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Algebraic Geometry — *An unbounded family of log Calabi–Yau pairs*, by Gilberto Bini and Filippo F. Favale, communicated on January 13, 2017.

Abstract. — We give an explicit example of log Calabi–Yau pairs that are log canonical and have a linearly decreasing Euler characteristic. This is constructed in terms of a degree two covering of a sequence of blow ups of three dimensional projective bundles over the Segre–Hirzebruch surfaces $\mathbb{F}_n$ for every positive integer n big enough.

Key words: Log Calabi–Yau pairs, geography of threefolds, projective bundles

Mathematics Subject Classification: 14J30, 14J32, 14J60

1. Introduction

A log Calabi–Yau pair (Y, D) consists of a proper variety Y and an effective $\mathbb{Q}$-divisor D such that (Y, D) is log canonical and $K_X + D$ is $\mathbb{Q}$-linearly equivalent to zero: see, for instance, [JK16]. A Calabi–Yau variety can be viewed as $(Y, 0)$. If Y is a Fano variety such that D is $\mathbb{Q}$-linearly equivalent to the anticanonical divisor, then (Y, D) is a log Calabi–Yau pair, provided it is log canonical.

Let us take into account three dimensional log Calabi–Yau pairs. As well known, there exist finitely many deformation types of Fano threefolds. As a result, there are finitely many possible values for their Euler characteristic. Conjecturally, this should be true for the collection of all Calabi–Yau threefolds too. Here by Calabi–Yau threefold we mean a complex Kähler compact manifold with trivial canonical bundle and no p-holomorphic forms for $p = 1, 2$. Since general log Calabi–Yau pairs interpolate between these two extremes, it is natural to wonder whether they are bounded or not. In this paper, we prove the following result.

Theorem 1. *There exists an integer N_0 such that, for every $n \geq N_0$ there exists a log Calabi–Yau threefold (Y, D) with the Euler characteristic of Y given by*

$$e(Y) = -48n - 46.$$

Moreover, Y is smooth and its Kodaira dimension is negative. Additionally, we have $K_Y + D = 0$, where D is a divisor isomorphic to a $K3$ surface.

Recently, Di Cerbo and Svaldi in [DS16] prove that log Calabi–Yau pairs are bounded. One of their assumption is that the pair are klt. Notice that there is no contradiction between their result and ours; indeed, the example in Theorem 1 is not klt but log canonical.

The proof of Theorem 1 is constructive. More specifically, we describe a collection of log Calabi–Yau threefolds with the properties mentioned above. First, take into account the Segre–Hirzebruch surface $\mathbb{F}_n$ for any positive integer n. Next, fix a suitable decomposable vector bundle on each $\mathbb{F}_n$, namely

$$\mathscr{V} := \mathscr{O}_{\mathbb{F}_n} \oplus \mathscr{O}_{\mathbb{F}_n}(2C_0 - F),$$

where C_0 is the unique effective divisor on $\mathbb{F}_n$ such that $C_0^2 = -n$ and F is the class of the fiber with respect to the $\mathbb{P}^1$-bundle structure on $\mathbb{F}_n$. For any n denote by X the scroll defined as $\mathbb{P}(\mathscr{V})$, the projective bundle of hyperplanes in $\mathscr{V}$. For futher information about these scrolls, see, for instance, [FF15].

If the linear system $|-2K_X|$ had a smooth member, then the double covering of X – branched along it – would be a smooth Calabi–Yau manifold. Unfortunately, this is not the case. The base locus of $|-2K_X|$ is given by a smooth rational curve. Luckily, the multiplicity of the generic section along the base locus is three. This requires a careful analysis of the cohomology group $H^0(X, -2K_X)$, which can be carried out more easily for n big enough.

If we blow up X along the smooth curve in the base locus of the bianticanonical system, we obtain a smooth threefold X_1. The linear system $|-2K_{X_1}|$ is not basepoint free. The base locus is given by a smooth rational curve γ_1. In order to resolve a generic section of the linear system $|-2K_X|$, we blow up X_1 along γ_1. We obtain a smooth threefold X_2. The degree two branched cover Y_2 along a smooth section of $-2K_X - 2E_1 - 4E_2$ is not normal. Taking the normalization of it is equivalent to taking the branched covering of X_2 along a smooth member of the linear system $-2K_{X_2} - 2E_2 = -2K_X - 2E_1 - 4E_2$.

Finally, in order to calculate the Euler characteristic of Y_2 for n big enough, it suffices to determine that of X_2 and that of a smooth surface in $|-2K_{X_2} - 2E_2|$. The former follows from the cohomology of blow ups along a submanifold and the latter from the Chern classes of it: see, for instance, [GH].

Our construction relies on the choice of the vector bundle $\mathscr{V} = \mathscr{O}_{\mathbb{F}_n} \oplus \mathscr{O}_{\mathbb{F}_n}(2C_0 - F)$. It is important to stress that this is only one of the possible choices in order to arrive at an unbounded family of log Calabi–Yau pairs. To be more precise, we analysed all the cases as $\mathscr{V}$ varies among the rank 2 vector bundle on $\mathbb{F}_n$ that are decomposable. Our method yields a double cover, which is a smooth Calabi–Yau threefold, only for a finite number of cases. We expect that for the great majority of the other cases the situation is similar to that presented in this paper: one can mimic the construction and obtain a log Calabi–Yau pair.

The paper is organized as follows. In Section 2 we recall some preliminary results. Section 3 is devoted to describing the bianticanonical system of the scroll X, in particular a desingularitazion of a generic section of it. At last, Section 4 concludes the exposition with the computation of the Euler characteristic, thus showing that it is in fact unbounded!

2. Some preliminary results

In this section we recall some basic facts and prove some results that will be applied in what follows. For further details, the reader is referred to [H], p. 369 ff.

Let S be a smooth projective surface and denote by X the projective bundle associated to a rank 2 vector bundle $\mathscr{V}$ on S. To avoid confusion we recall that X is the projective bundle $\mathbb{P}(\mathscr{V})$ over the base S, where $\mathbb{P}(\mathscr{V})$ is the projective bundle of hyperplanes in $\mathscr{V}$. In what follows we will set τ to be $c_1(\mathscr{O}_X(1))$.

Lemma 2. *Denote by $\varphi : X \to S$ the fibration given by the projective bundle structure. Then the following identities hold:*

$$\begin{aligned} c_1(X) &= 2\tau + \varphi^*(c_1(S) - c_1(\mathscr{V})); \\ c_2(X) &= \varphi^*(c_2(S) - c_1(\mathscr{V})c_1(S)) + 2\varphi^* c_1(S)\tau; \\ c_3(X) &= 2\varphi^*(c_2(S))\tau. \end{aligned}$$

Proof. We have the exact sequences

$$0 \to T_{X/S} \to T_X \to \varphi^* T_S \to 0, \tag{1}$$

$$0 \to \mathscr{O}_X \to (\varphi^* \mathscr{V}^\vee) \otimes \mathscr{O}_X(1) \to T_{X/S} \to 0. \tag{2}$$

Recall also that $H^*(X)$ is generated as an $H^*(S)$-algebra by τ with the single relation

$$\tau^2 - \varphi^* c_1(\mathscr{V})\tau = 0. \tag{3}$$

We have

$$\begin{aligned} c_1((\varphi^*\mathscr{V}^\vee) \otimes \mathscr{O}_X(1)) &= \varphi^* c_1(\mathscr{V}^\vee) + 2\tau = -\varphi^* c_1(\mathscr{V}) + 2\tau, \\ c_2((\varphi^*\mathscr{V}^\vee) \otimes \mathscr{O}_X(1)) &= \varphi^* c_2(\mathscr{V}^\vee) + \varphi^* c_1(\mathscr{V}^\vee)\tau + \tau^2 \\ &= \varphi^* c_2(\mathscr{V}) - \varphi^* c_1(\mathscr{V})\tau + \tau^2. \end{aligned}$$

By (3), this yields

$$c((\pi^*\mathscr{V}^\vee) \otimes \mathscr{O}_X(1)) = 1 + (2\tau - \varphi^* c_1(\mathscr{V})). \tag{4}$$

From the exact sequences (1) and (2), we get

$$\begin{aligned} c(X) &= c(T_{X/S})\varphi^* c(T_S) = c((\varphi^*\mathscr{V}^\vee) \otimes \mathscr{O}_X(1))\varphi^* c(T_S) \\ &= (1 + (2\tau - \varphi^* c_1(\mathscr{V})))\varphi^* c(S) \\ &= 1 + [2\tau - \varphi^* c_1(\mathscr{V}) + \varphi^* c_1(S)] \\ &\quad + [\varphi^* c_2(S) + \varphi^* c_1(S)(2\tau - \varphi^* c_1(\mathscr{V}))] + [2\varphi^* c_2(S)\tau] \\ &= 1 + [2\tau + \varphi^*(c_1(S) - c_1(\mathscr{V}))] \\ &\quad + [\varphi^*(c_2(S) - c_1(\mathscr{V})c_1(S)) + 2\varphi^* c_1(S)\tau] + [2\varphi^* c_2(S)\tau] \end{aligned} \tag{5}$$

□

In order to determine the cohomology of line bundles on X, we are going to apply the following result. We will recall it here for the sake of completeness: see, for instance, [H], page 253, Ex 8.4 (a).

LEMMA 3. *Let $\mathscr{V}$ be a vector bundle on a smooth surface S. Let $X = \mathbb{P}(\mathscr{V})$ and define τ as before. Then*

$$\begin{array}{ll} \varphi_*\mathscr{O}_X(a\tau) = 0 & \text{if } a < 0, \\ \varphi_*\mathscr{O}_X(a\tau) = S^a(\mathscr{V}) & \text{if } a \geq 0, \\ R^i\varphi_*\mathscr{O}_X(a\tau) = 0 \quad \forall a \in \mathbb{Z} & \text{if } 0 < i < \mathrm{Rk}(\mathscr{V}) - 1 \text{ or if } i \geq \mathrm{Rk}(\mathscr{V}), \\ R^{\mathrm{Rk}(\mathscr{V})-1}\varphi_*\mathscr{O}_X(a\tau) = 0 & \text{if } a > -\mathrm{Rk}(\mathscr{V}). \end{array}$$

3. A generic member of the bianticanonical linear system

From now onwards, S will be the Segre–Hirzebruch surface $\mathbb{F}_n = \mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-n))$, with n positive. Recall that $\mathrm{Pic}(\mathbb{F}_n)$ is generated by C_0, the only effective divisor on S such that $C_0^2 = -n$, and F, the class of a fiber of the $\mathbb{P}^1$-bundle. Hence, without loss of generality, any decomposable vector bundle of rank 2, up to tensor product with a line bundle, can be written as $\mathscr{V} = \mathscr{O}_{\mathbb{F}_n} \oplus \mathscr{O}_{\mathbb{F}_n}(-A)$, where $A = xC_0 + yF$ and x is nonnegative. For the sake of convenience, we will denote by the same symbol a divisor on S and its pullback on X. We will denote, as before, by X the projective bundle associated to $\mathscr{V}$

PROPOSITION 4. *Consider a divisor $D = a\tau + G$ on X where $G = bC_0 + cF$ is the pullback of a divisor on S. Then the following hold:*

i) *If $A = xC_0 + yF$ with $y \geq 0$ (i.e., if A is effective), D is effective if and only if $a, b, c \geq 0$.*
ii) *If $A = xC_0 - yF$ with $y > 0$, D is effective if and only if $a \geq 0$ and*

$$(b, c) \in \bigcup_{r=0}^{a} S_r \quad \text{with } S_r = \{(b, c) \mid b, c \geq 0\} + (rx, -ry).$$

iii) *If $A = xC_0 - yF$ with $y > 0$, the only prime and rigid divisors on X are τ, C_0 and $\tau + A$.*

PROOF. i) $D = a\tau + bC_0 + cF$ is effective if and only if $a \geq 0$; else we have

$$\varphi_*\mathscr{O}_X(a\tau + bC_0 + cF) = \varphi_*\mathscr{O}_X(a\tau) \otimes \mathscr{O}_S(bC_0 + cF) = 0.$$

Hence, we can assume $a \geq 0$. Doing so, we have

$$H^0(\mathscr{O}_X(D)) = \bigoplus_{r=0}^{a} H^0(\mathscr{O}_S(bC_0 + cF - rA)) \supset H^0(\mathscr{O}_S(bC_0 + cF)).$$

If $b, c \geq 0$ the divisor is effective.

ii) Assume, now, $A = xC_0 - yF$ with $y > 0$. In this case

$$V_r = H^0(\mathcal{O}_S(bC_0 + cF - rA)) = H^0(\mathcal{O}_S((b - rx)C_0 + (c + ry)F))$$

and $H^0(\mathcal{O}_X(D))$ is not zero exactly when at least one of these spaces is not zero. V_r is not zero exactly when $b \geq rx$ and $c \geq -ry$, i.e., when $(b, c) \in S_r$, so the second claim is proved.

iii) Finally assume $A = xC_0 - yF$ with $y > 0$, and consider the effective divisor $D = a\tau + bC_0 + cF$ with $a, b \geq 0$ and $-ay \leq c \leq 0$. If D is rigid then $(b, c) \in S_r$ for exactly one value or r (with $0 \leq r \leq a$). If $(b, c) \in S_r$ we can write D as $a\tau + rA + b'C_0 + c'F$ with $0 \leq c' < y$. If $r < a$ we can assume $0 \leq b' < x$ whereas, if $(b, c) \in S_a$, we can assume $b' \geq 0$. In both cases, the divisor $D = a\tau + rA + b'C_0$ is effective; hence, if $c' > 0$, we have $h^0(\mathcal{O}_X(D)) \geq 2$. This shows that we have to look for rigid divisors among the ones of the form

$$D = a\tau + rA + b'C_0,$$

where $a \geq 0, 0 \leq r < a$ and $0 \leq b' < x$ or with $a \geq 0, r = a$ and $b' \geq 0$. It is not difficult to see that, in this case, $h^0(\mathcal{O}_X(D)) = 1$, so we always get a rigid divisor. It is also easy to see that every such divisor can be written as a sum

$$a_1\tau + a_2C_0 + a_3(\tau + A),$$

which proves that τ, C_0 and $\tau + A$ are the only rigid prime divisors on X when $A = xC_0 - yF$ and $y > 0$. □

We will be interested in the case $\mathcal{V} = \mathcal{O}_{\mathbb{F}_n} \oplus \mathcal{O}_{\mathbb{F}_n}(-A)$ with $A = 2C_0 - F$. Recall that, in this case,

$$K_X = -2\tau - 2C_0 - (n + 2)F - 2C_0 + F = -2\tau - 4C_0 - (n + 1)F,$$

so we have

$$-2K_X = 4\tau + 8C_0 + (2n + 2)F.$$

As we will see, if n is big enough, the linear system $|-2K_X|$ does not have smooth members. Thus, we need to describe more closely the base locus and the type of singularities.

Proposition 5. *The base locus of the bianticanonical linear series is given by the complete intersection σ of the rigid divisors with class $\tau + A$ and C_0.*

Proof. Since $2C_0 - F$ is not effective, by Proposition 4 there are three rigid prime divisors, namely τ, C_0 and $\tau + A$. The intersections of these three divisors are

$$\tau(\tau + A) = 0, \quad \tau C_0 := \gamma, \quad (\tau + A)C_0 = (\tau - (2n + 1)F)_{|C_0} := \sigma.$$

The unique surface T with class given by τ is a Segre–Hirzebruch surface $\mathbb{F}_n$ with standard generators for $\mathrm{Pic}(\tau)$ given by

$$C_0|_T = \gamma_T, \quad F|_T = f_T.$$

By standard generators, we mean a basis of effective prime divisors under which the intersection product has representative matrix

$$\begin{bmatrix} -a & 1 \\ 1 & 0 \end{bmatrix}$$

where a is the (positive) index of the Segre–Hirzebruch surface. In particular, the class of the curve γ seen in T is given by γ_T.

Denote by R the only surface whose class is $\tau + A$. One can easily see that R is again a Segre–Hirzebruch surface $\mathbb{F}_n$ if one considers the vector bundle $\mathscr{V}' = \mathscr{V} \otimes \mathcal{O}_S(A)$ and uses the identification

$$X = \mathbb{P}(\mathscr{V}) = \mathbb{P}(\mathcal{O}_S \oplus \mathcal{O}_S(-A)) = \mathbb{P}(\mathcal{O}_S \oplus \mathcal{O}_S(A)) = \mathbb{P}(\mathscr{V}').$$

Indeed the class of $c_1(\mathcal{O}_{\mathbb{P}(\mathscr{V}')}(1)) = \tau'$ is $\tau + A$ under this identification. The standard generators for $\mathrm{Pic}(R)$ are

$$C_0|_R = \gamma_R, \quad F|_R = f_R.$$

The surface U, whose class is C_0, is also a Segre–Hirzebruch surface $\mathbb{F}_m$ with $m = 2n + 1$. The standard generators for the Picard lattice are

$$(\tau + A)|_U = \gamma_U, \quad F|_U = f_U.$$

Notice that

$$-2K_X = 4(\tau + A) + (2n + 6)F,$$

so, an eventual base point of $|-2K_X|$ cannot lie outside the surface R. In fact, $(2n + 6)F$ is globally generated. It is easy to prove that $\tau + A$ is not a component of $|-2K_X|$, so the base locus of the bianticanonical linear series is contained in R. In fact, let us restrict $-2K_X$ to R. This yields

$$(4\tau + 8C_0 + (2n + 2)F)|_R = 8\sigma + (2n + 2)f_R,$$

which shows that σ is contained in the base locus of the bianticanonical linear series. Conversely, given a point in such a base locus, it must belong to σ because it is in R and nowhere else than in σ because $|f_R|$ is globally generated in R. Therefore the claim is proved. □

Remark. The curves σ and γ are the intersection of C_0 with τ and $\tau + A$, respectively. We can also see them inside these surfaces and the following table describes their classes (a "–" simply means that the curve cannot be seen in that particular surface).

	T (τ)	R $(\tau + A)$	U (C_0)
$T \cap U = \gamma$	γ_T	–	$\gamma_U + (2n+1)f_U$
$R \cap U = \sigma$	–	γ_R	γ_U

From this description (as well as from adjunction) one can see that both γ and σ are smooth curves of genus 0. Moreover, σ is rigid in both R and U, whereas γ is rigid only in T. From now on, we will assume $n \gg 0$.

PROPOSITION 6. *The generic member of the bianticanonical system has multiplicity* 3 *along the base locus.*

PROOF. Define t, u and r to be the sections (uniquely determined up to scalar) such that

$$H^0(\mathcal{O}_X(\tau)) = \langle t \rangle \quad H^0(\mathcal{O}_X(C_0)) = \langle u \rangle \quad H^0(\mathcal{O}_X(\tau + A)) = \langle r \rangle$$

so the zero loci of t, u and r describe T, U and R, respectively. Define D_i to be $-2K_X - i(\tau + A)$. Therefore, there exists a positive integer N_0 big enough such that for $n \geq N_0$ the following hold:

$$\begin{array}{r|c}
h^0(D_0) = 14n + 61 & - \\
h^0(D_1) = 11n + 52 & h^0(\mathcal{O}_X(D_0)) - h^0(\mathcal{O}_X(D_1)) = 3n + 9 \\
h^0(D_2) = 8n + 40 & h^0(\mathcal{O}_X(D_1)) - h^0(\mathcal{O}_X(D_2)) = 3n + 12 \\
h^0(D_3) = 5n + 25 & h^0(\mathcal{O}_X(D_2)) - h^0(\mathcal{O}_X(D_3)) = 3n + 15 \\
h^0(D_4) = 2n + 7 & h^0(\mathcal{O}_X(D_3)) - h^0(\mathcal{O}_X(D_4)) = 3n + 18
\end{array}$$

Let us now describe the sections of $\mathcal{O}_X(-2K_X)) = \mathcal{O}_X(4(\tau + A) + (2n+6)F)$.

We have the exact sequence

$$(6) \qquad 0 \longrightarrow H^0(\mathcal{O}_X(D_1)) \xhookrightarrow{-\otimes r} H^0(\mathcal{O}_X(-2K_X)) \longrightarrow H^0(\mathcal{O}_R(-2K_X))$$

Notice that $-2K_X|_R = (8C_0 + (2n+2)F)|_R = 8\gamma_R + (2n+2)f_R$ hence, as $R = \mathbb{F}_n$, we have

$$\begin{aligned} h^0(\mathcal{O}_R(-2K_X)) &= h^0(\mathcal{O}_{\mathbb{F}_n}(8\gamma_R + (2n+2)f_R)) = 3n + 9 \\ &= h^0(\mathcal{O}_X(D_0)) - h^0(\mathcal{O}_X(D_1)). \end{aligned}$$

Thus, the restriction map $H^0(\mathcal{O}_X(-2K_X)) \to H^0(\mathcal{O}_R(-2K_X))$ in Equation 6 is indeed surjective. Denote by V_0 a subspace of $H^0(\mathcal{O}_X(-2K_X))$ such that

$$V_0 \oplus (H^0(\mathcal{O}_X(D_1)) \otimes \langle r \rangle) \simeq H^0(\mathcal{O}_R(-2K_X)).$$

If $s \in H^0(\mathcal{O}_X(-2K_X))$, we have a decomposition of s as

$$s = r\alpha_0 + \beta_0$$

with $\alpha_0 \in H^0(\mathcal{O}_X(D_1))$ and $\beta_0 \in V_0$. In particular, β_0 does not vanish identically on R (it vanishes on γ_R and some $\mathbb{P}^1$'s transversal to γ_R). We can iterate this process by restricting α_0 on R. As before, we have the following exact sequences, namely:

$$(7)\qquad \begin{array}{l} 0 \longrightarrow H^0(\mathcal{O}_X(D_2)) \xhookrightarrow{-\otimes r} H^0(\mathcal{O}_X(D_1)) \longrightarrow H^0(\mathcal{O}_R(D_1)) \longrightarrow 0 \\ 0 \longrightarrow H^0(\mathcal{O}_X(D_3)) \xhookrightarrow{-\otimes r} H^0(\mathcal{O}_X(D_2)) \longrightarrow H^0(\mathcal{O}_R(D_2)) \longrightarrow 0 \\ 0 \longrightarrow H^0(\mathcal{O}_X(D_4)) \xhookrightarrow{-\otimes r} H^0(\mathcal{O}_X(D_3)) \longrightarrow H^0(\mathcal{O}_R(D_3)) \longrightarrow 0, \end{array}$$

where the surjectivity follows as before by inspecting the dimension of

$$H^0(\mathcal{O}_R(D_i)) = H^0(\mathcal{O}_{\mathbb{F}_n}((8-2i)\gamma_R + (2n+2+i)f_R)$$

and observing that it equals $h^0(\mathcal{O}_X(D_i)) - h^0(\mathcal{O}_X(D_{i+1}))$. Then we can create the vector spaces V_i such that

$$V_i \oplus (H^0(\mathcal{O}_X(D_{i+1})) \otimes \langle r \rangle) \simeq H^0(\mathcal{O}_R(D_i))$$

and sections $\alpha_i \in H^0(\mathcal{O}_X(D_{i+1}))$, $\beta_i \in V_i$ such that

$$\alpha_i = r\alpha_{i+1} + \beta_{i+1}.$$

Finally, the section s has the following form:

$$(8)\qquad s = r^4\alpha_3 + r^3\beta_3 + r^2\beta_2 + r\beta_1 + \beta_0.$$

Notice that $D_0|_R$, $D_1|_R$ and $D_2|_R$ are divisors with σ_R as fixed components so β_i for $i = 0, 1, 2$ will vanish on it (with multiplicity greater than or equal to 4). But the same is not true for $D_3|_R$, which is very ample. In particular, β_3 can be chosen such that $\beta_3|_R$ vanishes at exactly 5 points of σ_R (this is equal to $\sigma_R \cdot D_3|_R$) which are free on σ_R and whose associated curve cut σ_R transversely at such points.

In particular, the generic element of $|-2K_X|$ has σ as base curve and the multiplicity of σ along the generic bianticanonical divisor is 3. □

4. Blowing up the projective bundle

In this section we will describe a resolution of a generic member of the linear system $|-2K_X|$.

Near a point P of σ we can choose local coordinates (x, y, z) such that $x = y = 0$ is the local equation of σ near P, $x = 0$ and $y = 0$ are the local equations of R and U respectively and z is a coordinate on σ. We can also use (y, z) as local coordinates on R. We write, locally

$$s = x^3 f + x^4 g + x^2 y^4 f_1 + x y^6 f_2 + y^8 f_3,$$

where f is the local expression for β_3 and g is the local expression for α_3. We can blow up σ in X and take the strict transform $\tilde{D}$ of $D := \{s = 0\}$. Near P the blow up X_1 looks like

$$\{(x, y, z) \times (l_0 : l_1) \mid x l_1 - y l_0 = 0\}.$$

In the local chart $U_0 = \{l_0 \neq 0\}$ we have coordinates (x, z, l_1) with $y = xl_1$ and the local equation for the exceptional divisor E which is $x = 0$. The total transform of D has equation

$$x^3(f + xg + x^3l_1^4f_1 + x^4l_1^6f_2 + x^5l_1^8f_3),$$

so

$$\bar{s} = f + xg + x^3l_1^4f_1 + x^4l_1^6f_2 + x^5l_1^8f_3$$

is a local equazion for $\tilde{D}$. Notice that $f(0, y, z)$ is not identically zero because f is the local expression of β_3. From the proof of Proposition 6 we have also that $\tilde{D}$ is smooth along σ and hence everywhere (since it is the strict transform of something that has base locus σ). Unfortunately, $\tilde{D}$ is not a bianticanonical divisors on X_1: the bianticanonical class is indeed $\tilde{D} + E$ so we can take a bianticanonical divisor on X_1 to be the union of $\tilde{D}$ and E. This is reduced, reducible and singular exactly along the intersection $E_1 \cdot \tilde{D}$.

Lemma 7. *The divisor E is a Segre–Hirzebruch variety $\mathbb{F}_{n+1}$.*

Proof. The curve σ is a complete intersection. More precisely, it is the intersection of the two rigid divisors C_0 and $\tau + A$. Thus, the normal bundle is $\mathcal{O}_\sigma(C_0) \oplus \mathcal{O}_\sigma(\tau + A)$. By direct computation, this is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-n) \oplus \mathcal{O}_{\mathbb{P}^1}(-2n-1)$, which proves the claim. □

The Picard group of X_1 is generated by τ, C_0, F and the exceptional divisor E_1. By construction, the restriction of τ to E_1 is zero. Moreover, the restriction of C_0 to the exceptional divisor is an integer multiple of $f_1 = F|_{E_1}$, the class of a fiber of E_1 seen as Segre–Hirzebruch surface. This follows from the intersection numbers that are calculated in the next section. Therefore, the Picard group of the exceptional divisor is generated by the restriction of E_1 and F, respectively. It is not difficult to check that the unique divisor γ_1 on E_1 such that $\gamma_1^2 = -n - 1$ is given by

$$\gamma_1 = -E_{1|E_1} - (2n+1)F_{|E_1}.$$

The strict transform of the divisor $-2K_X$ is equal to $-2K_X - 3E_1$. Its intersection with E_1 is given by $3\gamma_1 + 5f_1$. This is an effective divisor on E_1, which is made up of the unique curve of self-intersection $-n-1$ and 5 disjoint fibers. Since we have

$$-2K_{X_1} = -2K_X - 2E_1 = (-2K_X - 3E_1) + E_1,$$

the sections of the bianticanonical divisor $-2K_{X_1}$ pass through the curve γ_1, which is the complete intersection of $\tau + A - E_1$ (strict transform of the divisor $\tau + A$) and E_1.

Therefore, we blow up X_1 along the curve γ_1 and obtain a new variety X_2 with exceptional divisor E_2. To determine its structure, we compute the normal bundle of γ_1 which is given by

$$N_{\gamma_1/X_1} = \mathcal{O}_{\gamma_1}(E_1) \oplus \mathcal{O}_{\gamma_1}(\tau + A - E_1) \simeq \mathcal{O}_{\mathbb{P}^1}(-n-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-n).$$

Therefore, the exceptional divisor E_2 is isomorphic to $\mathbb{F}_1$. Let us denote by γ_2 and f_2 the generators of E_2 such that $\gamma_2^2 = -1$, $\gamma_2 f_2 = 1$ and $f_2^2 = 0$. As in the case of E_1, we can take f_2 to be the restriction of F to E_2. As for the other divisor, it is easy to check that

$$\gamma_2 = -E_{2|E_2} - (n+1)F_{|E_2}. \tag{9}$$

The bianticanonical divisor of X_2 is thus given by

$$-2K_{X_2} = -2K_X - 2E_1 - 2E_2 = (-2K_X - 2E_1 - 4E_2) + 2E_2.$$

Let us compute the restriction of the divisor $(-2K_X - 2E_1 - 4E_2)$ to E_2. An easy calculation shows that it is equal to $4\rho_2 + 6f_2$, which corresponds to the class of a smooth irreducible curve on $E_2 \simeq \mathbb{F}_1$. Thus, there is a smooth member of the linear system

$$2(-K_{X_2} - E_2) = -2K_X - 2E_1 - 4E_2.$$

Being $2(-K_{X_2} - E_2)$ even, we can consider the cyclic covering $\beta : Y_2 \to X_2$ of degree two with branch along a smooth member of $-2K_{X_2} - 2E_2 = 2K_X - 2E_1 - 4E_2$.

Lemma 8. *Y_2 is a smooth threefold and $\beta^* E_2$ is a $K3$ surface. Moreover the pair $(Y_2, \beta^* E_2)$ is a log Calabi–Yau.*

Proof. Y_2 is clearly smooth as the branch divisor has been chosen to be smooth. Moreover, by [BPHV] page 55, we have also

$$K_{Y_2} = \beta^*(K_{X_2} + B_2/2) = -\beta^*(E_2) \tag{10}$$

so $(Y_2, \beta^* E_2)$ is a log Calabi–Yau. Notice that $\beta^*(E_2)$ is a degree two covering of the Segre–Hirzebruch surface $\mathbb{F}_1$ branched along the intersection of E_2 with the branch divisor of the covering $\beta : Y_2 \to X_2$. We have already seen that this intersection can be written as the smooth curve $B_2 = 4\rho_2 + 6f_2$ on $E_2 \simeq \mathbb{F}_1$, i.e. it is a smooth bianticanonical curve on $\mathbb{F}_1$. This is enough to conclude that the canonical divisor of $\beta^* E_2$ is trivial. The Euler characteristic is of $\beta^* E_2$ can be calculated as $2e(E_2) - e(R_2)$, where R_2 is the ramification divisor of the restriction of β to $\beta^*(E_2)$. Since β is a degree two covering, the divisor R_2 is isomorphic to the branch divisor B_2. This is a curve of genus 9, so the Euler characteristic of R_2 is -16. We have hence $e(\beta^* E_2) = 24$ so we can conclude that $\beta^*(E_2)$ is a $K3$ surface. $\square$

In the next section, we are going to calculate the Euler characteristic of Y_2 for every $n \geq N_0$. To conclude this section, let us prove the following result.

THEOREM 9. *Let Y_2 be as above. Then we have:*

$$h^{1,0}(Y_2) = 0, \quad h^{2,0}(Y_2) = 0, \quad h^{3,0}(Y_2) = 0.$$

Moreover, Y_2 has negative Kodaira dimension.

PROOF. We need to determine $h^{q,0}(Y_2)$ for $q \geq 1$. Recall that

$$\beta_* \mathcal{O}_{Y_2} \simeq \mathcal{O}_{X_2} \oplus \mathcal{O}_{X_2}(-B_2/2) = \mathcal{O}_{X_2} \oplus \mathcal{O}_{X_2}(K_{X_2} + E_2)$$

and that $R^q \beta_* \mathcal{F} = 0$ for all $\mathcal{F}$ coherent on Y_2 and for all $q \geq 1$. Hence, by Leray spectral sequence, we have

$$H^q(\mathcal{O}_{Y_2}) \simeq H^q(\mathcal{O}_{X_2}) \oplus H^q(\mathcal{O}_{X_2}(K_{X_2} + E_2)).$$

X_2 is birational to X, which is a projective bundle over $\mathbb{F}_n$ so the Hodge numbers $h^{q,0}(X_2) = h^{q,0}(X)$ are zero for $q \geq 1$. Hence we need to prove that $h^q(\mathcal{O}_{X_2}(K_{X_2} + E_2))$ is zero for $q \geq 1$ in order to conclude the proof. If $q = 3$ this is straightforward: we have

$$h^3(\mathcal{O}_{X_2}(K_{X_2} + E_2)) = h^0(\mathcal{O}_{X_2}(-E_2)) = 0 \tag{11}$$

because E_2 is effective. We have $h^p(\mathcal{O}_{X_2}(K_{X_2})) = h^{3-p}(\mathcal{O}_{X_2}) = h^{3-p}(\mathcal{O}_X)$ so,

$$h^1(\mathcal{O}_{X_2}(K_{X_2})) = h^2(\mathcal{O}_{X_2}(K_{X_2})) = 0 \quad \text{and} \quad h^3(\mathcal{O}_{X_2}(K_{X_2})) = 1. \tag{12}$$

To compute $H^q(K_{X_2} + E_2)$ for $q = 1, 2$, let us consider the exact sequence

$$0 \to \mathcal{O}_{X_2}(K_{X_2}) \to \mathcal{O}_{X_2}(K_{X_2} + E_2) \to \mathcal{O}_{E_2}(K_{X_2} + E_2) \to 0,$$

which yields, using also Equations 11 and 12, the exact sequences

$$0 \to H^1(\mathcal{O}_{X_2}(K_{X_2} + E_2)) \to H^1(\mathcal{O}_{E_2}(K_{X_2} + E_2)) \to 0 \tag{13}$$

$$0 \to H^2(\mathcal{O}_{X_2}(K_{X_2} + E_2)) \to H^2(\mathcal{O}_{E_2}(K_{X_2} + E_2)) \to H^3(\mathcal{O}_{X_2}(K_{X_2})) \to 0 \tag{14}$$

By adjunction, $\mathcal{O}_{E_2}(K_{X_2} + E_2)$ is the canonical divisor of K_{E_2} so $H^1(\mathcal{O}_{E_2}(K_{E_2})) = H^{1,2}(\mathbb{F}_1) = 0$ (or, alternatively, by Lemma 2.9 of [CM02]). Hence, from the exact sequence 13, also $H^1(X_2, \mathcal{O}_{X_2}(K_{X_2} + E_2))$ is zero.

Both the second and the third term of the exact sequence 14 have dimension 1 so $h^2(\mathcal{O}_{X_2}(K_{X_2} + E_2)) = 0$.

In order to see that the Kodaira dimension is $-\infty$, it is enough to observe that $-K_{Y_2}$ is effective and this follows from Equation 10. □

5. THE EULER CHARACTERISTIC

In this section, we will calculate the Chern numbers of X_2. Recall that $X = \mathbb{P}(\mathcal{V})$ with $\mathcal{V} = \mathcal{O}_S \oplus \mathcal{O}_S(-A)$ and $A = 2C_0 - F$. If $X_1 = \mathrm{Bl}_\sigma X$, where σ is the rational curve cut out by R and U. If E_1 is the class of the exceptional divisor, we can consider the complete intersection curve cut out by the two divisors $\tau + A - E_1$ and E_1. As for the notation, denote by E_2 the exceptional divisor of the second blow up. We will apply the following lemma:

LEMMA 10. *Let Z be a smooth complex threefold and let*

$$C \overset{j}{\hookrightarrow} Z,$$

where C is a smooth curve. If $Z' = \mathrm{Bl}_C(Z)$ with exceptional divisor E and blow up map $\pi : Z' \to Z$. Then the following hold:

$$c_1(Z') = \pi^* c_1(Z) - E \tag{15}$$

$$c_2(Z') = \pi^*(c_2(Z) - \eta_C) - \pi^* c_1(Z) E \tag{16}$$

$$H^*(Z') = H^*(Z) \oplus H^*(E)/H^*(C), \tag{17}$$

where η_C is the class of C in $H^4(Z)$. Moreover, if $\alpha_p \in CH^p(Z)$ and $p + q = 3$ with $q \geq 1$, then

$$E \cdot (\pi^* \alpha_2) = 0 \quad E^2 \cdot (\pi^* \alpha_1) = -j^* \alpha_1 \quad E^3 = -c_1(N_{C/Z}). \tag{18}$$

PROOF. The first two identities can be found in [GH], p. 609. Let α_p be a class in $CH^p(Z)$ and consider the following commutative diagram

$$\begin{tikzcd} E \arrow[r, hook, "\imath"] \arrow[d, "\pi"'] & Bl_C(Z) \arrow[d, "\pi"] \\ C \arrow[r, hook, "j"'] & Z \end{tikzcd}$$

If we assume that $q \geq 1$ we can write $E^q = E^{q-1} \cdot E = E^{q-1} \imath_*(1)$ so that

$$\begin{aligned} E^q \cdot (\pi^* \alpha_p) &= (E^{q-1} \pi^* \alpha_p) \imath_*(1) = \imath^*(E^{q-1} \pi^* \alpha_p) \cdot 1 = \imath^*(E^{q-1})(\pi \circ \imath)^* \alpha_p \\ &= \imath^*(E^{q-1})(j \circ \pi)^* \alpha_p = \imath^*(E^{q-1}) \pi^*(j^* \alpha_p) = \pi_*(\imath^* E)^{q-1} \cdot (j^* \alpha_p). \end{aligned}$$

The restriction of the exceptional divisor to itself is the tautological class of E when seen as the total space of the projective bundle $\mathbb{P}(N_{C/Z}) \to C$. If we denote by $h = c_1(\mathcal{O}_{\mathbb{P}(N_{C/Z})}(1))$, we have $\imath^*(E)^{q-1} = (-1)^{q-1} h^{q-1}$. By definition we have also $\pi_*(h^{q-1}) = s_{q-2}(N_{C/Z})$, where $s_n(N_{C/Z})$ is the Segre class of level n of the vector bundle $N_{C/Z}$. To conclude, it is enough to observe that $s_1(N_{C/Z}) = -c_1(N_{C/Z})$ and $s_0(N_{C/Z}) = 1$. □

Recall that

$$c_1(X) = 2\tau + 4C_0 + (n+1)F, \tag{19}$$

$$c_2(X) = 4\tau C_0 + (2n+4)\tau F + (-2n+6) C_0 F, \tag{20}$$

$$c_3(X) = 8\tau C_0 F \tag{21}$$

and that σ, the center of the first blow up, is the complete intersection of $\tau + A$ and C_0. Hence

$$N_{\sigma/X} = \mathcal{O}_\sigma(\tau + A) \oplus \mathcal{O}_\sigma(C_0)$$

and the class η_σ of σ in $H^4(X)$ is simply the class of $(\tau + A)C_0$. In order to simplify notation, we will write α to indicate both a class in X and its pull-back to X_1 and X_2. The first Chern class of X_1 is simply given by $c_1(X) - E_1$ whereas

$$c_2(X_1) = c_2(X) - (\tau + A)C_0 - c_1(X)E_1.$$

We are blowing up a smooth rational curve so

$$E_1 = \mathbb{P}(N_{\sigma/X}) = \mathbb{P}(\mathcal{O}_\sigma(\tau + A) \oplus \mathcal{O}_\sigma(C_0))$$

is the Segre–Hirzebruch surface $\mathbb{F}_{n+1}$. By (15), we obtain that the Hodge structure of X_1 is pure and $h^{1,1}(X_1) = 4$. To recap, we have

$$c_1(X_1) = c_1(X) - E_1, \tag{22}$$

$$c_2(X_1) = c_2(X) - (\tau + A)C_0 - c_1(X)E_1, \tag{23}$$

$$c_3(X_1) = 10\tau C_0 F. \tag{24}$$

Moreover, the relations that characterize the intersection theory on X_1 are (here we don't report the ones coming from X) given by

$$E_1\tau = 0 \quad E_1C_0F = 0 \quad E_1^2C_0 = n \quad E_1^2F = -1 \quad E_1^3 = 3n + 1.$$

The first relation follows simply by observing that τ and σ are disjoint so τ and E_1 do not intersect. The others follow from Lemma 10 using

$$C_0 j^*\sigma = C_0^2(\tau + A) = -n \quad Fj^*\sigma = C_0(\tau + A)F = 1.$$

and

$$c_1(N_{\sigma/X}) = C_0^2(\tau + A) + C_0(\tau + A)^2 = -(3n + 1).$$

The curve γ_1 is smooth and rational. If we blow it up, we obtain an exceptional divisor E_2 that is isomorphic to $\mathbb{F}_1$. Indeed, the normal bundle of such a curve is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-n-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$. Using the same argument as before, we have

$$c_1(X_2) = c_1(X_1) - E_2, \tag{25}$$

$$c_2(X_2) = c_2(X_1) - (\tau + A - E_1)E_1 - c_1(X_1)E_2, \tag{26}$$

$$c_3(X_2) = 12\tau C_0 F. \tag{27}$$

Continuing as before we get

$$E_2\tau = 0 \quad E_2C_0F = 0 \quad E_2E_1C_0 = 0 \quad E_2E_1F = 0 \quad E_2E_1^2 = 0$$
$$E_2^2C_0 = n \quad E_2^2F = -1 \quad E_2^2E_1 = n \quad E_2^3 = 2n + 1$$

This is all we need to prove the following theorem

THEOREM 11. *For every positive integer n big enough there exists a pair (Y, D) such that*

- *Y is a smooth threefold of negative Kodaira dimension with*

$$e(Y) - 48n - 46 \quad \text{and} \quad h^{q,0}(Y) = 0 \quad \text{for } q \geq 1;$$

- *D is a smooth $K3$ surface;*
- *(Y, D) is a log canonical log Calabi–Yau pair;*

PROOF. Fix $n \geq N_0$ and consider the projective bundle $X = \mathbb{P}(\mathscr{V})$ over $\mathbb{F}_n$, where

$$\mathscr{V} = \mathscr{O}_{\mathbb{F}_n} \oplus \mathscr{O}_{\mathbb{F}_n}(-2C_0 + F).$$

First, blow up the projective bundle along the base locus of the bianticanonical divisor obtaining X_1. Next, blow up such a variety along the base locus of the bianticanonical divisor to obtain X_2. Take the degree two covering of X_2 with branch B_2 as described in the previous sections to finally obtain Y_2. Then one can take $Y = Y_2$ and $D = \beta^*E_2$. Everything, apart form the calculation for the Euler characteristics, have been done in the previous sections.

In order to compute the Euler characteristic, recall that if D is a smooth irreducible divisor on X_2, we have

$$c_2(D) = c_2(X_2) - c_1(X_2)D + D^2 \tag{28}$$

so

$$\mathrm{e}(D) = (c_2(X_2) - c_1(X_2)D + D^2)N_{D/X_2}. \tag{29}$$

Hence, being the branch locus B_2 a smooth element of $|-2K_{X_2} - 2E_2|$, we have that

$$e(B_2) = 48n + 70.$$

The Euler number of X_2 is given by 12 so we have, finally,

$$\mathrm{e}(Y_2) = 2\mathrm{e}(X_2) - \mathrm{e}(B_2) = 2 \cdot 12 - (48n + 70) = -48n - 46. \tag{30}$$

Although feasible by hands, we have done the last computation using Magma. □

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