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Calculus of Variations — *On the approximation of SBV functions*, by Guido de Philippis, Nicola Fusco and Aldo Pratelli, communicated on January 13, 2017.

This paper is dedicated to the memory of Professor Ennio De Giorgi.

Abstract. — In this paper we deal with the approximation of SBV functions in the strong BV topology. In particular, we provide three approximation results. The first one, Theorem A, concerns general SBV functions; the second one, Theorem B, concerns SBV functions with absolutely continuous part of the gradient in L^p, $p > 1$; and the third one, Theorem C, concerns SBV^p functions, that is, those SBV functions for which not only the absolutely continuous part of the gradient is in L^p, but also the jump set has finite $\mathscr{H}^{N-1}$-measure. The last result generalizes the previously known approximation theorems for SBV^p functions, see [5, 7]. As we discuss, the first and the third result are sharp. We conclude with a simple application of our results.

Key words: SBV functions, approximation, free discontinuity problems

Mathematics Subject Classification: 26B30

1. Introduction

SBV functions, first introduced in [3], arise as a natural tool in order to study free discontinuity problems, which are a wide class of variational problems appearing, for instance, in image analysis, fracture mechanics and liquid crystals theory. Typical energies involve bulk and surface densities and are often modeled by integral functionals of the form

$$F(u) = \int_\Omega f(x, \nabla u)\, dx + \int_{J_u} g(x, u^+, u^-, \nu_u)\, d\mathscr{H}^{N-1}. \tag{1.1}$$

Here, u is a scalar (or vectorial) function in $\mathrm{SBV}(\Omega)$, ∇u is the absolutely continuous part of its gradient Du, J_u and $u^\pm$ are the jump set and the traces of u on both sides of J_u, and ν_u is the approximate normal to J_u (all the relevant definitions are listed in Section 1.1).

Also in order to study functionals of the above type, it is clearly of primary importance to have compactness and approximation results for SBV functions. This paper deals with the question of the approximation. In the literature, there are two approximation results, quite known, one due to Braides and Chiadò Piat in 1996 (see [5]), and the other by Cortesani and Toader in 1999 (see [7], see also

the weaker result obtained in the earlier paper [9]); they both deal with the SBV^p functions, which are the SBV functions for which ∇u belongs to L^p, and the jump set J_u has finite $\mathscr{H}^{N-1}$ measure, see Section 1.1. Let us summarize the results in the following statement.

THEOREM 1.1 (Braides–Chiadò Piat [5], Cortesani–Toader [7]). *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded set with Lipschitz boundary, let $p > 1$, and let $u \in \mathrm{SBV}^p(\Omega) \cap L^\infty(\Omega)$. Then:*

[1 (Braides–Chiadò Piat)] *There exists a sequence $u_j \in \mathrm{SBV}^p(\Omega)$ such that $\|u_j - u\|_{\mathrm{BV}} \to 0$ and $\nabla u_j \to \nabla u$ in L^p, for every $j \in \mathbb{N}$ it is $\|u_j\|_{L^\infty} \le \|u\|_{L^\infty}$ and $u_j \in \mathrm{C}^1(\Omega \setminus R_j)$, being $R_j \supseteq J_{u_j}$ some closed rectifiable set, and $\mathscr{H}^{N-1}(J_{u_j} \Delta J_u) \to 0$.*

[2 (Cortesani–Toader)] *There exists a sequence $u_j \in \mathrm{SBV}^p(\Omega)$ such that $u_j \to u$ in L^1 and $\nabla u_j \to \nabla u$ in L^p, and for every $j \in \mathbb{N}$ it is $\|u_j\|_{L^\infty} \le \|u\|_{L^\infty}$, J_{u_j} is polyhedral (i.e., the intersection of Ω with a finite union of $(N-1)$-dimensional simplexes), and $u_j \in \mathrm{C}^\infty(\Omega \setminus \overline{J_{u_j}}) \cap W^{1,\infty}(\Omega \setminus \overline{J_{u_j}})$. Moreover,*

$$\limsup_{j\to\infty} \int_{J_{u_j} \cap \bar{\Omega}'} g(x, u_j^+, u_j^-, \nu_{u_j})\, d\mathscr{H}^{N-1} \le \int_{J_u \cap \bar{\Omega}'} g(x, u^+, u^-, \nu_u)\, d\mathscr{H}^{N-1} \tag{1.2}$$

for every open set $\Omega' \subset\subset \Omega$ and every u.s.c. function $g : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{S}^{N-1} \to [0, \infty)$ such that $g(x, a, b, \nu) = g(x, b, a, -\nu)$ for all $x \in \Omega$, $a, b \in \mathbb{R}$, $\nu \in \mathbb{S}^{N-1}$.

The above results suffice for many applications, nevertheless they are still not sharp; in fact, roughly speaking, one would like to find a sequence u_j converging to u in the strong BV topology and in L^p of the absolute continuous part of the gradient, with the functions u_j having a regular jump set and being smooth out of the jump set. Instead, in [5] there is no information on the shape of the jump set, while in [7] the BV convergence fails. In addition, both results are valid in SBV^p and not in SBV; this means that one assumes a higher integrability of ∇u and the finiteness of the $\mathscr{H}^{N-1}$-dimensional measure of the jump set. As we will discuss in Section 1.2, there are good practical reasons to do that; moreover, the L^p assumption on ∇u is actually satisfied in many applications (however, in Section 6 we will deal with a situation for which this is not the case). Nevertheless, for some important functionals of the form (1.1) there is no guarantee that minimizers (or at least minimizing sequences) have finite measure of the jump set, and in fact for such functionals a lot is still not known. For these reasons, it appears desirable to have approximation results dealing with completely general SBV functions, or with SBV functions having absolutely continuous part of the gradient in L^p, with no constraint about the measure of the jump set; we will call for brevity SBV^p_∞ the space of such functions (see also Section 1.1).

In this paper, we contribute to give an answer to these questions. More precisely, we present three approximation results, respectively for SBV, SBV^p_∞ and SBV^p functions; the last one generalizes Theorem 1.1. Our results read as follows.

THEOREM A (Approximation in SBV). *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, and let $u \in SBV(\Omega)$. Then, there exists a sequence of functions $u_j \in SBV(\Omega)$ and of compact, C^1, manifolds with (possibly empty) C^1 boundary $M_j \subset\subset \Omega$, such that $J_{u_j} \subseteq M_j \cap J_u$, $\mathscr{H}^{N-1}(\overline{J_{u_j}} \setminus J_{u_j}) = 0$, and*

$$\|u_j - u\|_{BV(\Omega)} \to 0, \quad u_j \in C^\infty(\Omega \setminus \overline{J_{u_j}}).$$

THEOREM B (Approximation in SBV^p_∞). *Let $\Omega \subseteq \mathbb{R}^N$ be a local extension domain (see Definition 4.4), and let $u \in SBV^p_\infty(\Omega)$. Then, there exists a sequence of functions $u_j \in SBV^p(\Omega)$ and of compact, C^1 manifolds with (possibly empty) C^1 boundary $M_j \subset\subset \Omega$, such that $J_{u_j} \subseteq M_j$ but $\mathscr{H}^{N-1}(M_j \setminus J_{u_j}) = 0$ and*

$$\|u_j - u\|_{BV(\Omega)} \to 0, \quad u_j \in C^\infty(\Omega \setminus \overline{J_{u_j}}), \quad \nabla u_j \xrightarrow[L^p(\Omega)]{} \nabla u. \tag{1.3}$$

THEOREM C (Approximation in SBV^p). *Let $\Omega \subseteq \mathbb{R}^N$ be an open set with locally Lipschitz boundary, and let $u \in SBV^p(\Omega)$. Then, there exists a sequence of functions $u_j \in SBV^p(\Omega)$ and of compact, C^1, manifolds with (possibly empty) C^1 boundary $M_j \subset\subset \Omega$, such that $J_{u_j} \subseteq M_j$ but $\mathscr{H}^{N-1}(M_j \setminus J_{u_j}) = 0$ and*

$$\begin{aligned} &\|u_j - u\|_{BV(\Omega)} \to 0, \quad u_j \in C^\infty(\Omega \setminus \overline{J_{u_j}}), \\ &\nabla u_j \xrightarrow[L^p(\Omega)]{} \nabla u, \quad \mathscr{H}^{N-1}(J_{u_j} \Delta J_u) \to 0. \end{aligned} \tag{1.4}$$

Notice that the only difference between the approximations given for SBV^p and for SBV^p_∞ functions consists in the validity of the convergence $\mathscr{H}^{N-1}(J_{u_j} \setminus J_u) \to 0$ (actually, the convergence of $\mathscr{H}^{N-1}(J_u \setminus J_{u_j})$ to 0 in Theorem C is a direct consequence of the BV convergence). However, as we will discuss in Section 1.2, this is a substantial difference, and it is precisely the lack of this convergence making Theorem B still not sharp. We remark that, in Theorems B and C, since the jump sets J_{u_j} are contained in the compact, C^1, $(N-1)$-dimensional manifolds M_j, then in particular they are essentially closed, that is, $\mathscr{H}^{N-1}(\overline{J_{u_j}} \setminus J_{u_j}) = 0$.

Let us note also that, in Theorem A, one can decide that the jump sets J_{u_j} of the functions u_j coincide $\mathscr{H}^{N-1}$-a.e. with the C^1 manifolds M_j, but in this case it is no more true that they are contained in J_u. Moreover, in Theorems B and C, one can remove the assumption of Ω to be locally an extension domain, or a set with Lipschitz boundary, but then the L^p convergences in (1.3) and (1.4) become L^p_{loc} convergences, see Theorem 4.7. We underline that our Definition 4.4 of extension domains is even weaker than the usual one, we only require $W^{1,p}(\Omega)$ to be dense in $W^{1,1}(\Omega)$. In our three results we do not need to assume that u is bounded; however, if $u \in L^\infty(\Omega)$, then we can always assume that

$\|u_j\|_{L^\infty} \le \|u\|_{L^\infty}$ for every $j \in \mathbb{N}$: this is an immediate consequence of Lemma 3.2. We remark that an approximation result for SBV functions, similar to our Theorem A, was also proved in [13]. Finally, through the paper we consider for simplicity of notations the case of scalar functions; however, the case of vector-valued functions is identical.

As an immediate application of the approximation result in SBV, we will consider in Section 6 a representation formula for the total variation recently obtained in [11] for functions $u \in \mathrm{SBV}(\Omega)$ for which $\mathscr{L}^N(\overline{J_u}) = 0$, and we show that the same formula still holds in general, with no additional assumptions on the jump set. The case of a general function in $\mathrm{BV}(\Omega)$ with non trivial Cantor part is also discussed.

1.1. Definitions and notations. Here we briefly give all the definitions and notations used in this paper, most of which are standard: one can refer for instance to the book [4] for a complete account of the subject. Given an open set $\Omega \subseteq \mathbb{R}^N$, the space of the *functions of bounded variation* is given by the set $\mathrm{BV}(\Omega)$ of all the L^1 functions over Ω whose distributional derivative Du is a finite Radon measure. For any function $u \in \mathrm{BV}(\Omega)$, one denotes by $\nabla u \in L^1(\Omega)$ the absolutely continuous part (with respect to the Lebesgue measure) of Du, and $D^s u$ the singular part. Hence, $Du = \nabla u \mathscr{L}^N + D^s u$, and $u \in W^{1,1}(\Omega)$ if and only if $D^s u = 0$. The measure $D^s u$ does not charge $\mathscr{H}^{N-1}$-negligible sets; moreover, one further decomposes $D^s u = D^j u + D^c u$, where $D^j u$ is called *jump part* and $D^c u$ *Cantor part*. While the Cantor part $D^c u$ does not charge $\mathscr{H}^{N-1}$-finite sets, the jump part $D^j u$ is concentrated on a $(N-1)$-dimensional set J_u, called the *jump set*, which is countably rectifiable: this means that there exist countably many sets M_i, $i \in \mathbb{N}$, each one being a C^1 image of the unit ball of $\mathbb{R}^{N-1}$, so that $\mathscr{H}^{N-1}(J_u \setminus \bigcup_{i\in\mathbb{N}} M_i) = 0$. In addition, for every point $x \in J_u$, there exist a direction $\nu_u = \nu_u(x) \in \mathbb{S}^{N-1}$, and two numbers $u^+ = u^+(x) \neq u^- = u^-(x)$, such that

$$\lim_{r\to 0} \fint_{B^+_{\nu_u}(x,r)} |u(y) - u^+|\, dy = \lim_{r\to 0} \fint_{B^-_{\nu_u}(x,r)} |u(y) - u^-|\, dy = 0,$$

where $B^\pm_\nu(x,r)$ are the two half-balls defined by

$$B^\pm_\nu(x,r) = \{y \in \mathbb{R}^N : |y-x| < r, (y-x)\cdot \nu \gtrless 0\}.$$

Moreover, one has $D^j u = (u^+ - u^-)\mathscr{H}^{N-1} \llcorner J_u$: this explains why this part of the derivative is called "jump part". In particular, the strictly positive quantity $|u^+(x) - u^-(x)|$ is called "jump". We recall that a sequence $\{u_j\} \subset \mathrm{BV}(\Omega)$ converges *strictly* to u if

$$\|u_j - u\|_{L^1(\Omega)} + \big|\, |D_j u|(\Omega) - |Du|(\Omega)\big| \to 0.$$

Note that this also trivially implies that $Du_j \overset{*}{\rightharpoonup} Du_j$. We will say that a sequence $\{u_j\} \subset \mathrm{BV}(\Omega)$ converges to u in the BV sense if it converges in the strong

norm topology:

$$\|u_j - u\|_{BV(\Omega)} \to 0.$$

The space $\mathrm{SBV}(\Omega)$ of the *special functions of bounded variation* is given by the set of all BV functions u for which the Cantor part $D^c u$ of the derivative vanishes, thus $Du = \nabla u \mathscr{L}^N + D^j u$. Despite the elementary definition, this space is extremely important, since it is the natural space in which functions live in several applications. It is important to notice that $\mathrm{SBV}(\Omega)$ is not a closed subspace of $\mathrm{BV}(\Omega)$ in the strict topology, because the strict limit of a bounded sequence of SBV functions can have a non-trivial Cantor part in the derivative, which can arise both from the absolutely continuous part and from the jump part of the derivatives. Also for this reason, in many applications one considers the space $\mathrm{SBV}^p(\Omega)$, see for instance [5, 6, 7]: given some $p > 1$, the space $\mathrm{SBV}^p(\Omega)$ is defined as the space of the SBV functions u for which the quantity $\|u\|_{\mathrm{BV}} + \|\nabla u\|_{L^p} + \mathscr{H}^{N-1}(J_u) < \infty$ is finite. As an immediate consequence of the well-known compactness Theorem for SBV functions (see [4, Theorem 4.8]), one obtains that limits of sequences in SBV^p for which the above quantity is uniformly bounded remain in SBV^p. Basically, the higher integrability of the absolutely continuous parts of the gradients prevents them to create Cantor part in the limit, while the boundedness of the measures of the jump sets prevents the jump parts to create Cantor part in the limit.

For reasons that will be discussed in the next section, we will also be interested in an intermediate space between SBV and SBV^p, that is, the space of SBV functions u for which the higher integrability $\nabla u \in L^p$ holds, but no constraint on the measure of J_u is assumed. Through this paper, we will denote by SBV^p_∞ this space. Notice that, as discussed above, this is *not* a closed subspace of SBV in the strict topology.

1.2. A brief discussion of our results and a comparison with Theorem 1.1. In this section we make a general discussion about the approximating issue in SBV, and then we comment our three results, and we compare them with Theorem 1.1.

First of all, let us consider a function $u \in \mathrm{SBV}$: the best approximation that one can hope to get, is to write u as a BV limit of SBV functions u_j, each of them having a "nice" jump set J_{u_j} and being smooth outside of $\Omega \backslash \overline{J_{u_j}}$. Notice that the BV convergence of u_j to u immediately implies that $\mathscr{H}^{N-1}(J_u \backslash J_{u_j})$ converges to 0 as soon as J_u has finite measure (otherwise it is of course infinite for every j, since $\mathscr{H}^{N-1}(J_{u_j})$ is finite). On the other hand, it could be in principle possible that $\mathscr{H}^{N-1}(J_{u_j} \backslash J_u)$ does not converge to 0, and this quantity could even blow up: it is enough that the functions u_j have a very large part of the jump set where the jump $|u^+ - u^-|$ is very small. With this considerations at hand, Theorem A appears completely satisfactory; in fact, not only we have that $\mathscr{H}^{N-1}(J_{u_j} \backslash J_u)$ converges to 0, but also that J_{u_j} is a subset of J_u.

As discussed above, not many applications use the space SBV, which is a non-closed subspace of BV (even though, we consider an application in Section 6). In

order to roughly understand the reason, let us consider again a functional as in (1.1); to keep the discussion simple, we restrict ourselves to the particular case (still very general) of a Mumford-Shah-like functional of the form

$$F(u) = \int_\Omega |\nabla u|^p + \int_{J_u} g(|u^+(x) - u^-(x)|),$$

where $p > 1$ and g is a positive, increasing, l.s.c. function. When studying the problem of minimizing this functional in SBV (under suitable assumptions), it is of course not restrictive to consider BV functions for which ∇u belongs to L^p, hence the functions belonging to the space that we denote by SBV^p_∞. On the other hand, depending on the function g, it is not obvious whether or not it is restrictive to assume also that the measure of the jump set is finite, that is, to consider functions in the space SBV^p. This is of course not a problem for the original Mumford-Shah case, corresponding to $g \equiv 1$, or more in general for functions for which $\lim_{t\to 0^+} g(t) > 0$, because in this case any function with finite energy belongs to SBV^p. Otherwise, for instance for the important case when $g(t) = t^q$ with some $q > 0$, restricting oneself to the space SBV^p might change the minimizers; and actually, the fact that the space SBV^p_∞ is not closed in BV (while so is SBV^p, as said) is the main reason why much less is known for functionals of this last type. For instance, it is not clear if, for these functionals, the minimizers (if any) should belong to SBV^p or not. This clarifies the need of an approximation result for the space SBV^p_∞, and we give a partial answer in the present paper with Theorem B: as far as we know, this is the first approximation result for SBV functions with higher integrability of ∇u but without any constraint on the measure of the jump set. Unlike Theorem A, one can still not say that our result is completely satisfactory. Indeed, in our result we get an approximating sequence which converges in the BV sense and in the L^p sense of the absolutely continuous parts of the gradients, and which is done by functions which have the nicest possible jump set, and which are smooth outside. However, the information that $\mathscr{H}^{N-1}(J_{u_j} \setminus J_u) \to 0$ is missing, and this can create troubles in some cases. To understand that, consider once again the case of $g(t) = t^q$: if $q < 1$, then the convergence of u_j to u provided by our Theorem B does not imply that $F(u_j) \to F(u)$, and this is of course unsatisfactory. Notice that, instead, the convergence of $F(u_j)$ to $F(u)$ is an immediate consequence of the BV convergence if $q \geq 1$ (at least when the functions u_j are equi-bounded, as one usually has in the applications): for functionals of this type, then, the claim of our result seems to be enough for the applications.

Let us finally consider the case of the SBV^p functions. As discussed above, not for all functionals this is the "right" space to consider. However, our Theorem C seems again to be completely satisfactory, since we obtain also the convergence missing in Theorem B, compare (1.4) with (1.3).

To conclude, we can make a quick comparison between our results and those of Theorem 1.1. As already said, for several applications the results of Theorem 1.1 are enough; nevertheless, in [5] there is no information about the possible shape of the jump sets of the functions u_j, except the fact that they are con-

tained in a closed rectifiable set; analogously, in [7] the strong BV convergence fails. Notice that, since the jump sets of the approximating functions in [7] are polyhedral (i.e., a finite union of $(N-1)$-dimensional simplexes), hence in general disjoint from the jump set of u, then of course there is no possibility to have strong BV convergence in that result. We also underline that, in our result, the jump set is a compact C^1 manifold: hence, it is the *disjoint* union of finitely many C^1 images of $(N-1)$-dimensional simplexes; obtaining the disjointness, which is not ensured by the result of [7], requires some care, and it is done in Lemma 5.2. A last comment can be done about the strategy of the proof. In [5, 7] the authors use the well-known existence and regularity results for the Mumford-Shah functional, see for instance [8]. Our strategy is instead quite different; more precisely, given a function $u \in SBV(\Omega)$, we single out a compact subset K of the jump set J_u contained in a C^1 manifold, and we construct a smooth function in $\Omega \backslash K$ having the same upper and lower traces of u on K by means of a simple mollification argument with variable kernel; this is enough to conclude in the case of SBV functions, Theorem A, while a careful modification is needed to treat the cases of SBV^p_∞ or SBV^p functions, in order to get also the L^p convergence of the ∇u_j.

2. Mollification with variable kernel

In our construction to prove Theorem A we will make use of a mollification with a variable kernel. Even though this is a well established technique, in this section we collect the relevant definitions and the properties that we are going to need, in order to keep our presentation self-contained.

Through this section, we will consider a given compact set $K \subset\subset \Omega$, and we will write $D = K \cup \partial\Omega$. Then, we arbitrarily fix a "regularized distance function" from D, that is, a function $\delta : \Omega \to \mathbb{R}$ such that

$$\|D\delta\|_{L^\infty} \leq 1, \quad \frac{\operatorname{dist}(x,D)}{2} \leq \delta(x) \leq \operatorname{dist}(x,D) \quad \forall x \in \Omega,$$

and that $\delta \in C^\infty(\Omega \backslash D)$. Moreover, we also take a function $f \in C^\infty([0,\infty))$ satisfying

$$\begin{aligned} &f^{(j)}(0) = 0 \ \forall j \in \mathbb{N}, \quad 0 < f(t) \leq 1 \ \forall t \in (0,+\infty), \\ &0 \leq f'(t) \leq 1 \ \forall t \in [0,+\infty). \end{aligned} \tag{2.1}$$

Given a number $0 < \sigma < 1$ and a vector $y \in B(1)$, we define the "generalized translation" as the function

$$T_{\sigma,y}(x) = x - \sigma f(\delta(x))\, y.$$

Here, and in the following, we denote by $B(x,r)$ the ball with center x and radius $r > 0$, and we simply write $B(r)$ in place of $B(0,r)$. Notice that, by the properties

(2.1) and the choice of $0 < \sigma < 1$, one has $T_{\sigma,y} : \Omega \to \Omega$, and $T_{\sigma,y}$ is the identity on D. Since

$$DT_{\sigma,y}(x) = \mathrm{Id} - \sigma f'(\delta(x))\, y \otimes D\delta(x),$$

(observe that $f'(\delta(x))D\delta(x)$ is continuous on the whole Ω by construction), keeping in mind that $\det(\mathrm{Id} + a \otimes b) = 1 + a \cdot b$ and recalling again (2.1) and the fact that $\sigma < 1$ and that $|y| < 1$ we obtain

$$\det DT_{\sigma,y}(x) = 1 - \sigma f'(\delta(x))\, y \cdot D\delta(x) \geq 1 - \sigma > 0. \tag{2.2}$$

In particular $T_{\sigma,y}$ is a local diffeomorphism, and since a quick look at the definition ensures that it is a bijection from Ω onto itself, it is also a global diffeomorphism. Finally, we fix a smooth positive function $\rho \in \mathrm{C}_c^\infty(B(1))$, such that $\int_{B(1)} \rho = 1$. We are then ready to give the definition of the mollification with variable kernel, for a L^1_{loc} function and for a Radon measure. Notice that both definitions reduce to the standard mollification if $T_{\sigma,y}$ is replaced by the standard translation $T_y(x) = x - y$.

DEFINITION 2.1. Let f, σ and ρ as above. For any $u \in L^1_{\mathrm{loc}}(\Omega)$ we define

$$u_\sigma(x) = \int_{B(1)} u(T_{\sigma,y}(x))\rho(y)\, dy = \int_{B(1)} u(x - \sigma f(\delta(x))\, y)\rho(y)\, dy.$$

Instead, for any Radon measure $\mu \in \mathscr{M}(\Omega)$, we let $\mu_\sigma \in \mathscr{M}(\Omega)$ be the unique measure such that

$$\int_\Omega \varphi(x)\, d\mu_\sigma(x) = \int_{B(1)} \Big(\int_\Omega \varphi(T^{-1}_{\sigma,y}(z)) \det(DT^{-1}_{\sigma,y}(z))\, d\mu(z) \Big) \rho(y)\, dy \quad \text{for all } \varphi \in \mathrm{C}_c(\Omega),$$

that is,

$$\mu_\sigma = \int_{B(1)} (T^{-1}_{\sigma,y})_\# [\det(DT^{-1}_{\sigma,y})\mu] \rho(y)\, dy.$$

It is very simple to deduce from the definition that, if $\mu = u\, d\mathscr{L}^N$, then $\mu_\sigma = u_\sigma\, d\mathscr{L}^N$, as well as that $\mu_\sigma = \mu$ if the measure μ is concentrated on K; moreover, if $\mu_j \overset{*}{\rightharpoonup} \mu$ then $(\mu_j)_\sigma \overset{*}{\rightharpoonup} \mu_\sigma$. Before proving the main properties of u_σ, we need to make a simple observation about the density (in the strict sense) of smooth functions in BV.

LEMMA 2.2. *Let $u \in \mathrm{BV}(\Omega)$ be such that $D^s u$ is concentrated on a compact set $K \subset\subset \Omega$ and ∇u belongs to $L^p(\Omega)$ for some $1 \leq p < \infty$. Then, there exists a*

sequence of smooth functions $u_j : \Omega \to \mathbb{R}$ such that $u_j \to u$ strictly, and for every $\varepsilon > 0$ one has that $Du_j \to Du$ strongly in $L^p(\Omega\backslash K_\varepsilon)$, being K_ε the ε-neighborhood of K.

PROOF. First of all, assume that $\Omega = \mathbb{R}^N$. In this case, it is immediate to observe that the sequence $u * \rho_{1/j}$ is as needed, where $\rho_{1/j}$ is a standard smooth kernel concentrated in the ball of radius $1/j$.

Let us now consider the general situation of an open set Ω. Let A_1 and A_2 be two open sets such that $K \subset\subset A_1 \subset\subset A_2 \subset\subset \Omega$. By means of a smooth cut-off function, we can write $u = u_1 + u_2$, being $u_1 \in \mathrm{BV}(\Omega)$ supported in $\overline{A_2}$, with $\nabla u_1 \in L^p(\Omega)$, while $u_2 \in W^{1,p}(\Omega)$ is supported in $\Omega\backslash A_1$. By Meyers and Serrin Theorem, we can take a sequence $u_{2,j}$ of smooth functions converging to u_2 strongly in $W^{1,p}(\Omega)$. Instead, concerning u_1, we can extend it by 0 outside of A_2, getting a function in $\mathrm{BV}(\mathbb{R}^N)$, and then we find the sequence $u_{1,j}$ with a convolution as before. If we now let $u_j = u_{1,j} + u_{2,j}$, this sequence is clearly as requested, since in the set $A_2\backslash\overline{A_1}$ both the convergences of $u_{1,j}$ and $u_{2,j}$ to u_1 and u_2 are strong in $W^{1,p}$. □

PROPOSITION 2.3. *Let $1 \le p < \infty$, $u \in L^p(\Omega)$, μ be a Radon measure, and let u_σ and μ_σ be as in Definition* 2.1 *for some $0 < \sigma \le 1/2$. Then*

(i) $u_\sigma \in C^\infty(\Omega\backslash K)$.

(ii) *The following estimates hold:*

$$\|u_\sigma\|_{L^p(\Omega)} \le 2\|u\|_{L^p(\Omega)}, \quad |\mu_\sigma|(\Omega) \le 2|\mu|(\Omega). \tag{2.3}$$

In particular, the map $u \mapsto u_\sigma$ is linear and continuous in L^p.

(iii) *One has $\|u_\sigma - u\|_{L^p(\Omega)} \to 0$ as $\sigma \to 0$ and, if $u \in \mathrm{C}(\Omega)$, then $\|u_\sigma - u\|_{L^\infty(\Omega')} \to 0$ as $\sigma \to 0$ for every $\Omega' \subset\subset \Omega$.*

(iv) *If $u \in BV(\Omega)$, then $u_\sigma \in BV(\Omega)$ and*

$$Du_\sigma = (Du)_\sigma + \sigma\xi^\sigma, \tag{2.4}$$

where ξ^σ is a Radon measure such that $\xi^\sigma \llcorner K = 0$ and $|\xi^\sigma|(\Omega) \le 2|Du|(\Omega)$. Moreover,

$$Du_\sigma \llcorner K = Du \llcorner K. \tag{2.5}$$

Finally, if $\nabla u \in L^p(\Omega)$ and J_u is contained in K, then

$$\nabla u_\sigma \xrightarrow[L^p(\Omega)]{} \nabla u.$$

PROOF. Point (i) follows from the fact that for $x \in \Omega\backslash K$

$$u_\sigma(x) = \frac{1}{(\sigma f(\delta(x)))^N} \int_{B(x,\sigma f(\delta(x)))} u(z)\rho\Big(\frac{x-z}{\sigma f(\delta(x))}\Big)\,dz,$$

and by the smoothness of f on $\mathbb{R}^+$, ρ on $B(1)$, and δ in $\Omega\backslash K$.

To prove point (ii), we start with an L^p function u. By Jensen inequality, Fubini Theorem and the change of variable $z = T_{\sigma,y}(x)$, also keeping in mind that $T_{\sigma,y}(\Omega) = \Omega$, we have

$$\begin{aligned}\int_\Omega |u_\sigma(x)|^p\,dx &\le \int_\Omega \int_{B(1)} |u(T_{\sigma,y}(x))|^p \rho(y)\,dy\,dx \\ &= \int_{B(1)} \Big(\int_\Omega |u(T_{\sigma,y}(x))|^p\,dx \Big) \rho(y)\,dy \\ &= \int_{B(1)} \int_\Omega |u(z)|^p \det DT^{-1}_{\sigma,y}(z)\,dz \rho(y)\,dy \\ &\le \|\det DT^{-1}_{\sigma,y}\|_{L^\infty(\Omega)} \|u\|^p_{L^p(\Omega)}.\end{aligned} \tag{2.6}$$

Since by (2.2) for $\sigma \le 1/2$ we have $\|\det DT^{-1}_{\sigma,y}\|_{L^\infty(\Omega)} \le (1-\sigma)^{-1} \le 2$, inequality (2.3) for an L^p function follows from (2.6). More in general, calling K_ε the ε-neighborhood of K, it is clear by construction that $T_{\sigma,y}(x)$ might belong to $K_{\varepsilon/2}$ only if x belongs to K_ε, thus we also have that

$$\int_{\Omega\setminus K_\varepsilon} |u_\sigma(x)|^p \le 2\|u\|^p_{L^p(\Omega\setminus K_{\varepsilon/2})}. \tag{2.7}$$

Let us now consider a Radon measure μ, and let u_j be a sequence of L^1 functions such that $u_j\,d\mathscr{L}^N \overset{*}{\rightharpoonup} \mu$. As noticed above, we get that

$$(u_j)_\sigma\,d\mathscr{L}^N = (u_j\,d\mathscr{L}^N)_\sigma \overset{*}{\rightharpoonup} \mu_\sigma,$$

thus the estimate (2.3) for the measure μ follows from the same estimate for the functions u_j.

Concerning point (iii), we note that the second part is an immediate consequence of the uniform continuity of u on compact sets and of the fact that $\|T_{\sigma,y} - \mathrm{Id}\|_{L^\infty(\Omega)} \to 0$ as $\sigma \to 0$. The fact that $\|u_\sigma - u\|_{L^p(\Omega)} \to 0$, then, follows by the density of $C^0_c(\Omega)$ in $L^p(\Omega)$ and by (2.3) exactly as in the classical case.

Let us now prove point (iv). We first assume that $u \in C^\infty(\Omega)$, then we get

$$Du_\sigma(x) = \int_{B(1)} (Du(T_{\sigma,y}(x)) - \sigma f'(\delta(x)) Du(T_{\sigma,y}(x)) \cdot y D\delta(x)) \rho(y)\,dy,$$

so that (2.4) holds with

$$\xi^\sigma = -f'(\delta(x)) D\delta(x) \int_{B(1)} Du(T_{\sigma,y}(x)) \cdot y \rho(y)\,dy. \tag{2.8}$$

Notice that in the present case, also by (2.1), the measure ξ^σ is actually a smooth function. Moreover, $\xi^\sigma \llcorner K = 0$ because for every $x \in K$ one has $\delta(x) = 0$ and

so $f'(\delta(x)) = 0$, and since $0 \le f' \le 1$ and $\|D\delta\|_{L^\infty} \le 1$, we have

$$|\xi^\sigma(x)| \le \int_{B(1)} |Du(T_{\sigma,y}(x)|\rho(y)\,dy = |Du|_\sigma(x), \tag{2.9}$$

so that applying part (ii) above to the function $|Du|$ we get

$$|\xi^\sigma|(\Omega) \le \| \, |Du|_\sigma \|_{L^1} \le 2\| \, |Du| \, \|_{L^1} = 2|Du|(\Omega).$$

In conclusion, (2.4) of point (iv) holds if u is a smooth function.

Let instead now $u \in \mathrm{BV}(\Omega)$ be a generic function, and let $u_j \in \mathrm{BV}(\Omega) \cap \mathrm{C}^\infty(\Omega)$ be a sequence such that $u_j \overset{*}{\rightharpoonup} u$ in the strict BV sense (with the additional property granted by Lemma 2.2 if $D^s u$ is concentrated on K and $\nabla u \in L^p$). First of all, note that $(u_j)_\sigma \to u_\sigma$ in L^1 by part (ii), hence $D(u_j)_\sigma \to Du_\sigma$ in the sense of distributions. Since for every j we have $D(u_j)_\sigma = (Du_j)_\sigma + \sigma\xi_j^\sigma$ according to (2.4), and since as already noticed $Du_j \overset{*}{\rightharpoonup} Du$ implies $(Du_j)_\sigma \overset{*}{\rightharpoonup} (Du)_\sigma$, we have to check the weak* limit of ξ_j^σ for $j \to \infty$. Let us then take a bounded and continuous function $\varphi \in \mathrm{C}_b(\Omega, \mathbb{R}^N)$, with compact support: applying (2.8) to each smooth function u_j, we get

$$\begin{aligned}
\langle -\xi_j^\sigma, \varphi \rangle &= \int_\Omega f'(\delta(x)) D\delta(x) \cdot \varphi(x) \Big(\int_{B(1)} Du_j(T_{\sigma,y}(x)) \cdot y\rho(y)\,dy \Big) dx \\
&= \int_{B(1)} y \cdot \Big(\int_\Omega Du_j(T_{\sigma,y}(x)) f'(\delta(x)) D\delta(x) \cdot \varphi(x)\,dx \Big) \rho(y)\,dy \\
&= \int_{B(1)} y \cdot \Big(\int_\Omega Du_j(z) f'(\delta(T_{\sigma,y}^{-1}(z))) D\delta(T_{\sigma,y}^{-1}(z)) \\
&\qquad\qquad \cdot \varphi(T_{\sigma,y}^{-1}(z)) \det(DT_{\sigma,y})^{-1}(z)\,dz \Big) \rho(y)\,dy \\
&= \int_{B(1)} y \cdot \Big(\int_\Omega Du_j(z) g(y,z)\,dz \Big) \rho(y)\,dy \\
&= \int_\Omega Du_j(z) \cdot \Big(\int_{B(1)} y g(y,z) \rho(y)\,dy \Big) dz,
\end{aligned}$$

where for any $y \in B(1)$ we have set

$$g(y,z) = f'(\delta(T_{\sigma,y}^{-1}(z))) D\delta(T_{\sigma,y}^{-1}(z)) \cdot \varphi(T_{\sigma,y}^{-1}(z)) \det(DT_{\sigma,y})^{-1}(z). \tag{2.10}$$

By construction, g is a continuous, compactly supported, scalar function, with $\|g\|_{L^\infty} \le 2\|\varphi\|_{L^\infty}$ and of course depending on φ, so we can define

$$h(z) = \int_{B(1)} y g(y,z) \rho(y)\,dy, \tag{2.11}$$

and the calculations above give

$$\langle -\xi_j^\sigma, \varphi\rangle = \int_\Omega Du_j(z)\cdot h(z)\,dz = \langle Du_j, h\rangle \to \langle Du, h\rangle.$$

Since the map $\varphi \mapsto h$ is easily seen to be linear and continuous, we have a measure ξ^σ such that $\xi_j^\sigma \overset{*}{\rightharpoonup} \xi^\sigma$. Summarizing, we have shown that for any $u \in \mathrm{BV}(\Omega)$ there is a measure ξ^σ such that (2.4) holds true, and this also implies that $u_\sigma \in \mathrm{BV}(\Omega)$. Moreover, since $\xi_j^\sigma \overset{*}{\rightharpoonup} \xi^\sigma$, the validity of $|\xi^\sigma|(\Omega) \le 2|Du|(\Omega)$ is straightforward, since we know it for every ξ_j^σ and u_j. In order to prove that $\xi^\sigma \llcorner K = 0$, let us take a function $\varphi \in \mathrm{C}_b(\Omega)$ supported in the ε-neighborhood K_ε of K. Thus, $\varphi(T_{\sigma,y}^{-1}(z)) = 0$ whenever the distance between $T_{\sigma,y}^{-1}(z)$ and K is bigger than ε. On the other hand, if it is smaller, then $\delta(T_{\sigma,y}^{-1}(z)) \le \varepsilon$, and this implies that $f'(\delta(T_{\sigma,y}^{-1}(z))) \le \varepsilon\|f''\|_{L^\infty}$. Recalling the definitions (2.10) and (2.11), we deduce that $\|h\|_{L^\infty} \le \|g\|_{L^\infty} \le 2\varepsilon\|f''\|_{L^\infty}\|\varphi\|_{L^\infty}$. Recalling that $\langle -\xi^\sigma, \varphi\rangle = \langle Du, h\rangle$ and sending $\varepsilon \to 0$, we have obtained that $\xi^\sigma \llcorner K = 0$. In other words, we have now proved the validity of (2.4).

Let us pass now to show (2.5). By (2.4) we have $Du_\sigma \llcorner K = (Du)_\sigma \llcorner K$, so to obtain (2.5) we have to show

$$(Du)_\sigma \llcorner K = Du \llcorner K. \tag{2.12}$$

Keeping in mind Definition 2.1, for any function $\varphi \in \mathrm{C}_b(\Omega)$ we have

$$\begin{aligned}
\langle (Du)_\sigma, \varphi\rangle &= \int_{B(1)} \langle (T_{\sigma,y}^{-1})_\# [\det(DT_{\sigma,y}^{-1})Du], \varphi\rangle \rho(y)\,dy \\
&= \int_{B(1)} \langle \det(DT_{\sigma,y}^{-1})Du, \varphi\circ T_{\sigma,y}^{-1}\rangle \rho(y)\,dy \\
&= \int_{B(1)} (\langle Du, \varphi\rangle + \langle Du, \varphi\circ T_{\sigma,y}^{-1} - \varphi\rangle \\
&\quad + \langle (\det(DT_{\sigma,y}^{-1}) - 1)Du, \varphi\circ T_{\sigma,y}^{-1}\rangle)\rho(y)\,dy \\
&= \langle Du, \varphi\rangle + \int_{B(1)} (\langle Du, \varphi\circ T_{\sigma,y}^{-1} - \varphi\rangle \\
&\quad + \langle (\det(DT_{\sigma,y}^{-1}) - 1)Du, \varphi\circ T_{\sigma,y}^{-1}\rangle)\rho(y)\,dy.
\end{aligned}$$

Let us now again restrict our attention to the case when φ is supported in K_ε. Since by construction the function $\varphi\circ T_{\sigma,y}^{-1}$ is concentrated on $K_{2\varepsilon}$, and moreover for every $x \in K$ one has $\varphi\circ T_{\sigma,y}^{-1}(x) - \varphi(x) = \varphi(x) - \varphi(x) = 0$, then we can evaluate

$$\begin{aligned}
&|\langle (Du)_\sigma, \varphi\rangle - \langle Du, \varphi\rangle| \\
&\quad \le \|\varphi\|_{L^\infty}(2|Du|(K_{2\varepsilon}\backslash K) + \|\det(DT_{\sigma,y}^{-1}) - 1\|_{L^\infty(K_{2\varepsilon})}|Du|(K_{2\varepsilon})).
\end{aligned}$$

By sending ε to 0, since $\|\det(DT^{-1}_{\sigma,y}) - 1\|_{L^\infty(K_{2\varepsilon})}$ goes to 0 by (2.2), we obtain (2.12).

To conclude the proof, let us now assume that the jump set J_u is contained in K, and that the function ∇u belongs to $L^p(\Omega)$: we have to prove that ∇u_σ converges to ∇u in $L^p(\Omega)$. Recalling (2.4) and by linearity, we have

$$Du_\sigma = (Du)_\sigma + \sigma\xi^\sigma = (\nabla u\, d\mathscr{L}^N + D^s u)_\sigma + \sigma\xi^\sigma = (\nabla u)_\sigma\, d\mathscr{L}^N + (D^s u)_\sigma + \sigma\xi^\sigma. \tag{2.13}$$

By point (iii) we know that $(\nabla u)_\sigma$ converges to ∇u in $L^p(\Omega)$, and on the other hand since $D^s u$ is concentrated in K then $(D^s u)_\sigma = D^s u$ is also concentrated in K. As a consequence, to deduce that ∇u_σ converges in $L^p(\Omega)$ to ∇u when $\sigma \to 0$, it is enough to observe that the measures ξ^σ are actually functions, uniformly bounded in $L^p(\Omega)$.

To do so, we fix some $\varepsilon \to 0$, and we consider the situation in $\Omega\backslash K_\varepsilon$: keeping in mind that $\xi^\sigma_j \overset{*}{\rightharpoonup} \xi^\sigma$, applying the estimate (2.9) to each function u_j, and recalling (2.7) and Lemma 2.2, we derive that

$$\begin{aligned}\|\xi^\sigma\|_{L^p(\Omega\backslash K_\varepsilon)} &\le \liminf_{j\to\infty} \|\xi^\sigma_j\|_{L^p(\Omega\backslash K_\varepsilon)} \le \liminf_{j\to\infty} \| \, |Du_j|_\sigma\|_{L^p(\Omega\backslash K_\varepsilon)} \\ &\le 2 \liminf_{j\to\infty} \|Du_j\|_{L^p(\Omega\backslash K_{\varepsilon/2})} \\ &\le 2\|\nabla u\|_{L^p(\Omega\backslash K_{\varepsilon/2})} \le 2\|\nabla u\|_{L^p(\Omega)}.\end{aligned}$$

By letting ε to 0, recalling also that $\xi^\sigma \llcorner K = 0$, we deduce that

$$\|\xi^\sigma\|_{L^p(\Omega)} \le 2\|\nabla u\|_{L^p(\Omega)} \tag{2.14}$$

and, as noticed above, this uniform estimate in $L^p(\Omega)$ concludes the proof. □

An immediate corollary of the above proposition is the following result, which basically says that in all the converge results in SBV (or SBV^p, or SBV^p_∞), the smoothness of the approximating functions out of their jump sets comes for free.

COROLLARY 2.4. *Let $u \in \mathrm{SBV}(\Omega)$ be a function with $\nabla u \in L^p(\Omega)$ and $J_u \subseteq K$ for some compact set $K \subset\subset \Omega$ and $p \ge 1$. Then, for every $\varepsilon > 0$, there exists $\tilde{u} \in \mathrm{SBV}(\Omega)$ with $D^s\tilde{u} = D^s u$ and*

$$J_{\tilde{u}} = J_u, \quad \tilde{u} \in \mathrm{C}^\infty(\Omega\backslash K), \quad \|u - \tilde{u}\|_{\mathrm{BV}(\Omega)} + \|\nabla u - \nabla\tilde{u}\|_{L^p(\Omega)} < \varepsilon.$$

If, in addition, Ω has finite measure and $u \in W^{1,\infty}(\Omega\backslash K)$, then also $\tilde{u} \in W^{1,\infty}(\Omega\backslash K)$.

PROOF. We apply Proposition 2.3 to the function u, finding the BV functions u_σ. By the proposition, each function u_σ belongs to $\mathrm{C}^\infty(\Omega\backslash K)$, so the measure $D^s u_\sigma$ is concentrated on K; recalling (2.13) and the fact that $\xi^\sigma \llcorner K = 0$, we derive that $D^s u_\sigma = (D^s u)_\sigma = D^s u$, which also implies that $J_{u_\sigma} = J_u$. Moreover, points (iii)

and (iv) ensure that $u_\sigma \to u$ and $\nabla u_\sigma \to \nabla u$ in $L^p(\Omega)$, so to obtain the first part of the thesis it is enough to set $\tilde{u} = u_\sigma$ for some $\sigma = \sigma(\varepsilon)$ small enough.

Let us now suppose that $u \in W^{1,\infty}(\Omega\backslash K)$. Since $Du_\sigma = \nabla u_\sigma + D^s u_\sigma$ with $D^s u_\sigma = D^s u$ concentrated in K, we have to show that $\nabla u_\sigma \in L^\infty(\Omega)$. By (2.13), $\nabla u_\sigma = (\nabla u)_\sigma + \sigma \xi^\sigma$, and by Definition 2.1 it is obvious that $\|(\nabla u)_\sigma\|_{L^\infty} \leq \|\nabla u\|_{L^\infty(\Omega)}$. To conclude, it is then enough to observe that the functions ξ^σ are uniformly bounded in $L^\infty(\Omega)$; but in fact, since the estimate (2.14) is true for every σ and every p, by letting $p \to \infty$ we directly find that $\|\xi^\sigma\|_{L^\infty} \leq 2\|\nabla u\|_{L^\infty}$ for every σ. The functions u_σ are then also in $W^{1,\infty}(\Omega\backslash K)$, and the proof is concluded. □

We want now to prove that the traces of u_σ on K coincide with those of u: recall that a function u is said to have *right and left traces* $u^\pm(x_0)$ with respect to a vector $v \in \mathbb{S}^{N-1}$ at a point x_0, if

$$\lim_{r\to 0} \fint_{B_v^\pm(x_0,r)} |u(x) - u^\pm(x_0)|\, dx = 0.$$

We can then prove what follows.

LEMMA 2.5. *Let $u \in L^1(\Omega)$ be a function, and let $x_0 \in K$ be a point such that u admits right and left traces with respect to a vector $v \in \mathbb{S}^{N-1}$. Then, for any $\sigma \leq 1/2$ we have that u_σ admits the same traces at x_0.*

PROOF. Without loss of generality, we assume that $x_0 = 0$, that the traces are $u^+(x_0) = 1$ and $u^-(x_0) = 0$, and we denote $B^\pm(r) = B_v^\pm(0,r)$. It is enough to show that

$$\lim_{r\to 0} \fint_{B^+(r)} |u_\sigma - 1| = 0. \tag{2.15}$$

Let us take any r such that $B(2r) = B(0,2r) \subset\subset \Omega$, and let us define v the restriction of u to $B(2r)$, extended to 0 outside, and w the function given by $w = 1$ in $B^+(2r)$ and 0 outside. By the definition of the left and right traces we have that

$$\frac{\|v - w\|_{L^1(\Omega)}}{r^N} = \frac{\|u - u^+\|_{L^1(B^+(2r))} + \|u - u^-\|_{L^1(B^-(2r))}}{r^N} \underset{r\to 0}{\longrightarrow} 0.$$

Hence, by (ii) of Proposition 2.3, one has also

$$\frac{\|v_\sigma - w_\sigma\|_{L^1(\Omega)}}{r^N} \underset{r\to 0}{\longrightarrow} 0.$$

Moreover, by construction and since $x_0 \in K$, for every $x \in B(r)$ we have that $T_{\sigma,y}(x) \in B(2r)$; as a consequence, recalling Definition 2.1, we get that $u_\sigma = v_\sigma$ in $B(r)$, then the last inequality implies

$$\frac{\|u_\sigma - w_\sigma\|_{L^1(B^+(r))}}{r^N} \underset{r\to 0}{\longrightarrow} 0. \tag{2.16}$$

We have then to evaluate $\|w_\sigma - 1\|_{L^1(B^+(r))}$. Keeping in mind that $w = 1$ on $B^+(2r)$ and 0 outside, and recalling the definition of w_σ, we immediately obtain that $0 \le w_\sigma \le 1$ everywhere. Let now $x \in B^+(r)$: as already noticed, for every $y \in B(1)$ one has $T_{\sigma,y}(x) \in B(2r)$; in particular, if $T_{\sigma,y}(x) \in B^+(2r)$ for each $y \in B(1)$ one has $w_\sigma(x) = 1$. By the properties of f and δ we get

$$|T_{\sigma,y}(x) - x| = |\sigma f(\delta(x))\, y| \le f(\delta(x)) \le f(r).$$

Summarizing, we know that $0 \le w_\sigma(x) \le 1$ for every $x \in B^+(r)$, and that $w_\sigma(x) = 1$ if the whole ball $B(x, f(r))$ is contained in $B^+(2r)$, that is, for every $x \in B^+(r)$ which does not belong to the set

$$\{x \in B^+(r) : x \cdot \nu \le f(r)\}.$$

Since a rough estimate ensures that the volume of this set is less than $\omega_{N-1} r^{N-1} f(r)$, we obtain

$$\frac{\|w_\sigma - 1\|_{L^1(B^+(r))}}{r^N} \le \omega_{N-1} \frac{f(r)}{r}.$$

Putting this inequality together with (2.16), and keeping in mind that $f(r)/r$ goes to 0, when $r \to 0$, since $f'(0) = 0$, we derive the validity of (2.15), and this concludes the proof. □

3. The proof of Theorem A

This section is devoted to show Theorem A.

Proof of Theorem A. Let us fix a small quantity ε. Then, since the jump set J_u of u is $(N-1)$-rectifiable, we can find a compact, C^1 manifold M with C^1 boundary and a compact set $K_\varepsilon \subseteq J_u \cap M$ satisfying

$$|Du|(J_u \backslash K_\varepsilon) \le \frac{\varepsilon}{4}; \tag{3.1}$$

actually, M can be chosen as a finite union of C^1 images of the closed unit disk in $\mathbb{R}^{N-1}$.

Let us now consider the functions u_σ defined in Section 2 with $K = K_\varepsilon$. First of all, by Proposition 2.3 we know that every u_σ is a BV function in Ω, of class C^∞ in $\Omega \backslash K$; this implies that every u_σ belongs to $SBV(\Omega)$. Moreover, since K is contained in the jump set J_u of u, by Lemma 2.5 we obtain that $J_{u_\sigma} = K$ up to $(N-1)$-negligible subsets. Hence, $\mathscr{H}^{N-1}(\overline{J_{u_\sigma}} \backslash J_{u_\sigma}) = 0$. Therefore, keeping in mind (3.1), we see that to conclude the proof we have to show that, for σ small enough, $\|u - u_\sigma\|_{BV(\Omega)} \le \varepsilon$. Since by (iii) in Proposition 2.3 we already have that $u_\sigma \xrightarrow[L^1(\Omega)]{} u$ for $\sigma \to 0$, we are reduced to check only that, for σ small enough,

$$|Du - Du_\sigma|(\Omega) \le \varepsilon. \tag{3.2}$$

By (2.4), we know that $Du_\sigma = (Du)_\sigma + \sigma\xi^\sigma$, with $|\xi^\sigma|(\Omega) \le 2|Du|(\Omega)$, thus

$$Du_\sigma = (\nabla u \mathscr{L}^N + Du \llcorner K + Du \llcorner (J_u \backslash K))_\sigma + \sigma\xi^\sigma.$$

Moreover, as already noticed after Definition 2.1, $\mu_\sigma = \mu$ for every measure μ concentrated on K; therefore, by linearity we can rewrite the last equality as

$$Du_\sigma = (\nabla u)_\sigma \mathscr{L}^N + Du \llcorner K + (Du \llcorner (J_u \backslash K))_\sigma + \sigma\xi^\sigma.$$

We derive, thanks to (2.3), (iv) of Proposition 2.3 and (3.1),

$$\begin{aligned} |Du - Du_\sigma|(\Omega) &\le \|\nabla u - (\nabla u)_\sigma\|_{L^1(\Omega)} + |Du \llcorner (J_u \backslash K)|(\Omega) \\ &\quad + |(Du \llcorner (J_u \backslash K))_\sigma|(\Omega) + \sigma|\xi^\sigma|(\Omega) \\ &\le \|\nabla u - (\nabla u)_\sigma\|_{L^1(\Omega)} + 3|Du \llcorner (J_u \backslash K)|(\Omega) + 2\sigma|Du|(\Omega) \\ &\le \|\nabla u - (\nabla u)_\sigma\|_{L^1(\Omega)} + \frac{3}{4}\varepsilon + 2\sigma|Du|(\Omega). \end{aligned}$$

By (iii) of Proposition 2.3 the validity of (3.2) for $\sigma \ll 1$ immediately follows, hence the proof is concluded. □

REMARK 3.1. As an immediate application of Lemma 2.5 we have that, if u admits an inner trace on $\partial\Omega$, then the same is true for u_σ (hence for every function u_j of Theorem A) and the two traces coincide.

A quick look to the above construction ensures that, if the function u is in L^∞, then the same is true for every function u_j, and in fact $\|u_j\|_{L^\infty(\Omega)} \le \|u\|_{L^\infty(\Omega)}$. We want now to observe something much stronger, which will be useful in the sequel; namely, that starting from *every* sequence $\{u_j\}$ as in Theorem A, one can construct by smooth truncation a new sequence $\{\tilde{u}_j\}$, still approximating u, satisfying the L^∞ bound. This is a straightforward consequence of the next general result, which we can directly prove for SBV or SBV^p_∞ functions. Notice that, instead of giving two different results for the case of an SBV, or of an SBV^p_∞, function, we present a single claim for a function $u \in \mathrm{SBV}$ with $\nabla u \in L^p$ for some $p \ge 1$: of course, these functions are simply the SBV functions if $p = 1$, and the SBV^p_∞ functions if $p > 1$.

LEMMA 3.2. *Let $u \in \mathrm{SBV}(\Omega) \cap L^\infty(\Omega)$ be a function such that $\nabla u \in L^p(\Omega)$ for some $p \ge 1$. Then, for every $\varepsilon > 0$ there exists $\delta > 0$ with the following property: whenever $v \in \mathrm{SBV}(\Omega)$ is a function satisfying*

$$\|u - v\|_{\mathrm{BV}(\Omega)} + \|\nabla u - \nabla v\|_{L^p(\Omega)} < \delta, \tag{3.3}$$

there is a modification $\tilde{v} \in \mathrm{SBV}(\Omega)$ of v such that

$$J_{\tilde{v}} = J_v, \quad \|\tilde{v}\|_{L^\infty} \le \|u\|_{L^\infty}, \quad \|u - \tilde{v}\|_{\mathrm{BV}(\Omega)} + \|\nabla u - \nabla\tilde{v}\|_{L^p(\Omega)} < \varepsilon.$$

In addition, if $v \in \mathrm{C}^\infty(\Omega \backslash \overline{J_v})$, then the same is true for $\tilde{v}$.

PROOF. Without loss of generality, let us assume that $\|u\|_{L^\infty} = 1$. Let moreover $\delta \ll \eta \ll 1$ be two fixed constants, depending on u and ε, to be specified later, and let $\tau : \mathbb{R} \to (-1-2\eta, 1+2\eta)$ be a smooth function satisfying

$$0 < \tau'(t) \le 1 \ \forall t \in \mathbb{R}, \quad \tau(t) = t \ \forall -1-\eta \le t \le 1+\eta.$$

Given now a function $v \in \mathrm{SBV}(\Omega)$ satisfying (3.3), we define $w = \tau \circ v$. Notice that of course $w \in \mathrm{SBV}(\Omega)$, $J_w = J_v$, and if $v \in \mathrm{C}^\infty(\Omega \backslash \overline{J_v})$ the same is true for w.

We want to estimate the deviation between w and u; first of all, it is obvious that

$$\|w-u\|_{L^1(\Omega)} \le \|v-u\|_{L^1(\Omega)} \le \delta. \tag{3.4}$$

Let us now concentrate ourselves on the singular parts of Du and Dw; this is very easy in the set $J_u \Delta J_v$, since

$$\begin{aligned} |D^s u - D^s w|(J_u \Delta J_v) &= |D^s u - D^s w|(J_u \backslash J_v) + |D^s u - D^s w|(J_v \backslash J_u) \\ &= |D^s u|(J_u \backslash J_v) + |D^s w|(J_v \backslash J_u) \\ &\le |D^s u|(J_u \backslash J_v) + |D^s v|(J_v \backslash J_u) \\ &= |D^s u - D^s v|(J_u \backslash J_v) + |D^s u - D^s v|(J_v \backslash J_u) \\ &= |D^s u - D^s v|(J_u \Delta J_v) \le \delta. \end{aligned} \tag{3.5}$$

Keep now in mind that J_u is countably rectificable; as a consequence, we can write $J_u = G \cup H$ in such a way that G is a finite union of Lipschitz $(N-1)$-dimensional graphs, while $|D^s u|(H) < \eta$. Since, on Lipschitz sets, the trace operator for BV functions is continuous with respect to the strong BV convergence (see [4, Theorem 3.86]), there exists a constant C such that

$$\|f^\pm\|_{L^1_{\mathscr{H}^{N-1}}(G)} \le C \|f\|_{\mathrm{BV}(\Omega)} \tag{3.6}$$

for every function $f \in \mathrm{BV}(\Omega)$, where $f^\pm$ are the two traces of f on the two sides of G. Notice that C only depends on the set G, hence on u and η, but not on δ.

We can then now evaluate $D^s u - D^s w$ on the set $J_u \cap J_v$. Within the set H we simply have

$$\begin{aligned} |D^s u - D^s w|(J_u \cap J_v \cap H) &\le |D^s u|(H) + |D^s w|(H) \\ &\le |D^s u|(H) + |D^s v|(H) \\ &\le 2|D^s u|(H) + \|v-u\|_{\mathrm{BV}(\Omega)} \le 2\eta + \delta. \end{aligned} \tag{3.7}$$

Instead, concerning the set $J_u \cap J_v \cap G$, we have to further subdivide it. More precisely, we write $J_u \cap J_v \cap G = G_1 \cup G_2$, where G_1 is the subset done by all the points where both the traces $v^\pm$ are in $(-1-\eta, 1+\eta)$, and G_2 are the remaining points. In G_1, we have by construction $D^s w = D^s v$; instead, since for every $x \in G_2$

one has either $|v^+(x)| \geq 1+\eta$ or $|v^-(x)| \geq 1+\eta$, so in particular at least one between $|v^+ - u^+|$ and $|v^- - u^-|$ is bigger than η, by (3.6) we deduce

$$\begin{aligned}\eta \mathscr{H}^{N-1}(G_2) &\leq \|v^+ - u^+\|_{L^1_{\mathscr{H}^{N-1}}(G_2)} + \|v^- - u^-\|_{L^1_{\mathscr{H}^{N-1}}(G_2)} \\ &\leq 2C\|v-u\|_{\mathrm{BV}(\Omega)} \leq 2C\delta.\end{aligned}$$

Moreover, the jump $|(w-u)^+ - (w-u)^-|$ is clearly at most $4+4\eta$ everywhere, so we get

$$\begin{aligned}|D^s u - D^s w|(J_u \cap J_v \cap G) &= |D^s u - D^s v|(G_1) + |D^s u - D^s w|(G_2) \\ &\leq \delta + (4+4\eta)\mathscr{H}^{N-1}(G_2) \leq \delta + \frac{8+8\eta}{\eta} C\delta.\end{aligned}$$

Putting this estimate together with (3.5) and 3.7), we obtain

$$|D^s u - D^s w|(\Omega) \leq 2\eta + 3\delta + \frac{8+8\eta}{\eta} C\delta. \tag{3.8}$$

Finally, we have to estimate $\nabla u - \nabla w$: calling $A = \{x \in \Omega : |v(x)| > 1+\eta\}$, we have that $\nabla w = \nabla v$ on $\Omega \backslash A$, while $|\nabla w| \leq |\nabla v|$ on A. Moreover, $|u-v| > \eta$ in A, hence $\eta|A| \leq \|v-u\|_{L^1(\Omega)} \leq \delta$, that is, $|A| \leq \delta/\eta$. Whatever η is, up to take δ small enough we have then that the measure $|A|$ is as small as we wish; in particular, since $\nabla u \in L^p(\Omega)$, we can take δ so small that $\|\nabla u\|_{L^p(A)} < \eta$. Consequently, we can evaluate

$$\begin{aligned}\|\nabla w - \nabla u\|_{L^p(\Omega)} &\leq \|\nabla v - \nabla u\|_{L^p(\Omega\backslash A)} + \|\nabla w - \nabla u\|_{L^p(A)} \\ &\leq \delta + \|\nabla w\|_{L^p(A)} + \|\nabla u\|_{L^p(A)} \\ &\leq \delta + \|\nabla v\|_{L^p(A)} + \|\nabla u\|_{L^p(A)} \\ &\leq \delta + 2\|\nabla u\|_{L^p(A)} + \|\nabla u - \nabla v\|_{L^p(A)} \\ &\leq 2\delta + 2\eta.\end{aligned}$$

Since this estimate holds for any $p \geq 1$, in particular the case $p=1$ and the estimates (3.8) and (3.4) give

$$\|w-u\|_{\mathrm{BV}(\Omega)} = \|w-u\|_{L^1(\Omega)} + |Du - Dw|(\Omega) \leq 4\eta + 6\delta + \frac{8+8\eta}{\eta} C\delta,$$

from which we further deduce

$$\|u-w\|_{\mathrm{BV}(\Omega)} + \|\nabla u - \nabla w\|_{L^p(\Omega)} \leq 6\eta + 8\delta + \frac{8+8\eta}{\eta} C\delta.$$

We are finally in position to conclude, by defining $\tilde{v} = \frac{1}{1+2\eta} w$. In fact, it is clear that $\tilde{v} \in \mathrm{SBV}(\Omega)$, that $J_{\tilde{v}} = J_w = J_v$, that $\|\tilde{v}\|_{L^\infty} \leq 1 = \|u\|_{L^\infty}$, and that $\tilde{v}$ belongs

to $C^\infty(\Omega\backslash\overline{J_v})$ as soon as so does v. Moreover,

$$\|w-\tilde{v}\|_{BV(\Omega)}+\|\nabla w-\nabla\tilde{v}\|_{L^p(\Omega)}=\frac{2\eta}{1+2\eta}(\|w\|_{BV(\Omega)}+\|\nabla w\|_{L^p(\Omega)}),$$

so we finally conclude the proof by evaluating

$$\begin{aligned}&\|u-\tilde{v}\|_{BV(\Omega)}+\|\nabla u-\nabla\tilde{v}\|_{L^p(\Omega)}\\&\quad\le 2(\|u-w\|_{BV(\Omega)}+\|\nabla u-\nabla w\|_{L^p(\Omega)})+2\eta(\|u\|_{BV(\Omega)}+\|\nabla u\|_{L^p(\Omega)})\\&\quad\le 12\eta+16\delta+\frac{16+16\eta}{\eta}C\delta+2\eta(\|u\|_{BV(\Omega)}+\|\nabla u\|_{L^p(\Omega)})<\varepsilon,\end{aligned}$$

where the last inequality holds true as soon as η has been chosen small enough depending on u and ε, and δ small enough depending on η (recall that C depends on u and η but not on δ). □

In the lemma above, we have considered the situation of a bounded function $u\in SBV$. Now we notice that, in fact, for our purposes it is always admissible to assume that an SBV function is bounded: this is a very simple observation, which will be useful later.

LEMMA 3.3. *Let $\Omega\subseteq\mathbb{R}^N$ be an open set, and let $u\in SBV(\Omega)$ be a function with $\nabla u\in L^p(\Omega)$ for some $p\ge 1$. Then, for every $\varepsilon>0$ there exists a function $u_\varepsilon\in SBV(\Omega)\cap L^\infty(\Omega)$ such that*

$$\|u-u_\varepsilon\|_{BV(\Omega)}+\|\nabla u-\nabla u_\varepsilon\|_{L^p(\Omega)}\le\varepsilon,\quad J_{u_\varepsilon}\subseteq J_u. \tag{3.9}$$

PROOF. Keep in mind that J_u is countably rectifiable, hence it is contained, up to $\mathcal{H}^{N-1}$-negligible subsets, in the union of C^1 compact manifolds M_i, $i\in\mathbb{N}$. Moreover, as already observed, by [4, Theorem 3.88] we know that the two traces $\tau_i^\pm: BV(\Omega)\to L^1(M_i)$ on the two sides of each manifold M_i are continuous. As a consequence, we can select a big constant K such that

$$\|u\|_{L^1(A_K)}+\|\nabla u\|_{L^1(A_K)}+\|\nabla u\|_{L^p(A_K)}+|D^s u|(B_K)<\varepsilon, \tag{3.10}$$

where we call

$$A_K=\{x\in\Omega:|u(x)|\ge K\},\quad B_K=\bigcup_{i\in\mathbb{N}}\{x\in M_i,\max\{|\tau_i^+(x)|,|\tau_i^-(x)|\}\ge K\}.$$

With such a choice of K, we then let u_ε be the standard truncation of u at level K, that is, $u_\varepsilon(x)=\operatorname{sgn}(u(x))\min\{K,|u(x)|\}$. It is clear that $u_\varepsilon\in SBV(\Omega)$ and that $\nabla u_\varepsilon\in L^p(\Omega)$, as well as that $J_{u_\varepsilon}\subseteq J_u$. Since, on the other hand, $D^s u_\varepsilon=D^s u$ on $J_u\backslash B_K$ and $|D^s u_\varepsilon|\le|D^s u|$ on B_K, then (3.9) comes directly from (3.10) since

$$\begin{aligned}&\|u-u_\varepsilon\|_{BV(\Omega)}+\|\nabla u-\nabla u_\varepsilon\|_{L^p(\Omega)}\\&\quad\le\|u\|_{L^1(A_K)}+\|\nabla u\|_{L^1(A_K)}+|D^s u|(B_K)+\|\nabla u\|_{L^p(A_K)}.\end{aligned}$$

□

4. The proof of Theorem B

This section is devoted to the proof of Theorem B; before doing that, we present three simple technical results. The first one is an extension lemma for smooth sets with a C^1 crack.

LEMMA 4.1. *Let $A \subseteq \mathbb{R}^N$ be a smooth, bounded, open set, and let $H \subset\subset A$ be a compact, $(N-1)$-dimensional, connected, C^1 manifold, with (possibly empty) C^1 boundary. Then, there exists a constant C, depending only A and on H, such that for any three functions $g \in L^1(\partial A)$ and $g^{\pm} \in L^1(H)$, there exists a function $\varphi \in W^{1,1}(A\backslash H)$ whose trace on ∂A coincides with g and whose two traces on (the two sides of) H are g^+ and g^-, satisfying*

$$\|\varphi\|_{W^{1,1}(A\backslash H)} \leq C(\|g\|_{L^1(\partial A)} + \|g^+\|_{L^1(H)} + \|g^-\|_{L^1(H)}). \tag{4.1}$$

If moreover $g \in C^1(\partial A)$ and $g^{\pm} \in C^1(H)$ with $g^+ = g^-$ on ∂H, then there exists a function $\psi \in W^{1,\infty}(A\backslash H)$, again with g as trace on ∂A and $g^{\pm}$ as traces on H, satisfying

$$\|\psi\|_{W^{1,\infty}(A\backslash H)} \leq C(\|g\|_{C^1(\partial A)} + \|g^+\|_{C^1(H)} + \|g^-\|_{C^1(H)}). \tag{4.2}$$

PROOF. Since H is a C^1 manifold with C^1 boundary, we can find an open, Lipschitz set $A_0 \subset\subset A$, contained in a small neighborhood of H, with the property that its boundary ∂A_0 consists of two parts, H^+ and H^-, so that H^+ and H^- are two C^1 manifolds with disjoint interiors and with the same $(N-2)$-dimensional boundary $\partial H^+ = \partial H^- = \partial H$. This is a very simple geometrical fact, Figure 1 depicts the situation for the two possible cases, namely, when H has non-empty boundary (H_1) and when it has empty boundary (H_2).

As a consequence, we can find a diffeomorphism $\Phi : A\backslash H \to A\backslash A_0$, bi-Lipschitz up to the boundary for the geodesic distance, which is the identity in a neighborhood of ∂A, and such that the images of (the two sides of) H under Φ are H^+ and H^-.

The standard extension result for Lipschitz sets ensures that there exists a constant C_1, depending only on A and on A_0, thus actually only on A and on H, such that for any two maps $g \in L^1(\partial A)$ and $g_0 \in L^1(\partial A_0)$ there exists a function $v \in W^{1,1}(A\backslash A_0)$ whose traces on ∂A and ∂A_0 coincide with g and g_0 respectively and such that

$$\|v\|_{W^{1,1}(A\backslash A_0)} \leq C_1(\|g\|_{L^1(\partial A)} + \|g_0\|_{L^1(\partial A_0)}). \tag{4.3}$$

Figure 1. Construction in Lemma 4.1: the shaded parts on the right are A_0.

To obtain the searched φ, then, it is then enough to define g_0 as $g^+ \circ \Phi^{-1}$ on H^+ and as $g^- \circ \Phi^{-1}$ on H^-, and then simply set $\varphi = v \circ \Phi$; the validity of (4.1) comes directly from (4.3) and by the fact that Φ is bi-Lipschitz up to the boundary.

A similar argument can be done to find the searched function ψ when g, g^+ and g^- are C^1. In fact, the function defined on $\partial(A \backslash A_0)$, which equals g on ∂A and g_0 on ∂A_0, is C^1 by construction, so there exists a function $w \in W^{1,\infty}(A \backslash A_0)$ with g and g_0 as traces, for which

$$\|w\|_{W^{1,\infty}(A \backslash A_0)} \leq C_1(\|g\|_{C^1(\partial A)} + \|g_0\|_{C^1(\partial A_0)}).$$

Thus, defining $\psi = w \circ \Phi$, we get a $W^{1,\infty}$ function on $A \backslash H$ with g, g^+ and g^- as traces and satisfying the estimate (4.2). □

Our second preliminary lemma is the estimate of how much a continuous function changes (in the BV sense) if we substitue its value in a ball with the average value on the ball itself. We will use the following notation: given a continuous function φ on an open set U, and given a ball B compactly contained in U, we set φ_B as the function

$$\varphi_B(x) = \begin{cases} \varphi(x) & \text{if } x \in U \backslash B, \\ \fint_B \varphi(y)\, dy & \text{if } x \in B. \end{cases} \tag{4.4}$$

LEMMA 4.2. *Let U be an open set, and B a ball compactly contained in U. Let moreover $\varphi \in W^{1,1}(U)$ be a continuous function. Then, the function φ_B defined in* (4.4) *belongs to* $\mathrm{SBV}(U)$*, its jump set satisfies $J_{\varphi_B} \subseteq \partial B$, and*

$$\|\varphi - \varphi_B\|_{\mathrm{BV}(U)} \leq C' \|\varphi\|_{W^{1,1}(B)}, \tag{4.5}$$

where C' is a purely geometrical constant, not depending on φ, U or B.

PROOF. Since the function φ_B coincides with the continuous, $W^{1,1}$ function φ in $U \backslash B$ and is constant in B, of course it belongs to $\mathrm{SBV}(U)$ with $J_{\varphi_B} \subseteq \partial B$, hence we only have to deal with (4.5).

First of all, calling $\kappa = \fint_B \varphi(y)\, dy$, by the trace inequality and the Sobolev–Poincarè inequality we get that

$$\begin{aligned} \int_{\partial B} |\varphi - \kappa| \leq C_T \|\varphi - \kappa\|_{W^{1,1}(B)} &= C_T \|\varphi - \kappa\|_{L^1(B)} + C_T \|D\varphi\|_{L^1(B)} \\ &\leq C_T(C_P + 1)\|D\varphi\|_{L^1(B)}. \end{aligned}$$

Notice that both the constants C_T and C_P depend on the radius of the ball B; nevertheless, if we define C_1 as the smallest constant such that for every φ

$$\int_{\partial B} |\varphi - \kappa| \leq C_1 \|D\varphi\|_{L^1(B)},$$

a trivial rescaling argument ensures that C_1 does not depend on the radius of B. Then, keeping in mind that φ is continuous, we can evaluate

$$\begin{aligned}\|\varphi-\varphi_B\|_{\mathrm{BV}(U)} &= |D\varphi_B|(\partial B)+\|\varphi-\varphi_B\|_{W^{1,1}(B)}\\ &= \int_{\partial B}|\varphi-\kappa|+\|\varphi-\varphi_B\|_{L^1(B)}+\|D\varphi\|_{L^1(B)}\\ &\le (C_1+1)\|D\varphi\|_{L^1(B)}+2\|\varphi\|_{L^1(B)}\le (C_1+2)\|\varphi\|_{W^{1,1}(B)},\end{aligned}$$

hence (4.5) is established with the purely geometric constant $C'=C_1+2$. □

Our last preliminary technical result allows to modify the jump set of a SBV function, in order to make it more regular.

LEMMA 4.3. *Let $u\in \mathrm{SBV}(\Omega)$, and let $M\subset\subset\Omega$ be a compact manifold, polyhedral or of class C^1, such that $J_u\subseteq M$. Then, for every $\delta>0$, there exists a function $v\in\mathrm{SBV}(\Omega)$ such that $J_v\subseteq M$ and $\mathscr{H}^{N-1}(M\backslash J_v)=0$, satisfying $\|u-v\|_{\mathrm{BV}(\Omega)}+\|u-v\|_{L^\infty(\Omega)}<\delta$. In addition, if $u\in\mathrm{SBV}^p(\Omega)$, then $v\in\mathrm{SBV}^p(\Omega)$ and also $\|\nabla u-\nabla v\|_{L^p(\Omega)}<\delta$. Finally, if u belongs to $\mathrm{C}^\infty(\Omega\backslash M)$, or to $W^{1,\infty}(\Omega\backslash M)$, then so does v.*

PROOF. Let us assume, for a moment, that M is connected, and let us consider a smooth open set A such that $M\subset\subset A\subset\subset\Omega$. Let $\varphi^+:M\to\mathbb{R}$ be a C^1 function with $\varphi^+(x)=0$ for every point x in the boundary of M, and $\varphi^+(x)>0$ for every other $x\in M$, and let $\varphi^-:M\to\mathbb{R}$ be identically 0. By Lemma 4.1, we get a function $\varphi\in W^{1,\infty}(A\backslash M)$, whose trace at ∂A is zero, while the traces on the two sides of M are φ^+ and φ^-. In particular, extending φ by 0 outside of A, we have $\varphi\in\mathrm{SBV}(\Omega)\cap W^{1,\infty}(\Omega\backslash M)$ with $J_\varphi=M$. By Corollary 2.4, we are allowed to assume that $\varphi\in\mathrm{SBV}(\Omega)\cap\mathrm{C}^\infty(\Omega\backslash M)\cap W^{1,\infty}(\Omega\backslash M)$. We want to define $v=u+\varepsilon\varphi$ for a suitable, small ε.

Of course, whatever ε is, we have that $v\in\mathrm{SBV}(\Omega)$, and v belongs to $\mathrm{SBV}^p(\Omega)$, or $\mathrm{C}^\infty(\Omega\backslash M)$, or $W^{1,\infty}(\Omega\backslash M)$, as soon as so does u. Moreover, $J_v\subseteq J_u\cup J_\varphi=M$. The fact that $\|u-v\|_{\mathrm{BV}(\Omega)}+\|u-v\|_{L^\infty(\Omega)}<\delta$ is clearly true for every ε small enough, as well as the fact that $\|\nabla u-\nabla v\|_{L^p(\Omega)}<\delta$, in case that $u\in\mathrm{SBV}^p(\Omega)$. Therefore, to conclude we only have to find a small ε such that $\mathscr{H}^{N-1}(M\backslash J_v)=0$.

But actually, any point $x\in M$ belongs to the jump set of v for all real ε except one; as a consequence, the values of ε for which $\mathscr{H}^{N-1}(M\backslash J_v)>0$ are only countably many, and then the existence of some ε as required is obvious and the proof is conclued when M is connected.

If M is not connected, by compactness it has anyway a finite number of connected components M_i; we can then consider disjoint, smooth sets A_i with $M_i\subset\subset A_i\subset\subset\Omega$, and repeat in each of them the above argument, so that the conclusion follows also in the general case. □

We give now the definition of the extension domains that we are going to need for Theorem B.

DEFINITION 4.4. Let $\Omega \subseteq \mathbb{R}^N$ be an open set. We say that Ω is a *local extension domain* if $W^{1,p}(\Omega)$ is dense in $W^{1,1}(\Omega)$.

Notice that this definition is even weaker than the usual one. In fact, given a function $u \in W^{1,1}(\Omega)$, we do not need a function $\tilde{u} \in W^{1,1}(\mathbb{R}^N)$ which coincides with u in Ω, we only want to find a function $v \in W^{1,p}(\Omega)$ such that $\|v-u\|_{W^{1,1}(\Omega)}$ is arbitrarily small (by Meyers and Serrin Theorem, this requirement is of course weaker). We are now ready to present the construction of the approximation required by Theorem B.

PROOF (OF THEOREM B). Let us take Ω and u as in the claim, and let us fix a very small constant $\varepsilon \ll \|u\|_{\mathrm{BV}(\Omega)}$. Notice that, thanks to Lemma 3.3, we can assume that $u \in L^\infty(\Omega)$. Moreover, we can also assume that the support of u is bounded, that is, $u(x) = 0$ for every $x \in \Omega$ with $|x|$ big enough: to achieve this, it is enough to multiply u by a smooth function $\eta : \mathbb{R}^N \to [0,1]$ such that $\eta(x) = 1$ for $|x| < R_1$, $\eta(x) = 0$ for $|x| > R_2$, $\|D\eta\|_{L^\infty} \leq 1$, and $R_2 \gg R_1 \gg 1$. We aim to find a function $u_\varepsilon \in \mathrm{SBV}^p(\Omega) \cap \mathrm{C}^\infty(\Omega \backslash \overline{J_{u_\varepsilon}})$ satisfying

$$\|u_\varepsilon - u\|_{\mathrm{BV}(\Omega)} < \varepsilon, \quad \|\nabla u_\varepsilon - \nabla u\|_{L^p(\Omega)} < \varepsilon. \tag{4.6}$$

Moreover, we will find a compact, $(N-1)$-dimensional manifold M, C^1 and with C^1 boundary, such that J_{u_ε} is contained in M and coincides with it up to $\mathscr{H}^{N-1}$-negligible subsets. Of course, once we do so we will have concluded the proof.

We start by selecting a sufficiently big constant K, depending on Ω, u and ε, so that both properties

$$\begin{aligned} &\int_{\{|\nabla u| > 2K\}} |\nabla u|^p < \frac{\varepsilon^p}{2^{p+2}}, \\ &\|\nabla u\|_{L^p(F)} < \frac{\varepsilon}{2 \cdot 2^{1/p}}, \quad \|u\|_{L^1(F)} + \|\nabla u\|_{L^1(F)} < \frac{\varepsilon}{2C'} \\ &\forall F \subseteq \Omega : |F| \leq \frac{2\|u\|_{\mathrm{BV}(\Omega)}}{K} \end{aligned} \tag{4.7}$$

hold, where C' is the constant of Lemma 4.2. We also fix a small constant δ, depending on Ω, u, ε and K, hence actually only on Ω, u and ε, satisfying

$$(6+5C')\delta < \frac{\varepsilon}{2}, \quad (3K)^{p-1} 5\delta < \frac{\varepsilon^p}{12}, \quad \delta < \varepsilon\Big(1 - \Big(\frac{5}{6}\Big)^{1/p}\Big). \tag{4.8}$$

We will define several approximating functions trough successive refinements, until we will reach the desired function u_ε. For the sake of clarity, we divide our construction in some steps.

Step I. The function $v_1 \in \mathrm{SBV}(\Omega)$ from Theorem A.
First of all, we apply Theorem A to get a first approximation $v_1 \in \mathrm{SBV}(\Omega)$ which satisfies

$$\|v_1 - u\|_{\mathrm{BV}(\Omega)} < \delta, \tag{4.9}$$

and so that v_1 is C^∞ in $\Omega \backslash \overline{J_{v_1}}$; moreover, there is a compact, C^1 manifold $M' \subset\subset \Omega$, with C^1 boundary, which contains J_{v_1}. Notice that the choice $u_\varepsilon = v_1$ does not work because Theorem A does not give information on $\|\nabla v_1 - \nabla u\|_{L^p(\Omega)}$, and we do not even know whether $\nabla v_1 \in L^p(\Omega)$.

Step II. The function $v_2 \in \mathrm{SBV}(\Omega) \cap \mathrm{C}^\infty(\Omega \backslash M')$, with C^1 traces on M' coinciding on $\partial M'$.
We want now to modify v_1 so to become smooth in $\Omega \backslash M'$, and in such a way that its traces on M' become C^1 and coincide on $\partial M'$. To do so, let us call M_i, for $1 \le i \le P$, the connected components of M', which are finitely many, and let A_i be disjoint smooth open sets, compactly contained in Ω, and each one compactly containing the corresponding manifold M_i.

We apply Lemma 4.1 to the set $A = A_i$ and with $H = M_i$, getting a constant C_i. Then, we set $g_i = 0$ on ∂A_i, and we let $g_i^\pm \in L^1(M_i)$ be two functions such that $v_1^\pm + g_i^\pm$ are two C^1 functions on M_i coinciding on ∂M_i, where $v_1^\pm$ denote the two traces of v_i on M_i, and satisfying

$$\|g_i^\pm\|_{L^1(M_i)} < \frac{\delta}{(C_i + 1)P}. \tag{4.10}$$

Lemma 4.1 provides then us with a function $\varphi_i \in W^{1,1}(A_i \backslash M_i)$, with zero as trace on ∂A_i and with $g_i^\pm$ as traces on M_i and satisfying the estimate (4.1), which by (4.10) becomes

$$\|\varphi_i\|_{W^{1,1}(A_i \backslash M_i)} \le \frac{2\delta C_i}{(C_i + 1)P}. \tag{4.11}$$

We can then define the function $\tilde{v}_2 \in \mathrm{SBV}(\Omega)$ as the function coinciding with $v_1 + \varphi_i$ on each A_i, and with v_1 in $\Omega \backslash \bigcup_i A_i$. Notice that the $J_{\tilde{v}_2} \subseteq M'$, the traces of $\tilde{v}_2$ are C^1 on M' and coincide on $\partial M'$, and by (4.10) and (4.11) we evaluate

$$\|\tilde{v}_2 - v_1\|_{\mathrm{BV}(\Omega)} = \sum_{i=1}^{P} \|\varphi_i\|_{W^{1,1}(A_i \backslash M_i)} + \|g_i^+ - g_i^-\|_{L^1(M_i)} \le 2\delta,$$

which by (4.9) implies

$$\|\tilde{v}_2 - u\|_{\mathrm{BV}(\Omega)} < 3\delta.$$

We can then apply Corollary 2.4 to $\tilde{v}_2 \in \mathrm{SBV}(\Omega)$, finding to $v_2 \in \mathrm{SBV}(\Omega) \cap \mathrm{C}^\infty(\Omega \backslash M')$ with $\|v_2 - \tilde{v}_2\|_{\mathrm{BV}(\Omega)} < 3\delta - \|\tilde{v}_2 - u\|_{\mathrm{BV}(\Omega)}$, so that

$$\|v_2 - u\|_{\mathrm{BV}(\Omega)} < 3\delta. \tag{4.12}$$

Notice that, by Lemma 2.5, the traces of v_2 on M' coincide with those of $\tilde{v}_2$.

Step III. The function $v_3 \in \mathrm{SBV}(\Omega) \cap \mathrm{C}^\infty(\Omega \backslash M') \cap W^{1,\infty}(\Omega' \backslash M')$ *for* $M' \subset\subset \Omega' \subset\subset \Omega$.

Our next goal is to modify v_2, so to become $W^{1,\infty}$ in $\Omega' \backslash M'$ for every open set $\Omega' \subset\subset \Omega$ compactly containing M'. Since the traces of v_2 on M' are C^1 and coincide on $\partial M'$, and since M' is a finite union of connected, C^1 manifolds, Lemma 4.1 provides us with a function $\psi \in W^{1,\infty}(\Omega \backslash M')$ which equals 0 outside of a neighborhood of M', and whose traces on M' coincide with those of v_2; considered on the whole Ω, ψ is of course an SBV function. Again by Corollary 2.4 and Lemma 2.5, we can assume without loss of generality that $\psi \in \mathrm{SBV}(\Omega) \cap W^{1,\infty}(\Omega \backslash M') \cap \mathrm{C}^\infty(\Omega \backslash M')$. Let us then write $v_2 = \psi + \omega$. By definition, $\omega \in \mathrm{SBV}(\Omega) \cap \mathrm{C}^\infty(\Omega \backslash M')$; however, both the traces of ω on M' are zero by construction, hence we derive $\omega \in W^{1,1}(\Omega) \cap \mathrm{C}^\infty(\Omega \backslash M')$. By Meyers and Serrin Theorem, we can find $\omega_\delta \in W^{1,1}(\Omega) \cap \mathrm{C}^\infty(\Omega)$ such that $\|\omega_\delta - \omega\|_{\mathrm{BV}(\Omega)} = \|\omega_\delta - \omega\|_{W^{1,1}(\Omega)} < \delta$. We can now simply define $v_3 = \psi + \omega_\delta$: this function clearly belongs to $\mathrm{SBV}(\Omega) \cap \mathrm{C}^\infty(\Omega \backslash M')$, and since $\psi \in W^{1,\infty}(\Omega \backslash M')$ and $\omega_\delta \in \mathrm{C}^\infty(\Omega)$ we have also $v_3 \in W^{1,\infty}(\Omega' \backslash M')$ for every $M' \subset\subset \Omega' \subset\subset \Omega$. Finally, by construction $\|v_3 - v_2\|_{\mathrm{BV}(\Omega)} < \delta$, which from (4.12) gives

$$\|v_3 - u\|_{\mathrm{BV}(\Omega)} < 4\delta. \tag{4.13}$$

Step IV. The function $v_4 \in \mathrm{SBV}^p(\Omega) \cap \mathrm{C}^\infty(\Omega \backslash M')$.

Observe now that the function v_3 is smooth in $\Omega \backslash M'$, but this does not necessarily mean that ∇v_3 is in $L^p(\Omega \backslash M')$. In this step we face with this problem, replacing v_3 with $v_4 \in \mathrm{SBV}^p(\Omega) \cap \mathrm{C}^\infty(\Omega \backslash M')$. Let $\eta : \Omega \to [0,1]$ be a smooth function with compact support such that $\eta \equiv 1$ on a neighborhood of M', and let us set $\varphi = (1-\eta)v_3$; by construction, $\varphi \in W^{1,1}(\Omega)$. Let us now use the assumption on Ω to be a local extension domain in the sense of Definition 4.4: then, we can approximate φ in $W^{1,1}(\Omega)$ with $W^{1,p}$ functions, so again by Meyers and Serrin we can take a function $\varphi_\delta \in W^{1,p}(\Omega) \cap \mathrm{C}^\infty(\Omega)$ with $\|\varphi_\delta - \varphi\|_{W^{1,1}(\Omega)} < \delta$. Let us then define $v_4 = \eta v_3 + \varphi_\delta$: since ∇v_3 is bounded in $A \backslash M'$, being $M' \subset\subset A = \{\eta \neq 0\} \subset\subset \Omega$, we derive that $\eta v_3 \in \mathrm{SBV}^p(\Omega) \cap \mathrm{C}^\infty(\Omega \backslash M')$, so it is also $v_4 \in \mathrm{SBV}^p(\Omega) \cap \mathrm{C}^\infty(\Omega \backslash M')$, and $\|v_4 - v_3\|_{\mathrm{BV}(\Omega)} = \|\varphi_\delta - \varphi\|_{\mathrm{BV}(\Omega)} = \|\varphi_\delta - \varphi\|_{W^{1,1}(\Omega)} < \delta$, which by (4.13) gives

$$\|v_4 - u\|_{\mathrm{BV}(\Omega)} < 5\delta. \tag{4.14}$$

Step V. The final function u_ε.

We are now ready to give our last two approximating functions, namely, the function v_5 and the final function u_ε. Let us consider the set

$$F = \{x \in \Omega \backslash M' : |\nabla v_4(x)| > K\},$$

where K is the constant in (4.7). The set F is open in $\Omega \backslash M'$, since v_4 is smooth there; moreover, it has small measure: indeed, also by (4.14), we have

$$K|F| \leq \int_F |\nabla v_4| \leq \|v_4\|_{\mathrm{BV}(\Omega)} \leq \|u\|_{\mathrm{BV}(\Omega)} + \|v_4 - u\|_{BV(\Omega)} \leq 2\|u\|_{\mathrm{BV}(\Omega)},$$

and by (4.7) this implies that

$$\|\nabla u\|_{L^p(F)} < \frac{\varepsilon}{2\cdot 2^{1/p}}, \quad \|u\|_{L^1(F)} + \|\nabla u\|_{L^1(F)} < \frac{\varepsilon}{2C'}. \tag{4.15}$$

Now, let us use the fact that ∇v_4 belongs to $L^p(\Omega\backslash M')$, hence in particular $\nabla v_4 \in L^p(F)$: as a consequence, we can take finitely many disjoint balls B_i, $1 \le i \le k$, compactly contained in F, with the property that

$$\|\nabla v_4\|_{L^p(F\setminus\bigcup_{i=1}^k B_i)} \le \frac{\varepsilon}{2\cdot 2^{1/p}}. \tag{4.16}$$

We now define $v_5 : \Omega \to \mathbb{R}$ as the function given by

$$v_5(x) = \begin{cases} (v_4)_{B_i} & \text{if } x \in B_i, \\ v_4(x) & \text{if } x \notin \bigcup_{i=1}^k B_i, \end{cases}$$

where $(v_4)_{B_i}$ denotes the average of v_4 in the ball B_i, according with the notation of Lemma 4.2. In particular, observe that $v_5 = v_4$ on $\Omega\backslash F$. It is clear by construction that $v_5 \in \mathrm{SBV}^p(\Omega)$, and its jump set is contained in

$$M = M' \cup \bigcup_{i=1}^k \partial B_i.$$

Observe that M is a C^1 and compact manifold, with C^1 boundary, and since v_4 was smooth on $\Omega\backslash M'$, then v_5 is smooth on $\Omega\backslash M$. We apply now Lemma 4.3 to get our final function $u_\varepsilon \in \mathrm{SBV}^p(\Omega)$, which belongs to $\mathrm{C}^\infty(\Omega\backslash M)$ and which satisfies

$$\|\nabla v_5 - \nabla u_\varepsilon\|_{L^p(\Omega)} < \delta, \quad \|v_5 - u_\varepsilon\|_{\mathrm{BV}(\Omega)} < \delta, \quad \mathscr{H}^{N-1}(M\backslash J_{u_\varepsilon}) = 0. \tag{4.17}$$

Hence, we have then only to take care of (4.6) to conclude.

Recalling the estimate (4.5) of Lemma 4.2, since v_4 is a continuous function in the open set $\Omega\backslash M$, and by (4.14) and (4.15), we have

$$\begin{aligned} \|v_4 - v_5\|_{\mathrm{BV}(\Omega)} &\le C'\|v_4\|_{W^{1,1}(\bigcup_{i=1}^k B_i)} \le C'\|v_4\|_{W^{1,1}(F)} \\ &\le C'(\|v_4 - u\|_{\mathrm{BV}(\Omega)} + \|u\|_{L^1(F)} + \|\nabla u\|_{L^1(F)}) \le 5C'\delta + \frac{\varepsilon}{2}, \end{aligned}$$

so that again by (4.14), by (4.17), and by (4.8)

$$\begin{aligned} \|u - u_\varepsilon\|_{\mathrm{BV}(\Omega)} &\le \|u - v_4\|_{\mathrm{BV}(\Omega)} + \|v_4 - v_5\|_{\mathrm{BV}(\Omega)} + \|v_5 - u_\varepsilon\|_{\mathrm{BV}(\Omega)} \\ &\le (6 + 5C')\delta + \frac{\varepsilon}{2} < \varepsilon, \end{aligned}$$

and the first estimate in (4.6) follows.

Let us then pass to estimate the L^p norm of $\nabla v_5 - \nabla u$ in Ω. In $\Omega\backslash F$, using (4.14), (4.7) and (4.8), and recalling that $v_5 = v_4$, we can evaluate

$$\begin{aligned}\int_{\Omega\backslash F} |\nabla v_5 - \nabla u|^p &= \int_{\Omega\backslash F\cap\{|\nabla u|\le 2K\}} |\nabla v_4 - \nabla u|^p + \int_{\Omega\backslash F\cap\{|\nabla u|>2K\}} |\nabla v_4 - \nabla u|^p \\ &\le (3K)^{p-1}\|\nabla v_4 - \nabla u\|_{L^1(\Omega)} + 2^p \int_{\{|\nabla u|>2K\}} |\nabla u|^p \\ &\le (3K)^{p-1} 5\delta + \frac{\varepsilon^p}{4} \le \frac{\varepsilon^p}{3}.\end{aligned}$$

Instead, in F, by (4.15), by (4.16) and by construction we have

$$\begin{aligned}\|\nabla v_5 - \nabla u\|_{L^p(F)} &\le \|\nabla v_5\|_{L^p(F)} + \|\nabla u\|_{L^p(F)} \\ &= \|\nabla v_4\|_{L^p(F\backslash\bigcup_{i=1}^k B_i)} + \|\nabla u\|_{L^p(F)} \le \frac{\varepsilon}{2^{1/p}}.\end{aligned}$$

Putting together the last two estimates, (4.17), and again (4.8), we get the second estimate in (4.6), therefore the proof is concluded. □

For later use, we now remark what we have found after Step III in the above proof, namely, the result below.

LEMMA 4.5. *Let Ω be an open set, $u \in \mathrm{SBV}(\Omega)$, and let M be a C^1 manifold with C^1 boundary such that, for some small ε, $|D^s u|(J_u\backslash M) < \varepsilon/4$. Then, there exists a function $v \in \mathrm{SBV}(\Omega) \cap \mathrm{C}^\infty(\Omega\backslash M)$ such that $\|v - u\|_{\mathrm{BV}(\Omega)} < 4\varepsilon$, $\mathscr{H}^{N-1}(M\backslash J_v) = 0$, both the traces of v on the two sides of M are C^1, and v belongs to $W^{1,\infty}(\Omega'\backslash M)$ for every $M \subset\subset \Omega' \subset\subset \Omega$. Moreover, if u is compactly supported in Ω, then so is also v (and then, one has $v \in W^{1,\infty}(\Omega\backslash M)$).*

In fact, in this lemma, to get that $\mathscr{H}^{N-1}(M\backslash J_v) = 0$ one has to rely also on Lemma 4.3; moreover, the last point comes directly from the construction.

REMARK 4.6. We remark that, in Theorem B, we do not have $J_{u_j} \subseteq J_u$, which was the case for Theorem A. In fact, in our construction of the functions u_j for the proof of Theorem B, we have enlarged the jump set in Step V.

Observe that the domain Ω could be any open set in $\mathbb{R}^N$ in Theorem A, while we have added the assumption on Ω to be a local extension domain for Theorem B. Nevertheless, it is also possible to consider any open set Ω, up to replace the L^p convergence by an L^p_{loc} convergence.

THEOREM 4.7. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, and let $u \in \mathrm{SBV}^p_\infty(\Omega)$. Then, there esists a sequence of functions $u_j \in \mathrm{SBV}(\Omega) \cap \mathrm{SBV}^p_{\mathrm{loc}}(\Omega)$ and of compact, C^1, manifolds with (possibly empty) C^1 boundary $M_j \subset\subset \Omega$, such that $J_{u_j} \subseteq M_j$, $\mathscr{H}^{N-1}(M_j\backslash J_{u_j}) = 0$, and so that*

$$\|u_j - u\|_{\mathrm{BV}(\Omega)} \to 0, \quad u_j \in \mathrm{C}^\infty(\Omega\backslash\overline{J_{u_j}}), \quad \nabla u_j \xrightarrow[L^p_{\mathrm{loc}}(\Omega)]{} \nabla u.$$

PROOF. It is enough to repeat the proof of Theorem B with few minor modifications. More precisely, we define the functions v_1, v_2 and v_3 exactly as in the steps I, II and III of that proof. In place of Step IV, which is the only point where we have used the assumption on Ω of being a local extension domain, we simply set $v_4 = v_3$. Of course, we do not know whether $v_4 \in \mathrm{SBV}^p(\Omega)$, but $v_4 \in \mathrm{SBV}^p_{\mathrm{loc}}(\Omega)$ for sure, since it is smooth outside the compact set M' and ∇v_4 is bounded around M'.

Keep now in mind Step V: the fact that ∇v_4 was in $L^p(\Omega)$ was used only to get the balls B_i satisfying (4.16). In the present case we cannot get such an estimate, but since $\nabla v_4 \in L^p_{\mathrm{loc}}(\Omega)$ we can find balls B_i for which

$$\|\nabla v_4\|_{L^p(F\cap\Omega_\varepsilon\setminus\bigcup_{i=1}^H B_i)} \leq \frac{\varepsilon}{2\cdot 2^{1/p}}, \tag{4.16$'$}$$

where $\Omega_\varepsilon \subset\subset \Omega$ is a smooth open set such that

$$M' \subset\subset \Omega_\varepsilon, \quad \{x \in \Omega : |x| < \varepsilon^{-1}, B(x,\varepsilon) \subset\subset \Omega\} \subseteq \Omega_\varepsilon.$$

Continuing Step V, we can notice that the proof of the fact that $\|u - u_\varepsilon\|_{\mathrm{BV}(\Omega)} < \varepsilon$ did not use (4.16), hence the validity of the estimate still holds. Instead, (4.16) was used to obtain that $\|\nabla u - \nabla u_\varepsilon\|_{L^p(\Omega)} < \varepsilon$, and using in the very same way (4.16$'$) we readily get $\|\nabla u - \nabla u_\varepsilon\|_{L^p(\Omega_\varepsilon)} < \varepsilon$, which implies the L^p_{loc} convergence stated above. The proof is then concluded. □

REMARK 4.8. A further generalization of Theorem B for the case of a generic open set $\Omega \subseteq \mathbb{R}^N$ is possible. Namely, we can have the sequence u_j in $\mathrm{SBV}^p(\Omega)$, instead of $\mathrm{SBV}(\Omega) \cap \mathrm{SBV}^p_{\mathrm{loc}}(\Omega)$. But, in this case, both the BV and the L^p convergences in (1.3) become a $\mathrm{BV}_{\mathrm{loc}}$ and a L^p_{loc} convergence. To prove this, just define the open sets Ω_ε as in the proof of Theorem 4.7, notice that v_4 is Lipschitz in a neighborhood of $\partial\Omega_\varepsilon$ by construction, and replace v_4 with some function coinciding with it in Ω_ε, and Lipschitz in $\Omega\setminus\Omega_\varepsilon$. Of course, the function v_4 belongs to $\mathrm{SBV}^p(\Omega)$, but we do not have any estimate of $u - u_\varepsilon$ in $\Omega\setminus\Omega_\varepsilon$, so that even the BV estimate of $u - u_\varepsilon$ remains valid only in Ω_ε, thus we have only a $\mathrm{BV}_{\mathrm{loc}}$ estimate.

5. THE PROOF OF THEOREM C

This section is devoted to present the proof of Theorem C. In our construction, we will make use of Theorem B, of Theorem 1.1 (in particular, the part 2 by Cortesani and Toader), and of the following two technical lemmas.

LEMMA 5.1. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and $M_0 \subset\subset \Omega$ a C^1 manifold (possibly with boundary). Given $\delta > 0$ and a neighborhood $A \subset\subset \Omega$ of M_0, there exist a diffeomorphism $\Phi : \Omega \mapsto \Omega$, with $\Phi(x) = x$ outside of A, and a relatively open, C^1 set $M \subseteq M_0$ without boundary such that*

$$\|\Phi - Id\|_{C^1(\bar{\Omega};\,\mathbb{R}^N)} + \|\Phi^{-1} - Id\|_{C^1(\bar{\Omega};\,\mathbb{R}^N)} < \delta,$$

$$\mathscr{H}^{N-1}(M_0 \backslash M) < \delta, \quad \Phi(M) = \bigcup_{i=1}^{k} Q_i,$$

where the Q_i are $(N-1)$-dimensional open cubes with pairwise disjoint closures.

LEMMA 5.2. *Let $u \in \mathrm{SBV}^p(\Omega) \cap \mathrm{C}^1(\Omega \backslash \overline{J_u}) \cap W^{1,\infty}(\Omega \backslash \overline{J_u})$ with a polyhedral jump set $J_u \subset\subset \Omega$ (i.e., J_u is the intersection of Ω with a finite union of $(N-1)$-dimensional simplexes). Given $\varepsilon > 0$ there exists a function $u_\varepsilon \in \mathrm{SBV}^p(\Omega) \cap \mathrm{C}^\infty(\Omega \backslash \overline{J_{u_\varepsilon}}) \cap W^{1,\infty}(\Omega \backslash \overline{J_{u_\varepsilon}})$ such that $J_{u_\varepsilon} \subset\subset \Omega$ is a C^1 manifold with C^1 boundary and*

$$\|u - u_\varepsilon\|_{\mathrm{BV}(\Omega)} < \varepsilon, \quad \|\nabla u - \nabla u_\varepsilon\|_{L^p(\Omega;\,\mathbb{R}^N)} < \varepsilon, \quad \mathscr{H}^{N-1}(J_u \Delta J_{u_\varepsilon}) < \varepsilon.$$

Moreover, if $\overline{\Pi}$ is any given hyperplane in $\mathbb{R}^N$, we can build the function u_ε in such a way that $J_{u_\varepsilon} \backslash \overline{\Pi} \subset\subset \mathbb{R}^N \backslash \overline{\Pi}$, that is, the part of J_{u_ε} which is not contained in $\overline{\Pi}$ is a strictly positive distance apart from it.

The first lemma is a variant of a well know result stated in [10, Th. 3.1.23], in particular it can be deduced at once from [1, Th. 3.1]. The second one, instead, is a technical approximation result; notice that the result is not trivial because a polyhedral set is not a C^1 manifold, since different simplexes might intersect with each other and with $\partial\Omega$. To keep this section simple, we postpone the proof of Lemma 5.2 to the Appendix.

PROOF OF THEOREM C. For the sake of clarity, we will divide this proof in some steps. First, we will consider the case when u is compactly supported, then we will deduce the general case.

Part A. The case of u compactly supported in Ω.

Step I. The set M and the function u_1 from Theorems A and B.
First of all, we fix an arbitrary $\varepsilon > 0$ and we select a C^1 manifold $M_0 \subset\subset \Omega$ with C^1 boundary in such a way that

$$\mathscr{H}^{N-1}(J_u \Delta M_0) < \frac{\varepsilon}{3}, \quad |Du|(J_u \backslash M_0) < \frac{\varepsilon}{5}. \tag{5.1}$$

Applying Lemma 5.1 with some constant $\delta \ll \varepsilon$ to be specified later, we obtain another C^1 manifold $M \subseteq M_0$ without boundary, a diffeomorphism $\Phi : \Omega \mapsto \Omega$ coinciding with the identity map outside a compact subset of Ω, and finitely many disjoint open $(N-1)$-dimensional simplexes $\{Q_i\}_{i=1,\dots,k}$ such that

$$\mathscr{H}^{N-1}(J_u \Delta M) < \frac{\varepsilon}{2}, \quad |Du|(J_u \backslash M) < \frac{\varepsilon}{4}, \tag{5.2}$$

as well as

$$\|\Phi - Id\|_{C^1(\bar{\Omega};\mathbb{R}^N)} + \|\Phi^{-1} - Id\|_{C^1(\bar{\Omega};\mathbb{R}^N)} < \delta, \quad \Phi(M) = \bigcup_{i=1}^{k} Q_i. \tag{5.3}$$

In fact, the properties (5.3) are directly given by Lemma 5.1, and the first inequality in (5.2) comes from the corresponding one in (5.1) as soon as $\delta < \varepsilon/6$, while the second inequality in (5.2) comes from the corresponding one in (5.1) if δ is small enough, because $D^s u$ is a finite measure, absolutely continuous with respect to $\mathscr{H}^{N-1}$.

Lemma 4.5 (which is nothing else than Theorem A and the first three steps of Theorem B) provides then us with a function $u_1 \in \mathrm{SBV}(\Omega) \cap \mathrm{C}^\infty(\Omega\backslash M) \cap W^{1,\infty}(\Omega\backslash M)$ such that

$$\mathscr{H}^{N-1}(M\backslash J_{u_1}) = 0, \quad \|u - u_1\|_{\mathrm{BV}(\Omega)} < 4\varepsilon. \tag{5.4}$$

Notice that, since $u_1 \in \mathrm{SBV}(\Omega) \cap W^{1,\infty}(\Omega\backslash M)$, then of course it is also $u_1 \in \mathrm{SBV}^p(\Omega)$.

Step II. The function v_1 from Theorem 1.1.
Let us now set

$$v := u \circ \Phi^{-1} - u_1 \circ \Phi^{-1}.$$

By the first estimate in (5.2) and the last one in (5.4) we have that, if δ is sufficiently small,

$$\mathscr{H}^{N-1}(J_v\backslash\Phi(M)) < \varepsilon, \quad \|v\|_{\mathrm{BV}(\Omega)} < 8\varepsilon. \tag{5.5}$$

Moreover, of course v is still compactly supported in Ω and $v \in \mathrm{SBV}^p(\Omega)$, however we have no a priori estimate on the L^p norm of ∇v.

Let us now denote by ν_i the normal vector to each of the simplexes Q_i, and set $P_i = \{x + t\nu_i : x \in Q_i, t \in (-\eta, \eta)\}$, where η is a sufficiently small parameter to be chosen. In particular, we take η so small that the parallelepipeds P_i are pairwise disjoint.

Since the function v is bounded by construction, we can apply to it the result by Cortesani and Toader, Theorem 1.1, finding a sequence $\{f_j\}$ of SBV^p functions with polyhedral jump sets, with $f_j \in \mathrm{C}^\infty(\Omega\backslash\overline{J_{f_j}}) \cap W^{1,\infty}(\Omega\backslash\overline{J_{f_j}})$, such that

$$f_j \xrightarrow[L^1(\Omega)]{} v, \quad \nabla f_j \xrightarrow[L^p(\Omega)]{} \nabla v \tag{5.6}$$

and satisfying property (1.2) for any admissible function g. Notice that, since v is compactly supported in Ω, again by multiplication by a smooth cut-off function we can assume without loss of generality that all the functions f_j are supported

inside some given open set $\Omega' \subset\subset \Omega$. We claim that, by setting $v_1 = f_j$ for some $j \gg 1$, one has

$$\|\nabla v - \nabla v_1\|_{L^p(\Omega)} < \varepsilon, \quad \|v_1\|_{\mathrm{BV}(\Omega)} \le 2\|v\|_{\mathrm{BV}(\Omega)} \le 16\varepsilon, \tag{5.7}$$
$$\mathscr{H}^{N-1}\Big(J_{v_1} \Big\backslash \bigcup_{i=1}^{k} P_i\Big) < \varepsilon.$$

In fact, the first inequality in (5.7) is obvious for j big enough by (5.6). Instead, the validity of the second inequality in (5.7) for $j \gg 1$ comes by (5.6) and applying (1.2) with the function $g(x,a,b,v) = |b-a|$, since this implies that

$$\begin{aligned}\limsup_{j\to\infty} \|f_j\|_{\mathrm{BV}(\Omega)} &= \limsup_{j\to\infty} \left(\|f_j\|_{L^1(\Omega)} + \|\nabla f_j\|_{L^1(\Omega)} + |D^s f_j|(\Omega)\right)\\ &= \|v\|_{L^1(\Omega)} + \|\nabla v\|_{L^1(\Omega)} + \limsup_{j\to\infty} \int_{J_{f_j}\cap\bar{\Omega}'} g(x, f_j^+, f_j^-, \nu_{f_j})\, d\mathscr{H}^{N-1}\\ &\le \|v\|_{\mathrm{BV}(\Omega)}.\end{aligned}$$

Finally, the third property of (5.7) for $j \gg 1$ comes by applying again (1.2), this time with the u.s.c. function $g(x,a,b,v)$ which coincides with 0 whenever $x \in \bigcup_{i=1,\dots,k} P_i$, and with 1 otherwise: indeed,

$$\begin{aligned}\limsup_{j\to\infty} \mathscr{H}^{N-1}\Big(J_{f_j} \Big\backslash \bigcup_{i=1}^{k} P_i\Big) &= \limsup_{j\to\infty} \int_{J_{f_j}\cap\bar{\Omega}'} g(x, f_j^+, f_j^-, \nu_{f_j})\, d\mathscr{H}^{N-1}\\ &\le \int_{J_v\cap\bar{\Omega}'} g(x, v^+, v^-, \nu_v)\, d\mathscr{H}^{N-1}\\ &= \mathscr{H}^{N-1}\Big(J_v \Big\backslash \bigcup_{i=1}^{k} P_i\Big) < \varepsilon,\end{aligned}$$

since $\bigcup_{i=1}^{k} P_i \supseteq \Phi(M)$ and recalling (5.5). For future reference we observe that for every $1 \le i \le k$

$$\|Tr(v_1; Q_i + \eta\nu_i) - Tr(v_1; Q_i - \eta\nu_i)\|_{L^1(Q_i)} \le |Dv_1|(\overline{P_i}), \tag{5.8}$$

where $Tr(v_1; Q_i + \eta\nu_i)(x)$ and $Tr(v_1; Q_i - \eta\nu_i)(x)$ denote the upper trace of v_1 on $Q_i + \eta\nu_i$, and its lower trace on $Q_i - \eta\nu_i$ (where "upper" and "lower" are intended in the direction of ν_i).

Step III. The function v_2.
Let $\Psi : \Omega \to \Omega$ be a piecewise affine function which is a bijection from $\Omega \backslash \bigcup_i \overline{P_i}$ to $\Omega \backslash \bigcup_i \overline{Q_i}$, and such that $\Psi(x) = x$ unless x has distance at most $\sqrt{N}\eta$ from $\bigcup_i \overline{P_i}$, while $\Psi(\overline{P_i}) = \Psi(\partial(\overline{P_i})) = \overline{Q_i}$ for every i. In particular, we can take such a function so that, for every i, the function Ψ maps each of the two simplexes

$Q_i \pm \eta\nu_i$ in an affine way onto $Q_i' \subseteq Q_i$, obtained from Q_i with an homothety of factor $1-\eta$. With a slight abuse of notation we denote by Ψ^{-1} the inverse of the restriction of Ψ to $\Omega\setminus\bigcup_i \overline{P}_i$, so Ψ^{-1} maps $\Omega\setminus\Phi(\overline{M}) = \Omega\setminus\bigcup_i \overline{Q_i}$ onto $\Omega\setminus\bigcup_i \overline{P}_i$. It is very simple to check that, as soon as η is small enough, one can find such a map Ψ so that $\|D\Psi\|_{L^\infty(\Omega)} \le 2$ and $\|D\Psi^{-1}\|_{L^\infty(\Omega\setminus\bigcup_i \overline{Q_i})} \le 2$. We now set

$$v_2 := v_1 \circ \Psi^{-1}.$$

From our construction it follows immediately that $v_2 \in \mathrm{SBV}^p(\Omega\setminus\Phi(\overline{M}))$, and clearly v_2 can be uniquely extended to a function in $\mathrm{SBV}^p(\Omega)$, which we still denote v_2. Notice that $J_{v_2} \subset\subset \Omega$ is a polyhedral set. Moreover from (5.7) it follows that

$$\|v_2\|_{\mathrm{BV}(\Omega\setminus\Phi(\overline{M}))} \le 2^{N+4}\varepsilon, \quad \mathscr{H}^{N-1}(J_{v_2}\setminus\Phi(M)) \le 2^{N-1}\varepsilon. \tag{5.9}$$

To conclude this step, we want an estimate of the BV norm of v_2 on the whole Ω, as well as of the L^p norm of $\nabla v - \nabla v_2$. The latter is very easy to obtain; indeed, by construction, (5.7) yields that if η is very small then

$$\|\nabla v - \nabla v_2\|_{L^p(\Omega)} < 2\varepsilon. \tag{5.10}$$

Concerning the BV norm of v_2, by recalling (5.9), (5.8) and (5.7), the definition of the Q_i', and the definition of Ψ on ∂P_i, we get

$$\begin{aligned}
\|v_2\|_{\mathrm{BV}(\Omega)} &= \|v_2\|_{\mathrm{BV}(\Omega\setminus\Phi(\overline{M}))} + \sum_{i=1}^k \int_{Q_i} |v_2^+ - v_2^-|\, d\mathscr{H}^{N-1} \\
&\le C\varepsilon + \sum_{i=1}^k \int_{Q_i'} |v_2^+ - v_2^-|\, d\mathscr{H}^{N-1} + \int_{Q_i\setminus Q_i'} |v_2^+ - v_2^-|\, d\mathscr{H}^{N-1} \\
&\le C\varepsilon + \sum_{i=1}^k \|Tr(v_1; Q_i + \eta\nu_i) - Tr(v_1; Q_i - \eta\nu_i)\|_{L^1(Q_i)} \\
&\quad + C'\eta\mathscr{H}^{N-1}(Q_i)\|v_1\|_{L^\infty} \\
&\le C\varepsilon + |Dv_1|\Big(\bigcup_i \overline{P_i}\Big) + C'\eta\mathscr{H}^{N-1}\Big(\bigcup_i Q_i\Big)\|v_1\|_{L^\infty} \le C'\varepsilon,
\end{aligned} \tag{5.11}$$

where C and C' are two constants depending only on N (that we do not write explicitely just for the sake of shortness), and the last inequality is true as soon as η is small enough (keep in mind that v_1 is bounded by construction).

Step IV. The final functions w, w_ε and u_ε.
Let us now define $w = u_1 \circ \Phi^{-1} + v_2$, which is by construction a function in $\mathrm{SBV}^p(\Omega)$. From Steps I and III we know that $J_w \subseteq \Phi(M) \cup J_{v_2}$ and the latter

set is a polyhedral set compactly contained in Ω. Moreover, from (5.2) and (5.9) we have that

$$\mathscr{H}^{N-1}((\Phi(M) \cup J_{v_2}) \Delta \Phi(J_u)) \leq 2^N \varepsilon. \tag{5.12}$$

We claim that

$$\|u \circ \Phi^{-1} - w\|_{\mathrm{BV}(\Omega)} \leq (C' + 5)\varepsilon, \quad \|\nabla(u \circ \Phi^{-1}) - \nabla w\|_{L^p(\Omega)} < 2\varepsilon. \tag{5.13}$$

The first inequality follows by (5.11), (5.4) and (5.3), while the second is simply (5.10), since by definition

$$u \circ \Phi^{-1} - w = u \circ \Phi^{-1} - u_1 \circ \Phi^{-1} - v_2 = v - v_2.$$

Notice that, by Lemma 4.3 and Corollary 2.4, up to an arbitrarily small modification both in the BV norm and in the L^p norm of the absolutely continuous part of the gradient, we can assume that $J_w = \Phi(M) \cup J_{v_2}$ (up to $\mathscr{H}^{N-1}$-negligible sets) and that $w \in \mathrm{C}^\infty(\Omega \backslash J_w)$.

Keep in mind that, since $u_1 \in \mathrm{C}^\infty(\Omega \backslash \overline{J_{u_1}}) \cap W^{1,\infty}(\Omega \backslash \overline{J_{u_1}})$, $v_1 \in \mathrm{C}^\infty(\Omega \backslash \overline{J_{v_1}}) \cap W^{1,\infty}(\Omega \backslash \overline{J_{v_1}})$, Φ is a diffeomorphism, and Ψ is piecewise affine, then w also belongs to $W^{1,\infty}(\Omega \backslash \overline{J_w})$. As a consequence, we can apply Lemma 5.2 to w, finding a function $w_\varepsilon \in \mathrm{SBV}^p(\Omega) \cap \mathrm{C}^\infty(\Omega \backslash \overline{J_{w_\varepsilon}}) \cap W^{1,\infty}(\Omega \backslash \overline{J_{w_\varepsilon}})$ such that $J_{w_\varepsilon} \subset\subset \Omega$ is a manifold of class C^1 with C^1 boundary and satisfying

$$\|w - w_\varepsilon\|_{\mathrm{BV}(\Omega)} < \varepsilon, \quad \|\nabla w - \nabla w_\varepsilon\|_{L^p(\Omega;\mathbb{R}^N)} < \varepsilon, \quad \mathscr{H}^{N-1}(J_w \Delta J_{w_\varepsilon}) < \varepsilon. \tag{5.14}$$

We can finally set the final function $u_\varepsilon = w_\varepsilon \circ \Phi$. Then, keeping in mind (5.13), (5.3), (5.14) and (5.12), as well as the fact that $J_w = \Phi(M) \cup J_{v_2}$, we immediately obtain

$$\|u - u_\varepsilon\|_{\mathrm{BV}(\Omega)} < C''\varepsilon, \quad \|\nabla u - \nabla u_\varepsilon\|_{L^p(\Omega;\mathbb{R}^N)} < C''\varepsilon, \quad \mathscr{H}^{N-1}(J_u \Delta J_{u_\varepsilon}) < C''\varepsilon,$$

for a suitable, purely dimensional constant C''. The thesis is then obtained in this case.

Part B. The general case.
Let us now pass to consider the general case, which only requires few simple arguments to be reduced to the preceding, particular one. We divide for simplicity also this part in few steps.

Step I. The case of $J_u \subset\subset \Omega$.
First of all, let us assume that u is not necessarily compactly supported in Ω, but the jump set of u is compactly contained in Ω. In this case, we can argue more or less as in Lemma 2.2; that is, we take two open sets A_1 and A_2 such that $J_u \subset\subset A_1 \subset\subset A_2 \subset\subset \Omega$, and we use a smooth cut-off function to write $u = u_1 + u_2$, with $u_1 \in \mathrm{SBV}^p(\Omega)$ supported in $\overline{A_2}$, and $u_2 \in W^{1,p}(\Omega)$ supported in $\Omega \backslash A_1$. The conclusion is then obtained just applying Part A to the function u_1, and Meyers and Serrin Theorem to u_2.

Step II. The case of $\Omega = \mathbb{R}^N_+$: separating the jump set from the boundary.
Let us now consider the case when $\Omega = \mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x_N > 0\}$: we aim to approximate a given $u \in \mathrm{SBV}^p(\Omega)$ with another function whose jump set is a positive distance apart from $\partial\Omega$, that is, we claim the existence of a function $v \in \mathrm{SBV}^p(\Omega)$ such that

$$\begin{aligned} &\mathrm{dist}(J_v, \partial\Omega) > 0, \quad \|v-u\|_{\mathrm{BV}(\Omega)} < \varepsilon, \\ &\|\nabla v - \nabla u\|_{L^p(\Omega)} < \varepsilon, \quad \mathscr{H}^{N-1}(J_v \Delta J_u) < \varepsilon. \end{aligned} \tag{5.15}$$

First of all, as already done at the beginning of the proof of Theorem B, we can assume without loss of generality that the support of u is bounded. Then, via multiplication with a smooth cut-off function, we can also write $u = u_1 + u_2$, with $u_1, u_2 \in \mathrm{SBV}^p(\Omega)$ and so that u_1 is compactly supported in Ω and

$$J_u = J_{u_1} \cup J_{u_2}, \quad \mathscr{H}^{N-1}(J_{u_2}) < \delta, \quad |D^s u_2|(J_{u_2}) < \delta, \tag{5.16}$$

for some $\delta = \delta(\varepsilon)$ to be specified later. Then, we let $u_3 \in \mathrm{SBV}^p(\mathbb{R}^N)$ be the extension of u_2 by symmetry through the hyperplane $\{x_N = 0\} = \partial\Omega$, that is, $u_3(x, y) = u_2(x, |y|)$. Notice that by definition

$$\mathscr{H}^{N-1}(J_{u_3}) < 2\delta, \quad |D^s u_3|(J_{u_3}) < 2\delta.$$

Notice also that, since the support of u is bounded, then u_3 is compactly supported in $\mathbb{R}^N$ (while u_2 is not compactly supported in Ω). Hence, we can apply as before Theorem 1.1 to the function u_3 so to find a function $\tilde{u}_3 \in \mathrm{SBV}^p(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N \setminus \overline{J_{u_3}}) \cap W^{1,\infty}(\mathbb{R}^N \setminus \overline{J_{u_3}})$ with polyhedral jump set J_{u_3} satisfying

$$\|u_3 - \tilde{u}_3\|_{L^1(\mathbb{R}^N)} < \delta, \quad \|\nabla u_3 - \nabla \tilde{u}_3\|_{L^1(\mathbb{R}^N)} < \delta, \quad \|\nabla u_3 - \nabla \tilde{u}_3\|_{L^p(\mathbb{R}^N)} < \delta, \tag{5.17}$$

and by (1.2) we have also

$$\mathscr{H}^{N-1}(J_{\tilde{u}_3}) \le 2\mathscr{H}^{N-1}(J_{u_3}) < 4\delta, \quad |D^s \tilde{u}_3|(J_{\tilde{u}_3}) \le 2|D^s u_3|(J_{u_3}) < 4\delta. \tag{5.18}$$

Putting together the last estimates, we immediately deduce

$$\|u_3 - \tilde{u}_3\|_{\mathrm{BV}(\mathbb{R}^N)} < 8\delta. \tag{5.19}$$

Let us now apply Lemma 5.2 to the function $\tilde{u}_3$ with the hyperplane $\Pi = \{x_N = 0\} = \partial\Omega$, finding a function $\tilde{u}_{3,\delta} \in \mathrm{SBV}^p(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N \setminus \overline{J_{\tilde{u}_{3,\delta}}}) \cap W^{1,\infty}(\mathbb{R}^N \setminus \overline{J_{\tilde{u}_{3,\delta}}})$, satisfying

$$\|\tilde{u}_3 - \tilde{u}_{3,\delta}\|_{\mathrm{BV}(\mathbb{R}^N)} < \delta, \quad \|\nabla \tilde{u}_3 - \nabla \tilde{u}_{3,\delta}\|_{L^p(\mathbb{R}^N)} < \delta, \quad \mathscr{H}^{N-1}(J_{\tilde{u}_3} \Delta J_{\tilde{u}_{3,\delta}}) < \delta. \tag{5.20}$$

and such that the part of the jump set of $\tilde{u}_{3,\delta}$ not contained in $\partial\Omega$ has positive distance from $\partial\Omega$ itself. To conclude, it is enough to define $v = u_1 + \tilde{u}_{3,\delta}$ on Ω.

Indeed, the jump set of v has positive distance from $\partial\Omega$ by construction, and of course

$$J_{u_1}\backslash J_{\tilde{u}_{3,\delta}} \subseteq J_v \subseteq J_{u_1} \cup J_{\tilde{u}_{3,\delta}}. \tag{5.21}$$

Moreover,

$$\begin{aligned}\|v-u\|_{\mathrm{BV}(\Omega)} &= \|\tilde{u}_{3,\delta}-u_2\|_{\mathrm{BV}(\Omega)} \le \|\tilde{u}_{3,\delta}-\tilde{u}_3\|_{\mathrm{BV}(\Omega)} + \|\tilde{u}_3-u_2\|_{\mathrm{BV}(\Omega)} \\ &\le \|\tilde{u}_{3,\delta}-\tilde{u}_3\|_{\mathrm{BV}(\mathbb{R}^N)} + \|\tilde{u}_3-u_3\|_{\mathrm{BV}(\mathbb{R}^N)} < 9\delta\end{aligned}$$

by (5.20) and (5.19), and in the very same way

$$\|\nabla v-\nabla u\|_{L^p(\Omega)} \le \|\nabla\tilde{u}_{3,\delta}-\nabla\tilde{u}_3\|_{L^p(\mathbb{R}^N)} + \|\nabla\tilde{u}_3-\nabla u_3\|_{L^p(\mathbb{R}^N)} < 2\delta$$

by (5.20) and (5.17). Finally, by (5.16) and (5.21) we have

$$J_v\Delta J_u \subseteq ((J_{u_1}\cup J_{\tilde{u}_{3,\delta}})\backslash(J_{u_1}\cup J_{u_2})) \cup ((J_{u_1}\cup J_{u_2})\backslash(J_{u_1}\backslash J_{\tilde{u}_{3,\delta}})) \subseteq J_{\tilde{u}_{3,\delta}} \cup J_{u_2},$$

so by (5.16), (5.20) and (5.18) we have

$$\mathscr{H}^{N-1}(J_v\Delta J_u) < 6\delta.$$

In conclusion, (5.15) holds as soon as soon as we have chosen $\delta = \varepsilon/9$, and the step is concluded.

Step III. Conclusion.
It is easy to conclude by putting together the last two steps. Indeed, let $u \in \mathrm{SBV}^p(\Omega)$ be a given function. First of all, as already done several times, we select $\tilde{u} \in \mathrm{SBV}^p(\Omega)$ with bounded support and such that

$$\|\tilde{u}-u\|_{\mathrm{BV}(\Omega)} < \varepsilon, \quad \|\nabla\tilde{u}-\nabla u\|_{L^p(\Omega)} < \varepsilon, \quad \mathscr{H}^{N-1}(J_{\tilde{u}}\Delta J_u) < \varepsilon. \tag{5.22}$$

Since Ω has locally Lipschitz boundary, we can find another set $\Omega' \subseteq \Omega$, bounded and with Lipschitz boundary, in such a way that $\tilde{u} \equiv 0$ in $\Omega\backslash\Omega'$, and of course $\tilde{u} \in \mathrm{SBV}^p(\Omega')$. By compactness, we can find finitely many smooth, bounded, open sets $\Omega_i \subseteq \mathbb{R}^N$, $0 \le i \le K$, so that $\Omega' \subseteq \Omega_0 \cup \Omega_1 \cup \cdots \cup \Omega_K$, $\Omega_0 \subset\subset \Omega$, and for every $1 \le i \le K$ there is a bi-Lipschitz homeomorphism $\Phi_i : \mathbb{R}^N \to \mathbb{R}^N$ such that

$$\begin{aligned}&\Phi_i(\Omega_i) = (-1,1)^N, \quad \Phi_i(\Omega_i\cap\Omega) = (-1,1)^{N-1}\times(0,1), \\ &\Phi_i(\Omega_i\cap\partial\Omega) = (-1,1)^{N-1}\times\{0\}.\end{aligned}$$

Moreover, we can select a smooth partition of unity $\{\eta_i\}_{i=0,1,\dots,K}$ associated with the covering of Ω'. We write then $\tilde{u} = u_0 + u_1 + \cdots + u_K$, where for every $0 \le i \le K$ we have set $u_i = \eta_i\tilde{u} \in \mathrm{SBV}^p(\Omega_i)$.

Let us now take any $i > 0$; we have that $u_i \circ \Phi_i^{-1} \in \mathrm{SBV}^p(\mathbb{R}^N_+)$, so by Step II and in particular (5.15) we find a function $v_i \in \mathrm{SBV}(\mathbb{R}^N_+)$ such that

$$\|v_i - u_i \circ \Phi_i^{-1}\|_{\mathrm{BV}(\mathbb{R}^N_+)} < \varepsilon, \quad \|\nabla v_i - \nabla u_i \circ \Phi_i^{-1}\|_{L^p(\mathbb{R}^N_+)} < \varepsilon,$$
$$\mathscr{H}^{N-1}(J_{v_i} \Delta J_{u_i \circ \Phi_i^{-1}}) < \varepsilon,$$

and the jump set of v_i is a positive distance apart from $\partial \mathbb{R}^N_+$; hence, the function $v_i \circ \Phi_i$ belongs to $\mathrm{SBV}^p(\Omega_i)$, and its jump set is a positive distance apart from $\partial\Omega_i \cap \partial\Omega'$, so in particular from $\partial\Omega'$. As a consequence, if we define

$$v = u_0 + \sum_{i=1}^{K} v_i \circ \Phi_i,$$

then we have $v \in \mathrm{SBV}^p(\Omega')$, and the jump set J_v of v is a positive distance apart from $\partial\Omega'$, so in particular $J_v \subset\subset \Omega$. Moreover, from the decomposition of $\tilde{\boldsymbol{u}}$, also recalling (5.22), we deduce that

$$\|u - v\|_{\mathrm{BV}(\Omega)} < C\varepsilon, \quad \|\nabla u - \nabla v\|_{L^p(\Omega)} < C\varepsilon, \quad \mathscr{H}^{N-1}(J_u \Delta J_v) < C\varepsilon,$$

where C is a geometric constant, only depending on the sets Ω_i and on the bi-Lipschitz constants of the functions Φ_i. We can then simply apply Step I to the function v, and the proof is concluded. □

6. An application of our result

In this last section we present an application of our first result, Theorem A. Let $\Omega \subseteq \mathbb{R}^N$ be an open set, and for $\varepsilon > 0$ let us denote by $\mathscr{G}_\varepsilon$ any finite collection of disjoint open cubes $Q \subseteq \Omega$ with side length ε and arbitrary orientation. Given a function $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\varepsilon > 0$, we consider the quantity

$$\kappa_\varepsilon(u) := \varepsilon^{N-1} \sup_{\mathscr{G}_\varepsilon} \sum_{Q \in \mathscr{G}_\varepsilon} \fint_Q |u - u_Q| \, dx,$$

denoting $u_Q = \fint_Q u$. This quantity was introduced in [2], where it was proved that in the special case of the characteristic function of a measurable set the following formula holds

$$\lim_{\varepsilon \to 0} \kappa_\varepsilon(\chi_E) = \frac{1}{2} P(E),$$

where $P(E)$ denotes the perimeter of the set E. This formula was then extended in [11] to the case of a function $u \in \mathrm{SBV}_{\mathrm{loc}}(\Omega)$ with "well behaved" jump set. More precisely, the following result holds.

THEOREM 6.1. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and $u \in \mathrm{SBV}_{\mathrm{loc}}(\Omega)$ such that $\mathscr{L}^N(\overline{J_u}) = 0$. Then*

$$\lim_{\varepsilon \to 0} \kappa_\varepsilon(u) = \frac{1}{4}\int_\Omega |\nabla u|\, dx + \frac{1}{2}|D^s u|(\Omega). \tag{6.1}$$

As a consequence of Theorem A, we can show that the above representation formula holds with no assumptions on J_u.

COROLLARY 6.2. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set, and let $u \in \mathrm{SBV}_{\mathrm{loc}}(\Omega)$. Then* (6.1) *holds.*

PROOF. Let us assume for a moment that $u \in \mathrm{SBV}(\Omega)$. Then, given any $\delta > 0$, Theorem A provides us with a function $v \in \mathrm{SBV}(\Omega)$ such that $\|u - v\|_{\mathrm{BV}(\Omega)} < \delta$, and $\mathscr{H}^{N-1}(\overline{J_v} \setminus J_v) = 0$, so that in particular $\mathscr{L}^N(\overline{J_v}) = 0$ and then (6.1) holds for v. Given now any cube Q of side ε, we can evaluate

$$\left| \fint_Q |u - u_Q| - \fint_Q |v - v_Q| \right| \le \fint_Q |(u - v) - (u_Q - v_Q)| = \fint_Q |(u - v) - (u - v)_Q|$$

$$\le \frac{1}{2\varepsilon^{N-1}} |D(u - v)|(Q),$$

where the last inequality comes by the Poincaré inequality in a cube of side ε, which holds with constant $\varepsilon/2$. For any finite family $\mathscr{G}_\varepsilon$ of cubes of side ε, then, we have

$$\varepsilon^{N-1} \left| \sum_{Q \in \mathscr{G}_\varepsilon} \fint_Q |u - u_Q| - \sum_{Q \in \mathscr{G}_\varepsilon} \fint_Q |v - v_Q| \right| \le \frac{1}{2} \sum_{Q \in \mathscr{G}_\varepsilon} |Du - Dv|(Q) \le \frac{\delta}{2},$$

which implies that $|\kappa_\varepsilon(u) - \kappa_\varepsilon(v)| \le \delta/2$. Applying (6.1) to v, and sending first ε and then δ to 0, we directly obtain the validity of (6.1) also for u.

Suppose now that $u \notin \mathrm{SBV}(\Omega)$, so that we have to show $\kappa_\varepsilon(u) \to \infty$. Fix any open set $\Omega' \subset\subset \Omega$: since $u \in \mathrm{SBV}(\Omega')$, the very same argument as above, only considering cubes in Q', implies that

$$\liminf_{\varepsilon \to 0} \kappa_\varepsilon(u) \ge \frac{1}{4}\int_{\Omega'} |\nabla u|\, dx + \frac{1}{2}|D^s u|(\Omega'),$$

and letting $\Omega' \uparrow \Omega$ the conclusion follows. □

It is actually possible to estimate the behaviour of $\kappa_\varepsilon(u)$ even for the case of a function $u \in \mathrm{BV}(\Omega)$, thus possibly with a non vanishing Cantor part. In this case, by means of Theorem 6.1 and of a suitable approximation argument (see [12]), one can show that

$$\frac{1}{4}|Du|(\Omega) \le \liminf_{\varepsilon \to 0} \kappa_\varepsilon(u) \le \limsup_{\varepsilon \to 0} \kappa_\varepsilon(u) \le \frac{1}{2}|Du|(\Omega). \tag{6.2}$$

Generalizing [11, Ex. 2.2], we can now show that this estimate is sharp: in fact, if u has a non-vanishing Cantor part, then any limit between $1/4|Du|(\Omega)$ and $1/2|Du|(\Omega)$ is possible.

EXAMPLE 6.3. Given any real sequence λ_n, with $0 < \lambda_n < 1/2$ for every n, we consider the following Cantor-like function. We are going to define inductively the intervals J_i^n for any $n \in \mathbb{N}$ and $0 < i \le 2^{n-1}$, and the intervals I_i^n for any $n \in \mathbb{N}$ and $0 < i \le 2^n$. For $n = 1$, we let $I_1^1 = [0, \lambda_1]$, $J_1^1 = (\lambda_1, 1 - \lambda_1)$, and $I_2^1 = [1 - \lambda_1, 1]$. Then, once we have defined any interval I_i^n, we subdivide it in three parts, namely, I_{2i-1}^{n+1}, J_i^{n+1} and I_{2i}^{n+1}: the open interval J_i^{n+1} has the same center as I_i^n, while the two closed intervals I_{2i-1}^{n+1} and I_{2i}^{n+1} are respectively on its left and on its right, and the measure of each of them is a portion λ_{n+1} of the measure of I_i^n.

We define then also a sequence of continuous functions u_n. More precisely, given any $n \ge 1$, we define $u_n(x) = 0$ for $x \le 0$, $u_n(x) = 1$ for $x \ge 1$,

$$u_n(x) = \frac{2i-1}{2^k} \quad \text{for } x \in J_i^k, \text{ with } k \le n \text{ and } 1 \le i \le 2^{k-1},$$

and u_n is affine in each interval I_i^n for $1 \le i \le 2^n$. It is easily checked that u_n uniformly converges to a function $u \in \mathrm{BV}(\mathbb{R})$, and moreover Du is purely Cantor (that is, the absolutely continuous part and the jump part of Du are both 0), and $|Du|(\mathbb{R}) = 1$.

Suppose for a moment that the sequence λ_n takes constantly the value $0 < \lambda < 1/2$. In this case, a simple calculation ensures that, defining

$$\kappa^-(\lambda) = \liminf_{\varepsilon \to 0} \kappa_\varepsilon(u), \quad \kappa^+(\lambda) = \limsup_{\varepsilon \to 0} \kappa_\varepsilon(u),$$

one has that $\lambda \mapsto \kappa^\pm(\lambda)$ are two continuous and decreasing functions in $(0, 1/2)$, satisfying

$$\lim_{\lambda \to 0} \kappa^-(\lambda) = \frac{1}{2}, \quad \lim_{\lambda \to 1/2} \kappa^+(\lambda) = \frac{1}{4}.$$

As a consequence, we have shown that the lim inf and the lim sup in (6.2) can take any value in the open interval

$$\left(\frac{|Du|(\Omega)}{4}, \frac{|Du|(\Omega)}{2} \right).$$

Finally, one can also build an example of $u \in \mathrm{SBV}$ for which $\lim_{\varepsilon \to 0} \kappa_\varepsilon(u) = |Du|(\Omega)/4$ (resp., $\lim_{\varepsilon \to 0} \kappa_\varepsilon(u) = |Du|(\Omega)/2$). This can be obtained by the same construction as above choosing the sequence λ_n converging fast enough to $1/2$ (resp., to 0).

Appendix A. Proof of Lemma 5.2

This final section is devoted to the proof of Lemma 5.2.

Proof of Lemma 5.2. By assumption, the jump set of u is made by finitely many $(N-1)$-dimensional open simplexes. Nevertheless, in order to perform our recursive construction, it is simpler to consider a more general situation, namely, when J_u is made by finitely many $(N-1)$-dimensional polyhedra. In our construction, a 1-dimensional polyhedron in $\mathbb{R}^N$ is simply a segment in $\mathbb{R}^N$, and for every $2 \le n \le N-1$ we recursively define a n-dimensional polyhedron in $\mathbb{R}^N$ as a bounded, connected set, contained in an n-dimensional subspace of $\mathbb{R}^N$, whose boundary is a finite union of $(n-1)$-dimensional polyhedra.

We assume then that J_u is made by K polyhedra of dimension $N-1$, possibly intersecting with each other, and we call Π and $\{\Pi_i\}_{i=1,\dots,K-1}$ the closures of these polyhedra. Since our aim is, roughly speaking, to "separate" these polyhedra, we aim to reduce ourselves to a situation in which one polyhedron is a strictly positive distance apart from the other $K-1$. For simplicity of notations, we assume that the polyhedron Π is contained in the hyperplane $\{x_N = 0\}$. For any $1 \le i \le K-1$, we want now to define a $(N-2)$-dimensional polyhedron $\Gamma_i \subseteq \Pi$; if the intersection between Π_i and Π is empty, we simply set $\Gamma_i = \emptyset$. Otherwise, let us call Θ_i the $(N-1)$-dimensional hyperplane containing Π_i, and let us consider $\Pi \cap \Theta_i$, which is a finite union of $(N-2)$-dimensional closed polyhedra: then, we call Γ_i the union of those which intersect Π_i, so $\Pi \cap \Pi_i \subseteq \Gamma_i \subseteq \Pi \cap \Theta_i$, and both inclusions can be strict. Since, in our construction, we will need to know that the first inclusion is in fact an equality, we make a slight modification of u. More precisely, we fix a constant $\bar{\alpha} > 0$ so small that the set

$$J_u^+ = J_u \cup \bigcup_{i=1}^{K-1} \{(x,t) \in \Theta_i : 0 \le t \le \bar{\alpha}, \mathrm{pr}_i(x,t) \in \Gamma_i\}, \tag{A.1}$$

where $\mathrm{pr}_i : \Theta_i \to \Theta_i \cap \Pi$ is the orthogonal projection, satisfies

$$\mathscr{H}^{N-1}(J_u^+ \Delta J_u) = \mathscr{H}^{N-1}(J_u^+ \backslash J_u) < \frac{\varepsilon}{2K}. \tag{A.2}$$

From Lemma 4.3 we get then $u_1 \in \mathrm{SBV}^p(\Omega) \cap \mathrm{C}^1(\Omega \backslash J_u^+) \cap W^{1,\infty}(\Omega \backslash J_u^+)$ so that

$$\|u_1 - u\|_{\mathrm{BV}(\Omega)} + \|u_1 - u\|_{L^\infty(\Omega)} + \|\nabla u_1 - \nabla u\|_{L^p(\Omega)} < \frac{\varepsilon}{2K}, \quad J_{u_1} = J_u^+. \tag{A.3}$$

Notice that J_{u_1} is not the same set as J_u, but by construction it is still the union of K polyhedra, that for ease of notation we still denote by Π and Π_i; we have only slightly enlarged some of the polyhedra Π_i (actually, we could have even

diminished the total number of polyhedra, since two different ones contained in a same hyperplane could have been glued). Observe that, now, the equality $\Gamma_i = \Pi \cap \Pi_i$ holds true.

Let us now consider Π, which is subdivided by the $(N-2)$-dimensional sets Γ_i in finitely many "zones" $Z_1, Z_2, \dots Z_M$. More precisely, Π is the union of finitely many $(N-1)$-dimensional closed polyhedra Z_j, $1 \le j \le M$, in such a way that

$$\bigcup_{j=1}^{M} \partial Z_j = \partial\Pi \cup \bigcup_{i=1}^{K-1} \Gamma_i.$$

Notice that these zones are uniquely determined.

Let us now fix a small quantity $0 < \alpha < \bar{\alpha}$, to be determined later. Consider the closed, N-dimensional set $\{(x,t) : (x,0) \in \Pi, 0 \le t \le \alpha\}$: thanks to our modification and since $\alpha < \bar{\alpha}$, the sets Π_i divide this set in finitely many N-dimensional polyhedra; in particular, for each $1 \le j \le M$ there is a N-dimensional polyhedron $Z_{j,\alpha}$, one $(N-1)$-dimensional face of which is Z_j. Notice that the union of these $Z_{j,\alpha}$ is not necessarily the whole $\{(x,t) : (x,0) \in \Pi, 0 \le t \le \alpha\}$, there could be also other very small zones appearing if two different Γ_i's have an intersection with positive $(N-2)$-dimensional measure; however, we will not need to take care of these new zones. Observe that, whenever a point (x,t) with $0 < t < \alpha$ belongs to the boundary of some $Z_{j,\alpha}$, then either this point is contained in some Π_i, or $(x,0)$ belongs to the boundary of Π.

We fix now a given polyhedron $Z_{j,\alpha}$, and we want to define a modification $\tilde{u}_j$ of u_1, such that $\tilde{u}_j = u_1$ outside $Z_{j,\alpha}$. First of all, we take a piecewise affine diffeomorphism $\Phi : Z_{j,\alpha} \to Z_j \times [0,\alpha]$, being the identity on Z_j and on the (possibly empty) intersection $Z_{j,\alpha} \cap (\partial\Pi \times [0,\alpha])$: notice that we can do this in such a way that the bi-Lipschitz constant of this diffeomorphism remains bounded when $\alpha \to 0$. It is then simpler to construct a function v on $Z_j \times [0,\alpha]$ and eventually to define $\tilde{u}_j$ as $v \circ \Phi$ on $Z_{j,\alpha}$ and u_1 outside.

Let $\beta \ll \alpha$ be another constant, still to be specified later, and let $Z_j^{INT} \subseteq Z_j$ be given by

$$Z_j^{INT} = \{x \in Z_j : \mathrm{dist}(x, \partial Z_j) \le \beta\}.$$

A simple geometric argument ensures that there exists a diffeomorphism $\Psi : \partial Z_j \times [0,\beta] \to Z_j^{INT}$ with bi-Lipschitz constant which remains bounded for $\beta \to 0$, and in such a way that for every point $P \in \partial Z_j$ the set $\Psi(\{P\} \times [0,\beta])$ is a segment, call it σ_P, with endpoints $\Psi(P,0) = P$ and $\Psi(P,\beta) \in \partial Z_j^{INT} \backslash \partial Z_j$.

The set $Z_j^{INT} \times [0,\alpha]$ is then the union of the rectangles $\sigma_P \times [0,\alpha]$, with P varying in ∂Z_j. Let us then fix a point $P \in \partial Z_j$; notice that the segment σ_P belongs to Π, so to the jump set J_{u_1}, and call $v_- : \sigma_P \to \mathbb{R}$ the lower trace of u_1 on the segment, that is, for every $(y,0) \in \sigma_P$ we have $v^-(y,0) = \lim_{t \nearrow 0} u_1(y,t)$. Notice that the limit exists since $u_1 \in W^{1,\infty}(\Omega \backslash \overline{J_{u_1}})$. Instead, by construction the set

$\Phi^{-1}((\sigma_P\backslash\{P\})\times(0,\alpha])$ does not intersect J_{u_1}, so we can set $v^+:(\sigma_P\backslash\{P\})\times(0,\alpha]\to\mathbb{R}$ as $v^+=u_1\circ\Phi^{-1}$. Notice that v^+ is Lipschitz, thus it extends naturally to the whole $\sigma_P\times[0,\alpha]$: in general, $v^+(y,0)$ and $v^-(y,0)$ do not coincide; they do so, however, if $y\in\partial\Pi$, again by the fact that $u_1\in W^{1,\infty}(\Omega\backslash\overline{J_{u_1}})$. We are then in position to define v on the rectangle $\sigma_P\times[0,\alpha]$, by setting

$$v(y,t\alpha)=(1-t)v^-(y,0)+tv^+(y,\alpha)+v^+(P,t\alpha)-tv^+(P,\alpha)-(1-t)v^+(P,0)$$

for every $(y,0)\in\sigma_P$ and $0\le t\le 1$. Notice that, on the horizontal sides of the rectangle one has

$$v(y,0)=v^-(y,0),\quad v(y,\alpha)=v^+(y,\alpha), \tag{A.4}$$

while on the vertical side touching ∂Z_j it is

$$v(P,t\alpha)=(1-t)(v^-(P,0)-v^+(P,0))+v^+(P,t\alpha). \tag{A.5}$$

Now, keep in mind that both v^- and v^+ are Lipschitz continuous, with Lipschitz constant at most $\|u_1\|_{W^{1,\infty}(\Omega\backslash J_{u_1})}$; as a consequence, by the definition, on the rectangle $\sigma_P\times[0,\alpha]$ the function v is Lipschitz continuous, with constant bounded by

$$\frac{5\|u_1\|_{W^{1,\infty}(\Omega\backslash J_{u_1})}}{\alpha}.$$

If we now repeat the same construction for every point $P\in\partial Z_j$, we end up with a function $v:Z_j^{INT}\times[0,\alpha]\to\mathbb{R}$, and this function satisfies

$$\|v\|_{W^{1,\infty}(Z_j\times[0,\alpha])}\le\frac{5\|u_1\|_{W^{1,\infty}(\Omega\backslash J_{u_1})}\operatorname{Lip}(\Psi)}{\alpha}. \tag{A.6}$$

We define then the function $\tilde{u}_j:\Omega\to\mathbb{R}$ as follows:

$$\tilde{u}_j(x)=\begin{cases}v(\Phi(x)) & \text{if } x\in\Phi^{-1}(Z_j^{INT}\times[0,\alpha]),\\ u_1(x) & \text{otherwise.}\end{cases}$$

By construction, the function $\tilde{u}_j$ belongs to $W^{1,\infty}$ in the set $\Phi^{-1}(Z_j^{INT}\times[0,\alpha])$, and it is a BV function outside, so it is globally a BV function on Ω. Thanks to the first equality in (A.4), $\tilde{u}_j$ is continuous across Z_j^{INT}, and by the second equality in (A.4) it is also continuous across $\Phi^{-1}(Z_j^{INT}\times\{\alpha\})$. Instead, $\tilde{u}_j$ is generally not continuous across $\Phi^{-1}((\partial Z_j^{INT}\backslash\partial Z_j)\times[0,\alpha])$, so we can expect this set to belong to $J_{\tilde{u}_j}$.

Finally, we want to determine whether $\tilde{u}_j$ is continuous across $\Phi^{-1}(\partial Z_j\times[0,\alpha])\subseteq\partial Z_{j,\alpha}$; more precisely, we intend to prove that, in $\Phi^{-1}(\partial Z_j\times[0,\alpha])$, the jump

set $J_{\tilde{u}_j}$ is contained in the jump set J_{u_1}. In fact, let us take a generic point $P \in \partial Z_j$ and $0 \le t \le 1$, and let us consider the point $Q = \Phi^{-1}(P, t\alpha)$: by construction, and keeping in mind (A.5), $\tilde{u}_j$ is continuous at Q if u_1 is continuous there, and $v^+(P,0) = v^-(P,0)$, and both things are generally false. Nevertheless, assume that $Q \notin J_{u_1}$: as noticed above, this means that Q is not contained in any of the Π_i, and then it is necessarily $Q = (P, \sigma)$ with $(P,0) \in \partial\Pi$ and some $0 \le \sigma \le \alpha$. And then, u_1 is continuous at Q because $Q \notin J_{u_1}$, and $v^+(P,0) = v^-(P,0)$ because the function u_1 is continuous on the boundary of Π. In conclusion, we have shown that if $Q \notin J_{u_1}$, then also $Q \notin J_{\tilde{u}_j}$; as a consequence, the jump set $J_{\tilde{u}_j}$ coincides with the jump set of u_1, except that in place of Z_j we have now the "L-shaped set"

$$\tilde{Z}_j = Z_j \backslash Z_j^{INT} \cup \Phi^{-1}((\partial Z_j^{INT} \backslash \partial Z_j) \times [0,\alpha]) = Z_j \backslash Z_j^{INT} \cup Z_j^L.$$

Taking α small enough, and keeping in mind that $\beta \ll \alpha$ has still to be chosen, and that the bi-Lipschitz constant of Φ does not explode when $\alpha \to 0$, we can then evaluate

$$\begin{aligned} \text{(A.7)} \qquad \mathscr{H}^{N-1}(J_{\tilde{u}_j} \Delta J_{u_1}) &= \mathscr{H}^{N-1}(Z_j^{INT}) + \mathscr{H}^{N-1}(Z_j^L) \\ &\le 2\mathscr{H}^{N-2}(\partial Z_j)\beta + 2\alpha \mathscr{H}^{N-2}(\partial Z_j)\,\mathrm{Lip}(\Phi^{-1}). \end{aligned}$$

Observe that the big achievement in passing from u_1 to $\tilde{u}_j$ is that $\tilde{Z}_j$ is a positive distance apart from $J_{\tilde{u}_j} \backslash \tilde{Z}_j$, so we have separated a piece of the jump set from all the rest.

Let us now estimate the distance between u_1 and $\tilde{u}_j$ in the BV sense, and in the L^p sense of the absolutely continuous part of the gradient. Calling $A = \{x \in \Omega : u_1(x) \ne \tilde{u}_j(x)\}$, we have by construction

$$\mathscr{H}^N(A) \le 2\mathscr{H}^{N-2}(\partial Z_j)\beta\alpha\,\mathrm{Lip}(\Phi^{-1}).$$

Hence, by construction, by (A.3), (A.6) and (A.7), by the fact that the bi-Lipschitz constants of Φ and Ψ do not explode when α and β go to 0, and up to choose $\beta \ll \alpha \ll 1$, we can evaluate

$$\text{(A.8)} \quad \mathscr{H}^{N-1}(J_{\tilde{u}_j} \Delta J_{u_1}) \le \frac{\varepsilon}{3MK}, \quad \|\tilde{u}_j - u_1\|_{\mathrm{BV}(\Omega)} + \|\nabla \tilde{u}_j - \nabla u_1\|_{L^p(\Omega)} \le \frac{\varepsilon}{3MK}.$$

It is now very simple to conclude: for each $1 \le j \le M$ we do the same construction, and we define the approximating function $\tilde{u}^L \in \mathrm{BV}(\Omega)$ as the function coinciding with $\tilde{u}_j$ on each $Z_{j,\alpha}$, and with u_1 outside the union of the different $Z_{j,\alpha}$. Thanks to (A.8), we have

$$\mathscr{H}^{N-1}(J_{\tilde{u}^L} \Delta J_{u_1}) \le \frac{\varepsilon}{3K}, \quad \|\tilde{u}^L - u_1\|_{\mathrm{BV}(\Omega)} + \|\nabla \tilde{u}^L - \nabla u_1\|_{L^p(\Omega)} \le \frac{\varepsilon}{3K}.$$

Moreover, by construction the jump set $J_{\tilde{u}^L}$ satisfies $J_{\tilde{u}^L} = \tilde{\Pi} \cup \bigcup_{i=1}^{K-1} \Pi_i$, where $\tilde{\Pi} = \bigcup_{j=1}^{M} \tilde{Z}_j$ is a stricitly positive distance apart from $J_{\tilde{u}^L} \setminus \tilde{\Pi}$. Notice that $\tilde{\Pi}$ is no more a connected polyhedron; in fact, it is a union of M pieces, and each piece is not a polyhedron, but an "L-shaped" set, not even contained in a $(N-1)$-dimensional hyperplane. Nevertheless, it is obvious by construction that there exists a bi-Lipschitz homeomorphism $F : \mathbb{R}^N \to \mathbb{R}^N$ which transforms $\tilde{\Pi}$ into a C^1 compact manifold with C^1 boundary, and which equals the identity outside of an arbitrarily small neighborhood U of $\tilde{\Pi}$; moreover, the bi-Lipschitz constant of F does not explode when U becomes smaller and smaller. Hence, we can assume that U is a strictly positive distance apart from the polyhedra Π_i, $1 \le i \le K-1$, so the function $\tilde{u} = \tilde{u}^L \circ F^{-1}$ satisfies

$$\mathcal{H}^{N-1}(J_{\tilde{u}} \Delta J_{u_1}) \le \frac{\varepsilon}{2K}, \quad \|\tilde{u} - u_1\|_{\mathrm{BV}(\Omega)} + \|\nabla \tilde{u} - \nabla u_1\|_{L^p(\Omega)} \le \frac{\varepsilon}{2K},$$

which by (A.3) and (A.2) become

$$\mathcal{H}^{N-1}(J_{\tilde{u}} \Delta J_u) \le \frac{\varepsilon}{K}, \quad \|\tilde{u} - u\|_{\mathrm{BV}(\Omega)} + \|\nabla \tilde{u} - \nabla u\|_{L^p(\Omega)} \le \frac{\varepsilon}{K}. \tag{A.9}$$

Summarizing, starting from the function $u \in \mathrm{BV}(\Omega)$ having K (possibly intersecting) polyhedra as jump set, we have chosen one of the polyhedra, Π, and constructed a function $\tilde{u}$ whose jump set is made by a C^1, compact manifold with C^1 boundary, together with $K-1$ polyhedra, and there is a strictly positive distance between the manifold and the polyhedra; in addition, each of the $K-1$ "new" polyhedra coincides with one of the "old" $K-1$ polyhedra, or with a small enlargement of it (at the beginning, we have added to each Π_i the small set $\{(x,t) \in \Theta_i : 0 \le t \le \bar{\alpha}, \mathrm{pr}_i(x,t) \in \Gamma_i\}$, recall the definition (A.1) of J_u^+). Finally, (A.9) holds and the set $\{u \ne \tilde{u}\}$ is an arbitrarily small neighborhood of the polyhedron Π. With an obvious recursion argument (and also using Corollary 2.4 to get the smoothness of u_ε for free), we obtain the first part of the conclusion. Notice that there is one polyhedron on which we never apply our construction: indeed, once we have done $K-1$ steps, and then transformed $K-1$ polyhedra into C^1 manifolds, each one a positive distance away from the remaining of the jump set, the last polyhedron is automatically isolated; hence, there is no need to apply our argument to this last polyhedron, it is enough to modify it so to become C^1, of course remaining away from the other manifolds.

Let us now prove the second part of the statement. Let $\overline{\Pi}$ be a given hyperplane; since J_u is compactly contained in Ω, we can select finitely many polyhedra $\overline{\Pi}_j \subset\subset \Omega$, $1 \le j \le H$, such that the intersection of J_u with $\overline{\Pi}$ is compactly contained in the union of the $\overline{\Pi}_j$. Thanks to Lemma 4.3, we can replace u with a function $\tilde{u}$ which is very close to u, whose jump set coincides with $J_u \cup \bigcup_{j=1}^{H} \overline{\Pi}_j \subset\subset \Omega$, and which is still smooth, bounded and with bounded differential outside of its jump set. Notice that the jump set of $\tilde{u}$ is still polyhedral; in particular, if J_u is the union of K polyhedra, then $J_{\tilde{u}}$ is done by $K+H$ ones.

We can then apply our construction above to the function $\tilde{u}$; more precisely, we perform K steps, in each of which we transform one of the K original polyhedra into a isolated C^1 manifold. In each of these steps we could have enlarged the polyhedra $\overline{\Pi}_j$, and it is also possible that some of these polyhedra have been glued together, so in the end we have polyhedra $\overline{\Pi}'_j$ for $1 \le j \le H'$ and a suitable $H' \le H$. Keep in mind that by construction the polyhedra $\overline{\Pi}'_j$ are still compactly contained in Ω, and inside the hyperplane $\overline{\Pi}$. Summarizing, after the K steps we have obtained a function v in $SBV^p(\Omega) \cap C^\infty(\Omega \setminus \overline{J_v}) \cap W^{1,\infty}(\Omega \setminus \overline{J_v})$, very close to $\tilde{u}$ and so to u, and whose jump set coincides with the union of $K + H'$ disconnected, compact pieces, namely, K connected C^1 manifolds, and H' polyhedra inside $\overline{\Pi}$. Notice also that, by construction, none of the manifolds can intersect $\overline{\Pi}$, since we have modified $\tilde{u}$ only in an arbitrarily small neighborhood of J_u, and the union of $\overline{\Pi}'_j$ is larger than that of $\overline{\Pi}_j$, which contains a neighborhood of $\overline{\Pi} \cap J_u$. Hence, we conclude by letting u_ε be a last, trivial modification of v which makes the polyhedra $\overline{\Pi}'_j$ become disjoint, compact, C^1 manifolds, still contained in $\overline{\Pi}$ and compactly contained in Ω. □

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