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Partial Differential Equations — *On the hydrodynamic and magnetohydrodynamic stability of an inclined layer heated from below*, by Paolo Falsaperla, Andrea Giacobbe and Giuseppe Mulone, communicated on January 13, 2017.

This paper is dedicated to the memory of Professor Giuseppe Grioli.

Abstract. — In this paper we investigate the stability of either the rest state (Bénard problem) or parallel laminar flows in hydrodynamics and magnetohydrodynamics for inclined layers heated from below. In particular, we numerically investigate the linear instability under general three-dimensional perturbations of the basic state, and we also give conditions for the nonlinear stability.

For particular basic states, a linear instability analysis shows that the critical Rayleigh numbers can be obtained for longitudinal perturbations (which can be studied analytically both, in the linear and nonlinear case) or for transversal perturbations depending on the inclination of the layer (and of course also on the velocity of the boundaries and the applied magnetic field).

Remarkable is the presence of points of codimension-two: particular values of the critical Rayleigh numbers obtained for specific angles of inclination for which there exist two equally destabilising perturbations, one longitudinal and the other transversal.

Key words: Hydrodynamics and magnetohydrodynamics, stability, inclined layers, laminar flows

Mathematics Subject Classification: 76E05, 76E06, 76E25

1. Introduction

With *inclined layer convection* one indicates the fluid-dynamical system in which a fluid layer is inclined at an angle with respect to the vertical and is heated from one side of the layer. This type of systems have been first investigated in the seventies [17, 19], and still are the subject of theoretical and experimental investigations which expose remarkable behaviours (see [34] and the bibliography therein).

The appearance of convection instability in such systems is important in many geophysical and industrial applications (many engineering applications require heating at the boundaries) [37, 18, 7, 5, 35]. Moreover, laminar flows of conducting fluids with or without an imposed magnetic field play an important role in many applications, for instance in geophysics, astrophysics, and biology [1, 2, 6].

In [11] Falsaperla et al. obtained the analytical expression of the basic stationary laminar solutions for an inclined layer filled with a hydromagnetic fluid heated from below and subject to the gravity field. In [12] the same authors studied the linear instability of such basic solutions *only for transverse perturbations*, and their nonlinear stability for special choices of the basic states described above. They also investigated the critical stability thresholds. In this article we extend the results of [12] to general three dimensional perturbations.

In [12], the authors investigate the effect of inclination for the Couette type basic state, and show how:

(a) the velocity of the Couette basic state at the upper boundary is strictly connected with stabilizing/destabilising effect of inclination (see [4, 18, 36] for similar phenomena);
(b) a coplanar magnetic field generates several closed disconnected neutral curves (islands) for some inclinations and non-zero velocities at the upper boundary (for transverse perturbations).

In this work we investigate the linear instability with respect to general three dimensional perturbations, which include the transverse (or spanwise) and the longitudinal (or streamwise) perturbations, and we numerically compute the critical linear Rayleigh numbers for some given parameters of the problem, such as the inclination, the velocity and the value of a coplanar magnetic field at the boundaries. We show that, given a basic flow, the linear instability is achieved with perturbations which are longitudinal for some inclinations and with transversal for other inclinations.

For three dimensional perturbations some interesting regions of critical Rayleigh numbers appear as a *trumpet-like surface*. The mouthpiece of this trumpet is connected to the islands obtained for transverse perturbations in [12].

We also give nonlinear stability conditions in the energy norm, and show that for longitudinal perturbations the critical linear and nonlinear thresholds coincide. In some cases these values are the critical Rayleigh numbers.

The paper is divided into five sections. In Section 2 we introduce the analytical problem and recall some classical stability/instability results for the Bénard problems and the laminar flows in hydrodynamics. In Section 3 we study the linear instability in the case of an inclined layer for general three dimensional perturbations. Our numerical results show that, in the Couette case, for particular values of the boundary conditions and for an angle of inclination in some interval, the more destabilizing perturbations are those longitudinal (similar to [13] for flows in bidispersive porous layer). Section 4 deals with nonlinear stability of longitudinal perturbations. Finally, in Section 5 we draw some conclusions and list some open problems.

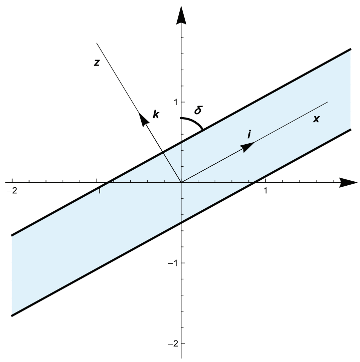
2. Position of the problem and particular cases

Consider the layer

$$\Omega_d = \mathbb{R}^2 \times (-d/2, d/2)$$

of width d filled with a hydromagnetic fluid and inclined of an angle δ with respect to the vertical. The fluid has temperature T and is subject to thermal expansion; its motion is also subject to and influences a magnetic field.

The fluid satisfies the boundary conditions of prescribed temperature $T^{\pm}$, that will be assumed lower on the top and higher on the bottom of the layer $T^- > T^+$ (this conditions is referred to as *heating from below*). For the velocity field and the magnetic field the boundary conditions depend on the physical nature of the bounding planes and can be stress-free and/or rigid, and electrically conducting or nonconducting.



The equations that model, in the Boussinesq approximation and in non-dimensional form, such a system are [12]:

$$
(1) \qquad \begin{cases} \mathbf{U}_t + \mathbf{U}\cdot\nabla\mathbf{U} = \mathrm{Pm}^{-1}\,\mathbf{H}\cdot\nabla\mathbf{H} - \nabla\Pi + RT(\cos\delta\,\mathbf{i} + \sin\delta\,\mathbf{k}) + \Delta\mathbf{U} \\ \nabla\cdot\mathbf{U} = 0 \\ \mathbf{H}_t + \mathbf{U}\cdot\nabla\mathbf{H} - \mathbf{H}\cdot\nabla\mathbf{U} = \mathrm{Pm}^{-1}\,\Delta\mathbf{H} \\ \nabla\cdot\mathbf{H} = 0 \\ T_t + \mathbf{U}\cdot\nabla T = \mathrm{Pr}^{-1}\,\Delta T, \end{cases}
$$

where $\mathbf{U}$, $\mathbf{H}$, T, Π are the velocity of the fluid, the magnetic field, the temperature, the pressure (including the magnetic pressure). Moreover,

- $\mathrm{Pr} = \nu/\kappa$ is the Prandtl number;
- $\mathrm{Pm} = \nu/\eta$ is the magnetic Prandtl number;
- $\mathrm{Ra} = R^2 = \dfrac{g\alpha\beta d^4}{\kappa\nu}$ is the Rayleigh number;
- κ is the thermal diffusivity;
- ν is the viscosity;
- α is the volume expansion coefficient;
- $\beta = (T^- - T^+)/d$ is the gradient of temperature.

We consider stationary and laminar solutions, in which all the fields are explicitly independent of time and the velocity field, the temperature field and the magnetic have the form

$$
(2) \qquad \mathbf{w} = U(z)\mathbf{i}, \quad T = T(z) \quad \mathbf{H} = \mathbf{H}(z) = (H(z), 0, H_3(z)).
$$

2.1. Particular cases

a) *The layer is horizontal, the fluid is isothermal and electrically nonconducting. This is the usual Navier-Stokes system.*

Let $m_0 = (\mathbf{w}(\mathbf{x}), p(\mathbf{x}))$ be a stationary flow of a viscous incompressible fluid $\mathscr{F}$ solution of Navier-Stokes equations. A *perturbation* to m_0, $(\mathbf{u}(\mathbf{x},t), \pi(\mathbf{x},t))$ satisfies (in a suitable non-dimensional form) the following IBVP:

$$(3)\quad \begin{cases} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \mathbf{w}\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{w} = -\nabla\pi + \mathrm{Re}^{-1}\Delta\mathbf{u} & \text{in } \Omega\times(0,\infty) \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega\times(0,\infty) \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) & \text{on } \Omega \\ \mathbf{u}(\mathbf{x},t) = 0 & \text{on } \partial\Omega\times[0,\infty), \end{cases}$$

where $\Omega = \mathbb{R}^2 \times [-1/2, 1/2]$, and Re is a Reynolds number.

Assume that $|\mathbf{u}|$ is so *small*, with $|\nabla\mathbf{u}|$ *small*, in such a way that we can neglect in (3) the nonlinear term $\mathbf{u}\cdot\nabla\mathbf{u}$. We thus obtain the *linearized* system

$$(4)\quad \begin{cases} \mathbf{u}_t + \mathbf{w}\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{w} = -\nabla\pi + \mathrm{Re}^{-1}\Delta\mathbf{u} & \text{in } \Omega\times(0,\infty) \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega\times(0,\infty) \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) & \text{on } \Omega \\ \mathbf{u}(\mathbf{x},t) = 0 & \text{on } \partial\Omega\times[0,\infty). \end{cases}$$

This system is linear and autonomous and therefore we may look for solutions of the following form:

$$(5)\quad \mathbf{u}(\mathbf{x},t) = e^{-\sigma t}\mathbf{q}(\mathbf{x}) \quad \pi(\mathbf{x},t) = e^{-\sigma t}\pi_0(\mathbf{x})$$

with σ *a priori* a complex number.

The eigenvalue problem (4) admits a nonempty set Σ of eigenvalues σ (which belong to a parabolic region of the complex plane, Prodi 1961 [29]).

As concerns the classical definitions of linear instability and nonlinear energy stability we refer to [14].

If the basic state is the rest state $\mathbf{w} = 0$, (4) gives its linear stability. It is easily seen that the basic state is also globally nonlinearly stable in the energy norm.

Plane parallel shear flows are characterized by the functional form

$$(6)\quad \mathbf{w} = \mathrm{Re}\begin{pmatrix} U(z) \\ 0 \\ 0 \end{pmatrix} = \mathrm{Re}\,U(z)\mathbf{i}.$$

The function $U(z) : [-1/2, 1/2] \to \mathbb{R}$ is assumed to be sufficiently smooth and is called the shear profile. Possible shear profiles are:

- Couette $U(z) = \frac{1}{2} + z$;
- Poiseuille $U(z) = 1 - 4z^2$.

Here we summarize some classical results:

- plane Poiseuille flow is linearly unstable for Re > 5772 (Orszag 1971 [28])
- pipe Poiseuille flow and plane Couette flow are linearly stable *for all* Reynolds numbers
- *laboratory experiments*: plane and pipe Poiseuille flows actually undergo transition to three-dimensional turbulence for Reynolds numbers on the order of 1000; plane Couette flow: the lowest Reynolds numbers at which turbulence can be produced and sustained has been shown to be between 300 and 400 both in numerical simulations and in experiments
- global asymptotic *energy-stability* for Reynolds numbers Re below some value Re_E which is of the order 10^2 (Joseph 1966), $\mathrm{Re} = \mathrm{Re}^y = 82.6$ ($\mathrm{Re}^x = 177$) for Couette, $\mathrm{Re} = \mathrm{Re}^y = 99.1$ ($\mathrm{Re}^x = 174$) for Poiseuille. Re^y is the streamwise or longitudinal number, Re^x is the spanwise or transverse number.

In [23] Kaiser et al. wrote the velocity field in terms of poloidal, toroidal and the mean field components. They used a generalized energy functional $\mathscr{E}$ (with some coupling parameters chosen in an optimal way) for *plane Couette* flow providing conditional nonlinear stability for Reynolds numbers Re below $\mathrm{Re}_{\mathscr{E}} := 177.2$, which is larger than the ordinary energy stability limit. The method allows the explicit calculation of so-called stability balls in the $\mathscr{E}$-norm; i.e., the system is stable with respect to any perturbation with respect to $\mathscr{E}$-norm in this ball.

Kaiser and Mulone (2005) [22] proved conditional nonlinear stability for *arbitrary plane parallel shear flows* up to some value Re_E which depends on the shear profile. They used a generalized functional E depending only on the poloidal component of the velocity field. As a consequence Re_E turns out to be Re_E^x, the ordinary energy stability limit for perturbations depending on x (transverse perturbations). In the case of the experimentally important profiles, viz. linear combinations of Couette and Poiseuille flow, this number is at least 174, the value for pure Poiseuille flow. For Couette flow it is at least 177.

Rionero and Mulone (1991) [31] studied the non-linear stability of parallel shear flows with the Lyapunov method in the (ideal) case of stress-free boundary conditions. In that paper they show that plane Couette flows and plane Poiseuille flows are conditionally asymptotically stable for all Reynolds numbers. A Lyapunov function (in terms of the *essential variables* ζ and w) has been introduced.

b) *The layer is horizontal, electrically nonconducting, and it is heated from below. The classical Bénard problem.*

The Bénard problem concerns with the stability/instability of an incompressible newtonian fluid $\mathscr{F}$ filling an infinite layer of thickness d, $\Omega_d = \mathbb{R}^2 \times (-d/2, d/2)$.

Let the fluid be subject to the vertical action of gravity $\mathbf{g} = -g\mathbf{k}$, and heated from below in such a way that an *adverse temperature gradient* $\beta > 0$ is maintained. Because of thermal expansion, the fluid at the bottom expands as it becomes hotter and heat is transported through the fluid by conduction. When the temperature gradient reaches a critical value β_c, the buoyancy overcomes gravity, the fluid gives rise to a regular cellular pattern and the motions take place within the cells.

This phenomenon is called *Bénard convection* after the experiments of Bénard (1900). The onset of convection depends on β and also on the depth d of the layer. In fact, the correct non-dimensional parameter for describing this threshold phenomenon is the Rayleigh number defined above.

Denoting by $\rho(T) = \rho_0(1 - \alpha_T(T - T^*))$, the equations of the fluid in the Oberbeck-Boussinesq approximation, in a non-dimensional form, are

$$\begin{cases} \mathbf{U}_t + \mathbf{U}\cdot\nabla\mathbf{U} = -\nabla\Pi + RT\mathbf{k} + \Delta\mathbf{U} \\ \nabla\cdot\mathbf{U} = 0 \\ T_t + \mathbf{U}\cdot\nabla T = \mathrm{Pr}^{-1}\,\Delta T. \end{cases} \tag{7}$$

A stationary solution of these equations with boundary conditions $T(-d/2) = T^-$, $T(d/2) = T^+$, is given by the *rest state conduction-solution*:

$$\mathbf{U} = 0 = 0, \quad T(z) = -RP_r^{-1}z + \tilde{T}_0, \quad \Pi = R\int_0^z T(z)\,dz + c_0 x,$$

with c_0 a suitable real number, and $\tilde{T}_0$ an adimensional form of T_0.

By writing as before the non-dimensional perturbation equations, it can be proved that the associated linear operator is autonomous and symmetric in $L^2(\mathscr{C})$ (where $\mathscr{C}$ is a suitable cell of periodicity), the strong principle of exchange of stabilities holds, and the instability arises as stationary convection. Symmetry implies the coincidence of linear and nonlinear critical Rayleigh numbers (Joseph 1966 [20]).

The critical Rayleigh numbers of linear instability are given by

$$R_L^2 = R_c^2 = R_B^2,$$

where $R_B^2 = 27/4\pi^4 \approx 657.511$ for stress free boundaries, $R_B^2 \approx 1707.76$, for rigid boundaries, and $R_B^2 \approx 1100.65$ for rigid-free boundaries.

In the Bénard problem many stabilizing effects can be considered, for instance a rotation field, a magnetic field, a gradient of concentration of mass for binary mixtures. In the last 30 years many stability results have been obtained for this type of systems [8, 9, 10, 15, 16, 24, 25, 27, 26, 32].

c) *The layer is inclined, electrically nonconducting, and heated from below.*

The equations that model, in the Boussinesq approximation and in non-dimensional form, such a system are:

$$\begin{cases} \mathbf{U}_t + \mathbf{U}\cdot\nabla\mathbf{U} = -\nabla\Pi + RT(\cos\delta\mathbf{i} + \sin\delta\mathbf{k}) + \Delta\mathbf{U} \\ \nabla\cdot\mathbf{U} = 0 \\ T_t + \mathbf{U}\cdot\nabla T = \mathrm{Pr}^{-1}\,\Delta T. \end{cases} \tag{8}$$

The basic motion, in the case of the Couette flow with velocity field $\mathbf{U}$ at the boundaries, $\mathbf{U}(-1/2) = 0$, $\mathbf{U}(1/2) = V\mathbf{i}$ is given by

$$
(9)\qquad \begin{cases} U(z) = R^2 P_r^{-1} \cos\delta \dfrac{z}{24}(4z^2 - 1) + V\Big(\dfrac{1}{2} + z\Big), \\ T(z) = -RP_r^{-1} z + \tilde{T}_0, \quad \Pi = R \displaystyle\int_0^z T(z) \sin\delta \, dz + c_0 x. \end{cases}
$$

Expression $(9)_1$ can be obtained from formula (13) in [11] by letting $\gamma \to 0$. It coincides with (2.2b) in [34].

The equations which govern the evolution of the perturbations $\mathbf{u}$, θ, $\bar{\pi}$ to the basic solution are:

$$
\begin{cases} \mathbf{u}_t + \mathbf{U} \cdot \nabla \mathbf{u} + w \mathbf{U}_z + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \bar{\pi} + R\theta(\cos\delta \mathbf{i} + \sin\delta \mathbf{k}) + \Delta \mathbf{u} \\ \theta_t + U(z)\theta_x - \Pr^{-1} Rw + \mathbf{u} \cdot \nabla\theta = \Pr^{-1} \Delta\theta \\ \nabla \cdot \mathbf{u} = 0. \end{cases}
$$

Observe that the boundary conditions on $z = \pm 1/2$ for $\mathbf{u}$, θ are

$$
\theta = 0, \quad u = v = w = 0.
$$

Linearizing the equations, we have

$$
(10)\qquad \begin{cases} \mathbf{u}_t + \mathbf{U} \cdot \nabla \mathbf{u} + w \mathbf{U}_z = -\nabla \bar{\pi} + R\theta(\cos\delta \mathbf{i} + \sin\delta \mathbf{k}) + \Delta \mathbf{u} \\ \theta_t + U(z)\theta_x - \Pr^{-1} Rw = \Pr^{-1} \Delta\theta \\ \nabla \cdot \mathbf{u} = 0. \end{cases}
$$

Taking into account the dependence of U only on z, this system becomes

$$
(11)\qquad \begin{cases} \mathbf{u}_t + U \mathbf{u}_x + w U' \mathbf{i} = -\nabla \bar{\pi} + R\theta(\cos\delta \mathbf{i} + \sin\delta \mathbf{k}) + \Delta \mathbf{u} \\ \theta_t + U(z)\theta_x - \Pr^{-1} Rw = \Pr^{-1} \Delta\theta \\ \nabla \cdot \mathbf{u} = 0. \end{cases}
$$

Since the system is autonomous, we consider solutions of the form $f(x, y, z, t) = f(z)e^{i(ax+by)+ct}$ (with $f = u, v, w, \theta$ or $\bar{\pi}$) in the domain $\mathscr{C} \times (0, +\infty)$, with $\mathscr{C} = [0, 2\pi/a] \times [0, 2\pi/b] \times [-1/2, 1/2]$ (the positive constants a, b are called *wave numbers*).

The generalized Orr-Sommerfeld equations are

$$
(12)\qquad \begin{cases} (c + iaU)(w'' - (a^2 + b^2)w) - iawU'' \\ \quad = -ia \,\mathrm{Ra}\, \theta' \cos\delta - (a^2 + b^2)\,\mathrm{Ra}\, \theta \sin\delta \\ \qquad + (w'''' - 2(a^2 + b^2)w'' + (a^2 + b^2)^2 w) \\ \Pr(c + iaU)\theta - w = \theta'' - (a^2 + b^2)\theta, \end{cases}
$$

where we have posed $Rw = \hat{w}$ and we have denoted $\hat{w}$ with w.

Falsaperla et al. [12], have numerically analysed, in some very specific conditions, system (12) to determine the critical Rayleigh number for a variety of boundary conditions and physical parameters, when $b = 0$ (*transverse perturbations*). Here we extend the investigation in [12] to longitudinal perturbations

($a = 0,\ b \neq 0$) and also to more general perturbations with both $a \neq 0,\ b \neq 0$, without an applied magnetic field. In the next section we will treat the coplanar magnetic case. In our numerical computations, we use the Chebyshev-tau method with a number of polynomials ranging form 15 to 25 for each unknown function. All computations have been performed with rigid and nonconducting boundaries, with fixed Prandtl number $\mathrm{Pr} = 6.7$. In the calculations and in the next figures, the boundary values $U(1/2) = u$ and $U(-1/2) = 0$ have been chosen.

The simplest case is that of an inclined Bénard layer with rigid boundary conditions. In this case, as studied in [34], longitudinal perturbations are the most destabilizing up to a certain inclination with respect to the horizontal. The angle at which transverse perturbations become the more destabilizing depends strongly on the Prandtl number of the fluid. For $\mathrm{Pr} = 1.07$ (carbon dioxide) in [34] the authors report an angle of 77.746°. In our work we consider a fluid with $\mathrm{Pr} = 6.7$ and find that this transition is obtained for an angle of 88.1° (for this value we have a *codimension-two* point).

The left panel of fig. 1 shows the instability thresholds for the longitudinal and transverse perturbations as a function of the angle $\varphi = \pi/2 - \delta$. We observe that the critical Rayleigh number for transverse perturbations has a jump discontinuity around $\varphi = 24.1^\circ$. This phenomenon is due to the presence of an island of instability which disappears for angles larger of 24.1°. In fact, the right panel, which describes the transverse critical curves, shows the critical curves in the plane (a, Ra) for angles near the critical value 24.1°. We note that for an angle in the interval $[22^\circ, 23^\circ]$ the instability region becomes disconnected and an island of instability appears. For angles greater than 24.1° the island disappears and a jump, shown in the left panel, arises (see also fig. 3).

A similar situation appears for Couette-type motions when the upper boundary is moving in the downward direction (at least for small velocities in the interval $[-5, 0]$). In this case the longitudinal perturbations are the most destabilizing up to a very large angle (near 90°) as in the Bénard case. Instead a different situ-

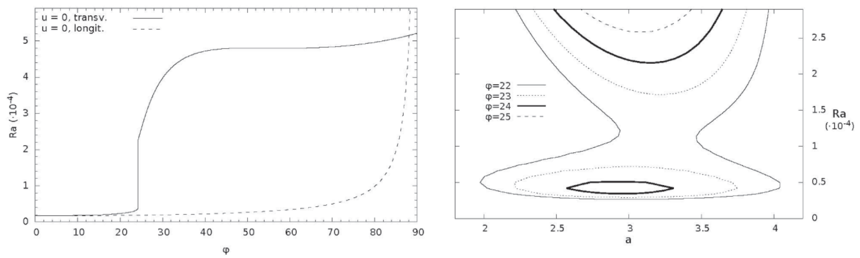


Figure 1. The critical Rayleigh number vs the angle with the horizontal $\varphi = \pi/2 - \delta$ for the inclined Bénard layer. The critical longitudinal curve (dotted) and transverse (continuous) are plotted. The codimension-two point is obtained for a critical angle $\varphi_c = 88.1^\circ$ corresponding to $\delta_c = 1.9^\circ$. The most destabilizing perturbations are longitudinal for all inclination angle $\delta \in [1.9^\circ, 90^\circ]$. The right panel shows the critical Rayleigh number vs the wave number a (transverse perturbations) for different values of the angle φ. The appearance of an instability island near 22°–23° is shown.

ation appears when the upper boundary is moving in the upward direction, as shown in fig. 2, for $u = 2$ and $u = 5$. We observe in this case that transverse perturbations prevail also for an interval of inclinations near to the horizontal case. Figure 2 shows that for $u = 2$ two codimension-two points are present ($\varphi_1 = 22.75°$ and $\varphi_2 = 88°$) different from the case $u = 0$ (the inclined Bénard) where only one codimension-two point is present. Left panel of fig. 2 shows that the most destabilizing critical curve can be associated to either longitudinal or transverse perturbations. As before we denote with Ra^x and Ra^y the critical Rayleigh numbers with respect to transverse and longitudinal perturbations, i.e. $\mathrm{Ra}^x = \min \mathrm{Ra}(0, b)$ and $\mathrm{Ra}^y = \min \mathrm{Ra}(a, 0)$. The right panel shows that the change of critical instabilities between Ra^x and Ra^y takes place in a different way for $u = 2$ and $u = 5$. In the first case the codimension-two point appears (i.e. $\mathrm{Ra}^x = \mathrm{Ra}^y$) for $\varphi = 22.75°$ and the critical Rayleigh number depends continuously on the angle. Instead for $u = 5$ the critical Rayleigh number is discontinuous at the angle $\varphi = 39.3°$ because Ra^x jumps above the value of Ra^y.

Figure 3 shows the critical Rayleigh surface as a function of both the wave numbers a and b. In all our calculations we see that the minimum of Ra with

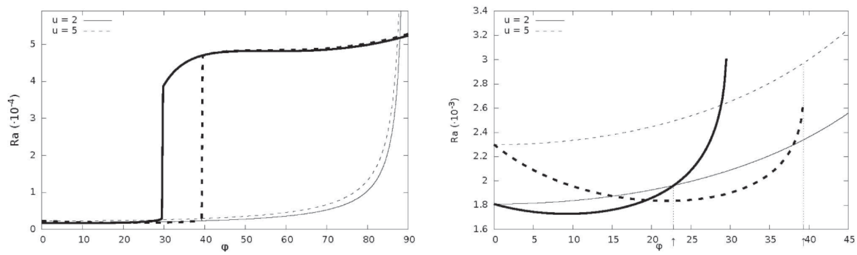


Figure 2. Both panels show the critical Rayleigh numbers vs the inclination angle φ, the right panel is a zoom of the left one. The thick plots represent the critical Rayleigh numbers for transverse perturbations while the thinner plots represent the critical Rayleigh numbers for longitudinal perturbations.

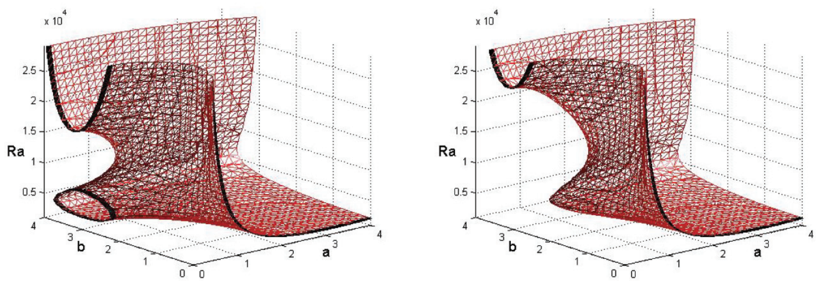


Figure 3. The graphs of the critical Rayleigh numbers as functions of a and b for the inclined Bénard problem with $\varphi = 23°$ (left) and $\varphi = 25°$ (right) are represented. The thick black curves are the intersection of the surface with the planes $a = 0$ and $b = 0$.

respect to a and b is achieved only for $a = 0$ or $b = 0$. (This is not a proof that the instability sets in via two-dimensional perturbations, in particular this is not a proof of a Squire-like theorem.) For this reason we have focussed our attention to longitudinal and transverse perturbations. The left panel represents the critical surface $\mathrm{Ra} = \mathrm{Ra}(a, b)$ for the inclined Bénard problem with an angle $\varphi = 22.75°$. In this case a *trumpet-like* surface appears. The right panel corresponds to an angle $\varphi = 25°$. In this case the surface is simply-connected.

3. The hydromagnetic case: linear instability

First we recall that, if the magnetic field has the form $(2)_3$, the solenoidality of $\mathbf{H}$ implies that the third component of the magnetic field must be a constant γ.

Moreover we recall that an interesting phenomenon appears when the layer is inclined. In fact, in the horizontal case (the classical magnetic Bénard problem), the magnetic field is constant and orthogonal to the layer. This is not possible when the layer is inclined (see the following lemma) because of the cubic dependence of the velocity field on z (see (9)). Moreover, equation $(1)_3$ becomes

$$-\gamma U'(z) = \mathrm{Pm}^{-1} H''.$$

In [11] the following lemma has been proved:

Lemma 1 ([11]). *If the velocity field and the magnetic field are regular functions and have the form* $\mathbf{w} = U(z)\mathbf{i}$, $\mathbf{H} = H(z)\mathbf{i} + H_3\mathbf{k}$, *with* $\mathbf{w}'$ *not identically zero, then* $H_3 = 0$ *if and only if* $H(z)$ *is linear function (in particular identically zero) of* z. *Then* $\mathbf{H} = H\mathbf{i}$, *i.e.* $\mathbf{H}$ *must be coplanar to the plane* $z = 0$.

From this lemma it follows that, in the inclined case, the magnetic field must have necessarily a component parallel to the layer (see below).

Some physically relevant boundary conditions are:

- (rigid, rigid, electrically nonconducting, electrically nonconducting) up to a uniform translation, one can assume that $U(-1/2) = 0$ and $U(1/2) = u$ for the velocity field, while the boundary conditions on the first two components of the magnetic field are $H(-1/2) = h_-$, $H(1/2) = h_+$. This case includes Couette and Poiseuille basic solutions;
- (rigid, rigid, electrically conducting, electrically nonconducting) the conditions on the velocity field are the same, that is $U(1/2) = u$, $U(-1/2) = 0$. For the magnetic field one has conditions on the first derivatives below $H'(-1/2) = h'$ and $H(1/2) = h$;
- (rigid, stress free, electrically conducting, electrically nonconducting) $U(-1/2) = 0$, $U'(1/2) = u'$, $H'(-1/2) = h'$, $H(1/2) = h$.

The general solutions to these equations are (see [11]),

$$U(z) = u_1 \cosh(\gamma z) + u_2 \sinh(\gamma z) - \frac{\mathrm{Ra}}{\mathrm{Pr}\,\gamma^2}\cos\delta z + \frac{b_1}{\mathrm{Pm}\,\gamma} + \frac{\sigma_1}{\gamma^2},$$

$$H(z) = -\mathrm{Pm}\,u_1 \sinh(\gamma z) - \mathrm{Pm}\,u_2 \cosh(\gamma z) + \frac{\mathrm{Ra}\,\mathrm{Pm}}{\mathrm{Pr}\,\gamma}\cos\delta\frac{z^2}{2} - \frac{\mathrm{Pm}\,\sigma_1}{\gamma}z + c_1,$$

with u_1, u_2, b_1, c_1, σ_1 integrating constants. Observe that the functions must satisfy 4 boundary conditions, and the constants of integrations are 5. Among them, the constant σ_1 is related to an exterior force field, exerted through a non-trivial "*pressure*" function. For simplicity, here we always assume that $\sigma_1 = 0$.

In the coplanar magnetic-Couette case, for rigid–rigid, electrically nonconducting–electrically nonconducting boundaries with velocity zero at the lower plane and u at the upper plane, and with boundary values of the first component of magnetic field h_- and h_+, we have, [11]

$$U(z) = R^2 P_r^{-1}\cos\delta\frac{z}{24}(4z^2 - 1) + V\left(\frac{1}{2} + z\right), \quad T(z) = -RP_r^{-1}z + \tilde{T}_0,$$

$$H(z) = \frac{h_+ + h_-}{2} + (h_+ - h_-)z.$$

The equations which govern the evolution of the perturbations $\mathbf{u}$, $\mathbf{h}$, θ, $\bar{\pi}$ to the basic solution are:

$$\begin{cases} \mathbf{u}_t + \mathbf{U}\cdot\nabla\mathbf{u} + w\mathbf{U}_z + \mathbf{u}\cdot\nabla\mathbf{u} = \mathrm{Pm}^{-1}(\mathbf{H}\cdot\nabla\mathbf{h} + \ell\mathbf{H}_z + \mathbf{h}\cdot\nabla\mathbf{h}) \\ \quad -\nabla\bar{\pi} + R\theta(\cos\delta\mathbf{i} + \sin\delta\mathbf{k}) + \Delta\mathbf{u} \\ \mathbf{h}_t + \mathbf{U}\cdot\nabla\mathbf{h} + w\mathbf{H}_z + \mathbf{u}\cdot\nabla\mathbf{h} - \mathbf{H}\cdot\nabla\mathbf{u} - \ell\mathbf{U}_z - \mathbf{h}\cdot\nabla\mathbf{u} = \mathrm{Pm}^{-1}\Delta\mathbf{h} \\ \theta_t + U(z)\theta_x - \mathrm{Pr}^{-1}Rw + \mathbf{u}\cdot\nabla\theta = \mathrm{Pr}^{-1}\Delta\theta \\ \nabla\cdot\mathbf{u} = 0, \quad \nabla\cdot\mathbf{h} = 0. \end{cases}$$

Observe that the boundary conditions on $z = \pm 1/2$ for $\mathbf{u}$, $\mathbf{h}$, θ are

$$\theta = 0, \quad u = v = w = 0, \quad h = k = \ell = 0.$$

Linearizing the equations, we have

$$\begin{cases} \mathbf{u}_t + \mathbf{U}\cdot\nabla\mathbf{u} + w\mathbf{U}_z = \mathrm{Pm}^{-1}(\mathbf{H}\cdot\nabla\mathbf{h} + \ell\mathbf{H}_z) - \nabla\bar{\pi} \\ \quad + R\theta(\cos\delta\mathbf{i} + \sin\delta\mathbf{k}) + \Delta\mathbf{u} \\ \mathbf{h}_t + \mathbf{U}\cdot\nabla\mathbf{h} + w\mathbf{H}_z - \mathbf{H}\cdot\nabla\mathbf{u} - \ell\mathbf{U}_z = \mathrm{Pm}^{-1}\Delta\mathbf{h} \\ \theta_t + U(z)\theta_x - \mathrm{Pr}^{-1}Rw = \mathrm{Pr}^{-1}\Delta\theta \\ \nabla\cdot\mathbf{u} = 0, \quad \nabla\cdot\mathbf{h} = 0. \end{cases} \tag{13}$$

Taking into account that, in the coplanar case $\gamma = 0$, this system becomes

$$(14)\quad \begin{cases} \mathbf{u}_t + U\mathbf{u}_x + wU'\mathbf{i} = \mathrm{Pm}^{-1}(H\mathbf{h}_x + \ell H'\mathbf{i}) - \nabla\bar{\pi} \\ \quad + R\theta(\cos\delta\mathbf{i} + \sin\delta\mathbf{k}) + \Delta\mathbf{u} \\ \mathbf{h}_t + U\mathbf{h}_x + wH'\mathbf{i} - H\mathbf{u}_x - \ell U'\mathbf{i} = \mathrm{Pm}^{-1}\Delta\mathbf{h} \\ \theta_t + U(z)\theta_x - \mathrm{Pr}^{-1} Rw = \mathrm{Pr}^{-1}\Delta\theta \\ \nabla\cdot\mathbf{u} = 0, \quad \nabla\cdot\mathbf{h} = 0. \end{cases}$$

Since the system is autonomous, as before we consider solutions of the form $f(x,y,z,t) = f(z)e^{i(ax+by)+ct}$ (with $f = u, v, w, h, k, \ell, \theta$ or $\bar{\pi}$) in the domain $\mathscr{C}\times(0,+\infty)$, with $\mathscr{C} = [0, 2\pi/a]\times[0, 2\pi/b]\times[-1/2, 1/2]$.

The generalized Orr-Sommerfeld equations are

$$(15)\quad \begin{cases} (c + iaU)(w'' - (a^2+b^2)w) - iawU'' \\ \quad = \mathrm{Pm}^{-1}(iaH(\ell'' - (a^2+b^2)\ell) - ia\ell H'') - ia\,\mathrm{Ra}\,\theta'\cos\delta \\ \qquad - (a^2+b^2)\,\mathrm{Ra}\,\theta\sin\delta + (w'''' - 2(a^2+b^2)w'' + (a^2+b^2)^2 w) \\ (c + iaU)\ell - iawH = \mathrm{Pm}^{-1}(\ell'' - (a^2+b^2)\ell) \\ \mathrm{Pr}(c + iaU)\theta - w = \theta'' - (a^2+b^2)\theta, \end{cases}$$

where we have posed $Rw = \hat{w}$ and we have denoted $\hat{w}$ with w.

As above, we note that Falsaperla et al. [12], have numerically analysed, in some very specific conditions, the system to determine the critical Rayleigh number for a variety of boundary conditions and physical parameters, when $b = 0$ (*transverse perturbations*). Here we extend the investigation in [12] to longitudinal perturbations ($a = 0$, $b \neq 0$) and also to more general perturbations with both $a \neq 0$, $b \neq 0$ in the presence of a coplanar magnetic field. The computations have been performed with rigid and electrically nonconducting boundaries, with fixed Prandtl number $\mathrm{Pr} = 6.7$ and magnetic Prandtl number $\mathrm{Pm} = 1$. In the calculations and in the next figures, $\varphi = 15^\circ$, the boundary values $U(1/2) = 1$ and $U(-1/2) = 0$, and $h^- = h^+ = 11.5$, $h^- = h^+ = 13$ have been chosen. These values were used also in [12] to show a peculiar phenomenon of the presence of two islands of instability. Here we report the three-dimensional instability surfaces corresponding to the previous values of the physical parameters to show the two trumpet-like surfaces.

Fig. 4 shows the disappearance of one mouthpiece by changing the values of h at the boundaries.

We have also verified that, by varying the inclination from $\varphi = 15^\circ$ to $\varphi = 16^\circ$ and $\varphi = 17^\circ$, the two mouthpieces disappear. This corresponds to the loss of the islands in the transverse perturbations. With the disappearance of each mouthpiece a jump discontinuity in the critical Rayilegh number takes place for transverse perturbations as a function of the inclination.

Remark 1. The calculations of linear instability for general three-dimensional perturbations, in the cases we have studied, show that for some given parameters (angle of inclination, velocity and magnetic fields at the boundaries) the critical Rayleigh numbers are obtained via longitudinal perturbations. For this reason it is interesting to study (analytically) the onset of instability with respect to these perturbations (both in the linear and nonlinear regime).

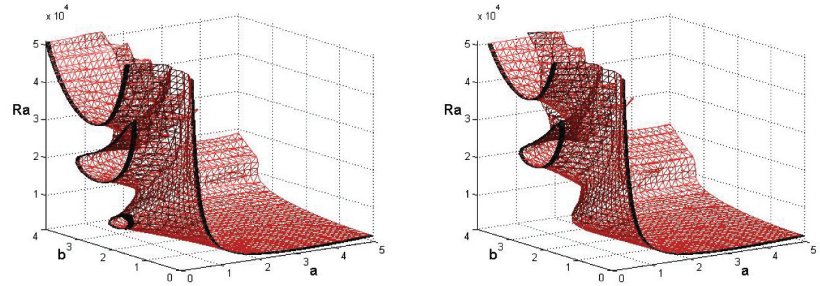


Figure 4. The graphs of the critical Rayleigh numbers for an inclined coplanar magnetic layer as function of a and b with $\varphi = 15°$ (left and right) are represented. The magnetic field assigned at the boundaries are $h^+ = h^- = 11.5$ (left) and $h^+ = h^- = 13$ (right). The thick black curves are the intersection of the surfaces with the planes $a = 0$ and $b = 0$. On the left panel two mouthpieces in the trumpet-like surface are present, in the right panel only one mouthpiece arises.

In the general case (with $\gamma \neq 0$), if we consider only *longitudinal perturbations*, i.e. we pose $a = 0$ in (15), we obtain

$$(16)\quad \begin{cases} c(w'' - b^2 w) = \mathrm{Pm}^{-1}\gamma(\ell'' - b^2\ell)' - b^2\,\mathrm{Ra}\,\theta\sin\delta + w'''' - 2b^2w'' + b^4 w \\ c\ell - \gamma w' = \mathrm{Pm}^{-1}(\ell'' - b^2\ell) \\ \mathrm{Pr}\,c\theta - w = \theta'' - b^2\theta. \end{cases}$$

These equations coincide with (dimensional) equations (113)–(115) of Chandrasekhar [3], pag. 163, with Ra replaced by $\mathrm{Ra}\sin\delta$ and $k_x = 0$, $k_y = b$.

Thus the critical Rayleigh number for longitudinal perturbations, Ra^y, is given by

$$\mathrm{Ra}^y = \frac{R_B^2}{\sin\delta},$$

where Ra^y is the classical linear critical threshold of the magnetic Bénard problem.

In particular, critical linear Rayleigh number Ra^y for stationary convection for stress-free boundaries, is given by:

$$\mathrm{Ra}^y = \frac{\pi^4}{\sin\delta}\min_{x>0}\frac{1+x}{x}\left((1+x)^2 + \frac{\gamma^2}{\pi^2}\right).$$

It is easy to see that Ra^y attains its minimum when

$$2x^3 + 3x^2 = 1 + \frac{\gamma^2}{\pi^2},$$

(see [3], chapter 3, pag. 171, formula (166), where $Q = \gamma^2$).

If we assume $\gamma = 0$ (magnetic coplanar case of pure fluid-dynamics case) and leave Pm and Pr to be arbitrary positive numbers, then

$$\mathrm{Ra}^{y} = \frac{R_B^2}{\sin\delta},$$

where R_B^2 is the classical linear critical threshold of the Bénard problem (Chen and Pearlstein 1989, [4]).

4. The hydromagnetic case, nonlinear stability, longitudinal perturbations

To study nonlinear stability we use as Lyapunov function the classical energy norm

$$E = \frac{1}{2}(\|\mathbf{u}\|^2 + \mathrm{Pm}^{-1}\|\mathbf{h}\|^2 + \mathrm{Pr}\|\theta\|^2).$$

The *energy identity* is

$$\dot{E} = I - D,$$

where

$$D = \|\nabla\mathbf{u}\|^2 + \mathrm{Pm}^{-2}\|\nabla\mathbf{h}\|^2 + \|\nabla\theta\|^2 \tag{17}$$

and

$$\begin{aligned} I = &-(\mathbf{u}, w\mathbf{U}') + \mathrm{Pm}^{-1}((\mathbf{u}, \ell\mathbf{H}') - (\mathbf{h}, w\mathbf{H}') + (\mathbf{h}, \ell\mathbf{U}')) \\ &+ R((u,\theta)\cos\delta + (1+\sin\delta)(w,\theta)). \end{aligned}$$

From the energy identity we obtain:

$$\dot{E} \le \left(\max_{\mathscr{S}} \frac{I}{D} - 1\right) D,$$

where $\mathscr{S}$ is the space of the kinematically admissible functions (see Rionero 1968 [30], Joseph 1976, [21], Straughan 2004, [33]). The Euler-Lagrange equations of this problem are

$$\begin{cases} -wU'\mathbf{i} - uU'\mathbf{k} + \mathrm{Pm}^{-1}(\ell H'\mathbf{i} - hH'\mathbf{k}) \\ \quad + R\theta(\cos\delta\mathbf{i} + (1+\sin\delta)\mathbf{k}) + 2\Delta\mathbf{u} = \nabla\pi \\ uH'\mathbf{k} - wH'\mathbf{i} + \ell U'\mathbf{i} + hU'\mathbf{k} + 2\,\mathrm{Pm}^{-1}\,\Delta\mathbf{h} = \nabla\sigma \\ R\cos\delta u + R(1+\sin\delta)w + 2\Delta\theta = 0 \\ \nabla\cdot\mathbf{u} = 0, \quad \nabla\cdot\mathbf{h} = 0. \end{cases} \tag{18}$$

By taking the third components of the double *curl* of $(18)_{1,2}$, we have

$$(19)\qquad \begin{cases} U'\Delta_1 u + P_m^{-1} H'\Delta_1 h - R(1+\sin\delta)\Delta_1\theta - 2\Delta\Delta w = 0 \\ -wU' + P_m^{-1}\ell H' + R\cos\delta\theta + 2\Delta u = 0 \\ H'\Delta_1 u + U'\Delta_1 h + 2P_m^{-1}\Delta\Delta\ell = 0 \\ \ell U' - wH' + 2P_m^{-1}\Delta h = 0 \\ R\cos\delta u + R(1+\sin\delta)w + 2\Delta\theta = 0. \end{cases}$$

This is a very difficult problem to solve in general. However, in the next subsection, we consider (for both the fluid dynamics and magnetohydrodynamics cases) only longitudinal perturbations.

4.1. *The longitudinal perturbations*

a) *Fluid-dynamics case*

Following Joseph 1976, [21], by introducing

$$\mathscr{E}(t) = \frac{1}{2}(\|v\|^2 + \|w\|^2 + P_r\|\theta\|^2)$$

we easily obtain

$$\mathscr{E}(t) < \mathscr{E}(0)\exp\Big(-\frac{2\pi^2}{\max(1,P_r)}\Big(1 - \frac{\mathrm{Ra}}{R_B^2}\sin\delta\Big)t\Big).$$

Thus, for *longitudinal perturbations*, we obtain the *same stability threshold* as in the linear case

$$R_{\mathscr{E}}^2 = R_{Long}^2 = \frac{R_B^2}{\sin\delta}.$$

It is possible to prove that whenever $R^2 < R_{Long}^2$ also $\|u\|^2$ exponential decays and we have the exponential decay of the energy

$$E(t) = \frac{1}{2}(\|\mathbf{u}\|^2 + \mathrm{Pr}\|\theta\|^2).$$

THEOREM 1. *Disturbances of laminar flows in an inclined layer, heated from below which are x-independent decay under the inequality*

$$\mathscr{E}(t) < \mathscr{E}(0)\exp\Big(-\frac{2\pi^2}{\max(1,P_r)}\Big(1 - \frac{\mathrm{Ra}}{R_B^2}\sin\delta\Big)t\Big)$$

whenever

$$\mathrm{Ra} < \frac{R_B^2}{\sin\delta}$$

independent of the basic solution $U(z)\mathbf{i}$. R_B^2 is the classical linear critical threshold of the Bénard problem. Moreover, the criterion is necessary for stability as well as sufficient for stability to x-independent disturbances.

b) *MHD-case*

In the MHD case, we introduce the function $\mathscr{E}(t)$

$$\mathscr{E}(t) = \frac{1}{2}(\|v\|^2 + \|w\|^2 + \mathrm{Pm}^{-1}(\|k\|^2 + \|\ell\|^2) + P_r\|\theta\|^2).$$

It is easy to see that

$$\dot{\mathscr{E}}(t) = 2R\sqrt{\sin\delta}(\theta, w) - (\|\nabla v\|^2 + \|\nabla w\|^2 + P_m^{-2}[\|\nabla k\|^2 + \|\nabla \ell\|^2] + \|\nabla\theta\|^2)$$

We obtain

$$\mathscr{E}(t) < \mathscr{E}(0)\exp\Big(-\frac{2\pi^2}{\max(1, P_m^{-1}, P_r)}\Big(1 - \frac{\mathrm{Ra}}{R_B^2}\sin\delta\Big)t\Big)$$

As in the fluid dynamics case, we can prove the estimates for $\|u\|^2$ and $\|h\|^2$.

THEOREM 2. *Let us consider a hydromagnetic laminar flow, in an inclined layer, heated from below, subject to a coplanar magnetic field or a magnetic field which has a constant component normal to the layer and non-linear coplanar components. The disturbances which are x-independent, decay under the inequality*

$$\mathscr{E}(t) < \mathscr{E}(0)\exp\Big(-\frac{2\pi^2}{\max(1, P_m^{-1}, P_r)}\Big(1 - \frac{\mathrm{Ra}}{R_B^2}\sin\delta\Big)t\Big)$$

whenever

$$\mathrm{Ra} < \frac{R_B^2}{\sin\delta}$$

independent of the basic solution $U(z)\mathbf{i}$, $H(z)\mathbf{i} + \gamma\mathbf{k}$. R_B^2 is the classical linear critical threshold of the magnetic Bénard problem. Moreover, the criterion is necessary for stability as well as sufficient for stability to x-independent disturbances.

In Falsaperla et al. (2016) [12], sufficient conditions for global nonlinear stability in the energy norm are given in many cases (different boundary conditions, magnetic field and angle of inclination).

5. Conclusions and some open problems

We have numerically studied linear instability of hydrodynamic and magnetohydrodynamic motions of an inclined layer heated from below in the Couette case and in the coplanar magnetic Couette case for three–dimensional perturbations.

In particular we have analytically investigated linear instability and nonlinear energy stability with respect to longitudinal perturbations.

The results obtained here generalize those of [12]. Our calculations of linear instability (in all the cases we have considered) show that the instability occurs only for longitudinal or transverse perturbations. For this reason we have focussed our investigation on transverse perturbations (numerically) and to longitudinal perturbations (analytically).

The critical Rayleigh number as a function of the wave numbers a, b is a 2-dimensional surface that can be simply connected or have one or more (in the magnetic case) holes. The consequence of this fact is a jump discontinuity for the function Ra^x, the transversal critical Rayleigh number.

In the case of the inclined Bénard problem we obtain, as in [34], one codimension-two bifurcation point that corresponds to a particular choice of the parameters (specifically the angle φ) for which Ra^x coincides with Ra^y. A *new phenomenon* described in this article is the fact that when $u > 0$ (the inclined Couette case and the inclined magnetic coplanar case) more than one codimension-two points appear. This means that, for a given positive velocity u, the instability arises with transverse rolls for small (like in classical Couette flows) and large inclinations (in the interval $[0°, 90°]$), it arises via longitudinal rolls for other inclinations. Moreover transverse-longitudinal bifurcations can take place with or without the presence of a codimension-two point depending on how Ra^x overtakes Ra^y (in a continuous or a discontinuous way, i.e. with a crossing or with a jump).

As concerns nonlinear stability, we recall some general results, and we analytically find the nonlinear stability threshold Ra^y with respect to longitudinal perturbations.

Some possible open problems are:

- the study of linear and nonlinear critical Rayleigh numbers in the hydromagnetic case with the normal component of magnetic field $\gamma \neq 0$ and the their dependence on the Prandtl numbers;
- the investigation of the Euler-Lagrange equations for the study of nonlinear stability;
- the validity, for particular physical parameters, of the Squire theorem for the inclined layers with and without magnetic field;
- the investigation of linear instability and nonlinear stability of stationary *laminar flows of porous layers* (in the case of bidispersive flow, see Falsaperla et al. (2016) [13]) heated from both below and above in particular the dependence on the effect of inclination (without a magnetic field);
- the effect of heating from above (see [34]);
- the investigation of the stability/instability of more general basic solutions (for instance with components in the y-direction).

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