Rend. Lincei Mat. Appl. 28 (2017), 479–494
DOI 10.4171/RLM/772

Continuum Mechanics — *Riemann problems for viscoelastic media*, by Natale Manganaro, communicated on January 13, 2017.

This paper is dedicated to the memory of Professor Giuseppe Grioli.

Abstract. — In this paper we considered a hyperbolic first order nonhomogeneous system of PDEs describing a viscoelastic medium where short memory effects are present. We solved a class of Riemann problems by integrating a suitable 2×2 reduced homogeneous model. The nonlinear interaction of two shock waves propagating in opposite directions are studied.

Key words: Viscoelasticity, Riemann problems, differential constraints

Mathematics Subject Classification: 35L40, 35L67, 74DXX

1. Introduction

Within the theoretical framework of nonlinear wave propagation, the Riemann problem (RP) represents one of the most interesting and celebrated problems of the applied mathematics. As it is well known, it consists in an initial value problem characterized by constant data with a discontinuity in $x = 0$. It takes its name from Riemann who in [39] (see also [40]) studied a situation where a gas is contained in a cylindrical tube which is divided in two parts by a thin layer. The gas is initially at rest while the pressure and the mass density are constant but with different values on the left and on the right of the layer. The problem is to study the evolution of the gas when at the time $t = 0$ the layer is broken. Riemann found the general solution in terms of constant states separated by rarefaction waves, contact discontinuities and shoch waves. Since then many efforts have been done for solving Riemann problems for hyperbolic systems of PDEs and starting from the fundamental paper of Lax [29] many results have been obtained for conservation laws (see [14, 43] and references therein quoted). In particular it can be proved that for systems of conservation laws, under the assumption of not large initial jumps, a RP admits an unique solution in terms of constant states separated by rarefaction waves, shock waves and/or contact discontinuities. Unfortunately the validity of such results are limited to the homogeneous case. In fact a rarefaction wave is characterized by a self-similar simple wave solution which, in general, is not admitted by nonhomogeneous systems. Thus only few cases of exact solution to RP for balance laws are known in the literature (see for instance [8, 11, 38]) and usually such a situation is studied from a numerical point of view.

A more hard task is to find exact solutions to a Generalized Riemann Problem (GRP) which consists of non-constant initial data with a discontinuity at $x = 0$. The main problem is to determine an exact solution to the initial non constant data as well as a rarefaction–like wave connecting the resulting left and right non constant states. Some contributions on GRP have been given concerning existence and uniqueness theorems [3, 24, 25, 31, 32, 42]. Some other contributions deal with asymptotic solutions [1, 30] while in some special cases exact solutions of GRP have been obtained by using a reduction procedure proposed within the framework of the theory of differential constraints [7, 12].

In this context, the main aim of this paper is to develop a reduction procedure for solving a RP for a first order quasilinear hyperbolic nonhomogeneous system describing a viscoelastic medium. More precisely, following the approach proposed in [13] for determining double wave solutions of first order hyperbolic homogeneous or nonhomogeneous systems, we show that for solving a RP for the governing system we are led to consider a corresponding RP for a suitable 2×2 reduced model. This makes easier the analysis under interest because for 2×2 hyperbolic homogeneous or nonhomogeneous systems a large body of results concerning exact solutions as well as nonlinear soliton-like wave interactions are known [9, 10, 28, 36, 37].

The paper is organized as follows. In section 2 we illustrate the viscoelastic model under interest as well as its main features. Moreover for further convenience we sketch some of the results obtained in [13] which will be useful for developing the reduction procedure at hand. In sections 3 and 4, under suitable hypotesis, we calculate rarefaction waves and shock waves admitted by the governing viscoelastic system previously considered. In section 5, we solve a RP through rarefaction and/or shock waves and we study a nonlinear waves interaction problem. Finally some conclusion and remarks will end the paper.

2. Rate–type viscoelastic model

Here we consider the following rate–type model which describes a viscoelastic medium:

$$v_t - \sigma_x = 0 \tag{1}$$

$$\varepsilon_t - v_x = 0 \tag{2}$$

$$\sigma_t - \Phi(\varepsilon, \sigma)\varepsilon_t = \Psi(\varepsilon, \sigma) \tag{3}$$

where x and t denote, respectively, space a time coordinates, v is the particle velocity, ε the strain and σ the stress. Moreover $\Phi(\varepsilon, \sigma)$ and $\Psi(\varepsilon, \sigma)$ are material response functions which measure, respectively, the instantaneous and non-instantaneous response of the material. The characteristic speeds of the system (1)–(3) are

$$\lambda^{(1)} = -\sqrt{\Phi}, \quad \lambda^{(2)} = 0, \quad \lambda^{(3)} = \sqrt{\Phi} \tag{4}$$

so that it results to be strictly hyperbolic provided that $\Phi > 0$.

The model (1)–(3) has been widely adopted in the litterature in order to describe viscoelastic processes where memory effects are present (see, for instance, the Cristescu's book [6] for an exhaustive review on this subject). In such a framework, relation (1) is the equation of motion, (2) characterizes a compatibility condition, while (3) denotes the stress-strain rate-type equation. Many results have been obtained for the system (1)–(3) concerning energy estimates and phase transformation fenomena [15, 16, 17, 18, 19, 45, 46, 47], moving boundary problems [20, 21], traveling waves and similarity solutions [44], reduction procedures [22, 23, 35], numerical experiments [5], [41].

The rate-type equation (3) generalizes different models proposed in litterature. For instance, if

$$\Phi = E, \quad \Psi = -\frac{1}{\tau}\sigma \tag{5}$$

it reduces to the pioneering Maxwell's model, while if

$$\Phi = E, \quad \Psi = -\frac{1}{\tau}(\sigma - \sigma_e(\varepsilon)) \tag{6}$$

it specializes to the Malvern's model [33, 34]. In (5) and (6) E is the Young's modulus, τ a relaxation time while $\sigma = \sigma_e(\varepsilon)$ denotes the equilibrium stress-strain curve. Furthermore in [27] Herrmann and Nunziato proved that the celebrated integro-differential equation proposed by Coleman and Noll [4] within the framework of finite linear viscoelasticity is equivalent to (3) with

$$\Phi = \sigma_i'(\varepsilon) + \alpha(\varepsilon)(\sigma - \sigma_i(\varepsilon)) \tag{7}$$

$$\Psi = -\frac{1}{\tau}(\sigma - \sigma_e(\varepsilon)) \tag{8}$$

where $\alpha(\varepsilon)$ is a constitutive function while $\sigma = \sigma_i(\varepsilon)$ characterizes the instantaneous stress-strain curve defined by

$$\frac{d\sigma_i}{d\varepsilon} = \Phi(\sigma_i(\varepsilon), \varepsilon). \tag{9}$$

The material response function characterized in (8) is widely used by many researchers. In particular in [26] Gurtin et al. proved that if Ψ is smooth in a neighborhood of a point $(\varepsilon_0, \sigma_0)$ of the equilibrium curve $\sigma = \sigma_e(\varepsilon)$, then there exists a constant $k(\varepsilon_0) \geq 0$ such that

$$\Psi = -k(\sigma - \sigma_e(\varepsilon)) + O(\delta)$$

as

$$\delta = |\varepsilon - \varepsilon_0| + |\sigma - \sigma_0|$$

approaches to zero. Furthermore it can be proved [26] that the model (1)–(3) admits a free energy $\psi(\varepsilon, \sigma)$ if and only if the following relations hold

$$\frac{\partial \psi}{\partial \varepsilon} + \Phi \frac{\partial \psi}{\partial \sigma} = \sigma, \quad \Psi \frac{\partial \psi}{\partial \sigma} \leq 0. \tag{10}$$

In passing we notice also that when $\tau \to 0$ system (1)–(3) supplementd by (8) reduces to the celebrated p-system

$$\begin{cases} \varepsilon_t - v_x = 0 \\ v_t + (p(\varepsilon))_x = 0 \end{cases} \tag{11}$$

where $p(\varepsilon) = -\sigma_e(\varepsilon)$ is the pressure-like function.

In [13] classes of double wave exact solutions of (1)–(3) have been determined by following a reduction procedure therein proposed. The main idea was to reduce the problem of integrating the full governing set of equations at hand to that of solving a suitable 2×2 reduced system. In particular, among the different cases therein considered, it was proved that if the material response functions Φ and Ψ adopt the form

$$\Phi = \varphi^2(\mu), \quad \Psi = \Psi_0(w)(\alpha_0^2 - \varphi^2), \tag{12}$$

under the variable transformation

$$\mu = \sigma - \alpha_0^2 \varepsilon \tag{13}$$

$$w = \sigma - F(\mu), \quad F(\mu) = \int \frac{\varphi^2}{\varphi^2 - \alpha_0^2} d\mu, \tag{14}$$

the equations (1)–(3) specialize to

$$\frac{\partial v}{\partial t} - \frac{\partial}{\partial x}(w + F(\mu)) = 0 \tag{15}$$

$$\frac{\partial}{\partial t}(F(\mu) - \mu) - \alpha_0^2 \frac{\partial v}{\partial x} = \alpha_0^2 \Psi_0(w) \tag{16}$$

$$\frac{\partial w}{\partial t} = \alpha_0^2 \Psi_0(w) \tag{17}$$

where $\varphi(\mu)$ and $\Psi_0(w)$ are unspecified functions while α_0 is a constant. Next, under the double wave's reduction

$$v = \frac{w}{\alpha_0} \pm G(\mu), \quad G(\mu) = \int \frac{\varphi}{\varphi^2 - \alpha_0^2} d\mu, \tag{18}$$

the equations (15) and (16) assume to the form

$$\mu_t \mp \varphi(\mu)\mu_x = 0 \tag{19}$$

$$w_t - \alpha_0 w_x = 0. \tag{20}$$

Therefore in order to determine exact solutions of (1)–(3) supplemented by (12) we are led to solve the uncoupled system (19), (20) along with (17) which plays the role of differential constraint. According to the method of differential constraints [48], relation (17) selects the class of initial value problems admitted by (19), (20). After some elementary algebra, the solution of (17), (19) and (20) is given by

$$\mu = \mu_0(z), \quad z = x \pm \varphi(\mu)t \tag{21}$$

$$w = w_0(\xi), \quad \xi = x + \alpha_0 t \tag{22}$$

provided that

$$\frac{dw_0}{d\xi} = \alpha_0 \Psi_0(w_0). \tag{23}$$

Finally, once the constitutive functions φ and Ψ_0 are specified, then exact solutions of (1)–(3) are determined from (13), (14) and (18) through (21)–(23).

3. Rarefaction waves

Let us consider the following Riemann problem

$$\mathbf{U}(x,0) = \begin{cases} \mathbf{U}_L & \text{for } x < 0 \\ \mathbf{U}_R & \text{for } x > 0 \end{cases}, \quad \text{where } \mathbf{U} = \begin{pmatrix} v \\ \varepsilon \\ \sigma \end{pmatrix} \tag{24}$$

where $\mathbf{U}_L$ and $\mathbf{U}_R$ are constant vectors characterizing equilibrium states of (1)–(3).

Here, in order to solve the initial value problem (24) through rarefaction waves, we consider the "reduced" Riemann problem

$$\mu(x,0) = \begin{cases} \mu_L & \text{for } x < 0 \\ \mu_R & \text{for } x > 0 \end{cases}, \quad w(x,0) = \begin{cases} w_L & \text{for } x < 0 \\ w_R & \text{for } x > 0 \end{cases} \tag{25}$$

for the equations (17), (19) and (20), where, owing to (13)–(14),

$$\mu_L = \sigma_L - \alpha_0^2 \varepsilon_L, \quad \mu_R = \sigma_R - \alpha_0^2 \varepsilon_R, \quad w_L = \sigma_L - F_L, \quad w_R = \sigma_R - F_R. \tag{26}$$

In (26) and in what follows $F_L = F(\mu_L)$ and $F_R = F(\mu_R)$. Moreover here and in the following we assume that $\varphi(\mu)$ is a monotone non-negative function.

In the case of $\lambda = \lambda_1 = -\varphi$, a straightforward integration of (19) leads to

$$\begin{aligned} &\mu = \mu_L \quad \text{for } x < -\varphi(\mu_L)t \\ &\varphi(\mu) = -\frac{x}{t} \quad \text{for } -\varphi(\mu_L)t \le x \le -\varphi(\mu_R)t \\ &\mu = \mu_R \quad \text{for } x > -\varphi(\mu_R)t \end{aligned} \tag{27}$$

with

$$\varphi(\mu_L) > \varphi(\mu_R) \tag{28}$$

while from (17) and (20) we find

$$w = w_L = w_R = w_0, \quad \text{provided that } \Psi_0(w_0) = 0. \tag{29}$$

Furthermore, from (18), the corresponding rarefaction curve in the (v, μ, w) space is easily obtained

$$v = R_1(\mu, v_L, \mu_L) = v_L + (G(\mu) - G(\mu_L)), \quad w = w_0. \tag{30}$$

Therefore owing to relations (27)–(29), from (13) and (14), a smooth solution of (1)–(3) along with (24) is given by $(30)_1$ and

$$\sigma = w_0 + F(\mu) \tag{31}$$

$$\varepsilon = \frac{1}{\alpha_0^2}(w_0 + F(\mu) - \mu) \tag{32}$$

along with the conditions

$$v_R = R_1(\mu_R, v_L, \mu_L) = v_L + (G(\mu_R) - G(\mu_L)) \tag{33}$$

$$w_0 = \sigma_R - F_R = \sigma_L - F_L. \tag{34}$$

As far as the rarefaction curve (30) is concerned, for further convenience we notice that

$$\frac{dR_1}{d\mu} \gtrless 0 \quad \Leftrightarrow \quad \varphi \gtrless \alpha_0 \tag{35}$$

$$\frac{d^2R_1}{d\mu^2} \gtrless 0 \quad \Leftrightarrow \quad \frac{d\varphi}{d\mu} \lessgtr 0. \tag{36}$$

Of course similar results can be obtained when $\lambda = \lambda_3 = \varphi$. In such a case the corresponding rarefaction curve is given by

$$v = R_3(\mu, v_L, \mu_L) = v_L - (G(\mu) - G(\mu_L)), \quad w = w_0. \tag{37}$$

with

$$\varphi(\mu_L) < \varphi(\mu_R) \tag{38}$$

and

$$\frac{dR_3}{d\mu} \gtrless 0 \quad \Leftrightarrow \quad \varphi \lessgtr \alpha_0 \tag{39}$$

$$\frac{d^2 R_3}{d\mu^2} \gtrless 0 \quad \Leftrightarrow \quad \frac{d\varphi}{d\mu} \gtrless 0. \tag{40}$$

In passing we remark that the functions $W_1^{(-)} = v - G(\mu)$ and $W_2^{(-)} = w$ characterizing the rarefaction curve (30) are Riemann invariants of (1)–(3) associated to the characteristic speed $\lambda = -\sqrt{\Phi}$ as well as the functions $W_1^{(+)} = v + G(\mu)$ and $W_2^{(+)} = w$ involved in (37) are Riemann invariants corresponding to $\lambda = \sqrt{\Phi}$.

4. Shock waves

Here we look for solution of the Riemann problem (24) in terms of shock waves and we write the Rankine-Hugoniot conditions of (15)–(17)

$$s[v] + [w + F(\mu)] = 0 \tag{41}$$

$$s[F(\mu) - \mu] + \alpha_0^2[v] = 0 \tag{42}$$

$$s[w] = 0 \tag{43}$$

where s is the shock speed while $[\cdot]$ denotes the jump of the corresponding quantity across the shock line.

A direct inspection of (41)–(43) leads to three shock's families. The 1-shocks and the 3-shocks are characterized by

$$s = -\frac{F - F_L}{v - v_L}, \quad w = w_L, \quad v = v_L \pm \frac{H(\mu, \mu_L)}{\alpha_0} \tag{44}$$

where

$$H = \sqrt{(F - F_L)(F - F_L + \mu_L - \mu)}, \tag{45}$$

while the 2-family is a contact discontinuity and it is determined by

$$s = 0, \quad v = v_L, \quad w = w_L + F_L - F. \tag{46}$$

In (44) and (46) the index L means that the concerning quantity is evaluated on the left of the shock.

Next, as far as the 1-shocks and the 3-shocks are concerned, by requiring the Lax conditions are satisfied, we are led to the following four cases:

i) if $\varphi' > 0$ and $\varphi > \alpha_0$ then

$$v = v_L + \frac{1}{\alpha_0} H(\mu, \mu_L), \quad \text{with} \begin{cases} \mu > \mu_L, & \text{for 1-shocks} \\ \mu < \mu_L, & \text{for 3-shocks} \end{cases} \tag{47}$$

ii) if $\varphi' > 0$ and $\varphi < \alpha_0$ then

$$v = v_L - \frac{1}{\alpha_0} H(\mu, \mu_L), \quad \text{with} \begin{cases} \mu > \mu_L, & \text{for 1-shocks} \\ \mu < \mu_L, & \text{for 3-shocks} \end{cases} \tag{48}$$

iii) if $\varphi' < 0$ and $\varphi > \alpha_0$ then

$$v = v_L - \frac{1}{\alpha_0} H(\mu, \mu_L), \quad \text{with} \begin{cases} \mu < \mu_L, & \text{for 1-shocks} \\ \mu > \mu_L, & \text{for 3-shocks} \end{cases} \tag{49}$$

iv) if $\varphi' < 0$ and $\varphi < \alpha_0$ then

$$v = v_L + \frac{1}{\alpha_0} H(\mu, \mu_L), \quad \text{with} \begin{cases} \mu < \mu_L, & \text{for 1-shocks} \\ \mu > \mu_L, & \text{for 3-shocks} \end{cases} \tag{50}$$

Moreover $s < 0$ for 1-shocks while $s > 0$ for 3-shocks.

Let us denote with

$$v = S_1(\mu, v_L, \mu_L), \quad v = S_3(\mu, v_L, \mu_L) \tag{51}$$

the Rankine-Hugoniot curves corresponding, respectively, to the 1-shocks and 3-shocks which are characterized in the previous i)–iv) cases.

Since

$$\frac{dH}{d\mu} = \operatorname{sgn}(\mu - \mu_L) \frac{(F'(\mu)(F'(\xi) - 1) + F'(\xi)(F'(\mu) - 1))}{\sqrt{F'(\xi)(F'(\xi) - 1)}}$$

$$\frac{d^2 H}{d\mu^2} = \frac{\operatorname{sgn}(\mu - \mu_L)}{2\sqrt{F'(\xi)(F'(\xi) - 1)}} \left\{ F''(\mu)(2F'(\xi) - 1) - \frac{(F'(\xi) - F'(\mu))^2}{4(F'(\xi)(F'(\xi) - 1))} \right\}$$

where $\xi \in (\mu_L, \mu)$ if $\mu_L < \mu$ or $\xi \in (\mu, \mu_L)$ if $\mu < \mu_L$, then it is easily to verify that $\frac{dS_1}{d\mu} > 0$ and $\frac{dS_3}{d\mu} < 0$ in the i) and iii) cases, $\frac{dS_1}{d\mu} < 0$ and $\frac{dS_3}{d\mu} > 0$ in the ii) and iv) cases, $\frac{d^2 S_1}{d\mu^2} < 0$ and $\frac{d^2 S_3}{d\mu^2} > 0$ in the i) and ii) cases while $\frac{d^2 S_1}{d\mu^2} > 0$ and $\frac{d^2 S_3}{d\mu^2} < 0$ in the iii) and iv) cases.

Furthermore, owing to (30) and (37), in all the four cases i)–iv) it follows soon that

$$\left(\frac{dR_1}{d\mu}\right)_{\mu_L} = \left(\frac{dS_1}{d\mu}\right)_{\mu_L}, \quad \left(\frac{dR_3}{d\mu}\right)_{\mu_L} = \left(\frac{dS_3}{d\mu}\right)_{\mu_L}, \tag{52}$$

$$\left(\frac{d^2 R_1}{d\mu^2}\right)_{\mu_L} = \left(\frac{d^2 S_1}{d\mu^2}\right)_{\mu_L}, \quad \left(\frac{d^2 R_3}{d\mu^2}\right)_{\mu_L} = \left(\frac{d^2 S_3}{d\mu^2}\right)_{\mu_L}. \tag{53}$$

Thererefore in the (v, μ, w) space the solution of the Riemann problem (24) in terms of shock waves is determined by (51) along with $w_R = w_L$ provided that the equilibrium state $(v_R, \mu_R) \in S_1(\mu, v_L, \mu_L)$ or $(v_R, \mu_R) \in S_3(\mu, v_L, \mu_L)$. Furthermore, taking (13) and (14) into account, the solution of (24) in the (v, ε, σ) space is given by

$$\sigma_R - \sigma_L = F(\sigma_R - \alpha_0^2 \varepsilon_R) - F(\sigma_L - \alpha_0^2 \varepsilon_L) \tag{54}$$

$$v_R - v_L = \pm\sqrt{(\varepsilon_R - \varepsilon_L)(\sigma_R - \sigma_L)} \tag{55}$$

$$s = -\frac{\sigma_R - \sigma_L}{v_R - v_L} \tag{56}$$

where in (55) the $\pm$ sign is determined according to the analysis previously developed in the i)–iv) cases.

REMARK 1. Let us consider the subsystem [2] of (15)–(17) determined by $w = w_0 = \text{const.}$ such that $\Psi_0(w_0) = 0$. By introducing the variable transformation

$$m = \frac{F(\mu) - \mu}{\alpha_0^2} \tag{57}$$

and setting

$$p(m) = -F(\mu(m)) = -\int \varphi^2(\mu(m))\, dm \tag{58}$$

the equations (15) and (16) reduce to

$$\frac{\partial v}{\partial t} + \frac{\partial p(m)}{\partial x} = 0 \tag{59}$$

$$\frac{\partial m}{\partial t} - \frac{\partial v}{\partial x} = 0. \tag{60}$$

Therefore, since

$$\frac{dp}{dm} = -\varphi^2, \quad \frac{d^2 p}{dm^2} = -2\varphi(\varphi^2 - \alpha_0^2)\frac{d\varphi}{d\mu} \tag{61}$$

in the ii) and iii) cases, the pair of equations (59) and (60) is the celebrated p-system where $p'(m) < 0$ and $p''(m) > 0$.

REMARK 2. Owing to (13) and (14), since for the 1-shocks and 3-shocks

$$[\sigma] = [F], \quad \alpha_0^2[\varepsilon] = [F - \mu], \tag{62}$$

it follows that in the i) and iv) cases $[\varepsilon] > 0$ and $[\sigma] > 0$ for 1-shocks while $[\varepsilon] < 0$ and $[\sigma] < 0$ for 3-shocks. The opposite situation is in the ii) and iii) cases where

$[\varepsilon] < 0$ and $[\sigma] < 0$ for 1-shocks while $[\varepsilon] > 0$ and $[\sigma] > 0$ for 3-shocks. Furthermore since in the 1-shocks $s < 0$, the indexes $_L$ and $_R$ denote, respectively, the unperturbed state $_0$ and the perturbed state while for the 3-shocks $s > 0$ so that the indexes $_L$ and $_R$ denote, respectively, the perturbed state and the unperturbed state $_0$. Therefore for both the 1-shocks and the 3-shocks $\varepsilon > \varepsilon_0$ and $\sigma > \sigma_0$ in the i) and iv) cases while $\varepsilon < \varepsilon_0$ and $\sigma < \sigma_0$ in the ii) and iii) cases.

REMARK 3. If in the material response functions (12) we assume $\varphi = \alpha_0$, then the governing system (1)–(3) specializes to the model describing linear elasticity where $\sigma = \alpha_0^2 \varepsilon$ so that we can set $\alpha_0^2 = E$ where E is the Young's modulus. In such a case the elastic waves propagate with velocities $\pm\sqrt{E}$. Since the viscoelastic waves described by the system (1)–(3) supplemented by (12) propagate with the chracteristic speeds (4), then in the cases i)–iv) the conditions $\varphi > \alpha_0$ or $\varphi < \alpha_0$ require that the viscoelastic waves propagate, respectively, faster or slower than the corresponding elastic ones.

5. RIEMANN PROBLEM AND WAVE INTERACTIONS

In the previous sections we solved the Riemann problem (24) through a rarefaction or a shock wave and we characterized four different cases. Here our main aim is to solve the initial value problem (24) in a more general form. In the following we concentrate our attention to the i) case.

Owing to the analysis developed in sections 2 and 3 and taking (52) and (53) into account, in figure 1 we represent in the (μ, v) plane the rarefaction curves $v = R_1$, $v = R_3$ as well as the shock curves $v = S_1$, $v = S_3$. Moreover we set

$$\mathscr{F} = \{v = T_3(\mu, \mu_0, v_0) : v_0 = T_1(\mu_0, \mu_L, v_L)\} \tag{63}$$

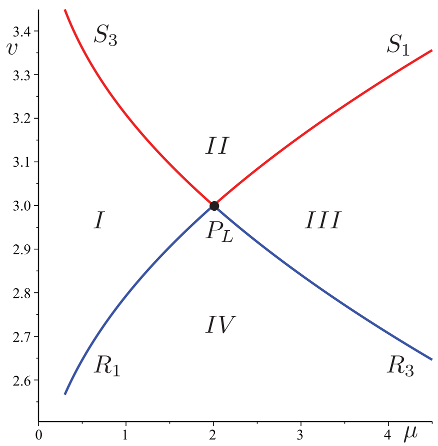

Figure 1. Rarefaction curves $v = R_1(\mu, \mu_L, v_L)$, $v = R_3(\mu, \mu_L, v_L)$ and shock curves $v = S_1(\mu, \mu_L, v_L)$ and $v = S_3(\mu, \mu_L, v_L)$ where $P_L = (\mu_L, v_L)$.

where $T_i = R_i \cup S_i$ with $i = 1$ or $i = 3$. The situation depicted in figure 1 is formally similar to what happens in the Riemann problem's solution for the p-system. Therefore the analysis there developed (see for instance [43]) can be extended to the present case. In particular it can be proved that through each point of the the regions I, II and III passes one and only one curve $T_3 \in \mathscr{F}$ so that if the right initial datum (μ_R, v_R) belongs to one of the first three regions, then the solution of (24) consists of three constant states separated by rarefaction and/or shock waves. In region IV, if (μ_R, v_R) is not far from (μ_L, v_L), the RP is solved in terms of constant states connected by the R_1 and R_3 rarefaction waves, while if (μ_R, v_R) is far from (μ_L, v_L), then not every point of the region can be reached by a curve T_3 and the "vacuum" situation could appear. Of course such an analysis can be carried on also in the remaining ii)–iv) cases.

Next we aim to show how the results obtained previously can be useful for solving more general initial data. For instance let us consider the two Riemann problems

$$\mu(x,0) = \begin{cases} \mu_1 & \text{for } x < 0 \\ \mu_2 & \text{for } 0 < x < x_0, \\ \mu_3 & \text{for } x > x_0 \end{cases} \quad v(x,0) = \begin{cases} v_1 & \text{for } x < 0 \\ v_2 & \text{for } 0 < x < x_0 \\ v_3 & \text{for } x > x_0 \end{cases} \tag{64}$$

where $\mu_2 < \mu_3 < \mu_1$ while $v_2 = S_3(\mu_2, v_1, \mu_1)$ (front shock) and $v_3 = S_1(\mu_3, v_2, \mu_2)$ (back shock). Owing to the analysis carried on above, the solution of (64) is determined by the constant states (μ_1, v_1), (μ_2, v_2) and (μ_3, v_3) which are separated, respectively, by a front shock and a back shock whose shock lines are

$$x = x_3(t) = s_3 t; \quad s_3 = -\frac{F_2 - F_1}{v_2 - v_1} > 0 \tag{65}$$

$$x = x_1(t) = s_1 t + x_0; \quad s_1 = -\frac{F_3 - F_2}{v_3 - v_2} < 0. \tag{66}$$

(67)

Such a solution is valid for $t \in (0, t_c)$ where

$$t_c = \frac{x_0}{s_3 - s_1}$$

is the instant time when the front shock and the back shock interact in the place $x_c = \frac{s_3 x_0}{s_3 - s_1}$. Therefore at $t = t_c$ we find the new Riemann problem

$$\mu(x,t_c) = \begin{cases} \mu_1 & \text{for } x < x_c \\ \mu_3 & \text{for } x > x_c \end{cases}, \quad v(x,t_c) = \begin{cases} v_1 & \text{for } x < x_c \\ v_3 & \text{for } x > x_c \end{cases} \tag{68}$$

Since the right "initial" state (μ_3, v_3) is in the II region of the (μ, v) plane (see figure 2), then the solution of (68) is given in terms of the constant states (μ_1, v_1), $(\bar{\mu}, \bar{v})$ and (μ_3, v_3) separated by a back shock and a front shock whose shock lines are given, respectively, by

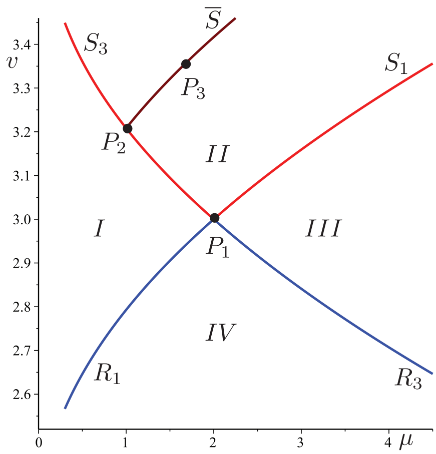

Figure 2. Rarefaction curves $v = R_1(\mu, \mu_1, v_1)$, $v = R_3(\mu, \mu_1, v_1)$ and shock curves $v = S_1(\mu, \mu_1, v_1)$, $v = S_3(\mu, \mu_1, v_1)$, $v = \overline{S}(\mu) = S_1(\mu, \mu_2, v_2)$. Moreover $P_1 = (\mu_1, v_1)$, $P_2 = (\mu_2, v_2)$, $P_3 = (\mu_3, v_3)$.

$$x = \bar{x}_1(t) = \bar{s}_1(t - t_c) + x_c; \quad \bar{s}_1 = -\frac{F(\bar{\mu}) - F_1}{\bar{v} - v_1} < 0 \tag{69}$$

$$x = \bar{x}_3(t) = \bar{s}_3(t - t_c) + x_c; \quad \bar{s}_3 = -\frac{F_3 - F(\bar{\mu})}{v_3 - \bar{v}} > 0. \tag{70}$$

(71)

Furthermore the state $\bar{\mu}$ and $\bar{v}$ can be calculated from

$$\begin{cases} \bar{v} = S_1(\bar{\mu}, v_1, \mu_1) \\ v_3 = S_3(\mu_3, \bar{v}, \bar{\mu}) \end{cases} \tag{72}$$

In figure 3 we show in the (x, t) plane the solution of (64) for $t > 0$.

6. Conclusions and final remarks

In this paper we considered nonlinear wave propagation problems for a quasilinear hyperbolic nonhomogeneous first order system describing a viscoelastic medium with short memory effects. In particular, following the results obtained in [13], we studied some Riemann problems for the viscoelastic governing model (1)–(3) by solving the reduced 2×2 homogeneous first order system (19), (20) along with the differential constraint (17). The general analysis carried on in section 5 permitted to solve a nonlinear shocks interaction problem.

The reduction procedure here considered was carried on under the hypotesis that the material response functions $\Phi(\varepsilon, \sigma)$ and $\Psi(\varepsilon, \sigma)$ adopt the form (12). It should be of a certain interest to notice that if we assume

$$\varphi^2(\mu) = \alpha_0^2 - c_0\mu, \quad \Psi_0(w) = \frac{1}{c_0\tau}\left(1 - k_0 e^{-\frac{c_0}{\alpha_0^2}w}\right), \tag{73}$$

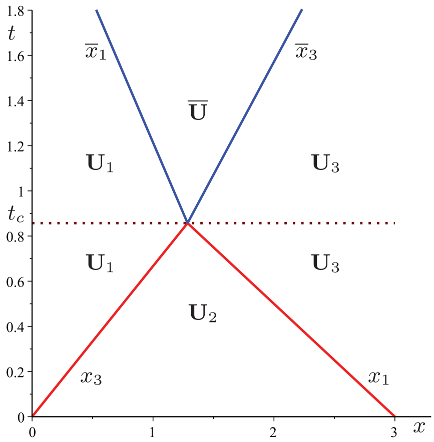

Figure 3. Sketch of the solution to (64) in the (x, t) plane. For $t < t_c$ the shock lines $x = x_3(t)$ and $x = x_1(t)$ separate, respectively, the constant states $(\mathbf{U}_1, \mathbf{U}_2)$ and $(\mathbf{U}_2, \mathbf{U}_3)$. After their interaction, at $t = t_c$ the new shock lines $x = \bar{x}_1(t)$ and $x = \bar{x}_3(t)$ separate, respectively, the constant states $(\mathbf{U}_1, \overline{\mathbf{U}})$ and $(\overline{\mathbf{U}}, \mathbf{U}_3)$. Here $\mathbf{U} = (\mu, v)$.

then the functions Φ and Ψ specialize to

$$\Phi = \alpha_0^2 - c_0(\sigma - \sigma_i(\varepsilon)), \quad \Psi = -\frac{1}{\tau}(\sigma - \sigma_e(\varepsilon)) \tag{74}$$

where

$$\sigma_i(\varepsilon) = \alpha_0^2\varepsilon, \quad \sigma_e(\varepsilon) = \alpha_0^2\varepsilon + k_0 e^{-c_0\varepsilon} \tag{75}$$

denote, respectively, the instantaneous and the equilibrium stress-strain curves. In (73) c_0 and k are arbitrary parameters while $\tau > 0$ is a relaxation time. Therefore the material response functions (74) are in a good agreement with the constitutive equations (7) and (8) related to the Coleman and Noll model. We note that the instantaneous stress-strain curve given in $(75)_1$ is the Hooke law so that we can identify again the parameter α_0^2 with the Young's modulus (see remark 3 of section 4). Finally if we assume $c_0 < 0$ and $k_0 < 0$, then the behaviour of the equilibrium stress-strain curve characterized in $(75)_2$ is in a good agreement with that obtained in [41] for polymethyl methacrylate [13].

Acknowledgments. This work was supported by Italian National Group of Mathematical Physics (GNFM-INdAM).

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Received 22 December 2016,
and in revised form 10 January 2017.

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