs provide a way to actually construct a set containing all the integral points in question. Thus, there are concrete applications to the study of Diophantine equations, more precisely to the problem of finding integral solutions to equations $F(x_0, \ldots, x_n) = c$, where c is a given nonzero value and F is a homogeneous form defining the branch locus D, i.e. a discriminant.

KEY WORDS: Integral points, diophantine equations, diophantine geometry

MATHEMATICS SUBJECT CLASSIFICATION: 11D72, 11G35, 11G99

1. INTRODUCTION

The study of integral points on varieties defined in a projective space as the complement of certain divisors is related to several Diophantine problems and it is a recurring and interesting problem in number theory. For the definition of integral points, for the concept of sets of S-integral points and quasi-S-integral points and for the connections with Diophantine equations we rely on [7] and [10].

Given an affine variety $V \subset \mathbb{P}_n$, we can consider its closure $\overline{V}$ in $\mathbb{P}_n$ and its divisor at infinity $D = \overline{V} \backslash V$. Many valuable thorems about integral points on V have been proved in the last century, but the majority of them requires the splitting of the divisor D in several components in order to be applied. There is a standard technique to bypass this requirement that consists in lifting integral points by means of a finite cover of the variety $\overline{V}$, unramified except possibly above points in D and such that the pull-back of D has more components than D itself. However, this method seldom applies if $\dim V > 1$ because, in general, the pull-back of D does not split as desired.

A remarkable exception to this is a result by Faltings, who proved the finiteness of integral points on the complements of certain irreducible singular curves in $\mathbb{P}_2$. In this case, the divisor D is the branch locus of a suitable projection from a smooth surface described in detail in the original paper [3]. The problem was also studied by Zannier who proved a similar result in [9] applying arithmetic

[1] Presented by U. Zannier.

considerations from [2] (and hence ultimately relying on the Schmidt's Subspace Theorem) to the same geometric setting introduced by Faltings. Zannier obtained the same conclusions under different hypotheses and, moreover, he proved that the fact that the projected surface has non-negative Kodaira dimension is a sufficient condition for the finiteness of integral points on $\mathbb{P}_2 \setminus D$. Later, both results were improved by Levin in [5], where the theorem is proved even for surfaces with negative Kodaira dimension.

In [9] Faltings' principle is also applied to the simpler case of a projection taken from a hypersurface in $\mathbb{P}_{n+1}$ and it leads to a bound for the dimension of any set of integral points on the complement in $\mathbb{P}_n$ of the branch locus of the projection. The analysis presented here will be similar but more general, as we will make no restrictive assumption on the projection. This will require some more care but it will also lead to stronger conclusions and more applications.

The geometric setting of the problem is described in detail in the second section of this paper along with the statement of our main result, proved in the third section. The fourth section contains some details, remarks and corollaries while in the last section we provide some examples and we observe that the results we have presented (which are effective, see §4) have a concrete application in the study of Diophantine equations $F(x_0, \ldots, x_n) = c$ for certain homogeneous irreducible forms F and non-zero values c (see also Proposition 2).

2. Setting of the problem

Let k be a number field and S a finite set of places of k which includes all the infinite ones. Let $\mathscr{X}$ and H be, respectively, an irreducible hypersurface of degree m and a hyperplane in the projective space $\mathbb{P}_{n+1}$, both defined over k. Let Q be a point in $\mathbb{P}_{n+1} \setminus H$ and consider the projection ϕ of $\mathscr{X}$ from the point Q to $H \simeq \mathbb{P}_n$.

Without any loss of generality we suppose $Q = (0 : \ldots : 0 : 1)$ and that H is defined by $X_{n+1} = 0$. The projection ϕ, takes then the form

$$\begin{array}{llcl} \phi: & \mathscr{X} & \to & \mathbb{P}_n \\ & (x_0 : \ldots : x_n : x_{n+1}) & \mapsto & (x_0 : \ldots : x_n). \end{array}$$

If $Q \in \mathscr{X}$ then $\phi(Q)$ is not defined unless we consider a blow-up. However, for our purposes, it will suffice to consider the restriction $\phi_{|\mathscr{X} \setminus Q}$ which, with a slight abuse of notation, will still be denoted by ϕ.

Let $f \in k[X_0, \ldots, X_{n+1}]$ be a homogeneous irreducible polynomial of degree m defining $\mathscr{X}$; we may view it as a univariate polynomial in X_{n+1} with coefficients in $k[X_0, \ldots, X_n]$

$$f(X_0, \ldots, X_n, X_{n+1}) = \sum_{l=0}^{d} f_l(X_0, \ldots, X_n) X_{n+1}^{d-l},$$

where $d = \deg_{X_{n+1}} f$ is the greatest integer such that the coefficient of X_{n+1}^d is not identically zero and every f_l is a homogeneous polynomial of degree $m - d + l$

(or the null polynomial). We suppose $d > 1$ and we remark that the geometrical request $Q \notin \mathscr{X}$ implies $d = m$, $\deg f_l = l$ when f_l is not the null polynomial and $f_0 \in k^*$ (this being the case discussed in [9]).

We consider the discriminant of f in respect of X_{n+1}, a polynomial in $k[X_0, \ldots, X_n]$ that we shall denote by $\Delta = \Delta(X_0, \ldots, X_n)$. Its zeroes are exactly the ramification points of ϕ, insofar as Q does not belong to $\mathscr{X}$, and in this case $\Delta = 0$ is the defining equation for the branch locus D. On the other hand, if $Q \in \mathscr{X}$, there are points in $\mathbb{P}_n$ where the polynomial f_0 vanishes: they may or may not belong to $\phi(\mathscr{X})$ or to $\{\Delta = 0\}$, but their preimages under ϕ surely have cardinality different from d. Hence, we define D as the union of the zero loci of f_0 and Δ.

We also consider a set T made by the points $(x_0 : \ldots : x_n) \in \mathbb{P}_n$ such that $f(x_0, \ldots, x_n, X)$, as a polynomial in $k[X]$, has exactly one root or none at all. If, for example, we require one root with multiplicity d, we must have $f_0(x_0, \ldots, x_n) \neq 0$ and we look for a factorization

$$\sum_{l=0}^{d} f_l(x_0, \ldots, x_n) X^{d-l} = f_0(x_0, \ldots, x_n) \cdot (X - \alpha)^d$$

for some $\alpha = \alpha(x_0, \ldots, x_n) \in \bar{k}$. We then turn the above requirement in d equations

$$f_l(x_0, \ldots, x_n) = f_0(x_0, \ldots, x_n) \cdot \binom{d}{l} (-\alpha)^l \quad l = 1, \ldots, d$$

and we observe that, in particular, we must have $f_1 = -d\alpha f_0$ or, equivalently, $-\alpha = f_1/(df_0)$. This leads to the following relations among the polynomials:

$$\begin{cases} f_0 \neq 0 \\ f_l = \binom{d}{l} \dfrac{f_1^l}{d^l f_0^{l-1}} \quad \forall l = 2, \ldots, d. \end{cases} \tag{1}$$

We denote by T_0 the set of points in $\mathbb{P}_n$ satisfying (1). We define in an analogous way the sets $T_1, \ldots, T_{d-1}$ consisting, respectively, of the points in $\mathbb{P}_n$ whose preimages *via* ϕ are made by single points with multiplicity, respectively, $d-1, d-2, \ldots, 1$. For example, the points in T_1 will satisfy $f_0 = 0$, $f_1 \neq 0$ and

$$f_l = \binom{d-1}{l-1} \frac{f_2^{l-1}}{(d-1)^{l-1} f_1^{l-2}} \quad \forall l = 3, \ldots, d.$$

Finally, we have $T = T_0 \cup \cdots \cup T_d$, where the last two sets involved are $T_{d-1} = \{f_0 = \cdots = f_{d-2} = 0, f_{d-1} \neq 0\}$ and the complement of $\phi(\mathscr{X})$ in $\mathbb{P}_n$, $T_d = \{f_0 = \cdots = f_{d-1} = 0, f_d \neq 0\}$.

We can now state our main result:

Theorem 1. *Assuming the hypotheses and notations discussed above in this section, the Zariski closure of any set of quasi-S-integral points for* $\mathbb{P}_n \setminus D$ *has dimension less than or equal to* $\dim T_0 + 1$.

This constitutes an improvement of Theorem 2.1 in [9] in two ways: it is more general because the hypotheses on the centre of projection Q (and thus on the leading coefficient $f_0(X_0, \ldots, X_n)$) have been removed and the bound provided is sharper as it is in terms of the dimension of T_0 (which is a subset of T). Moreover, not only it is possible that $\dim T_0 < \dim T$, but T_0 may be the empty set even if the degree d is small (compared to n), yielding a finiteness result for the integral points (see §4).

3. Proof of Theorem 1

We will make use of the following well-known fact (for a proof, see Proposition 2.3 in [9]):

Proposition 2. *Let* $L \subset \mathbb{P}_n$ *be an effective divisor defined by a form* $\Lambda \in k[X_0, \ldots, X_n]$ *and let* Σ *be a set of quasi-S-integral points for the affine variety* $\mathbb{P}_n \setminus L$. *Then there exists a finite set of places* $S' \supset S$ *of* k *such that each point of* Σ *has projective coordinates* $(x_0 : \ldots : x_n)$ *with* $x_i \in \mathcal{O}_{S'}$ *and* $\Lambda(x_0, \ldots, x_n) \in \mathcal{O}_{S'}^*$.

In order to give more emphasis to the underlying ideas and techniques leading to the result, we postpone the discussion of the "low degrees" case. More precisely, during the proof we will make the assumption that the degree $d = \deg_{X_{n+1}} f(X_0, \ldots, X_{n+1})$ is greater than or equal to 4. We will go back to that point in the next section (paragraph *"Low degrees"*) and complete the proof for $d = 2$ and $d = 3$.

First step – Let Σ be a set of quasi-S-integral points for $\mathbb{P}_n \setminus D$. By the above Proposition 2 there exists a finite set $S' \supset S$, made up of places of k, such that for every point in Σ there are projective coordinates $(x_0 : \ldots : x_n)$ such that every x_i belongs to $\mathcal{O}_{S'}$ and $\Delta(x_0, \ldots, x_n) \in \mathcal{O}_{S'}^*$; we choose $P \in \Sigma$ and projective coordinates $(x_0 : \ldots : x_n)$ for it so that the properties we just mentioned are satisfied.

Then we consider the equation $f(x_0, \ldots, x_n, X) = 0$ which has d distinct roots in $\overline{\mathbb{Q}}$ since $P \notin D$. We shall denote them by $\alpha_1, \ldots, \alpha_d$ and we consider the number field k' they generate over k, which depends on P: it has bounded degree and it is unramified except at places above S'. Hermite's Theorem then implies that there are at most a finite number of number fields with these properties, hence there exists a number field k'' such that it contains all the roots α_i regardless of the chosen point P. Finally, we may define a finite set S'' constituted by places of k'' that contains the extension of S' to a set of places of k'' and such that the polynomials $f_i(X_0, \ldots, X_n)$ have coefficients in $\mathcal{O}_{k'', S''}$ and $\mathcal{O}_{k'', S''}$ has class number 1.

We remark that proving the theorem after enlarging k or S is a stronger conclusion than the original claim and that, even if we consider sets with an infinite number of quasi-S-integral points, the number of required enlargements is finite. Thus, we assume in this proof $k = k''$ and $S = S''$.

SECOND STEP – We can now consider the usual factorization of the discriminant

$$\Delta(x_0, \dots, x_n) = f_0^{2d-2} \prod_{1 \le i < j \le d} (\alpha_i - \alpha_j)^2$$

which is valid because $f_0(x_0, \dots, x_n) \neq 0$ since $P \notin D$. Every root can be written as a product $\alpha_i = \mu_i \delta_i^{-1}$ with μ_i and δ_i coprime S-integers. We also note that every polynomial $\delta_i X - \mu_i$ divides $f(x_0, \dots, x_n, X)$ in $\mathcal{O}_S[X]$, hence $\delta_1 \dots \delta_d$ divides f_0 in $\mathcal{O}_S$. It follows that $\Delta(x_0, \dots, x_n)$ is divisible in $\mathcal{O}_S$ by $\prod_{i \neq j}(\delta_j \mu_i - \delta_i \mu_j)$ and, since the discriminant is an S-unit, we deduce that every factor $\delta_j \mu_i - \delta_i \mu_j$ belongs to $\mathcal{O}_S^*$.

We define $x_{ij} := \delta_j \mu_i - \delta_i \mu_j$ and we consider the identity

$$x_{i1} x_{23} + x_{i2} x_{31} + x_{i3} x_{12} = 0$$

where $i \in \{4, \dots, d\}$ and every summand is clearly in $\mathcal{O}_S^*$. Since we just produced solutions to the homogeneous S-unit equation, we may apply some finiteness result (see [6] or [10]) and obtain that, for example, the ratios $x_{i2}x_{31}/x_{i1}x_{32}$ lie in a finite set independent of the chosen point P. In order to write down algebraic relations among the roots α_i, we observe that we have just proved that for certain values $c_i = c_i(P)$ in a fixed finite set, we have

$$c_i = \frac{x_{i2} x_{31}}{x_{i1} x_{32}} = \frac{(\alpha_i - \alpha_2)(\alpha_3 - \alpha_1)}{(\alpha_i - \alpha_1)(\alpha_3 - \alpha_2)} \quad i \in \{4, \dots, d\}$$

and if we put $c_2 := 0$ and $c_3 := 1$ we have analogous relations for $i = 2$ and $i = 3$. After some easy manipulations, we can write the following expressions for the roots:

$$(2) \qquad \alpha_i = \begin{cases} \dfrac{\alpha_1(\alpha_2 - \alpha_3)c_i + \alpha_2(\alpha_3 - \alpha_1)}{(\alpha_2 - \alpha_3)c_i + \alpha_3 - \alpha_1} & i = 2, \dots, d \\[2ex] \dfrac{\alpha_4(\alpha_2 - \alpha_3)c_4 + \alpha_3(\alpha_4 - \alpha_2)}{(\alpha_2 - \alpha_3)c_4 + \alpha_4 - \alpha_2} & i = 1. \end{cases}$$

Finally, we can split Σ into finitely many subsets such that the c_i are fixed for every point in a given subset. Arguing separately with each subset we may then assume that the c_i do not depend on P.

THIRD STEP – We pause to outline how we will make use of the information obtained so far. We are going to define a quasi-projective variety in $\mathbb{P}_{n+4}$ and its projection on $\mathbb{P}_n$ will lead to the sought relation between Σ and T_0. Intuitively, $n + 1$ coordinates are required to define a point in $\Sigma \subset \mathbb{P}_n$ and four values are required to express all the roots α_i, see (2) above. The polynomials that we are about to introduce are defined following the relations (2) and then considering Viète's formulae to provide a link between the roots α_i and the polynomials f_i: they are essential in the definition of the quasi-projective variety above mentioned.

We start by defining some auxiliary polynomials in $k[Y_1, Y_2, Y_3, Y_4]$:

$$a_i(Y_1, Y_2, Y_3, Y_4) = \begin{cases} Y_1(Y_2 - Y_3)c_i + Y_2(Y_3 - Y_1) & i = 2, \dots, d \\ Y_4(Y_2 - Y_3)c_4 + Y_3(Y_4 - Y_2) & i = 1 \end{cases}$$

$$b_i(Y_1, Y_2, Y_3, Y_4) = \begin{cases} (Y_2 - Y_3)c_i + Y_3 - Y_1 & i = 2, \dots, d \\ (Y_2 - Y_3)c_4 + Y_4 - Y_2 & i = 1 \end{cases}$$

$$A_l(Y_1, Y_2, Y_3, Y_4) = \sum_{1 \le i_1 < \dots < i_l \le d} a_{i_1}(Y_1, Y_2, Y_3, Y_4) \dots a_{i_l}(Y_1, Y_2, Y_3, Y_4) \cdot \prod_{\substack{1 \le j \le d \\ j \ne i_1, \dots, i_l}} b_j(Y_1, Y_2, Y_3, Y_4) \quad l = 1, \dots, d$$

$$B(Y_1, Y_2, Y_3, Y_4) = \prod_{i=1}^{d} b_i(Y_1, Y_2, Y_3, Y_4).$$

If, as before, $P = (x_0 : \dots : x_n) \in \Sigma$ is the point in question and $f(x_0, \dots, x_n, X)$ has roots $\alpha_1, \dots, \alpha_d$, we observe that, because of (2),

$$\frac{a_i(\alpha_1, \alpha_2, \alpha_3, \alpha_4)}{b_i(\alpha_1, \alpha_2, \alpha_3, \alpha_4)} = \alpha_i \quad i = 1, \dots, d. \tag{3}$$

Furthermore, since the coefficients of a polynomial can be expressed as the product of the leading coefficient and the corresponding symmetric function calculated in its roots, we have for $l = 1, \dots, d$

$$f_l(x_0, \dots, x_n) = (-1)^l f_0(x_0, \dots, x_n) \cdot \frac{A_l(\alpha_1, \alpha_2, \alpha_3, \alpha_4)}{B(\alpha_1, \alpha_2, \alpha_3, \alpha_4)}. \tag{4}$$

After these remarks, we are ready to define and study a projective variety $V \subset \mathbb{P}_{n+4}$ given by the common zero locus of the d polynomials

$$Bf_l - (-1)^l f_0 A_l \tag{5}$$

where B and A_l belong to $k[Y_1, Y_2, Y_3, Y_4]$, f_0 and f_l are in $k[X_0, \dots, X_n]$ and l ranges from 1 to d.

Since our main interest is focused on $\Sigma \subset \mathbb{P}_n$, we are going to consider the projection of V to $\mathbb{P}_n$ by taking the first $n+1$ coordinates. To ensure that the projection is well-defined we have to remove points with nothing but zeroes in the first $n+1$ coordinates. In addition, we are going to ignore points that belong to V regardless of the first $n+1$ coordinates, only because the Y_i have special values. Furthermore, we would like, at some point, to get rid of the zeroes of $f_0(X_0, \ldots, X_n)$ in $\mathbb{P}_n$, because they cannot be in T_0. We accomplish these goals by defining in $\mathbb{P}_{n+4}$ the varieties

$$\begin{aligned}
U_0 &:= \{(z_0 : \ldots : z_n : y_1 : y_2 : y_3 : y_4) \in V : z_0 = \cdots = z_n = 0\}\\
U_1 &:= \{(z_0 : \ldots : z_n : y_1 : y_2 : y_3 : y_4) \in V : B(y_1, y_2, y_3, y_4) = 0 \text{ and}\\
&\qquad A_l(y_1, y_2, y_3, y_4) = 0\ \forall l = 1, \ldots, d\}\\
U_2 &:= \{(z_0 : \ldots : z_n : y_1 : y_2 : y_3 : y_4) \in V : f_0(z_0, \ldots, z_n) = 0\}\\
U &:= U_0 \cup U_1 \cup U_2
\end{aligned}$$

and a quasi-projective variety which is the complement of U in V:

$$W := V \backslash U.$$

Finally, we consider the projection from W to $\mathbb{P}_n$:

$$\begin{aligned}
\pi\colon \qquad & W \qquad\qquad \longrightarrow \qquad \mathbb{P}_n\\
(z_0 : \ldots : z_n &: y_1 : y_2 : y_3 : y_4) \mapsto (z_0 : \ldots : z_n).
\end{aligned}$$

FOURTH STEP – Once again we look at $P = (x_0 : \ldots : x_n) \in \Sigma$ and we observe that:

- $(x_0 : \ldots : x_n : \alpha_1 : \alpha_2 : \alpha_3 : \alpha_4) \in V$ because of (4)
- there exists $i \in \{0, \ldots, n\}$ such that $x_i \neq 0$ since $P \in \mathbb{P}_n$, hence $(x_0 : \ldots : x_n : \alpha_1 : \alpha_2 : \alpha_3 : \alpha_4) \notin U_0$
- $B(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \neq 0$ because the roots α_i are all pairwise distinct, hence $(x_0 : \ldots : x_n : \alpha_1 : \alpha_2 : \alpha_3 : \alpha_4) \notin U_1$
- $(x_0 : \ldots : x_n : \alpha_1 : \alpha_2 : \alpha_3 : \alpha_4) \notin U_2$ because $f_0(x_0, \ldots, x_n) \neq 0$ (since $P \notin D$).

It follows that $(x_0 : \ldots : x_n : \alpha_1 : \alpha_2 : \alpha_3 : \alpha_4)$ actually belongs to W, whence $\Sigma \subset \pi(W)$.

We investigate now what happens to W when intersected with the hyperplane $\{Y_2 = Y_3\} \subset \mathbb{P}_{n+4}$. First of all we notice that

$$\frac{a_i(y_1, y_2, y_2, y_4)}{b_i(y_1, y_2, y_2, y_4)} = y_2 \quad i = 1, \ldots, d$$

for every choice of y_1, y_2 and y_4 and, subsequently, we have

$$\frac{A_l(y_1, y_2, y_2, y_4)}{B(y_1, y_2, y_2, y_4)} = \sum_{1 \le i_1 < \cdots < i_l \le d} \frac{a_{i_1}}{b_{i_1}} \cdots \frac{a_{i_l}}{b_{i_l}} = \binom{d}{l} y_2^l \quad l = 1, \ldots, d.$$

From the defining equations of V and the ones displayed above, we have for every point $(z_0 : \ldots : z_n : y_1 : y_2 : y_2 : y_4) \in W \cap \{Y_2 = Y_3\}$ the following relation:

$$
\begin{aligned}
f_l(z_0, \dots, z_n) &= (-1)^l f_0(z_0, \dots, z_n) \frac{A_l(y_1, y_2, y_2, y_4)}{B(y_1, y_2, y_2, y_4)} \\
&= (-1)^l f_0(z_0, \dots, z_n) \binom{d}{l} y_2^l
\end{aligned}
\tag{6}
$$

which is valid for $l = 1, \dots, d$. In particular, we get $f_1 = -df_0 y_2$ and therefore

$$
f_l(z_0, \dots, z_n) = f_0(z_0, \dots, z_n) \binom{d}{l} \left(\frac{f_1(z_0, \dots, z_n}{df_0(z_0, \dots, z_n)} \right)^l \quad l = 1, \dots, d.
$$

Then, recalling the defining equations (1) for T_0, we have just proved that $\pi(W \cap \{Y_2 = Y_3\}) \subset T_0$.

We draw a diagram to help us clarify the role of the auxiliary objects we introduced in the proof:

$$
\begin{tikzcd}
 & W \arrow[d, "\pi"] & W \cap \{Y_2 = Y_3\} \arrow[l, hook'] \arrow[d, "\pi"] & \\
\Sigma \arrow[r, hook] & \pi(W) & \pi(W \cap \{Y_2 = Y_3\}) \arrow[l, hook'] \arrow[r, hook] & T_0
\end{tikzcd}
$$

Finally, we consider the Zariski closure of Σ and we readily have that its dimension is less than or equal to $\dim T_0 + 1$. This completes the proof for $d \geq 4$. $\square$

4. Remarks and details

Effectivity – A noteworthy feature of Theorem 1 is its effectivity. This is a consequence, essentially, of the fact that we obtained a finiteness result during the second step of the proof without the help of Schmidt's Theorem or other ineffective conclusions from Diophantine approximation. Instead, we used results about S-unit equations and it is known that a finite and complete set of non-proportional representatives can be effectively found (for example *via* Baker's theory, see [1]). Therefore it is possible to determine all the auxiliary objects introduced in the proof, assuming $\mathscr{X}$ is given, and we may actually exhibit the set $\pi(W)$ containing Σ.

We must point out that the set of solutions depends naturally on k and S and that they might have been enlarged with the application of Proposition 2. Thus, an explicit notion of quasi-S-integral points is also required to have a unique determination for the solutions of the S-unit equation: in other words, we are required to specify an affine model for $\mathbb{P}_n \backslash D$.

We also remark that another result of crucial importance in the proof is Hermite's Theorem, which is effective as well.

Analysis of the results – We would like to study the dimension of T_0, once the geometric setting is specified, and to compare it to the dimension of T. Obviously we have $\dim T_0 \leq \dim T$, as $T_0 \subset T$, but it is not hard to see that equality holds very often. In fact, T is the disjoint union of its $d + 1$ subsets T_i, each of them defined by an inequality and $d-1$ equations (save T_d which is defined by d equations) and $\dim T = \max\{\dim T_i\}_{i=0,\dots,d}$.

In order to study the difference between $\dim T$ and $\dim T_0$, it may be useful to have explicit conditions for the sets T_i. We see that a point $(x_0 : \dots : x_n) \in \mathbb{P}_n$ belongs to T_i if and only if the following conditions are satisfied (we denote $f_j(x_0, \dots, x_n)$ simply by f_j):

$$(7) \qquad \begin{cases} f_l = 0 & \forall l < i \\ f_i \neq 0 & \\ (d-i)^{l-i} f_i^{l-i-1} f_l = \binom{d-i}{l-i} f_{i+1}^{l-i} & \forall l \geq i+2. \end{cases}$$

We observe that if there is $l < i$ such that f_l divides f_i in $k[X_1, \dots, X_n]$ we have $T_i = \emptyset$. So, if we ask $Q \notin \mathcal{X}$ we have $f_0 \in k^*$ and so, for every $i = 1, \dots, d$, we have $f_0 \mid f_i$, whence $T_i = \emptyset$ for $i \geq 1$ and $T = T_0$.

Finally, we point out that the expected bound provided by Theorem 1 for the dimension of any set of quasi-S-integral points for $\mathbb{P}_n \backslash D$ is $n-d+2$. In fact, unless the polynomials f_i satisfy some special relations, the dimension of T_0 is lowered by one by every condition in (1), with the exception of the condition $f_0 \neq 0$.

Finiteness conditions – If T_0 is the empty set we obtain, as a result, the finiteness of the integral points for $\mathbb{P}_n \backslash D$. In order to find when $T_0 = \emptyset$, we have to check if the vanishing of the $d-1$ polynomials (see conditions (1))

$$r_i(X_0, \dots, X_n) := d^i f_0^{i-1} f_i - \binom{d}{i} f_1^i$$

implies $f_0 = 0$. In other words, we want to check if the variety defined by the polinomials r_i is contained in the variety defined by f_0, i.e. if $\bigcap_{i=2}^d V(r_i) \subset V(f_0)$ and this is equivalent to solve the radical membership problem $f_0 \in \sqrt{\langle r_2, \dots, r_d \rangle}$. This can be efficiently done computing a reduced Gröbner basis for the ideal $\langle r_2, \dots, r_d, 1 - Y f_0 \rangle$ in the ring $k[X_0, \dots, X_n, Y]$: if we get $\{1\}$ then $T_0 = \emptyset$ and the number of integral points for $\mathbb{P}_n \backslash D$ is finite.

On the other hand, an easily verifiable sufficient condition ensuring $T_0 = \emptyset$ could find some interesting applications. Suppose that there are $i, j \in \{1, \dots, d\}$ such that the polynomial f_i is the null polynomial and f_j vanishes only if f_0 does. Then, recalling conditions (1) for the set T_0, we get $f_0 \neq 0$ and $f_1 = 0$; this happens trivially if $i = 1$ and comes from the equation $d^i f_0^{i-1} f_i = \binom{d}{i} f_1^i$ otherwise. This yields the condition $f_l = 0$ for every $l \geq 1$, hence f_j must vanish and this contradicts the requirement $f_0 \neq 0$. Hence $T_0 = \emptyset$.

COROLLARY 3. *Notation being as in Section 2, suppose that there are $i, j \in \{1, \dots, d\}$ such that $f_i(X_0, \dots, X_n)$ is the null polynomial and $f_j(X_0, \dots, X_n) = 0$ implies $f_0(X_0, \dots, X_n) = 0$. Then every set of quasi-S-integral points for $\mathbb{P}_n \backslash D$ is a finite set.*

On the complement of $\{\Delta = 0\}$ – We state and prove a more general version of Theorem 1 which allows for the points of $\mathbb{P}_n$ where the leading coefficient of

$f(X_0, \dots, X_{n+1})$ as a polynomial in X_{n+1} vanishes. In other words, we investigate the quasi-S-integral points on the complement of the divisor defined by the discriminant.

THEOREM 4. *Notations being as in Section 2, if $d > 4$ then the Zariski closure of any set Σ of quasi-S-integral points for $\mathbb{P}_n \setminus \{\Delta = 0\}$ has dimension less than or equal to* $\dim(T_0 \cup T_1) + 1$.

Moreover, if $f_0(x_0, \dots, x_n) \neq 0$ (resp. $f_0(x_0, \dots, x_n) = 0$) for every $(x_0 : \dots : x_n) \in \Sigma$, we have that the dimension of the Zariski closure of Σ is less than or equal to $\dim T_0 + 1$ (*resp.* $\dim T_1 + 1$).

PROOF. Let Σ be a set of quasi-S-integral points for $\mathbb{P}_n \setminus \{\Delta = 0\}$ and consider a point $P = (x_0 : \dots : x_n) \in \Sigma$. As before, we look at the polynomial $f(x_0, \dots, x_n, X)$ which has d or $d-1$ roots: we denote these pairwise distinct roots by $\alpha_1, \dots, \alpha_{d-1}$ and, in case, α_d. The first thing we observe is that $f_0(x_0, \dots, x_n)$ and $f_1(x_0, \dots, x_n)$ cannot be both equal to zero for otherwise we would have $\Delta(x_0, \dots, x_n) = 0$. Again, we can apply Proposition 2 and enlarge k and S to ensure that every point of Σ has projective coordinates with entries in $\mathcal{O}_S$ and that Δ has values in $\mathcal{O}_S^*$.

If $f_0(x_0, \dots, x_n) \neq 0$ we follow the proof of Theorem 1 until we get the relations (2) among the roots. If $f_0(x_0, \dots, x_n) = 0$ we consider the discriminant $\Delta_{d-1}(\boldsymbol{x})$ of the polynomial $f(x_0, \dots, x_n, X) \in k[X]$ of degree $d-1$ and we observe that

$$\Delta(\boldsymbol{x}) = f_1(\boldsymbol{x})^2 \Delta_{d-1}(\boldsymbol{x}) = f_1(\boldsymbol{x})^{2d-2} \prod_{1 \le i < j \le d-1} (\alpha_i - \alpha_j)^2$$

and in a similar way we find relations among the $d-1$ roots like those in (2). Now we can split Σ into finitely many subsets such that the c_i's are fixed and that f_0 is either zero or non-zero for every point in a given subset. Arguing separately with each subset we may then assume we have d (or $d-1$) values c_i that do not depend on P.

We will handle these subsets in a different way depending on whether f_0 vanishes or not. We have already seen in the proof of Theorem 1 how to proceed in the second case and we define a quasi-projective variety $W \subset \mathbb{P}_{n+4}$ just as before. On the other hand, if f_0 vanishes, the path is the same but we need to slightly modify the polynomials $A_1, \dots, A_{d-1}$ and B in an obvious way to deal with the fact that we have only $d-1$ roots. For $l = 1, \dots, d-1$ we define

$$A'_l(Y_1, Y_2, Y_3, Y_4) = \sum_{1 \le i_1 < \dots < i_l \le d-1} a_{i_1}(Y_1, Y_2, Y_3, Y_4) \dots a_{i_l}(Y_1, Y_2, Y_3, Y_4) \cdot \prod_{1 \le j \le d-1, j \neq i_1, \dots, i_l} b_j(Y_1, Y_2, Y_3, Y_4)$$

$$B'(Y_1, Y_2, Y_3, Y_4) = \prod_{i=1}^{d-1} b_i(Y_1, Y_2, Y_3, Y_4).$$

Then, we consider the projective variety V' defined as the intersection of the zero loci of $f_0(x_0, \dots, x_n)$ and the $d-1$ polynomials

$$B'(Y_1, Y_2, Y_3, Y_4)f_{l+1}(X_0, \dots, X_n) - (-1)^l f_1(X_0, \dots, X_n)A'_l(Y_1, Y_2, Y_3, Y_4)$$

with l ranging from 1 to $d-1$. We define U'_0 exactly like U_0 in the proof of Theorem 1 and we denote by U'_1 and U'_2, the set of points in V with coordinates $(z_0 : \dots : z_n : y_1 : \dots : y_4)$ such that, respectively, $B'(y_1, y_2, y_3, y_4) = 0$ and $f_1(z_0, \dots, z_n) = 0$. Finally, we define a set $U' := U_0 \cup U'_1 \cup U'_2$ and its complement in V, the quasi-projective variety $W' := V' \backslash U'$. We notice that $W \cap W' = \emptyset$.

If we consider the projection $\pi : W \cup W' \to \mathbb{P}_n$ on the first $n+1$ coordinates, we observe that the subsets we have split Σ in are contained either in $\pi(W)$ or in $\pi(W')$ and therefore $\Sigma \subset \pi(W \cup W')$. As in the proof of Theorem 1, we have $\pi(W \cap \{Y_2 = Y_3\}) \subset T_0$ and, in a similar way, $\pi(W' \cap \{Y_2 = Y_3\}) \subset T_1$. Remembering that $W \cap W' = \emptyset$ as well as $T_0 \cap T_1 = \emptyset$, we can conclude as follows:

$$\begin{aligned}\dim \bar{\Sigma} &\leq \dim(\pi(W) \cup \pi(W')) \\ &\leq \dim((\pi(W) \cup \pi(W')) \cap \{Y_2 = Y_3\}) + 1 \\ &= \dim((\pi(W) \cap \{Y_2 = Y_3\}) \cup (\pi(W') \cap \{Y_2 = Y_3\})) + 1 \\ &\leq \dim(T_0 \cup T_1) + 1. \end{aligned}$$ □

We remark that in Theorem 4 there is an additional hypothesis on the degree d because when $d \leq 4$ the proof of Theorem 1 (see next paragraph, *Low degrees*) actually uses integrality with respect to D (i.e. that $f_0(x_0, \dots, x_n) \in \mathcal{O}_S^*$ when $(x_0, \dots, x_n) \in \Sigma$) and not just that $f_0(x_0, \dots, x_n) \neq 0$.

Low degrees – We now complete the proof of Theorem 1 by taking into account the cases of $d = 2$ and $d = 3$. When $d = 2$ we have two different roots α_1 and α_2 and we cannot apply results about S-unit equations: however we do not need them, since it is enough to use the trivial relations $\alpha_1 = \alpha_1$ and $\alpha_2 = \alpha_2$. Namely, we simply define the auxiliary polynomials $A_1(Y_1, Y_2) = Y_1 + Y_2$ and $A_2(Y_1, Y_2) = Y_1 Y_2$; then we consider the variety $V \subset \mathbb{P}_{n+2}$ defined by the polynomials

$$f_l(X_0, \dots, X_n) - (-1)^l f_0(X_0, \dots, X_n)A_l(Y_1, Y_2) \quad l = 1, 2$$

and the quasi-projective variety

$$W := V \backslash (\{f_0(X_0, \dots, X_n) = 0\} \cup \{X_0 = \dots = X_n = 0\}.$$

Subsequently, we consider the projection $\pi : W \to \mathbb{P}_n$ and everything will follow as in the proof of Theorem 1: if $(x_0, \dots, x_n) \in \Sigma$ then $(x_0, \dots, x_n, \alpha_1, \alpha_2) \in W$ and the points in $\pi(W \cap \{Y_1 = Y_2\})$ satisfy the defining relations for T_0.

When $d = 3$ the trivial relations considered above are no more sufficient to conclude, yet we lack the four different roots that enabled us to use results about the S-unit equation. If the hypersurface $\mathscr{X}$ is defined in $\mathbb{P}_{n+1}$ by the polynomial

$$f(X_0, \dots, X_{n+1}) = f_0 X_{n+1}^3 + f_1 X_{n+1}^2 + f_2 X_{n+1} + f_3,$$

where $f_i \in k[X_0, \dots, X_n]$ for $i = 0, 1, 2, 3$, let us suppose that $\deg f_0 \geq 2$ and put $\delta := \deg f_0 - 1$. We will introduce a subsidiary dimension and we consider the hypersurface $\mathscr{Z} \subset \mathbb{P}_{n+2}$ defined by

$$g(X_0, \dots, X_n, Z, X_{n+1}) = Z^\delta X_{n+1}^4 + f_0 X_{n+1}^3 + f_1 X_{n+1}^2 + f_2 X_{n+1} + f_3.$$

We keep the notations previously introduced, adding a superscript $^{\mathscr{X}}$ or $^{\mathscr{Z}}$ when a definition is related to the hypersurface (or relative polynomial and projection map) considered. We approach the problem thinking that $H^{\mathscr{X}} \simeq \mathbb{P}_n$, $H^{\mathscr{Z}} \simeq \mathbb{P}_{n+1}$ and $H^{\mathscr{X}} \simeq H^{\mathscr{Z}} \cap \{Z = 0\}$.

We arbitrarily choose a set of quasi-S-integral points Σ for $H^{\mathscr{X}} \backslash D^{\mathscr{X}}$ and we must prove that the dimension of its Zariski closure is less than or equal to $\dim T_0^{\mathscr{X}} + 1$. For every point $(x_0 : \dots : x_n) \in \Sigma \subset H^{\mathscr{X}}$ we consider a point $(x_0 : \dots : x_n : 0) \in H^{\mathscr{Z}}$ and we denote the set of all these points by Σ'. Namely, we define

$$\Sigma' = \{(x_0 : \dots : x_n : 0) \in H^{\mathscr{Z}} : (x_0 : \dots : x_n) \in \Sigma\}.$$

It turns out that Σ' is a set of quasi-S-integral points for the complement $H^{\mathscr{Z}} \backslash \{\Delta^{\mathscr{Z}} = 0\}$, since

$$\Delta^{\mathscr{Z}}(x_0, \dots, x_n, 0) = f_0(x_0, \dots, x_n)^2 \cdot \Delta^{\mathscr{X}}(x_0, \dots, x_n)$$

and both factors on the right-hand term are non-zero because for every point $(x_0 : \dots : x_n) \in \Sigma$ we have $(x_0 : \dots : x_n) \notin D^{\mathscr{X}}$. Now we parallel the proof of Theorem 4 to get that the dimension of the Zariski closure of Σ' is less than or equal to $\dim T_1^{\mathscr{Z}} + 1$. In fact, from the last displayed equation and denoting the roots of the polynomial f by α_1, α_2, α_3, we obtain

$$\begin{aligned} \Delta^{\mathscr{Z}}(x_0, \dots, x_n, 0) &= f_0(x_0, \dots, x_n)^6 \prod_{1 \leq i < j \leq 3} (\alpha_i - \alpha_j)^2 \\ &= \prod_{1 \leq i < j \leq 3} (f_0(x_0, \dots, x_n)(\alpha_i - \alpha_j))^2 \end{aligned} \tag{8}$$

which implies $(\alpha_i - \alpha_j) \in \mathcal{O}_S^*$ for every $i, j \in \{1, 2, 3\}$ such that $i \neq j$. This leads to a non-trivial fixed algebraic relation among the three roots and we outline the conclusion, since it follows similarly to what happens in the proof given for $d \geq 4$. In fact, $(\alpha_1 - \alpha_2, \alpha_2 - \alpha_3, \alpha_3 - \alpha_1)$ is a non-degenerate solution of the S-unit equation $x_1 + x_2 + x_3 = 0$ and we can write $\alpha_3 = (\alpha_1 - \alpha_2)c + \alpha_1$ with c in a finite set independent of the point chosen in Σ. Less auxiliary polynomials and only two variables are needed to define the variety V:

$$a_1(Y_1, Y_2) = Y_1, \quad a_2(Y_1, Y_2) = Y_2, \quad a_3(Y_1, Y_2) = (Y_1 - Y_2)c + Y_1$$
$$A_1(Y_1, Y_2) = a_1 + a_2 + a_3, \quad A_2(Y_1, Y_2) = a_1a_2 + a_1a_3 + a_2a_3$$
$$A_3(Y_1, Y_2) = a_1a_2a_3$$

The variety V is defined by the three polynomials

$$f_l(X_0, \dots, X_n) - (-1)^l f_0(X_0, \dots, X_n) A_l(Y_1, Y_2) \quad l = 1, 2, 3.$$

and we define the variety U as in the first part of this proof to obtain the quasi-projective variety W. As in the end of the proof of Theorem 4, the dimension of the Zariski closure of the set Σ' is less than or equal to $\dim T_1^{\mathscr{Z}} + 1$. The sought conclusion follows observing that $\Sigma' \simeq \Sigma$ and $T_1^{\mathscr{Z}} \simeq T_0^{\mathscr{X}}$.

We are left with the cases of $\deg f_0 = 0$ and $\deg f_0 = 1$ and we observe that the former has already a solution in [9], since we can assume $f_0 = 1$ without loss of generality. If $\deg f_0 = 1$, we follow the proof given in this subsection for $\deg f_0 \geq 2$ with the difference that the hypersurface $\mathscr{Z} \subset \mathbb{P}_{n+1}$ will be defined by the polynomial

$$g(X_0, \dots, X_n, Z, X_{n+1}) = ZX_{n+1}^4 + f_0^2 X_{n+1}^3 + f_0 f_1 X_{n+1}^2 + f_0 f_2 X_{n+1} + f_0 f_3.$$

Everything goes as before with the exception of (8) that becomes

$$\begin{aligned}\Delta^{\mathscr{Z}}(x_0, \dots, x_n, 0) &= f_0(x_0, \dots, x_n)^8 \Delta^{\mathscr{X}}(x_0, \dots, x_n) \\ &= f_0(x_0, \dots, x_n)^2 \prod_{1 \leq i < j \leq 3} (f_0(\alpha_i - \alpha_j))^2\end{aligned}$$

and the same conclusion follows. □

5. Applications and examples

The primary way to apply the results presented in the previous sections is to consider Diophantine equations $F(X_0, \dots, X_n) = c$, where F is a polynomial expressing the discriminant of another polynomial $f(X_0, \dots, X_n, X_{n+1})$ seen as a univariate polynomial in X_{n+1} and c is a non-zero element of the number field in question. Considering the hypersurface defined in $\mathbb{P}_{n+1}$ by the polynomial $f(X_0, \dots, X_n, X_{n+1})$, Theorem 1 provides information on the dimension of the set of solutions in terms of the dimension of the set T_0. Moreover, the set that actually bounds the integral points is $\pi(W)$ and we can find an explicit description for it following the proof of the theorem given in Section 3. It is important to remark that maybe $\dim \pi(W) < \dim T_0$, so, when a specific hypersurface is given, sometimes it is worth to write down the equations for the quasi-projective variety W and to study its projection on $\mathbb{P}_n$ (even if the numbers c_i have not been calculated).

Example – Let $\mathscr{X} \subset \mathbb{P}_3(k)$ be the nonsingular hypersurface defined by

$$XT^3 + X^2T^2 - \frac{X^4}{9} + \frac{Y^4}{27} - \frac{Z^4}{27} \in k[X, Y, Z, T].$$

We consider the projection of $\mathscr{X}$ from the point $(0:0:0:1)$ to the hyperplane $\{T = 0\} \simeq \mathbb{P}_2$ and we investigate the integral points in $\mathbb{P}_2 \backslash D$, where D is the ramification divisor of the projection. By Corollary 3, we may immediately conclude that any set of quasi-S-integral points for $\mathbb{P}_2 \backslash D$ is finite. We also exhibit the discriminant of the polynomial defining the hypersurface (seen as a univariate polynomial in T), since it is the relevant polynomial as it concerns applications to Diophantine equations:

$$\Delta(X, Y, Z) = \frac{X^2}{27}(3X^4 - Y^4 + Z^4)(X^4 + Y^4 - Z^4).$$

Trinomials – When the hypersurface $\mathscr{X}$ is defined by a trinomial

$$f(X_0, \dots, X_{n+1}) = aX_{n+1}^d + bX_{n+1}^r + c$$

where a, b and c are polynomials in $k[X_0, \dots, X_n]$, there is an explicit general formula for the discriminant of f. It can be calculated in many ways (see, for example, [8] or [4]) and, if d and r are coprime, it simplifies to the following:

$$\Delta = (-1)^{\frac{1}{2}d(d-1)} a^{d-r-1} c^{r-1} (d^d a^r c^{d-r} + (-1)^{d-1} (d-r)^{d-r} r^r b^d).$$

If, for example, we consider $\mathscr{X} \subset \mathbb{P}_3$ defined by the trinomial (in respect of T)

$$ZT^4 + 4XZT + 27(X^5 - Y^5 + Z^5) \in k[X, Y, Z, T],$$

we have the discriminant

$$\Delta(X, Y, Z) = 2^8 3^9 Z^3 (X^5 - Y^5 + Z^5)^2 (X^5 - Y^5 + Z^5 - X^4 Z).$$

and, therefore, the complement in $\mathbb{P}_2$ of the divisor D is the set

$$\{(x : y : 1) \in \mathbb{P}_2 : y^5 \neq 1 + x^5, y^5 \neq 1 - x^4 + x^5\}.$$

We can easily see that $T_0 = \{(0 : \zeta^i : 1) \in \mathbb{P}_2 : i = 0, \dots, 4\}$, where ζ is a fifth root of 1, so $\dim T_0 = 0$ and the dimension of any quasi-S-integral set for $\mathbb{P}_2 \backslash D$ is at most 1.

Acknowledgments. The author would like to express his gratitude to Umberto Zannier for his worthy insights and suggestions. The author also thanks Daniele Angella, Pietro Corvaja and Gabriele Ranieri for useful discussions and the referee for interesting and helpful comments and remarks.

References

[1] A. Baker, *Transcendental number theory*, Cambridge University Press, Cambridge (1976).

[2] P. Corvaja - U. Zannier, *On integral points on surfaces*, Ann. of Math. (2) 160 (2004); 705–726.

[3] G. Faltings, *A New Application of Diophantine Approximations*, in A panorama in Number Theory or The View from Baker's Garden (Proc. Conf. Zürich 1999) ed. by G. Wüstholz, Cambridge University Press (2002); 549–576.

[4] I. M. Gelfand - M. M. Kapranov - A. V. Zelevinsky, *Discriminants, Resultants and Multidimensional Determinants*, Birkhäuser, Boston (1994).

[5] A. Levin, *Generalizations of Siegel's and Picard's theorems*, Ann. of Math. (2) 170 (2009), no. 2; 609–655.

[6] W. M. Schmidt, *Diophantine Approximations and Diophantine Equations*, Springer-Verlag LNM 1467 (1996).

[7] J.-P. Serre, *Lectures on the Mordell-Weil theorem*, Vieweg, Braunschweig (1990).

[8] R. Swan, *Factorization of polynomials over finite fields*, Pacific J. Math. 12 (1962); 1099–1106.

[9] U. Zannier, *On the integral points on the complement of ramification-divisors*, J. Inst. Math. Jussieu (2005) 4(2); 317–330.

[10] U. Zannier, *Some applications of Diophantine Approximation to Diophantine Equations*, Forum Editrice Universitaria Udinese (2003).

Received 1 September 2015,
and in revised form 30 August 2016.

Andrea Ciappi
Scuola Normale Superiore
Piazza dei Cavalieri 7
56126 Pisa, Italy
andrea.ciappi@gmail.com