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Statistical Mechanics — *Continuum models in wealth distribution*, by Giuseppe Toscani, communicated on January 13, 2017.

This paper is dedicated to the memory of Professor Giuseppe Grioli.

Abstract. — We present and discuss the main analogies between continuum mechanics and the laws of an evolving economy. A sound basis for these connections is kinetic theory, which provides both mathematical and physical arguments to build up efficient mathematical models.

Key words: Wealth distribution, kinetic models, Fokker–Planck equations, local equilibria, Euler-type equations

Mathematics Subject Classification: 35Q84, 82B21, 91D10, 94A17

1. Introduction

The unexpected connections between continuum mechanics and the laws of an evolving economy were understood and fruitfully employed starting from the middle of the last century. In 1950, the New Zealand electrical engineer A. W. Phillips invented an ingenious hydraulic device (named MONIAC) useful to teach economical principles in a few easy and graphic operations based on fluid dynamics. Using the words of Phillips [24] "There has been an increasing use in economic theory of mathematical models, usually in the form of difference equations, sometimes of differential equations, for investigating the implications of systems of hypotheses. However, those students of economics who, like the present writer, are not expert mathematicians, often find some difficulty in handling these models effectively. [...] Mechanical models [...] may help non-mathematicians by enabling them to see the quantitative changes that occur in an interrelated system of variables following initial changes in one or more of them."

A similar remark was done by J. Meade (later Nobel Laureate for Economics) [21] "Once upon a time there was a student at the London School of Economics [...] who got into difficulties [...] with such questions as whether Savings are necessarily equal to Investment [...] but he realized that monetary flows and stocks of money could be thought of as tankfuls of water".

A sound basis for these connections started to be done around twenty years ago by a community of physicists, who gave a unifying theoretical framework to various social and economic phenomena. These investigations were identified under the name of *econophysics*. The neologism was introduced by H. E. Stanley during the Conference "Second Statphys–Kolkata", held in Kolkata (India) in 1995 (cf. [26]). Like biophysics, geophysics and astrophysics, this term is the result of the combination of the words *economy* and *physics*.

Among the various models present in the literature [22], mostly based on the approach furnished by statistical mechanics, kinetic models of socio-economic systems gained a lot of popularity, due to strong analogies between them and the classical kinetic theory of rarefied gases, described by the Boltzmann equation [6].

With respect to the classical kinetic theory of rarefied gases, where the equilibrium density is found to be a Gaussian (known as Maxwellian distribution [4, 8, 9]), maybe the main difference in kinetic theory of wealth distribution is related to the fact that, while the wealth variable w is assumed to be nonnegative, the corresponding equilibrium density, known as Pareto-type distribution [23], is represented by a curve that exhibits a polynomial decay at infinity. If in equilibrium the wealth in a multi-agent society is distributed according to a probability density $f(w)$, the distribution function of wealth, say $F(w)$ satisfies, for $w \gg 1$

$$1 - F(w) = \int_w^{+\infty} f(v)\, dv \cong w^{-p}, \quad p > 1.$$

The value of the positive constant p is usually called the Pareto index.

The equilibrium density of type (2.15) makes evident both the unequal distribution of wealth in the society, and the existence of a (small) class of extremely rich people. Various studies of the real data of western economies allowed to conclude that the Pareto index is varying between 1.5 e 3 (data referred to the year 2000: USA ~1.6, Japan ~1.8–2.2) [15]. The main consequence is that typically less than the 10% of the population possesses at least the 40% of thew total wealth of the country, and follows that law.

The differences between the equilibrium distribution of the Boltzmann equation and the Pareto-type equilibrium density appearing when studying wealth distribution reflect also at a macroscopic level, represented by the equations of fluid dynamics. A first step in this direction has been done few years ago by the present author with Bertram Düring [18], by deriving Euler-type equations for the distribution density of the propensity to invest in a society of agents trading according to a kinetic model with risk introduced in [14]. As we shall see in the following, the main ideas in [18] can be easily extended to construct equations of fluid dynamics for various traits of a human society naturally linked to the personal wealth. This allows to study social phenomena in a multi-agent society which are naturally varying in dependence of wealth.

2. The legacy of kinetic theory

The discussion that follows will be based on the kinetic Fokker–Planck equation [25]. The Fokker–Planck equation is a fundamental model in kinetic theories and statistical mechanics. It is a partial differential equation describing the time evolution of a density function $f(v,t)$, where $v \in \mathbb{R}^n$, $n \geq 1$ and $t \geq 0$, departing from a nonnegative initial density $f_0(v)$. The standard assumptions on $f_0(v)$ is that it possesses finite mass ρ, mean velocity u and temperature θ, where for any given density $g(v)$

$$\rho(g) = \int_{\mathbb{R}^n} g(v)\,dv \tag{2.1}$$

is the mass density,

$$u(g) = \frac{1}{\rho}\int_{\mathbb{R}^n} vg(v)\,dv \tag{2.2}$$

is the mean velocity, and θ is the temperature defined by

$$\theta(g) = \frac{1}{n\rho}\int_{\mathbb{R}^n} |v-u|^2 g(v)\,dv. \tag{2.3}$$

The general form of the equation reads

$$\frac{\partial f}{\partial t} = J(f) = \gamma \sum_{k=1}^{n}\left\{\frac{\partial^2 f}{\partial v_k^2} + \frac{1}{\theta(f)}\frac{\partial}{\partial v_k}[(v_k - u_k(f))f]\right\}. \tag{2.4}$$

The one-particle friction constant γ is usually assumed to be a function of ρ, u, θ. Equation (2.4) has a stationary solution of given mass ρ, mean velocity u and temperature θ given by the Maxwellian density function

$$M_{\rho,u,\theta}(v) = \rho\frac{1}{(2\pi\theta)^{n/2}}\exp\left\{-\frac{|v-u|^2}{2\theta}\right\}, \tag{2.5}$$

which is such that $J(M_{\rho,u,\theta}) = 0$. Note that mass density, mean velocity and temperature are preserved in time by the Fokker–Planck equation (2.4). It is moreover interesting to remark that, if the friction γ is taken to be proportional to the pressure $p = \rho\theta$, then $J(f)$ has the same kind of nonlinearity (quadratic) as the true Boltzmann equation.

In view of its conservations and its decay property towards the equilibrium Maxwellian distribution [2], the Fokker–Planck operator can be fruitfully used in place of the Boltzmann collision operator to describe the evolution of the rarefied gas phase space density $f(x,v,t)$ [8, 9]

$$\frac{\partial}{\partial t}f(x,v,t) = -v\cdot\nabla_x f(x,v,t) + \frac{1}{\varepsilon}J(f(x,v,t)). \tag{2.6}$$

This equation contains terms accounting for the two ways that the density can change. The

$$-v\cdot\nabla_x f(x,v,t)$$

term represents the effects of *transport*; that is, the motion

$$x_0 \mapsto x_0 + (t - t_0)v_0 \quad v_0 \mapsto v_0 \tag{2.7}$$

of molecules between interactions. The Fokker–Planck operator $J(f)$ represents the effects of interactions and describes relaxation to the local Maxwellian equilibrium [8, 9] as a function of the local mass $\rho(x,t)$, velocity $u(x,t)$ and temperature $\theta(x,t)$:

$$M(x,v,t) = \frac{\rho(x,t)}{(2\pi\theta(x,t))^{3/2}} \exp\left\{ -\frac{|v - u(x,t)|^2}{2\theta(x,t)} \right\}. \tag{2.8}$$

Last, ε in (2.6) is a suitable relaxation time.

Starting from the pioneering works of Mandelbrot [19], it is now commonly accepted by the kinetic community that in many aspects a trading market composed of a sufficiently large number of agents can be described using the laws of statistical mechanics, just like a rarefied gas, composed of many interacting particles. In fact, there is an almost literal translation of concepts: *molecules* are identified with the *agents*, the *particles' energy* correspond to the *agents' wealth*, and binary *collisions* translate into *trade interactions*.

Resorting to the proposed analogies between trading agents and colliding particles, various well established methods from kinetic theory and statistical physics are ready for application to the field of economy. Most notably, the numerous tools originally devised for the study of the time evolution of the density in a rarefied gas can now be used to analyze the evolution of wealth distribution. In this way, the kinetic description of market models via a Boltzmann-type equation provides one possible explanation for the development of universal profiles in wealth distributions of real economies.

A non secondary aspect of this analogy is the possibility to resort to the classical closure procedure around the local equilibrium density to recover the underlying equations of fluid dynamics. Indeed, the local Maxwellian equilibrium density (2.8) allows to obtain all moments of the distribution in terms of the principal ones, given by mass density, mean velocity and temperature. We remark that this procedure requires, as in the classical case, a transport term linear with respect to the velocity variable.

To be more precise, in agreement with the classical theory of rarefied gases, where the particle density depends on the space variable x, the velocity variable v and time t, in the framework of wealth distribution one can study the evolution of the distribution function of the agents which depends on a suitable trait $x \in I \subseteq \mathbb{R}$, wealth $w \in \mathbb{R}_+$ and time $t \in \mathbb{R}_+$, say $f = f(x,w,t)$. By analogy with the classical kinetic theory of rarefied gases, it is useful to emphasize the role of the different parameters by identifying the velocity with the wealth, and the position with the trait. By doing this, one assumes at once that the variation of the distribution $f(x,w,t)$ with respect to the wealth parameter w will depend on *trades* between agents, while the change of distributions in terms of the trait x depends on the *transport* term, which contains the equation of motion, namely the law of variation of x with respect to time. The most general law of transport that can be treated is given by

$$\frac{dx}{dt} = T(x, w) = A(x)L(w), \tag{2.9}$$

where $L(w)$ is a linear function of the wealth w. The time evolution of the distribution will then obey a non-homogeneous kinetic equation, given by

$$\frac{\partial}{\partial t} f(x, w, t) + T(x, w)\frac{\partial}{\partial x} f(x, w, t) = \frac{1}{\tau}\mathscr{J}(f(x, w, t)). \tag{2.10}$$

In (2.10), $T(\cdot\,, \cdot)$ describes the law of variation of the trait, while $\mathscr{J}$ represents the corresponding of the Fokker–Planck operator in (2.4) describing now the relaxation to the local wealth equilibrium density. Finally, τ is a suitable relaxation time, depending on the velocity of money circulation [28]. Note that in physical applications where no forces are present, the transport term is simply $T(x, w) = w$.

In wealth distribution models, the Fokker–Planck operator describing relaxation to equilibrium is assumed to be [7, 14]

$$\mathscr{J}(f(w)) = \frac{\sigma}{2}\frac{\partial^2}{\partial w^2}(w^2 f(w)) + \lambda\frac{\partial}{\partial w}(w - m(f))f(w), \tag{2.11}$$

where $w \geq 0$ and $m(f)$ is the mean wealth of $f(w)$. As before

$$\rho(f) = \int_{\mathbb{R}_+} f(w)\, dw, \quad m(f) = \frac{1}{\rho}\int_{\mathbb{R}_+} wf(w)\, dw. \tag{2.12}$$

In analogy with the classical kinetic theory, the homogeneous equation

$$\frac{\partial}{\partial t} f(w, t) = \mathscr{J}(f(w, t)) \tag{2.13}$$

is such that both the mass and the mean wealth $m(f)$ are conserved in time. Moreover, for any initial density $f(w, t = 0) = f_0(w)$ with mass ρ, mean m and finite second moment [27], the unique solution $f(w, t)$ to equation (2.13) converges towards its unique stationary state, the so-called Pareto-like state $M_{\rho,m}(w)$ given by (cf. also the original description by Amoroso [1])

$$M_{\rho,m}(w) = \rho\frac{(pm)^{p+1}}{\Gamma(p + 1)}\frac{1}{w^{p+2}}\exp\Big(-\frac{pm}{w}\Big), \tag{2.14}$$

where

$$p = \frac{2\lambda}{\sigma} > 0. \tag{2.15}$$

Therefore, the equilibrium density exhibits a Pareto power law tail for large w's. In particular, the existence of higher moments depends on the value of p, and their value at equilibrium is given in terms of mass ρ and mean m. Among other approaches, the Fokker–Planck equation (2.13) appears as the quasi-invariant trading limit of the Boltzmann type equation for wealth distribution introduced in [14]. This kinetic equation is obtained by resorting to the binary trade

$$\begin{aligned} v^* &= (1-\lambda)v + \lambda w + \eta v, \\ w^* &= \lambda v + (1-\lambda)w + \tilde{\eta} w. \end{aligned} \tag{2.16}$$

In (2.16), the result of the trade depends on the saving rate $\lambda \in (0,1)$, while the risks of the market are described by η and $\tilde{\eta}$, equally distributed random variables with zero mean and variance σ. Hence, Pareto tails in the Fokker–Planck equation depend on the balance between the saving and risk parameters in the microscopic trade (2.16).

If $p > 1$, the bounded second moment of the Pareto-type density can easily be evaluated by considering that in equilibrium, i.e. as $t \to \infty$, one has

$$\begin{aligned} 0 &= \frac{\sigma}{2}\int_{\mathbb{R}_+} w^2 \frac{\partial^2}{\partial w^2}(w^2 M_{\rho,m}(w))\,dw + \lambda \int_{\mathbb{R}_+} w^2 \frac{\partial}{\partial w}[M_{\rho,m}(w)(w-m)]\,dw \\ &= \sigma \int_{\mathbb{R}_+} w^2 M_{\rho,m}(w)\,dw - 2\lambda \int_{\mathbb{R}_+} w(w-m)M_{\rho,m}(w)\,dw \\ &= (\sigma - 2\lambda)\int_{\mathbb{R}_+} w^2 M_{\rho,m}(w)\,dw + 2\lambda m \int_{\mathbb{R}_+} w M_{\rho,m}(w)\,dw \\ &= (\sigma - 2\lambda)\int_{\mathbb{R}_+} w^2 M_{\rho,m}(w)\,dw + 2\lambda\rho m^2. \end{aligned}$$

Thus, if $p > 1$, the second moment of the equilibrium density is bounded, and

$$\int_{\mathbb{R}_+} w^2 M_{\rho,m}(w)\,dw = \frac{2\lambda}{2\lambda - \sigma}\rho m^2. \tag{2.17}$$

In analogy with the kinetic theory of rarefied gases, it appears natural to assume that in a closed economy the Pareto-type density $M_{\rho,m}$, equilibrium solution of the Fokker–Planck equation (2.13), plays the same role as played by the Maxwell distribution (2.5). However, contrary to what happens in classical kinetic theory, where the equilibrium Maxwellian has all moments bounded, in this case the number of moments bounded in the equilibrium depends on the value of the parameter p, as given in (2.15).

3. The Euler equations

In Section 2 we described the main properties of the Fokker–Planck equation (2.13), such as the existence of a unique equilibrium with tails, and the consequent possibility of obtaining higher-order moments from the first two (mass and mean wealth). As in the classical kinetic theory of rarefied gases, these properties are the basis of the construction of hydrodynamics. In kinetic theory, a direct and clear understanding of the derivation of macroscopic equations relies on the so-called fractional step method, very popular in the numerical approach to the Boltzmann equation. Let $f = f(x, w, t)$ be the distribution function of the agents, solving the Boltzmann-type equation (2.10). The fractional step method consists in considering separately and sequentially, in each (small) time step, the transport and relaxation operators. In consequence, during this short time interval one recovers the evolution of the density f from the joint action of the relaxation

$$\frac{\partial f}{\partial t} = \frac{1}{\tau}\mathscr{I}(f(x, w, t)) \tag{3.18}$$

and transport

$$\frac{\partial f}{\partial t} + T(x, w)\frac{\partial}{\partial x} f(x, w, t) = 0. \tag{3.19}$$

As in classical kinetic theory, where mass, mean velocity and energy are conserved by the Fokker–Planck operator, the conservation (for fixed x), of mass and mean wealth in the relaxation step is enough to guarantee that (3.18) pushes the solution towards the (local x-dependent) Pareto-type equilibrium with the same local mass and mean of the initial datum

$$M_{\rho,m}(x, w, t) = \rho(x, t)\frac{(pm(x, t))^{p+1}}{\Gamma(p + 1)}\frac{1}{w^{p+2}}\exp\Big(-\frac{pm(x, t)}{w}\Big), \tag{3.20}$$

In (3.20), $\rho(x, t)$, $m(x, t)$ are the macroscopic variables, namely the local density of agents with trait x at time t, given by

$$\rho(x, t) = \int_{\mathbb{R}_+} f(x, w, t)\, dw, \tag{3.21}$$

and the local mean

$$m(x, t) = \frac{1}{\rho(x, t)}\int_{\mathbb{R}_+} wf(x, w, t)\, dw. \tag{3.22}$$

Then, if τ is sufficiently small, one can easily argue that the solution to (3.18) is *sufficiently close* to the equilibrium (3.20), and this equilibrium can be used in the transport step (3.19) to close the equations. In detail, since the Fokker–Planck equation (3.18) is both mass and momentum preserving, integrating equation (3.19) with respect to the wealth velocity w, using as test functions $\varphi(w) = 1, w$ respectively, we obtain the conservation laws

$$\int_{\mathbb{R}_+}\Big(\frac{\partial f}{\partial t} + T(x, w)\frac{\partial}{\partial x} f(x, w, t)\Big)\, dw = 0, \tag{3.23}$$

and

$$\int_{\mathbb{R}_+} w\Big(\frac{\partial f}{\partial t} + T(x,w)\frac{\partial}{\partial x} f(x,w,t)\Big)\, dw = 0. \tag{3.24}$$

Let us fix the law $T(\cdot, w)$ to be linearly dependent on w,

$$T(x,w) = (w - \chi\overline{w})A(x), \tag{3.25}$$

where χ is a positive constant and $\overline{w}$ represents a suitable fixed value of the wealth. Then, one obtains from (3.23), (3.24) the equations

$$\frac{\partial \rho}{\partial t} + A(x)\frac{\partial}{\partial x}[\rho(m - \chi\overline{w})] = 0, \tag{3.26}$$

$$\frac{\partial(\rho m)}{\partial t} + A(x)\frac{\partial}{\partial x}\Big[\int_{\mathbb{R}_+} w^2 f(x,w,t)\, dw - \chi\overline{w}\rho m\Big] = 0. \tag{3.27}$$

Note that equation (3.27) depends on the second moment of the density. Using the equilibrium relation (2.17), however, we can express this second moment in terms of the local mass density and the mean as soon as $p > 1$. By this relationship we finally obtain the following system of equations:

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + T(x)\frac{\partial}{\partial x}[\rho(m - \chi\overline{w})] = 0,\\ &\frac{\partial(\rho m)}{\partial t} + T(x)\frac{\partial}{\partial x}\Big[\rho m\Big(\frac{2\lambda}{2\lambda - \sigma} m - \chi\overline{w}\Big)\Big] = 0, \end{aligned} \tag{3.28}$$

which have to be solved on $I \times (0, T)$ with appropriate boundary and initial conditions. Using (3.28) we can rewrite the second equation as

$$\frac{\partial m}{\partial t} + T(x)(m - \chi\overline{w})\frac{\partial m}{\partial x} + \frac{2\lambda}{2\lambda - \sigma}\frac{1}{\rho}\frac{\partial}{\partial x}[\rho m^2] = 0. \tag{3.29}$$

4. An example about personal satisfaction

Among the various applications of the system of macroscopic equations (3.28), we will present here one simple example that will help to clarify the method.

In classical kinetic theory of gases the relationship between position and velocity is stated by the law of dynamics, that leads to (2.7). In one space dimension, this law can be rephrased by saying that a positive velocity moves the particle on the right, while a negative one moves the particle to the left. Hence, the transport term is such that the value of the position will increase in presence of a positive velocity, and decrease in presence of a negative velocity.

This picture can be fruitfully used to represent any trait of the agents that is increasing or decreasing in dependence of the wealth. Such a situation appears when considering as trait x the measure of satisfaction about the personal status in a multi-agent society. It is commonly accepted that there is a level of wealth in the society, say $\overline{w}$, that is considered minimal to have a good quality of life. It is close to reality to say that a personal wealth less than $\overline{w}$ will move the agent towards a pessimistic view (so that x is decreasing), while a personal wealth bigger than $\overline{w}$ will move the agent towards an optimistic view (and in this case x will increase). By allowing x to vary on the whole real line, and by considering $x = 0$ as the border case of separation between optimists and pessimists, one can assume a transport law in the form

$$T(x, w) = \gamma(w - \overline{w}), \tag{4.30}$$

where γ is a positive constant which gives the rate of velocity at which the personal satisfaction is increasing (or decreasing) in terms of the wealth. The corresponding macroscopic equations then take the form

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \gamma \frac{\partial}{\partial x}[\rho(m - \overline{w})] = 0, \\ &\frac{\partial(\rho m)}{\partial t} + \gamma \frac{\partial}{\partial x}\left[\rho m\left(\frac{2\lambda}{2\lambda - \sigma} m - \overline{w}\right)\right] = 0. \end{aligned} \tag{4.31}$$

These equations, solved in $\mathbb{R} \times (0, T)$ give, for every fixed time $t > 0$, the local density $\rho(x, t)$ of people with degree of satisfaction x and the local flux $m(x, t)$ of people in the state x. Note that the solution depends on the key parameters of the model, namely the chosen minimal wealth $\overline{w}$, the rate γ, as well as the saving parameter λ and the risk parameter σ that characterize the equilibrium density (3.20).

5. Conclusions

In this note we tried to explain how the strong analogies between kinetic theory of ideal rarefied gases and the distribution of wealth in a conservative economy remain at the level of the derivation of macroscopic equations. By this methodology, it appears possible to study the time evolution of social phenomena in which the personal wealth is the main responsible for changes of a well specified trait of the agents. We gave in Section 4 an easy illustration of this idea. In this example the transport term takes a very simple form, similar to the classical transport term appearing in molecular dynamics. The interested reader can however take a look of a different more involved situation [18], in which the considered trait x is the propensity to invest, and the transport term depends on the trait itself. As discussed in [18] it is interesting to remark that, despite the huge variety of kinetic models which have been introduced so far to describe the evolution of the wealth distribution density [3, 10, 11, 12, 13], the analysis in [16, 17, 20] indicates that, even if the analytical form of the equilibrium density is missing, the closure relation (2.17) appears to be a universal one. This fact constitutes a strong validation of the Euler system (3.28).

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