Rend. Lincei Mat. Appl. 28 (2017), 495–513
DOI 10.4171/RLM/773

Mathematical Physics — *Galilean invariance and entropy principle for a system of balance laws of mixture type*, by Takashi Arima, Tommaso Ruggeri, Masaru Sugiyama and Shigeru Taniguchi, communicated on January 13, 2017.

This paper is dedicated to the memory of Professor Giuseppe Grioli.

Abstract. — After defining, in analogy with a mixture of continuous media, a *system of balance laws of mixture type*, we study the general properties obtained by imposing the Galilean invariance principle. For constitutive equations of local type we study also the entropy principle and we prove the compatibility between the two principles. These general results permit us to construct, from a single constituent theory, the corresponding theory of mixtures in an easy way. As an illustrative example of the general theory, we write down the hyperbolic system of balance laws of mixtures in which each component has 6 fields (mass density, velocity, temperature and dynamic pressure, among which only the last one is a nonequilibrium variable). This is the simplest system after Eulerian mixtures. Global existence of smooth solutions for small initial data is also proved.

Key words: Mixture of fluids, non-equilibrium thermodynamics, Galilean invariance, entropy principle

Mathematics Subject Classification: 82C35, 76R50, 76N15, 35L60

1. Introduction

Several decades ago, Ruggeri [9] studied a general structure in a system of balance laws compatible with the Galilean invariance. He proved that the densities, fluxes and productions in a system admit a unique dependence on the velocity field for any constitutive equations. If there exists natural order in the balance equations as in the case, for example, of the moment theory associated with the kinetic equation, the velocity dependence can be expressed in the form of a polynomial. In the same paper, the author also studied the requirement of the entropy principle for hyperbolic systems with constitutive equations of local type. It was proved that a perfect compatibility exists between two principles: the Galilean invariance dictates the velocity dependence and the entropy principle imposes constraints on the constitutive equations.

In this paper, starting from a single system of balance laws, we define, in analogy with a simple mixture of continuous media, a *system of balance laws of mixture type*. Then we firstly adapt the general properties of the Galilean invariance in a generic system of balance laws with a single constituent [9] to a system of mixture type with any number of constituents. The Galilean invariance dictates the velocity dependence of the field equations for any kind of constitutive equations and moreover it permits a construction of the resultant fields of a mixture. Secondly, for a system of balance laws of hyperbolic type, we discuss the consequence derived from the entropy principle and its compatibility with the Galilean invariance. After the general discussion, we study, as a special case, a system of balance laws in extended thermodynamics (ET) of polyatomic dissipative gases. In particular, we write down explicitly the hyperbolic system of mixtures in which each component has 6 fields (ET_6): mass density, velocity, temperature and dynamic pressure, among which only the last one is a nonequilibrium variable. This is the simplest system after Eulerian mixtures. The *main field* for which the differential system becomes symmetric hyperbolic is shown. Characteristic velocities, K-condition and the existence of global smooth solutions are also discussed.

2. System of balance laws of mixture type

Systems in continuum mechanics represent the balance laws, and under a regularity assumption these are expressed in the divergence form in space-time:

$$\frac{\partial \mathbf{F}}{\partial t} + \frac{\partial \mathbf{F}_i}{\partial x_i} = \mathbf{P}; \quad \mathbf{F}_i = \mathbf{F} v_i + \mathbf{G}_i, \tag{1}$$

where the density vector $\mathbf{F}$, the flux vector $\mathbf{F}_i$, the non-convective flux vector $\mathbf{G}_i$, and the production vector $\mathbf{P}$ are vectors of R^N and are expressed through constitutive equations in terms of a field $\mathbf{U}$. The vector $\mathbf{v}$ is the flow velocity. For the moment we consider generic constitutive equations, i.e., the above quantities may depend on $\mathbf{U}$ either locally or non-locally in space and/or in time. Usually this is a system of balance laws for a single-constituent material like a single fluid.

In the case of a simple mixture, following Truesdell [17], we usually adopt the assumption, which is also motivated by the kinetic theory, that the form of the system of field equations for any constituent is the same as that for a single fluid except that there appear new production terms due to the interchange of mass, momentum, energy, and other quantities between constituents. In analogy with such mixtures, we adopt the following:

Definition 1 (Systems of balance laws of mixture type). *A system of balance laws of mixture type is a system of the form:*

$$\frac{\partial \mathbf{F}^\alpha}{\partial t} + \frac{\partial \mathbf{F}_i^\alpha}{\partial x_i} = \mathbf{P}^\alpha + \mathbf{f}^\alpha; \quad \mathbf{F}_i^\alpha = \mathbf{F}^\alpha v_i^\alpha + \mathbf{G}_i^\alpha, \quad (\alpha = 1, 2, \ldots, n) \tag{2}$$

where the density vector $\mathbf{F}^\alpha$*, the flux vector* $\mathbf{F}_i^\alpha$*, and the production vector* $\mathbf{P}^\alpha$ *have, respectively, the same forms as* $\mathbf{F}$, $\mathbf{F}_i$ *and* $\mathbf{P}$ *of the single constituent, and are obtained by the formal substitution of* $\mathbf{U}^\alpha$ *instead of* $\mathbf{U}$. *The new production vector*

$\mathbf{f}^\alpha$ *emerges due to the interchange of the densities between the constituents. The number of constituents is n.*

The production term $\mathbf{P}^\alpha$ indicates the source of the dissipative fluxes that appears also in the single-constituent fluid. On the other hand, $\mathbf{f}^\alpha$ is the production originated from the effect of the mixture, such as chemical reaction and diffusion. In other words, it represents the interchange of mass, momentum, energy and other fields between the constituents. The system treated as a whole, which takes all components of (2) into account, must appear as a system of a single material. It is, however, more complex than (1) because diffusion contributes to the objective-variables:

$$\frac{\partial \mathscr{F}}{\partial t} + \frac{\partial \mathscr{F}_i}{\partial x_i} = \mathscr{P}; \quad \mathscr{F}_i = \mathscr{F} v_i + \mathscr{G}_i, \tag{3}$$

$$\mathscr{F} = \sum_{\alpha=1}^{n} \mathbf{F}^\alpha, \quad \mathscr{F}_i = \sum_{\alpha=1}^{n} \mathbf{F}_i^\alpha, \quad \mathscr{P} = \sum_{\alpha=1}^{n} \mathbf{P}^\alpha, \quad \sum_{\alpha=1}^{n} \mathbf{f}^\alpha = 0. \tag{4}$$

As a consequence of $(2)_2$, we have

$$\mathscr{G}_i = \sum_{\alpha=1}^{n} (\mathbf{F}^\alpha u_i^\alpha + \mathbf{G}_i^\alpha)$$

with the diffusion velocity u_i^α defined by

$$u_i^\alpha = v_i^\alpha - v_i \quad \Big(\sum_{\alpha=1}^{n} \rho^\alpha u_i^\alpha = 0\Big).$$

As the production vectors $\mathbf{f}^\alpha$ are not independent of each other due to the last condition of (4), it is sometimes convenient to consider another equivalent system that is composed of global balance laws and the component equations with indexes $b = 1, 2, \ldots, n-1$:

$$\begin{aligned} &\frac{\partial \mathscr{F}}{\partial t} + \frac{\partial \mathscr{F}_i}{\partial x_i} = \mathscr{P}, \\ &\frac{\partial \mathbf{F}^b}{\partial t} + \frac{\partial \mathbf{F}_i^b}{\partial x_i} = \mathbf{P}^b + \mathbf{f}^b; \quad \mathbf{F}_i^b = \mathbf{F}^b v_i^b + \mathbf{G}_i^b. \end{aligned} \tag{5}$$

An example: Mixture of Eulerian fluids

Here it is instructive, instead of continuing the general discussion, to give a concrete example of the above definition, that is, a mixture of Eulerian fluids. We start with a single Eulerian fluid that is in the form of (1). The conservation laws of mass, momentum and energy multiplied by 2 are expressed as

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) = 0,\\
&\frac{\partial}{\partial t}(\rho v_j) + \frac{\partial}{\partial x_i}(\rho v_i v_j + p\delta_{ij}) = 0,\\
&\frac{\partial}{\partial t}(\rho v^2 + 2\rho\varepsilon) + \frac{\partial}{\partial x_i}\{(\rho v^2 + 2\rho\varepsilon + 2p)v_i\} = 0,
\end{aligned} \tag{6}
$$

where ρ, p and ε are, respectively, the mass density, the pressure and the specific internal energy density, and δ_{ij} is the Kronecker delta.

Then according with the definition, the system of balance laws for a mixture of Eulerian fluids becomes in the form (2):

$$
\begin{aligned}
&\frac{\partial \rho^\alpha}{\partial t} + \frac{\partial}{\partial x_i}(\rho^\alpha v_i^\alpha) = \tau^\alpha,\\
&\frac{\partial}{\partial t}(\rho^\alpha v_j^\alpha) + \frac{\partial}{\partial x_i}(\rho^\alpha v_i^\alpha v_j^\alpha + p^\alpha\delta_{ij}) = m_i^\alpha,\\
&\frac{\partial}{\partial t}(\rho^\alpha (v^\alpha)^2 + 2\rho^\alpha\varepsilon^\alpha) + \frac{\partial}{\partial x_i}\{(\rho^\alpha (v^\alpha)^2 + 2\rho^\alpha\varepsilon^\alpha + 2p^\alpha)v_i^\alpha\} = e^\alpha,
\end{aligned} \tag{7}
$$

where the quantities with suffix α are corresponding quantities for the α-constituent, and τ^α, m_i^α, and e^α are the new production terms that represent the interchange of mass, momentum and energy, respectively. If there exists no chemical reaction τ^α vanishes.

The global system obtained by summing up all equations in (7) is in the form (3):

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) = 0,\\
&\frac{\partial}{\partial t}(\rho v_j) + \frac{\partial}{\partial x_i}(\rho v_i v_j - t_{ij}) = 0,\\
&\frac{\partial}{\partial t}(\rho v^2 + 2\rho\varepsilon) + \frac{\partial}{\partial x_i}\{(\rho v^2 + 2\rho\varepsilon)v_i - 2t_{ij}v_j + 2q_i\} = 0,
\end{aligned} \tag{8}
$$

where the global quantities are given by [12, 14]:

$$
\begin{aligned}
&\rho = \sum_{\alpha=1}^{n} \rho^\alpha && : \text{total mass density},\\
&\mathbf{v} = \frac{1}{\rho}\sum_{\alpha=1}^{n} \rho^\alpha \mathbf{v}^\alpha && : \text{mixture velocity},\\
&\mathbf{u}^\alpha = \mathbf{v}^\alpha - \mathbf{v} \quad \left(\sum_{\alpha=1}^{n} \rho^\alpha \mathbf{u}^\alpha = \mathbf{0}\right) && : \text{diffusion velocity},
\end{aligned} \tag{9}
$$

$$t_{ij} = -\sum_{\alpha=1}^{n}(p^{\alpha}\delta_{ij} + \rho^{\alpha}u_i^{\alpha}u_j^{\alpha}) \qquad : \text{stress tensor},$$

$$\varepsilon = \frac{1}{\rho}\sum_{\alpha=1}^{n}\rho^{\alpha}\left(\varepsilon^{\alpha} + \frac{1}{2}(u^{\alpha})^2\right) \qquad : \text{specific internal energy}, \tag{9}$$

$$q_i = \sum_{\alpha=1}^{n}\left\{\rho^{\alpha}\left(\varepsilon^{\alpha} + \frac{1}{2}(u^{\alpha})^2\right) + p^{\alpha}\right\}u_i^{\alpha} : \text{heat flux},$$

and

$$\sum_{\alpha=1}^{n}\tau^{\alpha} = \sum_{\alpha=1}^{n}\mathbf{m}_i^{\alpha} = \sum_{\alpha=1}^{n}e^{\alpha} = 0.$$

We notice the difference between two systems (6) and (8) (in general, between two systems (1) and (3)) due to the diffusion velocity $\mathbf{u}^{\alpha}$. In the whole mixture of Eulerian fluids, the stress tensor t_{ij} that is not isotropic anymore and the heat flux q_i have emerged.

3. Galilean invariance

It was proved [9] that the Galilean invariance of a system of balance laws (1) results in explicit dependence of the densities, the fluxes and the productions on the velocity. In fact in [9] it was proved that, if we split the field such that $\mathbf{U} \equiv (\mathbf{v}, \mathbf{w})$, where $\mathbf{v}$ is the velocity $\in \mathbf{R}^3$ and $\mathbf{w}$ corresponds to the other objective fields $\in \mathbf{R}^{N-3}$, there exists an $(N \times N)$-matrix operator $\mathbf{X}(\mathbf{v})$ such that:

$$\begin{cases} \mathbf{F}(\mathbf{v}, \mathbf{w}) = \mathbf{X}(\mathbf{v})\mathbf{F}(0, \mathbf{w}), \\ \mathbf{G}_i(\mathbf{v}, \mathbf{w}) = \mathbf{X}(\mathbf{v})\mathbf{G}_i(0, \mathbf{w}), \\ \mathbf{P}(\mathbf{v}, \mathbf{w}) = \mathbf{X}(\mathbf{v})\mathbf{P}(0, \mathbf{w}). \end{cases} \tag{10}$$

The operator $\mathbf{X}(\mathbf{v})$, which determines the velocity dependence of the fields, has the following feature:

$$\mathbf{X}(\mathbf{a} + \mathbf{b}) = \mathbf{X}(\mathbf{a})\mathbf{X}(\mathbf{b}) = \mathbf{X}(\mathbf{b})\mathbf{X}(\mathbf{a}),$$

and is expressed as an exponential matrix:

$$\mathbf{X}(\mathbf{v}) = e^{\mathbf{A}^r v_r} = \mathbf{I} + \mathbf{A}^r v_r + \frac{1}{2}\mathbf{A}^r\mathbf{A}^s v_r v_s + \cdots$$

with $\mathbf{A}^r$ being three $(N \times N)$ constant matrices such that

$$\mathbf{A}^r\mathbf{A}^s = \mathbf{A}^s\mathbf{A}^r, \quad \forall r, s = 1, 2, 3.$$

Usually there exists a natural order in the balance laws. For example, in the moment theory, the densities are tensors with increasing number of index. In such a case, the matrix $\mathbf{X}$ is sub-diagonal and the matrices $\mathbf{A}^r$ are nilpotent of order m, where m denotes the largest tensorial order. In this case the matrix $\mathbf{X}$ is polynomial in the velocity [9].

By Definition 1, the left side member of a balance-law system of mixture type has blocks, each of which corresponds to a constituent and has the same structure of (1). Therefore this system is also Galilean invariant:

$$\begin{cases} \mathbf{F}^\alpha(\mathbf{v}^\alpha, \mathbf{w}^\alpha) = \mathbf{X}(\mathbf{v}^\alpha)\mathbf{F}^\alpha(\mathbf{v}^\alpha = 0, \mathbf{w}^\alpha), \\ \mathbf{G}_i^\alpha(\mathbf{v}^\alpha, \mathbf{w}^\alpha) = \mathbf{X}(\mathbf{v}^\alpha)\mathbf{G}_i^\alpha(\mathbf{v}^\alpha = 0, \mathbf{w}^\alpha), \\ \mathbf{P}^\alpha(\mathbf{v}^\alpha, \mathbf{w}^\alpha) = \mathbf{X}(\mathbf{v}^\alpha)\mathbf{P}^\alpha(\mathbf{v}^\alpha = 0, \mathbf{w}^\alpha), \end{cases} \tag{11}$$

As a consequence, we have also

$$\mathbf{f}^\alpha(\mathbf{v}^\alpha, \mathbf{w}^\alpha) = \mathbf{X}(\mathbf{v}^\alpha)\mathbf{f}^\alpha(\mathbf{v}^\alpha = 0, \mathbf{w}^\alpha). \tag{12}$$

Now we want to prove the following theorem:

THEOREM 1. *The global system* (3) *is also Galilean invariant with the same matrix* $\mathbf{X}(\mathbf{v})$*:*

$$\begin{cases} \mathscr{F}(\mathbf{v}, \bar{\mathbf{w}}) = \mathbf{X}(\mathbf{v})\mathscr{F}(0, \bar{\mathbf{w}}), \\ \mathscr{G}_i(\mathbf{v}, \bar{\mathbf{w}}) = \mathbf{X}(\mathbf{v})\mathscr{G}_i(0, \bar{\mathbf{w}}), \\ \mathscr{P}(\mathbf{v}, \bar{\mathbf{w}}) = \mathbf{X}(\mathbf{v})\mathscr{P}(0, \bar{\mathbf{w}}), \end{cases} \tag{13}$$

where $\bar{\mathbf{w}}$ *denotes objective quantities that are, in general, different from* $\mathbf{w}$*, and the global intrinsic density, flux and production are given by:*

$$\begin{aligned} \mathscr{F}(0, \bar{\mathbf{w}}) &= \sum_{\alpha=1}^{n} \mathbf{F}^\alpha(\mathbf{v}^\alpha = \mathbf{u}^\alpha, \mathbf{w}^\alpha), \\ \mathscr{G}_i(0, \bar{\mathbf{w}}) &= \sum_{\alpha=1}^{n} \mathbf{F}_i^\alpha(\mathbf{v}^\alpha = \mathbf{u}^\alpha, \mathbf{w}^\alpha), \\ \mathscr{P}(0, \bar{\mathbf{w}}) &= \sum_{\alpha=1}^{n} \mathbf{P}^\alpha(\mathbf{v}^\alpha = \mathbf{u}^\alpha, \mathbf{w}^\alpha). \end{aligned} \tag{14}$$

PROOF. From (4) and $(2)_2$ we have

$$\begin{aligned} \mathscr{F}(\mathbf{v}, \bar{\mathbf{w}}) &= \sum_{\alpha=1}^{n} \mathbf{F}^\alpha(\mathbf{v}^\alpha, \mathbf{w}^\alpha), \\ \mathscr{G}_i(\mathbf{v}, \bar{\mathbf{w}}) &= \sum_{\alpha=1}^{n} [\mathbf{F}^\alpha(\mathbf{v}^\alpha, \mathbf{w}^\alpha)u_i^\alpha + \mathbf{G}_i^\alpha(\mathbf{v}^\alpha, \mathbf{w}^\alpha)], \\ \mathscr{P}(\mathbf{v}, \bar{\mathbf{w}}) &= \sum_{\alpha=1}^{n} \mathbf{P}^\alpha(\mathbf{v}^\alpha, \mathbf{w}^\alpha). \end{aligned} \tag{15}$$

Evaluating (15) at $\mathbf{v} = 0$ we obtain (14). Taking into account (15) and the property:

$$\mathbf{X}(\mathbf{v}^\alpha) = \mathbf{X}(\mathbf{v} + \mathbf{u}^\alpha) = \mathbf{X}(\mathbf{v})\mathbf{X}(\mathbf{u}^\alpha) = \mathbf{X}(\mathbf{u}^\alpha)\mathbf{X}(\mathbf{v}),$$

we obtain, from (11) and (12),

$$\begin{cases} \mathbf{F}^\alpha(\mathbf{v}^\alpha, \mathbf{w}^\alpha) = \mathbf{X}(\mathbf{v})\mathbf{F}^\alpha(\mathbf{v}^\alpha = \mathbf{u}^\alpha, \mathbf{w}^\alpha), \\ \mathbf{G}_i^\alpha(\mathbf{v}^\alpha, \mathbf{w}^\alpha) = \mathbf{X}(\mathbf{v})\mathbf{G}_i^\alpha(\mathbf{v}^\alpha = \mathbf{u}^\alpha, \mathbf{w}^\alpha), \\ \mathbf{P}^\alpha(\mathbf{v}^\alpha, \mathbf{w}^\alpha) = \mathbf{X}(\mathbf{v})\mathbf{P}^\alpha(\mathbf{v}^\alpha = \mathbf{u}^\alpha, \mathbf{w}^\alpha), \\ \mathbf{f}^\alpha(\mathbf{v}^\alpha, \mathbf{w}^\alpha) = \mathbf{X}(\mathbf{v})\mathbf{f}^\alpha(\mathbf{v}^\alpha = \mathbf{u}^\alpha, \mathbf{w}^\alpha). \end{cases} \tag{16}$$

Then (13) is obtained, and the theorem is proved.

In the case of Eulerian mixtures we have

$$\mathbf{X}(\mathbf{v}) = \begin{pmatrix} 1 & 0 & 0 \\ v_j & \delta_{jk} & 0 \\ v^2 & 2v_j & 1 \end{pmatrix}, \quad \mathbf{A}^r = \begin{pmatrix} 0 & 0 & 0 \\ \delta_{jr} & 0 & 0 \\ 0 & 2\delta_{jr} & 0 \end{pmatrix}.$$

It is easy to obtain the global fields (9) by applying the general expressions (14) to the present case.

4. Entropy principle and main field

4.1. Entropy principle

Until now we have studied the Galilean invariance for arbitrary constitutive equations, local or nonlocal, and, as a consequence, for parabolic or hyperbolic systems. From now we consider only local constitutive equations, i.e., we assume that the densities, fluxes and productions depend on the value of the field $\mathbf{U}(\mathbf{x}, t)$ in the present position $\mathbf{x}$ and at the present time t. In this case the constitutive equations must be compatible with the entropy principle that requires that every solution of the system of balance laws plus the constitutive equations satisfies a supplementary entropy law with non-negative entropy production:

$$\frac{\partial h}{\partial t} + \frac{\partial h_i}{\partial x_i} = \Sigma \geq 0; \quad h_i = hv_i + \varphi_i,$$

where h, φ_i and Σ are, respectively, the entropy density, intrinsic flux and production of the whole system and are velocity independent. The restriction that all thermodynamic processes satisfy the entropy law is expressed by introducing the main field $\mathbf{U}'^\alpha$ as follows [9, 14]:

$$\frac{\partial h}{\partial t} + \frac{\partial h_i}{\partial x_i} - \Sigma = \sum_{\alpha=1}^{n} \mathbf{U}'^\alpha \cdot \left(\frac{\partial \mathbf{F}^\alpha}{\partial t} + \frac{\partial \mathbf{F}_i^\alpha}{\partial x_i} - \mathbf{P}^\alpha - \mathbf{f}^\alpha\right).$$

As a consequence, the following relations hold:

$$
\begin{aligned}
&\mathrm{d}h = \sum_{\alpha=1}^{n} \mathbf{U}'^{\alpha} \cdot \mathrm{d}\mathbf{F}^{\alpha}, \quad \mathrm{d}h_i = \sum_{\alpha=1}^{n} \mathbf{U}'^{\alpha} \cdot \mathrm{d}\mathbf{F}_i^{\alpha}, \\
&\Sigma = \sum_{\alpha=1}^{n} \mathbf{U}'^{\alpha} \cdot (\mathbf{P}^{\alpha} + \mathbf{f}^{\alpha}) \geq 0,
\end{aligned} \tag{17}
$$

and therefore, choosing as field variables $\mathbf{F}^{\alpha}$, we obtain

$$
\mathbf{U}'^{\alpha} = \frac{\partial h}{\partial \mathbf{F}^{\alpha}}.
$$

On the other hand, the total specific entropy density $s = h/\rho$ is assumed to be the sum of the specific entropy densities of the constituent s^{α}, that is,

$$
h = \rho s = \sum_{\alpha=1}^{n} h^{\alpha} = \sum_{\alpha=1}^{n} \rho^{\alpha} s^{\alpha}. \tag{18}
$$

The entropy density of a constituent $h^{\alpha} = \rho^{\alpha} s^{\alpha}$ is a concave function of the densities $\mathbf{F}^{\alpha}$. As the sum of concave functions is also a concave function of the whole system, the entropy density h is a concave function of $\mathbf{F}^{\alpha}$ $(\alpha = 1, \ldots, N)$. Because of this, the original balance laws could be transformed in a symmetric form with respect to $\mathbf{U}'^{\alpha}$. In fact, by introducing the four potential:

$$
h' = \sum_{\alpha=1}^{n} \mathbf{U}'^{\alpha} \cdot \mathbf{F}^{\alpha} - h, \quad h_i' = \sum_{\alpha=1}^{n} \mathbf{U}'^{\alpha} \cdot \mathbf{F}^{\alpha} - h_i,
$$

the original system (2) can be put into the symmetric form:

$$
\sum_{\beta=1}^{n} \left(\frac{\partial^2 h'}{\partial \mathbf{U}'^{\alpha} \partial \mathbf{U}'^{\beta}} \frac{\partial \mathbf{U}'^{\beta}}{\partial t} + \frac{\partial^2 h_i'}{\partial \mathbf{U}'^{\alpha} \partial \mathbf{U}'^{\beta}} \frac{\partial \mathbf{U}'^{\beta}}{\partial x_i} \right) = \mathbf{P}^{\alpha} + \mathbf{f}^{\alpha}.
$$

4.2. Main field corresponding to the fields $\{\mathscr{F}, \mathbf{F}^b\}$

Instead of the system in the form (2) we can use the system in the form (5), which is sometimes more convenient to deal with. Therefore we have now another main field corresponding to the fields $\{\mathscr{F}, \mathbf{F}^b\}$. Let $\{\mathbf{V}', \mathbf{V}'^b\}$ denote such main field. Following the procedure above, we can also express the entropy density, instead of $(17)_1$, as follows:

$$
\mathrm{d}h = \mathbf{V}' \, \mathrm{d}\mathscr{F} + \sum_{b=1}^{n-1} \mathbf{V}'^b \, \mathrm{d}\mathbf{F}^b.
$$

By comparing this with $(17)_1$, we obtain

$$\mathbf{V}' = \mathbf{U}'^n, \quad \mathbf{V}'^b = \mathbf{U}'^b - \mathbf{U}'^n. \tag{19}$$

Therefore it is possible to determine $\{\mathbf{V}', \mathbf{V}'^b\}$ from the given $\mathbf{U}'^\alpha$ of the single-component fluid. The residual inequality is also expressed, instead of $(17)_3$, by $\{\mathbf{V}', \mathbf{V}'^b\}$ as

$$\Sigma = \mathbf{V}' \cdot \mathscr{P} + \sum_{b=1}^{n-1} \mathbf{V}'^b (\mathbf{P}^b + \mathbf{f}^b) \geq 0. \tag{20}$$

5. Compatibility between Galilean invariance and entropy principle

Following the procedure given in [9], it is possible to verify the following Galilean property of the main field:

$$\hat{\mathbf{U}}'^\alpha = \mathbf{U}'^\alpha \mathbf{X}(\mathbf{v}), \tag{21}$$

where, from now on, the hat indicates a quantity evaluated at zero velocity ($\mathbf{v} = 0$). Thanks to (21) and (17) with (16), we obtain

$$\begin{aligned} \mathrm{d}h &= \sum_{\alpha=1}^{n} \hat{\mathbf{U}}'^\alpha \, \mathrm{d}\hat{\mathbf{F}}^\alpha, \\ \mathrm{d}\varphi_i &= \sum_{\alpha=1}^{n} \hat{\mathbf{U}}'^\alpha (\mathrm{d}\hat{\mathbf{G}}_i^\alpha + u_i^\alpha \, \mathrm{d}\hat{\mathbf{F}}^\alpha + \hat{\mathbf{F}}^\alpha \, \mathrm{d}u_i^\alpha) \end{aligned} \tag{22}$$

with constraints

$$\begin{aligned} &\sum_{\alpha=1}^{n} \hat{\mathbf{U}}'^\alpha \mathbf{A}^r \hat{\mathbf{F}}^\alpha = \mathbf{0}, \\ &\sum_{\alpha=1}^{n} \hat{\mathbf{U}}'^\alpha \mathbf{A}^r (\hat{\mathbf{G}}_i^\alpha + u_i^\alpha \hat{\mathbf{F}}^\alpha) = \left(h - \sum_{\alpha=1}^{n} \hat{\mathbf{U}}'^\alpha \hat{\mathbf{F}}^\alpha\right) \delta_{ri}. \end{aligned}$$

Similarly, if we adopt the main field corresponding to the system (5), we have

$$\hat{\mathbf{V}}' = \mathbf{V}' \mathbf{X}(\mathbf{v}), \quad \hat{\mathbf{V}}'^b = \mathbf{V}'^b \mathbf{X}(\mathbf{v}).$$

Then we obtain

$$
\begin{aligned}
\mathrm{d}h &= \hat{\mathbf{V}}' \,\mathrm{d}\hat{\mathscr{F}} + \sum_{b=1}^{n-1} \hat{\mathbf{V}}'^{b} \,\mathrm{d}\hat{\mathbf{F}}^{b}, \\
\mathrm{d}\varphi_i &= \hat{\mathbf{V}}' \,\mathrm{d}\hat{\mathscr{G}}_i + \sum_{b=1}^{n-1} \hat{\mathbf{V}}'^{b} (\mathrm{d}\hat{\mathbf{G}}_i^{b} + u_i^{b} \,\mathrm{d}\hat{\mathbf{F}}^{b} + \hat{\mathbf{F}}^{b} \,\mathrm{d}u_i^{b})
\end{aligned} \tag{23}
$$

with constraints:

$$
\hat{\mathbf{V}}'\mathbf{A}^r\hat{\mathscr{F}} + \sum_{b=1}^{n-1} \hat{\mathbf{V}}'^{b}\mathbf{A}^r\hat{\mathbf{F}}^{b} = \mathbf{0},
$$

$$
\hat{\mathbf{V}}'\mathbf{A}^r\hat{\mathscr{G}}_i + \sum_{b=1}^{n-1} \hat{\mathbf{V}}'^{b}\mathbf{A}^r(\hat{\mathbf{G}}_i^{b} + u_i^{b}\hat{\mathbf{F}}^{b}) = \Big(h - \hat{\mathbf{V}}'\hat{\mathscr{F}} - \sum_{b=1}^{n-1} \hat{\mathbf{V}}'^{b}\hat{\mathbf{F}}^{b}\Big)\delta_{ri}.
$$

From (19) and by comparing (22) with (23), we obtain the relationship between the two sets of intrinsic main field:

$$
\begin{aligned}
\hat{\mathbf{V}}' &= \hat{\mathbf{U}}'^{n}, \\
\hat{\mathbf{V}}'^{b} &= \hat{\mathbf{U}}'^{b} - \hat{\mathbf{U}}'^{n}.
\end{aligned}
$$

These results were obtained firstly for an Eulerian mixture in [13].

Finally, the residual inequality is also expressed in terms of the intrinsic main field:

$$
\Sigma = \hat{\mathbf{V}}'\hat{\mathscr{P}} + \sum_{b=1}^{n-1} \hat{\mathbf{V}}'^{b}(\hat{\mathbf{P}}^{b} + \hat{\mathbf{f}}^{b}) \geq 0.
$$

To sum up, we have proved that there exists perfect compatibility between the Galilean invariance and the entropy principle also in the case of mixture. As seen above, the first dictates the velocity dependence in the balance laws and the second is the restrictions for obtaining the true constitutive objective equations.

6. ET_6 POLYATOMIC GAS MIXTURE

The system of balance laws of ET_6 mixture belongs to the form (1) with

$$
\mathbf{F} = \begin{pmatrix} \rho \\ \rho v_i \\ \rho v^2 + 2\rho\varepsilon \\ \rho v^2 + 3(p+\Pi) \end{pmatrix}, \quad
\mathbf{G}_i = \begin{pmatrix} 0 \\ (p+\Pi)\delta_{ij} \\ 2(p+\Pi)v_i \\ 2(p+\Pi)v_i \end{pmatrix}, \quad
\mathbf{P} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ P_{ll} \end{pmatrix}, \tag{24}
$$

where Π denotes the dynamic pressure [3, 2, 1, 10]. In this model, the shear viscous stress and the heat flux are neglected.

According with Definition 1, the system of balance laws of ET_6 mixtures is given in the form (2) with

$$
\mathbf{F}^\alpha = \begin{pmatrix} \rho^\alpha \\ \rho^\alpha v_i^\alpha \\ \rho^\alpha (v^\alpha)^2 + 2\rho^\alpha \varepsilon^\alpha \\ \rho^\alpha (v^\alpha)^2 + 3(p^\alpha + \Pi^\alpha) \end{pmatrix}, \quad \mathbf{G}_i^\alpha = \begin{pmatrix} 0 \\ (p^\alpha + \Pi^\alpha)\delta_{ij} \\ 2(p^\alpha + \Pi^\alpha) v_i^\alpha \\ 2(p^\alpha + \Pi^\alpha) v_i^\alpha \end{pmatrix},
$$

$$
\mathbf{P}^\alpha = \begin{pmatrix} 0 \\ 0 \\ 0 \\ P_{ll}^\alpha \end{pmatrix}, \quad \mathbf{f}^\alpha = \begin{pmatrix} \tau^\alpha \\ m_j^\alpha \\ e^\alpha \\ \omega^\alpha \end{pmatrix}. \tag{25}
$$

The global system is in the form (3) with

$$
\mathscr{F} = \begin{pmatrix} \rho \\ \rho v_i \\ \rho v^2 + 2\rho\varepsilon \\ \rho v^2 + 3(p + \Pi) \end{pmatrix}, \quad \mathscr{G}_i = \begin{pmatrix} 0 \\ -t_{ij} \\ -2t_{ij}v_j + 2q_i \\ -2t_{ij}v_j + 2Q_i \end{pmatrix}, \quad \mathscr{P} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathscr{P}_{ll} \end{pmatrix}.
$$

We notice that there appear (i) the global stress tensor $t_{ij} = -(p + \Pi)\delta_{ij} + \sigma_{\langle ij \rangle}$ where p is the total pressure, Π is the global dynamic pressure, and $\sigma_{\langle ij \rangle}$ is the global deviatoric shear stress tensor, (ii) the global heat flux q_i, and (iii) a new global quantity Q_i with the same dimension as q_i. The explicit expressions of these quantities will be given below.

6.1. *Galilean invariance of* ET_6 *mixture*

From (10) and (24), we obtain

$$
\mathbf{X}(\mathbf{v}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ v_j & \delta_{jk} & 0 & 0 \\ v^2 & 2v_j & 1 & 0 \\ v^2 & 2v_j & 0 & 1 \end{pmatrix}, \quad P_{ll} = \hat{P}_{ll}. \tag{26}
$$

Then, using the result (14), we obtain the expressions of the global quantities:

$$
t_{ij} = -\sum_{\alpha=1}^{n} \{\rho^\alpha u_i^\alpha u_j^\alpha + (p^\alpha + \Pi^\alpha)\delta_{ij}\},
$$

$$
\Pi = -\frac{1}{3} t_{ll} - p = \sum_{\alpha=1}^{n} \left(\Pi^\alpha + \frac{1}{3} \rho^\alpha (u^\alpha)^2 \right),
$$

$$
\sigma_{\langle ij \rangle} = t_{\langle ij \rangle} = -\sum_{\alpha=1}^{n} \rho^\alpha u_{\langle i}^\alpha u_{j \rangle}^\alpha,
$$

$$q_i = \sum_{\alpha=1}^{n} \left\{ \frac{1}{2} \rho^\alpha (u^\alpha)^2 + \rho^\alpha \varepsilon^\alpha + p^\alpha + \Pi^\alpha \right\} u_i^\alpha,$$

$$Q_i = \sum_{\alpha=1}^{n} \left\{ \frac{1}{2} \rho^\alpha (u^\alpha)^2 + \frac{5}{2} (p^\alpha + \Pi^\alpha) \right\} u_i^\alpha,$$

where the total pressure p is defined as follows:

$$p = \sum_{\alpha=1}^{n} p^\alpha.$$

For production terms, we have

$$\mathbf{P}^\alpha(\mathbf{v}, \mathbf{w}^\alpha) = \mathbf{X}(\mathbf{v}) \hat{\mathbf{P}}^\alpha, \quad \mathbf{f}^\alpha(\mathbf{v}, \mathbf{w}^\alpha) = \mathbf{X}(\mathbf{v}) \hat{\mathbf{f}}^\alpha$$

that imply

$$\begin{aligned} &P_{ll}^\alpha = \hat{P}_{ll}^\alpha, \quad \tau^\alpha = \hat{\tau}^\alpha, \quad \mathbf{m}^\alpha = \hat{\tau}^\alpha \mathbf{v} + \hat{\mathbf{m}}^\alpha, \\ &e^\alpha = \hat{\tau}^\alpha v^2 + 2\hat{\mathbf{m}}^\alpha \cdot \mathbf{v} + \hat{e}^\alpha, \quad \omega^\alpha = \hat{\tau}^\alpha v^2 + 2\hat{\mathbf{m}}^\alpha \cdot \mathbf{v} + \hat{\omega}^\alpha. \end{aligned} \tag{27}$$

From $(4)_3$, we have

$$\sum_{\alpha=1}^{n} \tau^\alpha = \sum_{\alpha=1}^{n} \mathbf{m}^\alpha = \sum_{\alpha=1}^{n} e^\alpha = 0, \quad \mathscr{P}_{ll} = \sum_{\alpha=1}^{n} P_{ll}^\alpha, \tag{28}$$

which, by taking into account (27) and $(26)_2$, are equivalent to

$$\sum_{\alpha=1}^{n} \hat{\tau}^\alpha = \sum_{\alpha=1}^{n} \hat{\mathbf{m}}^\alpha = \sum_{\alpha=1}^{n} \hat{e}^\alpha = 0, \quad \hat{P}_{ll} = \sum_{\alpha=1}^{n} \hat{\mathscr{P}}_{ll}.$$

In the present case, the system (5) becomes

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_k}{\partial x_k} = 0, \\ &\frac{\partial \rho v_i}{\partial t} + \frac{\partial (\rho v_i v_k - t_{ik})}{\partial x_k} = 0, \\ &\frac{\partial \left(\frac{1}{2} \rho v^2 + \rho \varepsilon\right)}{\partial t} + \frac{\partial}{\partial x_k} \left\{ \left(\frac{1}{2} \rho v^2 + \rho \varepsilon \right) v_k - t_{kj} v_j + q_k \right\} = 0, \\ &\frac{\partial}{\partial t} \{ \rho v^2 + 3(p + \Pi) \} + \frac{\partial}{\partial x_k} \{ [\rho v^2 + 3(p + \Pi)] v_k - 2 t_{kj} v_j + 2 Q_k \} = \hat{\mathscr{P}}_{ll}, \\ &\frac{\partial \rho^b}{\partial t} + \frac{\partial \rho^b v_k^b}{\partial x_k} = \hat{\tau}^b, \end{aligned} \tag{29}$$

$$\begin{aligned}
&\frac{\partial \rho^b v_i^b}{\partial t} + \frac{\partial}{\partial x_k}\{\rho^b v_i^b v_k^b + (p^b + \Pi^b)\delta_{ik})\} = \hat{m}_i^b + \hat{\tau}^b v_i,\\
&\frac{\partial}{\partial t}(\rho^b (v^b)^2 + 2\rho^b \varepsilon^b) + \frac{\partial}{\partial x_k}\{[\rho(v^b)^2 + 2\rho^b\varepsilon^b + 2p^b + 2\Pi^b]v_k^b\}\\
(29)\qquad &\quad = \hat{e}^b + \hat{\tau}^b v^2 + 2\hat{\mathbf{m}}^b\cdot\mathbf{v},\\
&\frac{\partial}{\partial t}\{\rho^b(v^b)^2 + 3(p^b + \Pi^b)\} + \frac{\partial}{\partial x_k}\{[\rho^b(v^b)^2 + 5(p^b+\Pi^b)]v_k^b\}\\
&\quad = \hat{P}_{ll}^b + \hat{\omega}^b + \hat{\tau}^b v^2 + 2\hat{\mathbf{m}}^b\cdot\mathbf{v}.
\end{aligned}$$

6.2. *Entropy density near equilibrium and main field in ET_6 mixture*

To avoid complications, we study the system near equilibrium where we can adopt linear constitutive equations with respect to the dynamic pressure. In this case, starting with the results of a single constituent [14], we obtain for each species:

$$h^\alpha = \rho^\alpha s_E^\alpha - \Psi^\alpha(\rho^\alpha, \varepsilon^\alpha)(\Pi^\alpha)^2, \tag{30}$$

where s_E^α is the specific entropy density in an equilibrium state, and

$$\Psi^\alpha = \frac{1}{2T^\alpha\Gamma^\alpha}, \tag{31}$$

with the absolute temperature of the α-constituent T^α and

$$\begin{aligned}
(32)\qquad &\Gamma^\alpha = p^\alpha\left(\frac{5}{3} - \frac{p_\varepsilon^\alpha}{\rho^\alpha}\right) - \rho^\alpha p_\rho^\alpha > 0,\\
&p_\varepsilon^\alpha = \left(\frac{\partial p^\alpha}{\partial \varepsilon^\alpha}\right)_{\rho^\alpha}, \quad p_\rho^\alpha = \left(\frac{\partial p^\alpha}{\partial \rho^\alpha}\right)_{\varepsilon^\alpha}.
\end{aligned}$$

The inequality (32) guarantees the concavity of h^α in (30). And, as the overall entropy h is the sum of h^α as shown in (18), h is a concave function and has a maximum in equilibrium. Therefore the system can be put in a symmetric form in the main field. We denote the main field associated with the system (25) as follows:

$$\mathbf{U}'^\alpha \equiv (\tilde{\lambda}^\alpha, \tilde{\Lambda}_i^\alpha, \tilde{\mu}^\alpha, \tilde{\zeta}^\alpha).$$

Taking into account that, for a fixed α, the main field are those for a single constituent system and that, for ET_6, these were already calculated in [3], we obtain

$$\begin{aligned}
&\tilde{\lambda}^\alpha = -\frac{g^\alpha}{T^\alpha} + \frac{(v^\alpha)^2}{2T^\alpha} + \left\{\frac{2}{\rho^\alpha}(p_\rho^\alpha\rho^\alpha - p_\varepsilon^\alpha\varepsilon^\alpha) - (v^\alpha)^2\left(\frac{2}{3} - \frac{p_\varepsilon^\alpha}{\rho^\alpha}\right)\right\}\Psi^\alpha\Pi^\alpha,\\
&\tilde{\Lambda}_i^\alpha = -\frac{v_i^\alpha}{T^\alpha} - 2v_i^\alpha\left(\frac{p_\varepsilon^\alpha}{\rho^\alpha} - \frac{2}{3}\right)\Psi^\alpha\Pi^\alpha,
\end{aligned}$$

$$\tilde{\mu}^\alpha = \frac{1}{2T^\alpha} + \frac{p^\alpha_\varepsilon}{\rho^\alpha}\Psi^\alpha\Pi^\alpha,$$
$$\tilde{\zeta}^\alpha = -\frac{2}{3}\Psi^\alpha\Pi^\alpha.$$

Let

$$\mathbf{V}' \equiv (\lambda, \Lambda_i, \mu, \zeta), \quad \mathbf{V}'^b \equiv (\lambda^b, \Lambda^b_i, \mu^b, \zeta^b),$$

be the corresponding main fields with respect to the equivalent system (29). From (19), we obtain

$$\lambda = \tilde{\lambda}^n, \quad \Lambda_i = \tilde{\Lambda}^n_i, \quad \mu = \tilde{\mu}^n, \quad \zeta = \tilde{\zeta}^n,$$
$$\lambda^b = \tilde{\lambda}^b - \tilde{\lambda}^n, \quad \Lambda^b_i = \tilde{\Lambda}^b_i - \tilde{\Lambda}^n_i, \quad \mu^b = \tilde{\mu}^b - \tilde{\mu}^n, \quad \zeta^b = \tilde{\zeta}^b - \tilde{\zeta}^n.$$

The explicit expression of these are as follows:

$$\lambda = -\frac{g^n}{T^n} + \frac{(v^n)^2}{2T^n} + \left\{\frac{2}{\rho^n}(p^n_\rho\rho^n - p^n_\varepsilon\varepsilon^n) - (v^n)^2\left(\frac{2}{3} - \frac{p^n_\varepsilon}{\rho^n}\right)\right\}\Psi^n\Pi^n,$$
$$\Lambda_i = -\frac{v^n_i}{T^n} - 2v^n_i\left(\frac{p^n_\varepsilon}{\rho^n} - \frac{2}{3}\right)\Psi^n\Pi^n,$$
$$\mu = \frac{1}{2T^n} + \frac{p^n_\varepsilon}{\rho^n}\Psi^n\Pi^n,$$
$$\zeta = -\frac{2}{3}\Psi^n\Pi^n,$$

(33)
$$\lambda^b = -\frac{g^b}{T^b} + \frac{(v^b)^2}{2T^b} + \left\{\frac{2}{\rho^b}(p^b_\rho\rho^b - p^b_\varepsilon\varepsilon^b) - (v^b)^2\left(\frac{2}{3} - \frac{p^b_\varepsilon}{\rho^b}\right)\right\}\Psi^b\Pi^b$$
$$- \left\{-\frac{g^n}{T^n} + \frac{(v^n)^2}{2T^n} + \left\{\frac{2}{\rho^n}(p^n_\rho\rho^n - p^n_\varepsilon\varepsilon^n) - (v^n)^2\left(\frac{2}{3} - \frac{p^n_\varepsilon}{\rho^n}\right)\right\}\Psi^n\Pi^n\right\},$$
$$\Lambda^b_i = -\frac{v^b_i}{T^b} - 2v^b_i\left(\frac{p^b_\varepsilon}{\rho^b} - \frac{2}{3}\right)\Psi^b\Pi^b - \left\{-\frac{v^n_i}{T^n} - 2v^n_i\left(\frac{p^n_\varepsilon}{\rho^n} - \frac{2}{3}\right)\Psi^n\Pi^n\right\},$$
$$\mu^b = \frac{1}{2T^b} + \frac{p^b_\varepsilon}{\rho^b}\Psi^b\Pi^b - \left(\frac{1}{2T^n} + \frac{p^n_\varepsilon}{\rho^n}\Psi^n\Pi^n\right),$$
$$\zeta^b = -\frac{2}{3}\Psi^b\Pi^b - \left(-\frac{2}{3}\Psi^n\Pi^n\right).$$

The residual inequality (20) is now given by

(34)
$$\Sigma = \sum_{b=1}^{n-1}(\hat{\lambda}^b\hat{\tau}^b + \hat{\lambda}^b_i\hat{m}^b_i + \hat{\mu}^b\hat{e}^b + \hat{\pi}^b\hat{\zeta}^b + \hat{P}^b_{ll}\hat{\zeta}^b) + \hat{P}_{ll}\hat{\zeta} > 0.$$

According to the single constituent ET_6 theory, the linear constitutive equation for the production term for a constituent α is given by [14]

$$\hat{P}^{\alpha}_{ll} = -\frac{3}{\tau^{\alpha}}\Pi^{\alpha}.$$

Therefore, from $(28)_3$, we have

$$\hat{P}_{ll} = -\sum_{\alpha=1}^{n}\frac{3}{\tau^{\alpha}}\Pi^{\alpha},$$

where τ^{α} is the relaxation time of the dynamic pressure of a constituent α. These relaxation times are positive. Taking into account that

$$\sum_{b=1}^{n-1}\hat{P}^{b}_{ll}\hat{\zeta}^{b} + \hat{P}_{ll}\hat{\zeta} = 2\sum_{\alpha=1}^{n}\frac{\Psi^{\alpha}}{\tau^{\alpha}}(\Pi^{\alpha})^2 > 0,$$

because $\tau^{\alpha} > 0$ and $\Psi^{\alpha} > 0$, and that the inequality (34) should be satisfied independently from the production terms, we have the remaining inequality:

$$\Sigma = \sum_{b=1}^{n-1}(\hat{\lambda}^{b}\hat{\tau}^{b} + \hat{\lambda}^{b}_{i}\hat{m}^{b}_{i} + \hat{\mu}^{b}\hat{e}^{b} + \hat{\pi}^{b}\hat{\zeta}^{b}) \geq 0.$$

This allows us to express the production terms by the main field as follows:

$$\begin{aligned}
\hat{\tau}^{b} &= \sum_{c}^{n-1}(\varphi_{bc}\hat{\lambda}^{c} + \beta_{bc}\hat{\mu}^{c} + \gamma_{bc}\hat{\zeta}^{c}),\\
\hat{m}^{b}_{i} &= \sum_{c}^{n-1}\psi_{bc}\hat{\Lambda}^{c}_{i},\\
\hat{e}^{b} &= \sum_{c}^{n-1}(\beta_{bc}\hat{\lambda}^{c} + \theta_{bc}\hat{\mu}^{c} + \kappa_{bc}\hat{\zeta}^{c}),\\
\hat{\pi}^{b} &= \sum_{c}^{n-1}(\gamma_{bc}\hat{\lambda}^{c} + \kappa_{bc}\hat{\mu}^{c} + \omega_{bc}\hat{\zeta}^{c}),
\end{aligned} \tag{35}$$

where the matrix

$$\begin{pmatrix} \varphi_{bc} & \beta_{bc} & \gamma_{bc} \\ \beta_{bc} & \theta_{bc} & \kappa_{bc} \\ \gamma_{bc} & \kappa_{bc} & \omega_{bc} \end{pmatrix}$$

and ψ_{bc} are positive definite. If these coefficients are given, the system is closed.

6.3. Characteristic velocities, K-condition and existence of global smooth solutions

Let V and $\delta\mathbf{U}$ be, respectively, the generic characteristic velocity and the right eigenvector of the hyperbolic system (1). For a single constituent of ET_6, in particular, these quantities are evaluated in Chapter 14 of the book [14]. Therefore, according to the structure of the mixture-type system of balance laws given by Definition 1, the characteristic velocities and the right eigenvectors for a mixture have the same expressions as those of the single constituent except for the change of the field: $\mathbf{U} \to \mathbf{U}^\alpha$. Then we obtain the following two kinds of waves with $\alpha = 1, \ldots, n$:

(i) Contact waves:

$$V = v_n^\alpha,$$

where $v_n^\alpha = \mathbf{v}^\alpha \cdot \mathbf{n}$ with $\mathbf{n}$ being a unit normal vector to the wave front. In this case, $\delta\rho^\alpha$, $\delta\mathbf{v}_T^\alpha$, δp^α are arbitrary. Then, for any α, the contact wave has multiplicity 4, and $\delta v_n^\alpha = 0$, $\delta\Pi^\alpha = -\delta p^\alpha$, where $\mathbf{v}_T^\alpha$ denotes the tangential velocity;

(ii) Sound waves:

$$V = v_n^\alpha \pm \sqrt{\frac{5}{3}\frac{p^\alpha + \Pi^\alpha}{\rho^\alpha}}.$$

In this case, for arbitrary $\delta\rho^\alpha$, we have

$$\delta\mathbf{v}^\alpha = \pm\mathbf{n}\left(\sqrt{\frac{5}{3}\frac{p^\alpha + \Pi^\alpha}{\rho^\alpha}}\right)\frac{\delta\rho^\alpha}{\rho^\alpha}, \quad \delta\varepsilon^\alpha = \frac{\delta\rho^\alpha}{\rho^\alpha}\left(\frac{p^\alpha + \Pi^\alpha}{\rho^\alpha}\right), \quad \delta\Pi^\alpha = \frac{\delta\rho^\alpha}{\rho^\alpha}\Gamma^\alpha, \tag{36}$$

where Γ^α is given by (32).

Concerning the qualitative analysis, there exists a particular rule called *K-condition* or *genuine coupling condition*, which was firstly given by Shizuta and Kawashima in [15] (see also [8]).

DEFINITION 2 (K-condition). *A system* (1) *satisfies the K-condition if, in the equilibrium manifold, any right characteristic eigenvectors* $\delta\mathbf{U}$ *are not in the null space of* $\nabla\mathbf{P}$, *where* $\nabla \equiv \partial/\partial\mathbf{U}$:

$$\delta\mathbf{P}|_E = (\nabla\mathbf{P}\delta\mathbf{U})|_E \neq 0 \quad \forall\delta\mathbf{U} \neq 0. \tag{37}$$

For dissipative one-dimensional systems (1) satisfying the K-condition, there exists the following global existence theorem by Hanouzet and Natalini [6]:

THEOREM 2 (Global existence). *Assume that the system* (1) *is strictly dissipative, a concave entropy exists, and the K-condition is satisfied. Then there exists*

$\delta > 0$, *such that, if* $\|\mathbf{U}(x,0)\|_2 \leq \delta$, *there is a unique global smooth solution, which verifies*

$$\mathbf{U} \in C^0([0,\infty); H^2(R) \cap C^1([0,\infty); H^1(R)).$$

This global existence theorem was generalized to a higher-dimensional case by Yong [18] and successively by Bianchini, Hanouzet and Natalini [4]. Moreover Ruggeri and Serre [11] proved that the constant equilibrium state is stable.

We now study the case of ET_6 polyatomic gas mixture. As only the last component of the production $\mathbf{P}$ is non-zero (see $(24)_3$), the K-condition (37) for a single constituent is satisfied if $\delta\Pi^\alpha \neq 0$ in equilibrium ($\Pi^\alpha = 0$). This is true for both contact waves and for sound waves (see (36) and the inequality (32) due to the concavity condition for (30) with (31)). In a mixture, the K-condition becomes

$$\delta\mathbf{P}^\alpha|_E + \delta\mathbf{f}^\alpha|_E \neq 0.$$

This is satisfied because the first term is non-null since it is non-zero for a single component and moreover we notice that $\delta\mathbf{f}^\alpha|_E \neq 0$ by simple calculations using $(25)_4$, (35), (33) and (36).

Therefore the K-condition for a mixture is satisfied. Then, as we have proved before the concavity of the entropy (see Section 6.2), we have reached the following theorem:

THEOREM 3. *The* ET_6 *mixture system* (29) *is symmetric huperbolic in the main field components and has global smooth solutions for all time, which converge to the equilibrium one provided that the initial data are sufficiently smooth according to the assumptions in Theorem* 2.

7. Conclusion

Following the idea of Truesdell, we have defined a system of balance laws of mixture type, which is constructed by the system of a single constituent. For this kind of system, we have studied the Galilean invariance, which not only dictates the velocity dependence in the field equations but also permits us to identify the global quantities of a mixture. Consequence of the entropy principle was also discussed in general by proving how it is possible to construct the main field of a mixture starting from the main field of a single constituent. If the system of a single constituent has a convex entropy, the system of mixture type preserves the convexity and is symmetric hyperbolic. As a consequence, we have a well-posed Cauchy problem and there exist global solutions for smooth initial data. The general theory was tested by constructing, in a simple and direct way, the system of a mixture of polyatomic gases that have a unique dissipative nonequilibrium quantity, that is, the dynamic pressure.

In the following papers, using the present model of ET_6 mixtures, we will study some intriguing problems: (i) shock wave phenomena [16], (ii) the parabolic limit of the present theory using the procedure of the so-called *Maxwellian Iteration* [7]. The comparison between the present theory with the previous Eulerian mixture theory [12, 5, 14] will also be made.

ACKNOWLEDGMENTS. This work was partially supported by JSPS KAKENHI Grant Numbers JP15K21452 (T.A.), JP16K17555 (S.T.) and by National Group of Mathematical Physics GNFM-INdAM (T.R.).

References

[1] T. Arima - T. Ruggeri - M. Sugiyama - S. Taniguchi, *Nonlinear extended thermodynamics of real gases with 6 fields*, Int. J. Non-Linear Mech., 72 (2015) 6–15.

[2] T. Arima - T. Ruggeri - M. Sugiyama - S. Taniguchi, *On the six-field model of fluids based on extended thermodynamics*, Meccanica, 49 (2014) 2181–2187.

[3] T. Arima - S. Taniguchi - T. Ruggeri - M. Sugiyama, *Extended thermodynamics of real gases with dynamic pressure: An extension of Meixner's theory*, Phys. Lett. A, 376 (2012) 2799–2803.

[4] S. Bianchini - B. Hanouzet - R. Natalini, *Asymptotic Behavior of Smooth Solutions for Partially Dissipative Hyperbolic Systems with a Convex Entropy*, Comm. Pure Appl. Math. 60 (2007) 1559–1622.

[5] H. Gouin - T. Ruggeri, *Identification of an average temperature and a dynamical pressure in a multitemperature mixture of fluids*, Phys. Rev. E 78 (2008) 016303.

[6] B. Hanouzet - R. Natalini, *Global existence of smooth solutions for partially dissipative hyperbolic systems with a convex entropy*, Arch. Rational Mech. Anal. 169 (2003) 89–117.

[7] E. Ikenberry - C. Truesdell, *On the pressure and the flux of energy in a gas according to Maxwell's kinetic theory, I*, J. Rational Mech. Anal, 5 (1956) 1–54.

[8] J. Lou - T. Ruggeri, *Acceleration Waves and Weak Shizuta-Kawashima Condition*, Suppl. Rend. Circ. Mat. Palermo, Non Linear Hyperbolic Fields and Waves. A tribute to Guy Boillat, 78 (2006) 187–200.

[9] T. Ruggeri, *Galilean invariance and entropy principle for systems of balance laws. The structure of extended thermodynamics*, Continuum Mech. Thermodyn., 1 (1989) 3–20.

[10] T. Ruggeri, *Non-linear maximum entropy principle for a polyatomic gas subject to the dynamic pressure*, Bull. Inst. Math. Acad. Sin., 11 (2016) 1–22.

[11] T. Ruggeri - D. Serre, *Stability of constant equilibrium state for dissipative balance laws system with a convex entropy*, Quart. Appl. Math. 62 (2004) 163–179.

[12] T. Ruggeri - S. Simić, *Average temperature and Maxwellian iteration in multi temperature mixtures of fluids*, Phys. Rev. E, 80 (2009) 026317.

[13] T. Ruggeri - S. Simić, *On the hyperbolic system of a mixture of Eulerian fluids: a comparison between single- and multi-temperature models*, Mathematical Methods in the Applied Sciences, 30 (2007) 827–849.

[14] T. Ruggeri - M. Sugiyama, *Rational Extended Thermodynamics beyond the Monatomic Gas*, Springer, Heidelberg, New York, Dordrecht, London (2015).

[15] Y. Shizuta - S. Kawashima, *Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation*, Hokkaido Math. J. 14 (1985) 249–275.

[16] S. Taniguchi - T. Arima - T. Ruggeri - M. Sugiyama, *Effect of dynamic pressure on the shock wave structure in a rarefied polyatomic gas*, Phys. Fluids, 26 (2014) 016103.

[17] C. Truesdell, *Rational Thermodynamics*, McGraw-Hill, New York (1969).
[18] W-A. Yong, *Entropy and global existence for hyperbolic balance laws*, Arch. Rational Mech. Anal. 172 (2004) 247–266.

Received 30 December 2016,
and in revised form 9 January 2017.

Takashi Arima
Department of Mechanical Engineering
Faculty of Engineering
Kanagawa University
Yokohama 221-8686, Japan
arima@kanagawa-u.ac.jp

Tommaso Ruggeri
Department of Mathematics
Alma Mater Research Center on Applied Mathematics
University of Bologna
Via Saragozza 8,
40123 Bologna, Italy
tommaso.ruggeri@unibo.it

Masaru Sugiyama
Nagoya Institute of Technology
Nagoya 466-8555, Japan
sugiyama@nitech.ac.jp

Shigeru Taniguchi
Department of Creative Engineering
National Institute of Technology
Kitakyushu College
Kitakyushu 802-0985, Japan
taniguchi.shigeru@kct.ac.jp