Rend. Lincei Mat. Appl. 28 (2017), 349–368
DOI 10.4171/RLM/767

Mathematical Analysis — *Modular interpolation and modular estimates of the Fourier transform and related operators*, by Kwok-Pun Ho, communicated on December 16, 2016.

Abstract. — We introduce an approach to modular interpolation in this paper. By using this interpolation, we establish the modular inequalities for the Fourier transform, the Laplace transform, the Hankel transform and the oscillatory integral operators. Moreover, we also obtain the modular Fourier restriction theorem.

Key words: Modular inequality, Fourier transform, interpolation, Hankel transform, oscillatory integrals, Fourier restriction

Mathematics Subject Classification: 42A38, 46A80, 46M35, 46E30, 42B20

1. Introduction

The main theme of this paper is the modular interpolation theory and its application on the modular inequalities for the Fourier transform, the Laplace transform, the Hankel transform and the oscillatory integral operators.

For the classical interpolation such as the Marcinkiewicz interpolation, the norm of the function spaces is generated via the K-functional and the real interpolation functor. Therefore, the norm inequalities for operators are obtained.

Other than the norm inequalities, the modular inequalities also play an important role in Analysis. For instance, the modular inequalities for the Hardy–Littlewood maximal function and the Fourier transform are obtained in [2, 19] and [18], respectively. Furthermore, the modular inequalities for the Calderón operators and the Hardy operators are established in [3].

There are several methods to obtain the modular inequalities. For instance, the distributional inequalities [19] and the extrapolation [7, 8] can be used to generate the modular inequalities.

In this paper, we use the idea of interpolation to obtain our modular inequalities for the Fourier transform, the Hankel transform and the oscillatory integral operators. We introduce an interpolation functor that generates the modular

$$\rho(f) = \int \Phi(f(x))\, dx$$

instead of the norm of a function space. Therefore, as a result of the action of this interpolation functor, we obtain modular inequalities instead of norm inequalities.

The interpolation functor used in this paper is defined via the K-functional. We obtain our results by applying our interpolation functor to Lebesgue spaces. As a well known fact, the K-functional of a measurable function f under a pair of Lebesgue spaces is given in term of the decreasing rearrangement of f [17]. Therefore, our modular inequalities involves the decreasing rearrangement of f.

Notice that one of the essential components for the interpolation theory is the Hardy inequalities. The modular Hardy inequalities are established in [3, 4]. The results in [3, 4] are also one of the main motivations of this paper.

There are some modular interpolation theories given in [3, 4, 20, 21]. On the other hand, the results given in [3, 4, 20, 21] cannot be applied to the Fourier transform. For instance, the validity of [3, Corollary 3.6] requires that the linear operator T is bounded on L^∞ or L^0 where $L^0 = \{f : |\operatorname{supp} f| < \infty\}$. Therefore, [3, Corollary 3.6] does not apply to the Fourier transform.

In addition, some modular inequalities of the Fourier transform are obtained in [18]. Notice that the results in [18] play special interest on linear operator of strong type $(2, 2)$ and $(1, \infty)$. Therefore, the results in [18] are not necessary applied to some other transforms related to Fourier transform such as the Hankel transform and the oscillatory integral operators. Especially, our method also gives the modular Fourier restriction theorem.

Our method is based on the ideas from [13, 15]. The results obtained in [13, 15, 16] show that we have the norm inequalities for the Fourier transforms, the Hankel transform and the oscillatory integral operators on rearrangement-invariant quasi-Banach function spaces. In this paper, we modify the method given in [13] to obtain the corresponding modular inequalities.

This paper is organized as follows. In Section 2, we recall some basic facts about r.i.q.B.f.s. and present the modular Hardy inequalities. Our modular interpolation theory is presented in Section 3. The modular estimates of the Fourier transform, the Laplace transform, the Hankel transform, the oscillatory integral operators and the modular Fourier restriction theorem are established in Section 4.

2. Definitions and preliminaries

Let $\mathscr{M}(0, \infty)$ and $\mathscr{M}(\mathbb{R}^n)$ be the sets of Lebesgue measurable functions on $(0, \infty)$ and $\mathbb{R}^n$, respectively.

Let $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}'(\mathbb{R}^n)$ denote the class of Schwartz functions and tempered distributions, respectively. Let L^p and $L^p(0, \infty)$ be the Lebesgue spaces on $\mathbb{R}^n$ and $(0, \infty)$, respectively.

For any $f \in \mathscr{M}(\mathbb{R}^n)$ and $s > 0$, write

$$d_f(s) = |\{x \in \mathbb{R}^n : |f(x)| > s\}|$$

and

$$f^*(t) = \inf\{s > 0 : d_f(s) \le t\}, \quad t > 0.$$

We call f and g are equimeasurable if $d_f(s) = d_g(s)$ for all $s > 0$. We write $f \approx g$ if

$$Bf \le g \le Cf,$$

for some constants $B, C > 0$ independent of appropriate quantities involved in the expressions of f and g.

DEFINITION 2.1. A Lebesgue measurable function $\Phi : [0, \infty) \to [0, \infty)$ is called a modular function if Φ is a non-decreasing function with $\lim_{t \to 0^+} \Phi(t) = 0$.

The subsequent result is well known. We present the proof again for completeness.

PROPOSITION 2.1. *Let Φ be a modular function. For any $f \in \mathscr{M}(\mathbb{R}^n)$, we have*

$$\int_0^\infty \Phi(f^*(t))\,dt = \int_{\mathbb{R}^n} \Phi(|f(x)|)\,dx. \tag{2.1}$$

PROOF. Let $f = \sum_{j=1}^N a_j \chi_{E_j}(x)$ be a simple function where $E_j \cap E_i = \emptyset$ when $i \ne j$ and $a_i \ge a_j$ when $i \le j$. We find that $f^*(t) = \sum_{j=1}^N a_j \chi_{[m_{j-1}, m_j)}(t)$ where $m_j = \sum_{i=1}^j |E_i|$. Consequently,

$$\int_0^\infty \Phi(f^*(t))\,dt = \sum_{j=1}^N \Phi(a_j)(m_j - m_{j-1}) = \sum_{j=1}^N \Phi(a_j)|E_j| = \int_{\mathbb{R}^n} \Phi(|f(x)|)\,dx$$

For general $f \in \mathscr{M}(\mathbb{R}^n)$, there exists a family of non-negative simple function f_j such that $f_j \uparrow |f|$. In view of [1, Chapter 2, Proposition 1.7], we also have $f_j^* \uparrow f^*$. As Φ is non-decreasing, we have $\Phi(f_j) \uparrow \Phi(|f|)$ and $\Phi(f_j^*) \uparrow \Phi(f^*)$. The monotone convergence theorem yields (2.1). □

A modular function Φ is said to satisfy the $\triangle_2$ condition if there exists a constant $C > 0$ such that

$$\Phi(2t) \le C\Phi(t), \quad t > 0.$$

We write $\Phi \in \triangle_2$ if it satisfies the $\triangle_2$ condition.

Therefore, whenever $\Phi \in \triangle_2$, we have

$$\Phi(a + b) \le \Phi(2\max(a, b)) \le C\max(\Phi(a), \Phi(b)) \le C(\Phi(a) + \Phi(b)). \tag{2.2}$$

A modular function Φ is said to satisfy the ∇_2 condition if there exists a constant $0 < H < 1$ such that

$$\Phi(t) \le H\Phi(2t), \quad t > 0.$$

We write $\Phi \in \nabla_2$ if it satisfies the ∇_2 condition.

DEFINITION 2.2. Let $0 < p < \infty$ and Φ be a modular function. Define

$$\Phi_p(t) = \Phi(t^{1/p}).$$

It is easy to see that whenever Φ is a modular function, Φ_p is also a modular function. Furthermore, $\Phi_p \in \triangle_2$ provided that $\Phi \in \triangle_2$.

We are ready to state one of the crucial supporting result for our modular interpolation theory, the modular Hardy inequalities. We first recall the definitions of two Hardy type operators used in [3].

Let $0 < a, b < \infty$. For any $f \in \mathscr{M}[0, \infty)$, write

$$S_a f(t) = \frac{1}{t^{1/a}} \int_0^t f(s) s^{1/a} \frac{ds}{s}, \tag{2.3}$$

$$\tilde{S}_b f(t) = \frac{1}{t^{1/b}} \int_t^\infty f(s) s^{1/b} \frac{ds}{s}. \tag{2.4}$$

We now present the modular inequalities for the Hardy type operators S_a and $\tilde{S}_b$.

THEOREM 2.2. *Let $0 < a \le 1$ and Φ be a modular function. There exist constants $B, C > 0$ such that for any decreasing nonnegative function f,*

$$\int_0^\infty \Phi(S_a f(t))\, dt \le B \int_0^\infty \Phi(Cf(t))\, dt \tag{2.5}$$

if and only if there exist constants $H, D > 0$ such that

$$t^a \int_0^t \frac{\Phi(y)}{y^{a+1}} dy \le H\Phi(Dt), \quad \forall t > 0. \tag{2.6}$$

For the proof of the above result, the reader is referred to [3, Theorems 2.1 and 2.3].

THEOREM 2.3. *Let $a > 1$ and Φ be a modular function. There exists a constant $C > 0$ such that for any decreasing nonnegative function f,*

$$\int_0^\infty \Phi(S_a f(t))\, dt \le C \int_0^\infty \Phi(f(t))\, dt \tag{2.7}$$

if and only if $\Phi \in \triangle_2$ and there exists a constant $B > 0$ such that

$$t^a \int_0^t \frac{\Phi(y)}{y^{a+1}} dy \le B\Phi(t), \quad \forall t > 0. \tag{2.8}$$

When $\Phi(t) = t^p$, $1 \le p < \infty$, we find that if $a < p$

$$t^a \int_0^t \frac{\Phi(y)}{y^{a+1}}\,dy = t^a \frac{t^{p-a}}{p-a} = \frac{1}{p-a} t^p.$$

We see that $\Phi(t) = t^p$ fulfills (2.6) and (2.8) if and only if $a < p$.

THEOREM 2.4. *Let $0 < b < \infty$ and Φ be a modular function. There exists a constant $C > 0$ such that for any decreasing nonnegative function f,*

$$\int_0^\infty \Phi(\tilde{S}_b f(t))\,dt \le C \int_0^\infty \Phi(f(t))\,dt \tag{2.9}$$

if and only if there exists a constant $B > 0$ such that

$$t^b \int_t^\infty \frac{\Phi(y)}{y^{b+1}}\,dy \le B\Phi(t), \quad \forall t > 0. \tag{2.10}$$

When $\Phi(t) = t^p$, $1 \le p < \infty$, we find that if $b > p$

$$t^b \int_t^\infty \frac{\Phi(y)}{y^{b+1}}\,dy = t^b \frac{t^{p-b}}{b-p} = \frac{1}{b-p} t^p.$$

Thus, $\Phi(t) = t^p$ satisfies (2.10) if and only if $b > p$.

For the proofs of Theorems 2.3 and 2.4, the reader is referred to [3, Theorem 4.2 (iii)] and [3, Theorem 4.5 (iii)], respectively.

In addition, Theorems 2.2, 2.3 and 2.4 are special cases of the general results in [3, Theorems 2.1, 2.3, 4.2 and 4.5].

Note that whenever Φ satisfies

$$\Phi(st) \le r\Phi(t), \quad \forall t \ge 0 \tag{2.11}$$

for some $1 < s^a < r$, then Φ fulfills (2.6). In fact, (2.11) gives

$$\begin{aligned}
\int_0^u \frac{\Phi(y)}{y^{a+1}}\,dy &= \sum_{i=-\infty}^{0} \int_{s^{i-1}u}^{s^i u} \frac{\Phi(y)}{y^{a+1}}\,dy = \sum_{i=-\infty}^{0} \int_{s^{-1}u}^{u} \frac{\Phi(s^i t)}{(s^i t)^{a+1}} s^i\,dt \\
&\le \sum_{i=-\infty}^{0} \left(\frac{r}{s^a}\right)^i \int_{s^{-1}u}^{u} \frac{\Phi(t)}{t^{a+1}}\,dt \le \sum_{i=-\infty}^{0} \left(\frac{r}{s^a}\right)^i \Phi(u) \int_{s^{-1}u}^{u} \frac{1}{t^{a+1}}\,dt \\
&\le Cu^{-a}\Phi(u)
\end{aligned}$$

for some $C > 0$ because Φ is non-decreasing. Hence, Φ satisfies (2.6).

Similarly, when Φ satisfies

$$\Phi(st) \le r\Phi(t), \quad \forall t \ge 0 \tag{2.12}$$

for some $1 < r < s^b$, then Φ fulfills (2.10). Precisely, we have

$$\int_u^\infty \frac{\Phi(y)}{y^{b+1}}\,dy = \sum_{i=1}^\infty \int_{s^{i-1}u}^{s^i u} \frac{\Phi(y)}{y^{b+1}}\,dy = \sum_{i=1}^\infty \int_{s^{-1}u}^{u} \frac{\Phi(s^i t)}{(s^i t)^{b+1}} s^i\,dt$$
$$\leq \sum_{i=1}^\infty \left(\frac{r}{s^b}\right)^i \int_{s^{-1}u}^{u} \frac{\Phi(t)}{t^{b+1}}\,dt \leq \sum_{i=1}^\infty \left(\frac{r}{s^b}\right)^i \Phi(u) \int_{s^{-1}u}^{u} \frac{1}{t^{b+1}}\,dt$$
$$\leq C u^{-b}\Phi(u)$$

which assures the validity of (2.10).

3. Modular interpolation

We recall the definition of the K-functional from [1, Section 3.1] and [27, Section 1.3.1]. The following definition involves the notion of compatible couple of quasi-normed spaces, for brevity, we refer the reader to [27, Section 1.2] for the definition of compatible couple of quasi-normed spaces.

Definition 3.1. Let (X_0, X_1) be a compatible couple of quasi-normed spaces. For any $f \in X_0 + X_1$, the K-functional is defined as

$$K(f, t, X_0, X_1) = \inf\{\|f_0\|_{X_0} + t\|f_1\|_{X_1} : f = f_0 + f_1\}$$

where the infimum is taking over all $f = f_0 + f_1$ for which $f_i \in X_i$, $i = 0, 1$.

The following is the modular interpolation functor used in this paper.

Definition 3.2. Let $0 < \theta, r < \infty$ and Φ be a modular function. Let (X_0, X_1) be a compatible couple of quasi-normed spaces on $\mathbb{R}^n$. The modular interpolation functor $\rho_{(X_0,X_1)_{r,\theta,\Phi}}$ is defined by

$$\rho_{(X_0,X_1)_{r,\theta,\Phi}}(f) = \int_0^\infty \Phi(t^{-\frac{1}{r}} K(f, t^{\frac{1}{\theta}}, X_0, X_1))\,dt.$$

We are now ready to present the modular interpolation theorem for linear operators.

Theorem 3.1. *Let $0 < \theta, r < \infty$ and Φ be a modular function. If (X_0, X_1) and (Y_0, Y_1) are compatible couples of quasi-normed spaces on $\mathbb{R}^n$ and T is a linear operator such that*

$$\|Tf\|_{Y_i} \leq C_i \|f\|_{X_i}, \quad i = 0, 1.$$

Then, we have

$$\int_0^\infty \Phi(t^{-\frac{1}{r}}K(Tf, t^{\frac{1}{\theta}}, Y_0, Y_1))\,dt \le C_0^\theta C_1^{-\theta}\int_0^\infty \Phi(C_0^{1-\frac{\theta}{r}}C_1^{\frac{\theta}{r}}t^{-\frac{1}{r}}K(f, t^{\frac{1}{\theta}}, X_0, X_1))\,dt. \tag{3.1}$$

PROOF. In view of the definition of the K-functional, we find that

$$\begin{aligned} K(Tf, t, Y_0, Y_1) &\le \inf\{\|Tf_0\|_{Y_0} + t\|Tf_1\|_{Y_1} : f = f_0 + f_1, f_i \in Y_i, i = 0, 1\} \\ &\le C_0 K(f, C_1 C_0^{-1} t, X_0, X_1). \end{aligned}$$

Multiplying $t^{-1/r}$ and then applying the modular $\rho_{(X_0, X_1)_{r,\theta,\Phi}}$ on both sides of the above inequality, we obtain

$$\int_0^\infty \Phi(t^{-\frac{1}{r}}K(Tf, t^{\frac{1}{\theta}}, Y_0, Y_1))\,dt \le \int_0^\infty \Phi(C_0 t^{-\frac{1}{r}}K(f, C_1 C_0^{-1} t^{\frac{1}{\theta}}, X_0, X_1))\,dt.$$

Next, by using the change of variable $t = C_0^\theta C_1^{-\theta} s$, we obtain

$$\int_0^\infty \Phi(t^{-\frac{1}{r}}K(Tf, t^{\frac{1}{\theta}}, Y_0, Y_1))\,dt \le \int_0^\infty C_0^\theta C_1^{-\theta}\Phi(C_0^{1-\frac{\theta}{r}}C_1^{\frac{\theta}{r}}s^{-\frac{1}{r}}K(f, s^{\frac{1}{\theta}}, X_0, X_1))\,ds.$$

Thus, we establish (3.1). □

To apply the above result, we have to show that the expression in the right hand side of (3.1) gives the modular

$$\rho(f) = \int \Phi(|f(x)|)\,dx. \tag{3.2}$$

This is precisely the result of the following theorem which asserts that the modular (3.2) can be generated by the K-functional of Lebesgue spaces.

THEOREM 3.2. *Let $0 < p_0 < p_1 \le \infty$ and Φ be a modular function. Suppose that $\Phi \in \triangle_2$, Φ_{p_0} satisfies*

$$t\int_0^t \frac{\Phi_{p_0}(y)}{y^2}\,dy \le H\Phi_{p_0}(Kt), \quad \forall t > 0 \tag{3.3}$$

for some constants $H, K > 0$ and Φ_{p_1} satisfies

$$t\int_t^\infty \frac{\Phi_{p_1}(y)}{y^2}\,dy \le B\Phi_{p_1}(t), \quad \forall t > 0 \tag{3.4}$$

for some $B > 0$.

Let $r = p_0$ *and*

$$\frac{1}{\theta} = \frac{1}{p_0} - \frac{1}{p_1}.$$

Then, there exist constants $C, D > 0$ *such that*

$$C\int_{\mathbb{R}^n} \Phi(f(x))\,dx \le \int_0^\infty \Phi(t^{-\frac{1}{r}} K(f, t^{\frac{1}{\theta}}, L^{p_0}, L^{p_1}))\,dt \le D\int_{\mathbb{R}^n} \Phi(f(x))\,dx. \tag{3.5}$$

Proof. The Holmstedt formulas for the K-functionals of Lebesgue spaces [17] state that

$$K(f, t, L^{p_0}, L^{p_1}) \approx \Big(\int_0^{t^\theta} (f^*(s))^{p_0}\,ds\Big)^{\frac{1}{p_0}} + t\Big(\int_{t^\theta}^\infty (f^*(s))^{p_1}\,ds\Big)^{\frac{1}{p_1}}.$$

Therefore,

$$\begin{aligned} &t^{-\frac{1}{r}} K(f, t^{\frac{1}{\theta}}, L^{p_0}, L^{p_1}) \\ &\quad\approx t^{-\frac{1}{p_0}}\Big(\int_0^t (f^*(s))^{p_0}\,ds\Big)^{\frac{1}{p_0}} + t^{-\frac{1}{p_1}}\Big(\int_t^\infty (f^*(s))^{p_1}\,ds\Big)^{\frac{1}{p_1}}. \end{aligned}$$

Consequently, (2.2) gives

$$\begin{aligned} &\int_0^\infty \Phi(t^{-\frac{1}{r}} K(f, t^{\frac{1}{\theta}}, L^{p_0}, L^{p_1}))\,dt \\ &\quad\le D\Big(\int_0^\infty \Phi\Big(Et^{-\frac{1}{p_0}}\Big(\int_0^t (f^*(s))^{p_0}\,ds\Big)^{\frac{1}{p_0}}\Big)\,dt \\ &\qquad + \int_0^\infty \Phi\Big(Et^{-\frac{1}{p_1}}\Big(\int_t^\infty (f^*(s))^{p_1}\,ds\Big)^{\frac{1}{p_1}}\Big)\,dt\Big) \\ &\quad\le D\int_0^\infty \Phi_{p_0}(S_1((f^*)^{p_0})(t))\,dt + D\int_0^\infty \Phi_{p_1}(\tilde{S}_1((f^*)^{p_1})(t))\,dt = I + II \end{aligned}$$

for some constants $D, E > 0$ because Φ is non-decreasing and $\Phi \in \triangle_2$.

Since Φ satisfies (3.3) and $(f^*)^{p_0}$ is non-increasing, we find that

$$I \le D\int_0^\infty \Phi_{p_0}((f^*(t))^{p_0})\,dt = D\int_0^\infty \Phi(f^*(t))\,dt = D\int_{\mathbb{R}^n} \Phi(f(x))\,dx$$

for some $D > 0$.

Similarly, as Φ fulfills (3.4), we obtain

$$II \leq D \int_0^\infty \Phi_{p_1}((f^*(t))^{p_1})\, dt = D \int_0^\infty \Phi(f^*(t))\, dt = D \int_{\mathbb{R}^n} \Phi(f(x))\, dx$$

for some $D > 0$.

Therefore, $\Phi \in \triangle_2$ yields

$$\int_0^\infty \Phi(t^{-\frac{1}{r}} K(f, t^{\frac{1}{\theta}}, L^{p_0}, L^{p_1}))\, dt \leq D \int_{\mathbb{R}^n} \Phi(f(x))\, dx \tag{3.6}$$

for some $D > 0$.

For the reverse inequality, since f^* is non-increasing and Φ is non-decreasing, we have

$$\begin{aligned} \int_0^\infty \Phi(t^{-\frac{1}{r}} K(f, t^{\frac{1}{\theta}}, L^{p_0}, L^{p_1}))\, dt &\geq C \int_0^\infty \Phi\Big(t^{-\frac{1}{p_0}}\Big(\int_0^t (f^*(s))^{p_0}\, ds\Big)^{\frac{1}{p_0}}\Big)\, dt \\ &\geq C \int_0^\infty \Phi(f^*(t))\, dt = C \int_{\mathbb{R}^n} \Phi(f(x))\, dx. \end{aligned} \tag{3.7}$$

Finally, (3.6) and (3.7) yield (3.5). □

Notice that even though we present our results for Lebesgue spaces on $\mathbb{R}^n$, they are still valid for Lebesgue spaces on σ-finite measures such as $L^p(0, \infty)$. For brevity, we skip the details.

In view of the estimates after Theorems 2.3 and 2.4, we find that if $p_0 < p$ and $p < p_1$, $\Phi(t) = t^p$ fulfills (3.3) and (3.4), respectively. Therefore, the function $\Phi(t) = \sum_{k=1}^n a_k t^{q_k}$ where $p_0 < q_k < p_1$ and $a_k > 0$, $1 \leq k \leq n$, satisfies (3.3) and (3.4).

4. Modular estimates

In this section, we use the results obtained in the previous section to establish several modular inequalities on the Fourier transform, the Laplace transform, the Hankel transform and the oscillatory integral operators. Especially, we also have the modular Fourier restriction theorem at the end of this section.

4.1. *Fourier transform*

For any $f \in \mathscr{S}'(\mathbb{R}^n)$, let $\mathscr{F}f$ denote the Fourier transform of f.

THEOREM 4.1. *Let Φ be a modular function. Suppose that $\Phi \in \triangle_2$, Φ satisfies*

$$t \int_0^t \frac{\Phi(y)}{y^2}\, dy \leq H\Phi(Kt), \quad \forall t > 0 \tag{4.1}$$

for some constants $H, K > 0$ *and* Φ_2 *satisfies*

$$t \int_t^\infty \frac{\Phi_2(y)}{y^2}\, dy \le B\Phi_2(t), \quad \forall t > 0 \tag{4.2}$$

for some $B > 0$*. Then, there exists constant* $C > 0$ *such that*

$$\int_0^\infty \Phi(s(\mathscr{F}f)^*(s))\frac{ds}{s^2} \le C\int_{\mathbb{R}^n} \Phi(f(x))\, dx. \tag{4.3}$$

PROOF. It is well known that the Fourier transform is bounded from L^1 to L^∞ and from L^2 to L^2.

From the assumptions on Φ, we are allowed to apply Theorem 3.2 with $p_0 = 1$ and $p_1 = 2$. We obtain $\theta = 2$ and

$$\begin{aligned} C_1 \int_{\mathbb{R}^n} \Phi(f(x))\, dx &\le \int_0^\infty \Phi(t^{-1}K(f, t^{\frac{1}{2}}, L^1, L^2))\, dt \\ &\le C_0 \int_{\mathbb{R}^n} \Phi(f(x))\, dx \end{aligned} \tag{4.4}$$

for some $C_1, C_0 > 0$.

The definition of the K-functional and the Holmstedt formulas [17] yield

$$K(\mathscr{F}f, t, L^\infty, L^2) = tK(\mathscr{F}f, t^{-1}, L^2, L^\infty) \approx t\Big(\int_0^{t^{-2}} ((\mathscr{F}f)^*(s))^2\, ds\Big)^{\frac{1}{2}}.$$

Since $(\mathscr{F}f)^*$ is non-increasing, we have

$$t^{-1}K(\mathscr{F}f, t^{\frac{1}{2}}, L^\infty, L^2) \ge Dt^{-\frac{1}{2}}\Big(\int_0^{t^{-1}} ((\mathscr{F}f)^*(s))^2\, ds\Big)^{\frac{1}{2}} \ge Dt^{-1}(\mathscr{F}f)^*(t^{-1})$$

for some $D > 0$.

Applying the modular $\int_0^\infty \Phi(\cdot)\, dt$ on both sides of the above inequality, we obtain

$$\begin{aligned} \int_0^\infty \Phi(t^{-1}K(\mathscr{F}f, t^{\frac{1}{2}}, L^\infty, L^2))\, dt &\ge \int_0^\infty \Phi(Dt^{-1}(\mathscr{F}f)^*(t^{-1}))\, dt \\ &\ge C\int_0^\infty \Phi(t^{-1}(\mathscr{F}f)^*(t^{-1}))\, dt \end{aligned}$$

for some $C > 0$ because $\Phi \in \triangle_2$. By using the change of variable $s = t^{-1}$, we find that

$$\int_0^\infty \Phi(t^{-1}K(\mathscr{F}f, t^{\frac{1}{2}}, L^\infty, L^2))\, dt \ge C\int_0^\infty \Phi(s(\mathscr{F}f)^*(s))\frac{ds}{s^2}. \tag{4.5}$$

Therefore, Theorem 3.1, (4.4) and (4.5) give (4.3). □

When $\Phi(t) = t^p$, $1 < p < 2$, conditions (4.1) and (4.2) are fulfilled and (4.3) becomes

$$\int_0^\infty s^{\frac{p}{p'}}((\mathscr{F}f)^*(s))^p \frac{ds}{s} = \int_0^\infty s^{p-2}((\mathscr{F}f)^*(s))^p \, ds \leq C \int_{\mathbb{R}^n} |f(x)|^p \, dx.$$

Thus, we have

$$\|\mathscr{F}f\|_{L^{p',p}} \leq C\|f\|_{L^p}$$

for some $C > 0$ where $L^{p',p}$ denote the Lorentz space. As $1 < p < 2$, in view of the embedding $L^{p',p} \hookrightarrow L^{p'}$ [1, Chapter 4, Proposition 4.2], we recover the Hausdorff–Young inequality

$$\|\mathscr{F}f\|_{L^{p'}} \leq C\|f\|_{L^p}$$

for some $C > 0$ when $1 \leq p \leq 2$.

Therefore, Theorem 4.1 can be considered as the modular Hausdorff–Young inequality.

Similar to the discussion at the end of Section 3, when $1 < q_k < 2$ and $a_k > 0$, $1 \leq k \leq n$, the function $\Phi(t) = \sum_{k=1}^n a_k t^{q_k}$ satisfies (4.1) and (4.2). Consequently, we have the corresponding modular Hausdorff–Young inequality.

The reader is referred to [18] for some other results on the modular inequalities of the Fourier transform.

4.2. Laplace transform

For any $1 \leq p \leq \infty$, let $L^p(0, \infty)$ denote the Lebesgue space on $(0, \infty)$.

For any $f \in \mathscr{M}(0, \infty)$, the Laplace transform of f is given by

$$\mathscr{L}f(s) = \int_0^\infty e^{-st} f(t) \, dt.$$

It is easy to see that $\mathscr{L} : L^1(0, \infty) \to L^\infty(0, \infty)$ is bounded. According to [10, p. 189], $\mathscr{L}$ is bounded on $L^2(0, \infty)$. Thus, similar to the modular estimate of the Fourier transform, we obtain the modular estimate of the Laplace transform.

THEOREM 4.2. *Let Φ be a modular function. Suppose that $\Phi \in \triangle_2$, Φ satisfies*

$$t \int_0^t \frac{\Phi(y)}{y^2} dy \leq H\Phi(Kt), \quad \forall t > 0 \tag{4.6}$$

for some constants $H, K > 0$ and Φ_2 satisfies

$$t \int_t^\infty \frac{\Phi_2(y)}{y^2} dy \leq B\Phi_2(t), \quad \forall t > 0 \tag{4.7}$$

for some $B > 0$. Then, there exists constant $C > 0$ such that

$$\int_0^\infty \Phi(s(\mathscr{L}f)^*(s))\,\frac{ds}{s^2} \le C\int_0^\infty \Phi(|f(t)|)\,dt. \tag{4.8}$$

Since the proof of the above theorem follows from the proof of Theorem 4.1, for simplicity, we leave it to the reader.

4.3. *Hankel transform*

We study the Hankel transform and its generalizations in this subsection.

Let $\alpha \ge -\frac{1}{2}$ and $v, \mu \in \mathbb{R}$. The operator $\mathscr{L}^\alpha_{v,\mu}$ is defined by

$$\mathscr{L}^\alpha_{v,\mu} f(y) = y^\mu \int_0^\infty (xy)^v f(x) J_\alpha(xy)\,dy. \tag{4.9}$$

where $J_\alpha(r)$ is the Bessel function of the first kind.

The family of operators $\{\mathscr{L}^\alpha_{v,\mu}\}$ contains a number of operators used in harmonic analysis. If we denote the Fourier transform of $f(|x|)$ by $\mathscr{F}f(|\xi|)$, then

$$\mathscr{F}f(|\xi|) = (2\pi)^{n/2}\mathscr{L}^{\frac{n}{2}-1}_{\frac{n}{2},1-n} f(|\xi|). \tag{4.10}$$

In [11], for any $\alpha > -1$, the operator $\mathscr{L}^\alpha_{\alpha+1,-2\alpha-1}$ is called as the Hankel transform.

In [6], $\mathscr{L}^\alpha_{\alpha+1,-2\alpha-1} = \tilde{\mathscr{H}}_\alpha$ is named as the Fourier–Bessel transform of order α. Moreover,

$$\mathscr{H}_\alpha f = \mathscr{L}^\alpha_{\frac{1}{2},0} f = \int_0^\infty f(t)(xt)^{\frac{1}{2}} J_\alpha(xt)\,dt$$

is the so-called Hankel transform of order α.

We recall the $L^p - L^q$ estimates of $\mathscr{L}^\alpha_{v,\mu}$ from [9, Theorem 1.1].

Theorem 4.3. *Let $\mu, v \in \mathbb{R}$, $\alpha \ge -\frac{1}{2}$ and $1 \le p \le q \le \infty$. The operator $\mathscr{L}^\alpha_{v,\mu}$ is bounded from $L^p(0,\infty)$ to $L^q(0,\infty)$ if and only if*

$$\mu = 1 - \frac{1}{p} - \frac{1}{q}, \quad \text{and} \quad -\alpha - 1 + \frac{1}{p} < v \le \frac{1}{2} - \max\{\mu, 0\}. \tag{4.11}$$

In view of the assumptions from the previous result, we have $1 \le p \le q$ and

$$\frac{1}{p} + \frac{1}{q} = 1 - \mu.$$

Therefore, the conditions in Theorem 4.3 impose a range for μ. That is, μ fulfills $-1 \le \mu \le 1$. Furthermore, for any fixed μ, we have

$$\frac{1}{p} \le \frac{1}{p} + \frac{1}{q} = 1 - \mu.$$

Since $1 \le p \le q$, we also have

$$\frac{2}{p} \ge \frac{1}{p} + \frac{1}{q} = 1 - \mu.$$

Therefore, the conditions in Theorem 4.3 show that μ and p satisfy

$$\frac{1}{1-\mu} \le p \le \frac{2}{1-\mu}. \tag{4.12}$$

We are now ready to obtain the modular inequalities for the family of operators $\{\mathscr{L}^{\alpha}_{v,\mu}\}$.

Theorem 4.4. *Let $-1 < \mu < 1$, $\alpha \ge -\frac{1}{2}$ and Φ be a modular function on $(0, \infty)$. Let $v \in \mathbb{R}$ satisfy $v \le \frac{1}{2} - \max\{\mu, 0\}$.*

Suppose that there exist $\frac{1}{1-\mu} \le q_0 < p_0 \le \frac{2}{1-\mu}$ such that $-\alpha - 1 + \frac{1}{q_0} < v$. If $\Phi \in \triangle_2$, Φ_{q_0} satisfies

$$t \int_0^t \frac{\Phi_{q_0}(y)}{y^2} dy \le H \Phi_{q_0}(Kt), \quad \forall t > 0 \tag{4.13}$$

for some constants $H, K > 0$ and Φ_{p_0} satisfies

$$t \int_t^{\infty} \frac{\Phi_{p_0}(y)}{y^2} dy \le B \Phi_{p_0}(t), \quad \forall t > 0 \tag{4.14}$$

for some $B > 0$. Then, there exists constant $C > 0$ such that

$$\int_0^{\infty} \Phi(s^{1-\mu}(\mathscr{L}^{\alpha}_{v,\mu} f)^*(s)) \frac{ds}{s^2} \le C \int_0^{\infty} \Phi(f(x))\, dx. \tag{4.15}$$

Proof. Let p_1, q_1 satisfy

$$\frac{1}{p_1} = 1 - \frac{1}{p_0} - \mu, \quad \text{and} \quad \frac{1}{q_1} = 1 - \frac{1}{q_0} - \mu. \tag{4.16}$$

Notice that $p_1 < q_1$.

Since $\frac{1-\mu}{2} < \frac{1}{p_0}$, we find that $\frac{2}{p_0} > 1 - \mu$. Thus,

$$\frac{1}{p_0} > 1 - \frac{1}{p_0} - \mu = \frac{1}{p_1}.$$

That is, $p_0 < p_1$. Similar, as $\frac{1-\mu}{2} < \frac{1}{q_0}$, we also have

$$\frac{1}{q_0} > 1 - \frac{1}{q_0} - \mu = \frac{1}{q_1}.$$

That is, $q_0 < q_1$.

Theorem 4.3 guarantees the boundedness of $\mathscr{L}^{\alpha}_{\nu,\mu} : L^{q_0}(0,\infty) \to L^{q_1}(0,\infty)$ and $\mathscr{L}^{\alpha}_{\nu,\mu} : L^{p_0}(0,\infty) \to L^{p_1}(0,\infty)$.

Let $\frac{1}{\theta} = \frac{1}{q_0} - \frac{1}{p_0}$ and $r = q_0$. Theorem 3.2 yields that

$$\text{(4.17)} \qquad D_1 \int_0^{\infty} \Phi(f(x))\,dx \le \int_0^{\infty} \Phi(t^{-\frac{1}{r}} K(f, t^{\frac{1}{\theta}}, L^{q_0}(0,\infty), L^{p_0}(0,\infty)))\,dt$$
$$\le D_0 \int_0^{\infty} \Phi(f(x))\,dx.$$

for some $D_1, D_0 > 0$.

Next, we find that

$$K(\mathscr{L}^{\alpha}_{\nu,\mu} f, t, L^{q_1}(0,\infty), L^{p_1}(0,\infty)) = tK(\mathscr{L}^{\alpha}_{\nu,\mu} f, t^{-1}, L^{p_1}(0,\infty), L^{q_1}(0,\infty)).$$

The Holmstedt formulas give

$$K(\mathscr{L}^{\alpha}_{\nu,\mu} f, t, L^{q_1}(0,\infty), L^{p_1}(0,\infty)) \ge Ct\Big(\int_0^{t^{\theta}} ((\mathscr{L}^{\alpha}_{\nu,\mu} f)^*(s))^{p_1}\,ds\Big)^{\frac{1}{p_1}}$$

for some $C > 0$.

As $(\mathscr{L}^{\alpha}_{\nu,\mu} f)^*$ is non-increasing and (4.16) gives

$$-\frac{1}{r} + \frac{1}{\theta} - \frac{1}{p_1} = -\frac{1}{q_0} + \frac{1}{q_0} - \frac{1}{p_0} - \frac{1}{p_1} = -\frac{1}{p_0} - \frac{1}{p_1} = \mu - 1,$$

we find that

$$t^{-\frac{1}{r}} K(\mathscr{L}^{\alpha}_{\nu,\mu} f, t^{\frac{1}{\theta}}, L^{q_1}(0,\infty), L^{p_1}(0,\infty)) \ge Ct^{-\frac{1}{r}+\frac{1}{\theta}}\Big(\int_0^{t^{-1}} ((\mathscr{L}^{\alpha}_{\nu,\mu} f)^*(s))^{p_1}\,ds\Big)^{\frac{1}{p_1}}$$
$$\ge Ct^{\mu-1}(\mathscr{L}^{\alpha}_{\nu,\mu} f)^*(t^{-1})$$

for some $C > 0$.

Therefore,

$$\int_0^{\infty} \Phi(t^{-\frac{1}{r}} K(\mathscr{L}^{\alpha}_{\nu,\mu} f, t^{\frac{1}{\theta}}, L^{q_1}(0,\infty), L^{p_1}(0,\infty)))\,dt$$
$$\ge \int_0^{\infty} \Phi(Ct^{\mu-1}(\mathscr{L}^{\alpha}_{\nu,\mu} f)^*(t^{-1}))\,dt.$$

By using the change of variable $s = t^{-1}$, we obtain

$$\int_0^\infty \Phi(t^{-\frac{1}{r}} K(\mathscr{L}^\alpha_{v,\mu} f, t^{\frac{1}{\theta}}, L^{q_1}(0,\infty), L^{p_1}(0,\infty)))\, dt \geq \int_0^\infty \Phi(Cs^{1-\mu}(\mathscr{L}^\alpha_{v,\mu} f)^*(s))\frac{ds}{s^2}. \tag{4.18}$$

Therefore, Theorem 3.1, (4.17) and (4.18) yield (4.15). □

Particularly, we have the modular inequality for the Fourier–Bessel transform $\tilde{\mathscr{H}}_\alpha = \mathscr{L}^\alpha_{\alpha+1,-2\alpha-1}$. In view of Theorem 4.4, if there exist $\frac{1}{2+2\alpha} \leq q_0 < p_0 \leq \frac{1}{1+\alpha}$, then for any $\Phi \in \triangle_2$ satisfying

$$t\int_0^t \frac{\Phi_{q_0}(y)}{y^2}\, dy \leq H\Phi_{q_0}(Kt), \quad t > 0$$
$$t\int_t^\infty \frac{\Phi_{p_0}(y)}{y^2}\, dy \leq B\Phi_{p_0}(t), \quad t > 0$$

for some $H, K, B > 0$, there exists constant $C > 0$ such that

$$\int_0^\infty \Phi(s^{2+2\alpha}(\tilde{\mathscr{H}}_\alpha f)^*(s))\frac{ds}{s^2} \leq C\int_0^\infty \Phi(f(x))\, dx.$$

4.4. Oscillatory integral operators

Let $a(x, y) \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. We call $\phi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ the phase function associated with $a(x, y)$ if there is an open cone $\Gamma \subset \mathbb{R}^n \backslash \{0\}$ such that $\operatorname{supp}_\theta a \subseteq \Gamma$ and for any $(x, \theta) \in \mathbb{R}^n \times \Gamma$

$$\phi(x, \lambda\theta) = \lambda\phi(x, \theta), \quad \lambda > 0$$

and $d\phi \neq 0$ where $d\phi$ denotes the differential of ϕ with respect to all of the variables, see [24, Section 0.5].

We say that ϕ satisfies the non-degeneracy condition if

$$\det\Big(\frac{\partial^2\phi}{\partial x_j \partial y_k}\Big) \neq 0$$

on the support of $a(x, y)$.

The oscillatory integral operator associated with $a(x, y) \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and $\phi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is given by

$$(T_\lambda f)(x) = \int_{\mathbb{R}^n} e^{i\lambda\phi(x,y)} a(x, y) f(x)\, dy, \quad \lambda > 0.$$

The reader is referred to [24, 25] for the studies and applications of oscillatory integral operators.

We now obtain the modular estimate of oscillatory integral operators.

THEOREM 4.5. *Let* Φ *be a modular function. Suppose that* $\Phi \in \triangle_2$, Φ *satisfies* (4.1) *and* (4.2). *Then, there exists constant* $C > 0$ *such that for any* $\lambda > 0$,

$$\int_0^\infty \Phi(s(T_\lambda f)^*(s))\frac{ds}{s^2} \le C\lambda^n \int_{\mathbb{R}^n} \Phi(\lambda^{-n} f(x))\, dx. \tag{4.19}$$

PROOF. In view of [24, Theorem 2.1.1], for any $\lambda > 0$, we have

$$\|T_\lambda f\|_{L^2} \le E\lambda^{-n/2}\|f\|_{L^2}$$

for some $E > 0$. Moreover, we also have

$$\|T_\lambda f\|_{L^\infty} \le H\|f\|_{L^1}$$

for some $H > 0$ [24, p. 56].

Let $r = 1$ and $\frac{1}{\theta} = 1 - \frac{1}{2} = \frac{1}{2}$. By applying Theorem 3.1 with $X_0 = L^1$, $X_1 = L^2$, $Y_0 = L^\infty$, $Y_1 = L^2$, $C_0 = E$ and $C_1 = H\lambda^{-n/2}$, we find that

$$\begin{aligned}\int_0^\infty & \Phi(t^{-1}K(Tf, t^{\frac{1}{2}}, L^\infty, L^2))\, dt \\ &\le C\lambda^n \int_0^\infty \Phi(B\lambda^{-n}t^{-1}K(f, t^{\frac{1}{2}}, L^1, L^2))\, dt.\end{aligned} \tag{4.20}$$

for some $B, C > 0$.

Similar to the proof of Theorem 4.1, Theorem 3.2 gives with $p_0 = 1$ and $p_1 = 2$, we obtain $\theta = 2$ and

$$\begin{aligned}D_0 \int_{\mathbb{R}^n} \Phi(\lambda^{-n} f(x))\, dx &\le \int_0^\infty \Phi(\lambda^{-n}t^{-1}K(f, t^{\frac{1}{2}}, L^1, L^2))\, dt \\ &\le D_1 \int_{\mathbb{R}^n} \Phi(\lambda^{-n} f(x))\, dx\end{aligned} \tag{4.21}$$

for some $D_0, D_1 > 0$.

Furthermore, similar to (4.5), we also have

$$\int_0^\infty \Phi(t^{-1}K(T_\lambda f, t^{\frac{1}{2}}, L^\infty, L^2))\, dt \ge \int_0^\infty \Phi(Ds(T_\lambda f)^*(s))\frac{ds}{s^2}. \tag{4.22}$$

Consequently, (4.21) and (4.22) yield (4.19). □

When $\Phi(t) = t^p$, $1 < p < 2$, conditions (4.1) and (4.2) are satisfied and we find that

$$\lambda^n \int_{\mathbb{R}^n} \Phi(B\lambda^{-n} f(x))\, dx = C\lambda^{n(1-p)} \int_{\mathbb{R}^n} |f(x)|^p\, dx$$

for some $B, C > 0$.

Hence, (4.19) becomes

$$\int_0^\infty s^{\frac{p}{p'}}((T_\lambda f)^*(s))^p \frac{ds}{s} = \int_0^\infty s^{p-2}((T_\lambda f)^*(s))^p\, ds \le C\lambda^{n(1-p)} \int_{\mathbb{R}^n} |f(x)|^p\, dx.$$

Consequently, we obtain

$$\|T_\lambda f\|_{L^{p',p}} \le C\lambda^{n\frac{1-p}{p}} \|f\|_{L^p}$$

for some $C > 0$.

In view of the embedding $L^{p',p} \hookrightarrow L^{p'}$, we get

$$\|T_\lambda f\|_{L^{p'}} \le C\lambda^{-\frac{n}{p'}} \|f\|_{L^p}$$

which recovers the L^p estimates of oscillatory integral operators [24, Corollary 2.1.2].

4.5. Fourier restriction

The Fourier restriction theorem plays a significant role in the estimates of solutions of partial differential equations, especially, on the wave equation and the Schrödinger equation. The reader is referred to [26] and [25, Chapter VIII, Sections 5.18–5.20] for details.

As an application of our modular interpolation functor, we now obtain the modular Fourier restriction theorem.

THEOREM 4.6. *Let $M \subset \mathbb{R}^n$ be a compact manifold of dimension $n-1$ whose Gaussian curvature is nonzero everywhere. Let Φ be a modular function. Suppose that $\Phi \in \triangle_2$, Φ satisfies*

$$t \int_0^t \frac{\Phi(y)}{y^2}\, dy \le H\Phi(Kt), \quad \forall t > 0 \tag{4.23}$$

for some constants $H, K > 0$ and Φ_2 satisfies

$$t \int_t^\infty \frac{\Phi_{\frac{2n+2}{n+3}}(y)}{y^2}\, dy \le B\Phi_{\frac{2n+2}{n+3}}(t), \quad \forall t > 0 \tag{4.24}$$

for some $B > 0$. Then, there exists constant $C > 0$ such that

$$\int_0^\infty \Phi(s^{\frac{n+3}{2n-2}}(\mathscr{F}f)^*_M(s))\frac{ds}{s^{\frac{2n}{n-1}}} \le C\int_{\mathbb{R}^n}\Phi(f(x))\,dx \tag{4.25}$$

where $(\mathscr{F}f)^*_M$ denote the non-increasing rearrangement of $\mathscr{F}f$ with respect to the Lebesgue measure on M.

Proof. For any $1 \le p \le \infty$, let $L^p(M)$ be the Lebesgue space on M. In view of [25, Chapter IX, Proposition 2.1], the operators $\mathscr{F} : L^1 \to L^\infty(M)$ and $\mathscr{F} : L^{\frac{2n+2}{n+3}} \to L^2(M)$ are bounded.

Let $r = 1$ and $\theta = \frac{2n+2}{n-1}$. Theorem 3.2 yields

$$\begin{aligned} C\int_{\mathbb{R}^n}\Phi(f(x))\,dx &\le \int_0^\infty \Phi(t^{-1}K(f, t^{\frac{n-1}{2n+2}}, L^1, L^{\frac{2n+2}{n+3}}))\,dt \\ &\le B\int_{\mathbb{R}^n}\Phi(f(x))\,dx \end{aligned}$$

for some $B, C > 0$.

Next, recall that the Holmstedt formulas [17] give

$$\begin{aligned} K(\mathscr{F}f, t^{\frac{n-1}{2n+2}}, L^\infty(M), L^2(M)) &= t^{\frac{n-1}{2n+2}}K(\mathscr{F}f, t^{-\frac{n-1}{2n+2}}, L^2(M), L^\infty(M)) \\ &\approx t^{\frac{n-1}{2n+2}}\left(\int_0^{t^{-\frac{n-1}{n+1}}}((\mathscr{F}f)^*_M(s))^2\,ds\right)^{\frac{1}{2}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_0^\infty \Phi(t^{-1}K(\mathscr{F}f, t^{\frac{n-1}{2n+2}}, L^\infty(M), L^2(M)))\,dt &\ge \int_0^\infty \Phi(Dt^{-\frac{n+3}{2n+2}}(\mathscr{F}f)^*_M(t^{-\frac{n-1}{n+1}}))\,dt \\ &\ge E\int_0^\infty \Phi(t^{-\frac{n+3}{2n+2}}(\mathscr{F}f)^*_M(t^{-\frac{n-1}{n+1}}))\,dt \end{aligned}$$

for some constants $E, D > 0$ because $\Phi \in \triangle_2$.

By using the change of variable $s = t^{-\frac{n-1}{n+1}}$, Theorem 3.1 gives (4.25). □

Acknowledgments. The author would like to thank the reviewer for his/her valuable suggestions which enhance the exposition of this paper.

References

[1] C. Bennett - R. Sharpley, *Interpolation of Operators*, Academic Press, 1988.

[2] B. Bongioanni, *Modular inequalities of maximal operators in Orlicz spaces*, Rev. Un. Mat. Argentina 44 no. 2 (2003), 31–47.

[3] M. Carro - H. Heinig, *Modular inequalities for the Calderón operator*, Tohoku Math. J. 52 (2000), 31–46.

[4] M. Carro - L. Nikolova, *Some extensions of the Marcinkiewicz interpolation theorem in terms of modular inequalities*, J. Math. Soc. Japan 55 (2003), 385–394.

[5] A. Cianchi, *Hardy inequalities in Orlicz spaces*, Trans. Amer. Math. Soc. 351 (1999), 2459–2478.

[6] L. Colzani - A. Crespi - G. Travaglini - M. Vignati, *Equiconvergence theorems for Fourier–Bessel expansions with applications to the harmonic analysis of radical functions in Euclidean and non-Euclidean spaces*, Trans. Amer. Math. Soc. 338 (1975), 43–55.

[7] D. Cruz-Uribe - J. Martell - C. Pérez, *Weights, Extrapolation and the Theory of Rubio de Francia*, Operator Theory: Advance and Applications, Volume 215, Birkhäuser Basel, (2011).

[8] G. Curbera - J. García-Cuerva - J. Martell - C. Pérez, *Extrapolation with weights, rearrangement-invariant function spaces, modular inequalities and applications to singular integrals*, Adv. Math., 203 (2006), 256–318.

[9] L. De Carli, *On the $L^p - L^q$ norm of the Hankel transform and related operators*, J. Math. Anal. Appl. 348 (2008), 366–382.

[10] G. Folland, *Real Analysis: Modern Techniques and Their Applications*, John Wiley & Sons, U.S.A., 1984.

[11] H. Hankel, *Die Fourier'schen reihen und integrale der cylinderfunctionen*, Math. Ann. 8 (1875).

[12] K.-P. Ho, *Vector-valued maximal inequalities on weighted Orlicz-Morrey spaces*, Tokyo J. Math. 36 (2013), 499–512.

[13] K.-P. Ho, *Fourier integrals and Sobolev embedding on rearrangement-invariant quasi-Banach function spaces*, Ann. Acad. Sci. Fenn. Math. 41 (2016), 897–922.

[14] K.-P. Ho, *Hardy's inequalities on Hardy spaces*, Proc. Japan Acad. Ser. A Math. Sci. 92 (2016), 125–130.

[15] K.-P. Ho, *Linear operators*, Fourier integral operators and k-plane transforms on rearrangement-invariant quasi-Banach function spaces (preprint).

[16] K.-P. Ho, *Fourier type transforms on rearrangement-invariant quasi-Banach function spaces* (preprint).

[17] T. Holmstedt, *Interpolation of quasi-normed spaces*, Math. Scand. 26 (1970), 177–199.

[18] M. Jodeit - A. Torchinsky, *Inequalities for Fourier transforms*, Studia Math. 37 (1971), 245–276.

[19] V. Kokilashvili - M. Krbec, *Weighted inequalities in Lorentz and Orlicz spaces*, World Scientific, Singapore, 1991.

[20] M. Krbec, *Modular interpolation spaces. I*, Z. Anal. Anwendungen 1 (1982), 25–40.

[21] T. Miyamoto, *On some interpolation theorems of quasilinear operators*, Math. Japonica 42 (1995), 545–556.

[22] J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Mathematics, Springer-Verlag Berlin Heidelberg, 1983.

[23] M. Rao - Z. Ren, *Theory of Orlicz Spaces*, Dekker, New York, 1991.

[24] C. Sogge, *Fourier integrals in classical analysis*, Cambridge University Press, 1993.

[25] E. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, 1993.

[26] R. Strichartz, *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations*, Duke Math. J. 44 (1977), 705–713.

[27] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, Amsterdam, New York, Oxford, North-Holland, (1978).

Received 27 September 2016,
and in revised form 12 December 2016.

Kwok-Pun Ho
Department of Mathematics and Information Technology
The Education University of Hong Kong
10 Lo Ping Road, Tai Po
Hong Kong, China
vkpho@eduhk.hk