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Group Theory — *On some permutable embeddings of subgroups of finite groups*, by Adolfo Ballester-Bolinches and James C. Beidleman, communicated on December 16, 2016.

Abstract. — In this survey paper several subgroup embedding properties related to some types of permutability are introduced and studied.

Key words: Finite group, permutability, S-permutability, S-semipermutability

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1. Introduction

In the following, G always denotes a finite group.

The purpose of this survey paper is to show how the embedding of certain types of subgroups of G can determine the structure of G. The subgroup embedding properties we are going to consider are natural extensions of the permutability.

Recall that a subgroup H of a group G is said to be *permutable* in G if H permutes with all subgroups of G, i.e. HK is a subgroup of G for all subgroups K of G. Clearly H is permutable in G if and only if H permutes with every p-subgroup of G for every prime p (see for instance [7, Theorem 1.2.2]).

Sometimes, the requirement for a subgroup is not to permute with all p-subgroups for all primes p, but just only with the maximal ones, that is, the Sylow p-subgroups for all primes p. This embedding property, called *S-permutability*, was introduced and studied by Kegel in [19].

Definition 1. A subgroup H of G is said to be *S-permutable* in G if H permutes with every Sylow p-subgroup of G for every prime p.

It is clear that normal subgroups are always permutable and permutable subgroups are S-permutable. One of the earliest results about S-permutable subgroups is due to Kegel, who proved in [19] that *every S-permutable subgroup is subnormal* (see [7, Theorem 1.2.14(3)]). Clearly the extent to which a subnormal subgroup can differ from being S-permutable, permutable or normal is of interest and so the description of the groups in which normality, permutability and S-permutability is transitive could help.

Definition 2.

1. A group G is a *T-group* if normality is a transitive relation in G, that is, if every subnormal subgroup of G is normal in G.
2. A group G is a *PT-group* if permutability is a transitive relation in G, that is, if H is permutable in K and K is permutable in G, then H is permutable in G.
3. A group G is a *PST-group* if S-permutability is a transitive relation in G, that is, if H is S-permutable in K and K is S-permutable in G, then H is S-permutable in G.

According to Kegel's result, G is a PST-group (respectively a PT-group) if and only if every subnormal subgroup is S-permutable (respectively permutable) in G.

Note that T implies PT and PT implies PST. On the other hand, PT does not imply T (non-Dedekind modular p-groups) and PST does not imply PT (non-modular p-groups). The reader is referred to [7, Chapter 2] for basic results about these classes of groups. Other characterisations based on subgroup embedding properties can be found in [5, 9].

Agrawal ([7, 2.1.8]) characterised soluble PST-groups. He proved that a soluble group G is a PST-group if and only if the nilpotent residual in G is an abelian Hall subgroup of G on which G acts by conjugation as power automorphisms. In particular, the class of soluble PST-groups is subgroup-closed.

Let G be a soluble PST-group with nilpotent residual L. Then G is a PT-group (respectively T-group) if and only if G/L is a modular (respectively Dedekind) group ([7, 2.1.11]).

Another interesting and less restrictive class of groups containing all nilpotent groups is the class of T_0-groups which has been studied in [1, 10, 8, 25, 27].

Definition 3. A group G is called a *T_0-group* if the Frattini factor group $G/\Phi(G)$ is a T-group.

A theorem of Ragland ([25]) shows that soluble T_0-groups are closely related to PST: every soluble T_0-group G is supersoluble and its nilpotent residual L is a Hall subgroup. If L is abelian, then G is a PST-group.

We now describe an example of a soluble T_0-group which is not a PST-group. It also shows that the class of all soluble T_0-groups is not subgroup-closed.

Example 4 ([10]). Let $G = \langle a, x, y \mid a^2 = x^3 = y^3 = [x, y]^3 = [x, [x, y]] = [y, [x, y]] = 1, x^a = x^{-1}, y^a = y^{-1} \rangle$. Then $H = \langle x, y \rangle$ is an extraspecial group of order 27 and exponent 3. Let $z = [x, y]$, so $z^a = z$. Then $\Phi(G) = \Phi(H) = \langle z \rangle = Z(G) = Z(H)$. Note that $G/\Phi(G)$ is a T-group so that G is a T_0-group. The maximal subgroups of H are normal in G. Let $K = \langle x, z, a \rangle$. Then $\langle xz \rangle$ is a maximal subgroup of $\langle x, z \rangle$, the Sylow 3-subgroup of K. Note that $\Phi(K) = 1$ and so K is not a T_0-subgroup of G. Note that also G is a soluble group which is not a PST-group.

It is proved in [8] that a soluble T_0-group G is a PST-group if and only if all subgroups of G are T_0-groups.

Another interesting extensions of the permutability which have been study intensively in recent years are the semipermutability and S-semipermutability introduced by Chen in [12].

DEFINITION 5. A subgroup H of a group G is said to be *semipermutable* (respectively, *S-semipermutable*) provided that it permutes with every subgroup (respectively, Sylow subgroup) K of G such that $\gcd(|H|, |K|) = 1$.

Unfortunately semipermutable subgroups are not subnormal in general. It is enough to consider a Sylow 2-subgroup of the symmetric group of degree 3.

Note that if D is an S-permutable π-subgroup of a group G, then D is subnormal in G and so its normal closure D^G is a π-group. In particular, if π is consists of a single prime p, then D^G is nilpotent. Moreover, if D is normal p-subgroup of G, the following statements are pairwise equivalent:

- D is S-semipermutable in G.
- D is S-permutable in G.
- D is normalised by $O^p(G)$.

Isaacs [18] showed that remnants of above results survive in the case of S-semipermutable subgroups.

THEOREM 6. *Let D be an S-semipermutable subgroup of a group G. Then D^G has a nilpotent Hall π'-subgroup. Also, if π consists of a single prime, then D^G is soluble.*

The second statement of the above result is a direct consequence of the following.

THEOREM 7 ([17]). *Let p be a prime and D a nilpotent subgroup of G. If D permutes with every Sylow p-subgroup of G, then $O^{p'}(D^G)$ is soluble.*

Therefore, the normal closure of an S-permutable nilpotent Hall subgroup of G is soluble.

It is known that S-semipermutability is not transitive. Hence it is natural to consider the following class of groups.

DEFINITION 8. A group G is called a BT-group if S-semipermutability is a transitive relation in G, that is, if H is S-semipermutable in K and K is S-semipermutable in G, then H is S-semipermutable in G.

This class was introduced and characterized by Wang, Li and Wang in [28]. Further contributions were presented in [1].

The following important theorem shows that soluble BT-groups form a subclass of PST-groups:

THEOREM 9 ([28]). *Let G be a group with nilpotent residual L. The following statements are equivalent:*

1. *G is a soluble BT-group;*
2. *every subgroup of G of prime power order is S-semipermutable;*
3. *every subgroup of G of prime power order is semipermutable;*
4. *every subgroup of G is semipermutable;*
5. *G is a soluble PST-group and if p and q are distinct primes not dividing the order of L with G_p a Sylow p-subgroup of G and G_q a Sylow q-subgroup of G, then $[G_p, G_q] = 1$.*

The following example describes a soluble PST-group which is not a BT-group.

Example 10. Let L be a cyclic group of order 7 and $A = C_3 \times C_2$ be the automorphism group of L. Here C_3 (respectively, C_2) is the cyclic group of order 3 (respectively, 2). Let $G = [L]A$ be the semidirect product of L by A. Let $L = \langle x \rangle$, $C_3 = \langle y \rangle$ and $C_2 = \langle z \rangle$ and note that $[\langle y \rangle^x, \langle z \rangle] \neq 1$. Now G is a PST-group by Agrawal's theorem, but G is not a BT-group by Theorem 9.

We want to continue to extend the knowledge of the S-semipermutability by considering the effect of imposing S-semipermutability to some subgroups of the Sylow subgroups.

2. S-permutability of subgroups of prime power order

We say that a group a group G is a MS-*group* if the maximal subgroups of all the Sylow subgroups of G are S-semipermutable in G. One of the aims of this section is to characterise MS-groups.

Unfortunately not every subgroup of an MS-group is an MS-group as the group G in Example 4 shows: G is an MS-group, but K is not an MS-group.

The first remarkable fact concerning the structure of an MS-group was proved by Ren in [26]. He showed that every MS-group is supersoluble.

Note that Ren's theorem is a consequence of a more general result proved by Li, Qiao, Su and Wang in [23].

Theorem 11. *Let p be a prime dividing the order of a group G and P a Sylow p-subgroup of G. Assume that d is a power of p such that $1 < d < |P|$. If all subgroups H of P with order d and all cyclic subgroups of P of order 4 (if P is a nonabelian 2-group and $d = 2$) are S-semipermutable in G, then G is p-supersoluble.*

The above result has been extended by Berkovich and Isaacs in [11].

Theorem 12. *Fix an integer $e \geq 3$, and let P be a noncyclic Sylow p-subgroup of a group G with $|P| > p^e = d$. If every noncyclic subgroup of order d is S-semipermutable in G, then G is p-supersoluble.*

This theorem is a consequence of the following results proved recently by Miao, the first author, Esteban-Romero and Li in [24].

THEOREM 13. *Let $P \in \mathrm{Syl}_p(G)$ and let d be a power of p such that $1 \le d < |P|$. Assume that $H \cap \mathrm{O}^p(G)$ is S-semipermutable in G for all noncyclic subgroups H of P with $|H| = d$. Then either $|P \cap \mathrm{O}^p(G)| > d$, or $P \cap \mathrm{O}^p(G)$ is cyclic, or else G is p-supersoluble.*

THEOREM 14. *Fix an integer $e \ge 3$, and let P be a normal p-subgroup of a group G with $|P| > p^e = d$. Assume that $H \cap \mathrm{O}^p(G)$ is S-semipermutable in G for all noncyclic subgroups H of P with $|H| = d$. Then P is contained in $\mathrm{Z}_{\mathfrak{U}}(G)$, the supersoluble hypercentre of G.*

From now on we prepare the way for the characterisation of MS-groups. The following result is the first step.

THEOREM 15 ([10]). *Let G be an MS-group with nilpotent residual L. Then:*

1. *if N is a normal subgroup of G, then G/N is an MS-group.*
2. *L is a nilpotent Hall subgroup of G;*
3. *G is a soluble T_0-group.*

The following notation is needed in the presentation of the next theorem which characterises MS-groups. Let G be a group whose nilpotent residual L is a Hall subgroup of G. Let $\pi = \pi(L)$ and let $\theta = \pi'$, the complement of π in the set of all prime numbers. Let θ_N denote the set of all primes p in θ such that if P is a Sylow p-subgroup of G, then P has at least two maximal subgroups. Further, let θ_C denote the set of all primes q in θ such that if Q is a Sylow q-subgroup of G, then Q has only one maximal subgroup, or, equivalently, Q is cyclic.

THEOREM 16 ([6]). *Let G be a group with nilpotent residual L. Then G is an MS-group if and only if G satisfies the following:*

1. *G is a T_0-group.*
2. *L is a nilpotent Hall subgroup of G.*
3. *If $p \in \pi$ and $P \in \mathrm{Syl}_p(G)$, then a maximal subgroup of P is normal in G.*
4. *Let p and q be distinct primes with $p \in \theta_N$ and $q \in \theta$. If $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$, then $[P, Q] = 1$.*
5. *Let p and q be distinct primes with $p \in \theta_C$ and $q \in \theta$. If $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ and M is the maximal subgroup of P, then $QM = MQ$ is a nilpotent subgroup of G.*

The group given in Example 4 is an MS-group which is not a PST-group. Example 17 presents an example of a soluble PST-group which is not an MS-group (see [10]).

EXAMPLE 17. Let $C = \langle x \rangle$ be a cyclic group of order 19^2, $D = \langle y \rangle$ a cyclic group of order 3^2, and $E = \langle z \rangle$ is a cyclic group of order 2 such that $D \times E \le \mathrm{Aut}(C)$. Then $G = [C](D \times E)$ is a soluble PST-group and G is not an MS-group since $[\langle y^2 \rangle^x, z] \ne 1$.

There are MS-groups which are not BT-groups either.
Two questions that seem natural are

1. When is a soluble PST-group an MS-group?
2. When is a soluble PST-group which is also an MS-group a BT-group?

Using Theorem 16 we are able to answer the first question and provide a partial answer to the second.

THEOREM 18 ([6]). *Let G be a soluble PST-group. Then G is an MS-group if and only if G satisfies* 4 *and* 5 *of Theorem* 16.

THEOREM 19 ([6]). *Let G be a soluble PST-group which is also an MS-group. If θ_C is the empty set, then G is a BT-group.*

A set $\mathfrak{Z}$ of Sylow subgroups of a group G is a *complete set of Sylow subgroups of G* if $\mathfrak{Z}$ contains exactly one Sylow subgroup for each prime dividing the order of G; $\mathfrak{Z}$ is called a *Sylow basis of G* if the Sylow subgroups in $\mathfrak{Z}$ are pairwise permutable. Sylow basis were introduced and studied by Hall (see [13]. They play a central role in the study of soluble groups as their existence characterises solubility ([13, Chapter I, Sections 3 and 4]).

Asaad and Heliel [2] introduced and studied the notion of a $\mathfrak{Z}$-*permutable* subgroup, where $\mathfrak{Z}$ is a complete set of Sylow subgroups of a group G. A subgroup of G is called $\mathfrak{Z}$-*permutable* if it permutes with every member of a complete set $\mathfrak{Z}$ of Sylow subgroups of G. It is clear that S-permutability implies $\mathfrak{Z}$-permutability but the converse does not hold in general. In fact, $\mathfrak{Z}$-permutable subgroups are not subnormal in general, and subnormal $\mathfrak{Z}$-permutable subgroups are not S-permutable either as the following example in [15] shows.

EXAMPLE 20. Let $E = \langle x, y \rangle$ be an extraspecial group of order 27 and exponent 3. Let a be the automorphism of order 2 of E given by $x^a = x^{-1}$, $y^a = y^{-1}$. Let $G = E \rtimes \langle a \rangle$ be the corresponding semidirect product. Then $\mathfrak{Z} = \{E, \langle a \rangle\}$ is a complete set of Sylow subgroups of G. The subgroup $H = \langle x \rangle$ is $\mathfrak{Z}$-permutable, but it does not permute with the Sylow 2-subgroup $\langle ay \rangle$. Therefore, H is not S-permutable. However, H is a subnormal subgroup of G. Note that G is a $\mathfrak{Z}$-MS-group but not an *MS*-group.

The structural impact of $\mathfrak{Z}$-permutability has been studied in [2, 14, 15, 16, 22, 21, 29, 20].

Next we are concerned with the role of a new subgroup embedding property in the structural study of the groups. More precisely, we are interested in obtaining global information about a group by assuming that the maximal subgroups (respectively all subgroups) of the Sylow subgroups in $\mathfrak{Z}$ are $\mathfrak{Z}$-S-semipermutable.

DEFINITION 21. Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a group G. A subgroup A of G is said to be $\mathfrak{Z}$-*S-semipermutable* if A is S-permutative with the members of $\mathfrak{Z}$, that is, if A permutes with all Sylow q-subgroups of $\mathfrak{Z}$ for all primes q not dividing $|A|$.

A group G is said to be a $\mathfrak{Z}$-*MS-group* if every maximal subgroup of every Sylow subgroup in $\mathfrak{Z}$ is $\mathfrak{Z}$-S-semipermutable in G.

Example 20 shows that a $\mathfrak{Z}$-MS-group is not an MS-group in general. The results above mentioned suggest a natural question: What is the structure of the $\mathfrak{Z}$-MS-groups? The results in [4] show that surprisingly the structure is quite similar to the one in the class *MS* despite the fact that the number of Sylow subgroups to play with is significantly lower. They also allow us to recover many of the earlier results as particular cases.

At this point it should be noted that a group G is an $\mathfrak{Z}$-MS-group if and only if every maximal subgroup of every Sylow subgroup in $\mathfrak{Z}$ is $\mathfrak{Z}$-permutable. Therefore, by [2, Theorem 3.1], G is supersoluble.

The following theorem gives more precise information.

THEOREM 22. *Let $\mathfrak{Z}$ be a complete system of Sylow subgroups of a group G. Assume that all maximal subgroups of every Sylow subgroup in $\mathfrak{Z}$ are $\mathfrak{Z}$-permutable. Then:*

1. *The nilpotent residual L of G is a Hall subgroup of G.*
2. *G is a supersoluble T_0-group.*

The following theorem provides a complete characterisation of $\mathfrak{Z}$-MS-groups.

THEOREM 23. *Let $\mathfrak{Z}$ be a complete system of Sylow subgroups of a group G and let L be the nilpotent residual of G. Then G is an $\mathfrak{Z}$-MS-group if and only if G satisfies the following properties:*

1. *G is a soluble T_0-group.*
2. *L is a nilpotent Hall subgroup of G.*
3. *If $p \in \pi$ and $P \in \mathrm{Syl}_p(G) \cap \mathfrak{Z}$, then a maximal subgroup of P is normal in G.*
4. *Let p and q be distinct primes with $p \in \theta_N$ and $q \in \theta$. If $P \in \mathrm{Syl}_p(G) \cap \mathfrak{Z}$ and $Q \in \mathrm{Syl}_q(G) \cap \mathfrak{Z}$, then $[P, Q] = 1$.*
5. *Let p and q be distinct primes with $p \in \theta_C$ and $q \in \theta$. If $P \in \mathrm{Syl}_p(G) \cap \mathfrak{Z}$ and $Q \in \mathrm{Syl}_q(G) \cap \mathfrak{Z}$ and M is the maximal subgroup of P, then $[M, Q] = 1$.*

COROLLARY 24. *Let $\mathfrak{Z}$ be a complete system of Sylow subgroups of a group G and let L be the nilpotent residual of G. If G is a $\mathfrak{Z}$-MS-group and L is abelian, then G is a soluble PST-group.*

We bring the paper to a close characterising groups G in which all subgroups of every Sylow subgroup in a complete system of Sylow subgroups $\mathfrak{Z}$ of G are $\mathfrak{Z}$-permutable ([3]).

THEOREM 25. *Let L be the nilpotent residual of a group G and let $\mathfrak{Z}$ be a complete set of Sylow subgroups of G. Then all the subgroups of $G_p \in \mathfrak{Z}$, for all $p \in \pi(G)$, are $\mathfrak{Z}$-permutable if and only if G satifies the following conditions:*

1. *G is a supersoluble T_0-group.*
2. *L is an abelian Hall subgroup of G.*
3. *G is a soluble PST-group.*
4. *If p and q are distinct primes from $\pi(G)\backslash\pi(L)$ with G_p and G_q contained in $\mathfrak{Z}$, then $[G_p, G_q] = 1$.*

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Adolfo Ballester-Bolinches
Departament de Matemàtiques
Universitat de València
Dr. Moliner 50
46100 Burjassot, València, Spain
Adolfo.Ballester@uv.es

James C. Beidleman
Department of Mathematics
University of Kentucky
Lexington
KY 40506-0027, USA
james.beidleman@uky.edu