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Partial Differential Equations — *The time-periodic unfolding operator and applications to parabolic homogenization*, by Micol Amar, Daniele Andreucci and Dario Bellaveglia, communicated on February 10, 2017.

This paper is dedicated to the memory of Ennio De Giorgi.

Abstract. — We apply the method of periodic unfolding to a classical homogenization problem for a parabolic equation. With respect to previous literature, we allow for capacity-like coefficients in the diffusion equation oscillating both in space and time, with general independent scales. Our approach relies upon a generalization of the unfolding technique to the time-periodic case.

Key words: Homogenization, unfolding technique, time oscillations, parabolic differential equations

Mathematics Subject Classification: 35B27, 35K10

1. Introduction

In this paper we develop an approach to the homogenization of parabolic problems with oscillating coefficients based on the method of periodic unfolding. Specifically we introduce operators of time-periodic unfolding modeled after the operators of space-periodic unfolding introduced and applied in [9, 10, 11, 8]. The first part of the paper contains results of more general interest which may possibly be applied in different frameworks from the one dealt with here.

Our interest in problems exhibiting oscillations in time originally arised from mathematical models with boundary conditions involving alternating pores (see [20]). Such conditions switch between a closed state and an open one, either periodically or according to a random scheme. As shown in [4], the limiting behavior of problems of this kind sharply depends on the relative scalings of the time and space variables; see also [6] for a MonteCarlo test of the model. In [5] oscillations in the boundary conditions have been coupled to time periodic changes in the diffusivity coefficient as a device to reproduce the selection capability exhibited by biological membranes.

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Let us compare our approach here to previous literature. The unfolding operator we define is essentially a suitable extension of its purely spatial counterpart in [9, 10], and some of the theory already established in the quoted papers carries over to our case. However some significant differences appear in the limiting behavior of the operator, due to the degenerate character of the available estimates of the time derivative of the unknown in the approximating differential problem. As a technical remark, we note that in this connection the need arises for both the space oscillation operator introduced in [10] and the space-time oscillation operator, see Definition 2.4. In [19, 21] the authors use unfolding in the space variables with a parametric dependence on time, to study a parabolic problem.

The papers [14, 12, 13] study problems similar to the one investigated in this paper, by means of two-scale convergence. With respect to those papers we cover more general cases in the following instances: First, we allow the space and time oscillation periods, respectively τ and ε, to vanish in the limit according to any rate, instead of assuming that $\tau = \varepsilon^r$ for some $r > 0$. Second, we can handle the case when the time derivative in the diffusion equation is multiplied by a coefficient oscillating in time, usually arising as a capacity coefficient in applications. On the other hand we require more regularity for the diffusion matrix. This assumption however enables us to partially answer a question raised in [12], see Remark 4.3 below, and is used only in the estimation procedure and not in the homogenization process. We also quote [15, 18] where the two-scales technique is applied to multiscale problems.

The cases when the capacity term is constant and $\tau = \varepsilon^r$, $r > 0$ were investigated also in [7, Chapter II] by means of asymptotic expansion techniques. There one can find also some formal comments on the case of an oscillating capacity coefficient. Here we deal rigorously with some of such problems.

A case of sign changing capacity oscillating in space was also treated in [2, 3] again by means of asymptotic expansions. We also quote [17] for the nonlinear case when $\tau = \varepsilon$.

In Section 2 we introduce the basic definitions and properties of the time-periodic unfolding operator, which are of general scope. We identify two possible limiting behaviors depending on the relative magnitude of τ and ε, which we call fast and slow oscillations in time (subsections 2.4 and 2.5 respectively). Notice that this classification relies on the degeneracy of the estimate of the time derivative, whose L^2 norm we assume to behave in the limit as τ^{-m} (see (2.37)). The parameter m provides, roughly speaking, the threshold value of ε as τ^{1-m}. As a matter of fact, $m = 1/2$ in the rest of the paper excepting Section 6.

In Section 3 we introduce the diffusion problem and obtain the relevant estimates needed for the homogenization process. Here we state precisely our assumptions.

In Section 4 we deal with the homogenization in the case of fast oscillations. Actually one has to discriminate the two subcases

$$\frac{\tau}{\varepsilon^2} \to 0, \quad \frac{\tau}{\varepsilon^2} \to \ell > 0, \quad \text{as } \varepsilon, \tau \to 0.$$

In Section 5 we consider the case of slow oscillations, where

$$\frac{\varepsilon^2}{\tau} \to 0, \quad \text{as } \varepsilon, \tau \to 0.$$

Section 6 is devoted to a case where a stable estimate in the L^2 norm of the time derivative is available, i.e., $m = 0$, and

$$\frac{\tau}{\varepsilon} \to \ell > 0, \quad \text{as } \varepsilon, \tau \to 0.$$

As a consequence a greater generality is possible in the choice of the capacity term multiplying the time derivative in the equation.

In Sections 4, 5, 6 we determine weak formulations of the homogenized problems. Finally in Section 7 we provide a more precise formulation of such problems, relying upon factorization and cell functions.

2. The time-periodic unfolding operator

2.1. Notation

Throughout the paper $\varepsilon > 0$ denotes the space period of the microstructure, and likewise $\tau > 0$ denotes its time period. Though this is not explicitly stressed by the notation for the sake of simplicity, we always assume that two sequences are given: $\varepsilon_i \to 0$, $\tau_i \to 0$ as $i \to \infty$. The limiting behavior of quantities depending on ε and τ is denoted by

$$\lim_{\varepsilon, \tau \to 0}$$

In the other notation for the sake of simplicity we drop as a general rule the dependence on ε, and write u_τ, $\mathcal{T}_\tau$ and so on. The symbol γ denotes a generic positive constant independent of ε, τ.

2.2. Definitions

Let $\Omega \subset \boldsymbol{R}^N$ be a bounded connected open set with Lipschitz boundary, and set

$$Y = (0,1)^N, \quad \Sigma = (0,1), \quad Q = Y \times \Sigma, \quad \Omega_T = \Omega \times (0,T).$$

Considering the tiling of $\boldsymbol{R}^N$ given by the sets $\varepsilon(\xi + Y)$, $\xi \in \boldsymbol{Z}^N$ we define

$$\Xi_\varepsilon = \{\xi \in \boldsymbol{Z}^N \mid \varepsilon(\xi + Y) \subset \Omega\}, \quad \hat{\Omega}_\varepsilon = \text{interior}\left\{ \bigcup_{\xi \in \Xi_\varepsilon} \varepsilon(\xi + \overline{Y}) \right\};$$

$$\hat{T}_\tau = \left\{ t \in (0,T) \mid \tau\left(\left[\frac{t}{\tau} \right] + 1 \right) \leq T \right\}, \quad \Lambda_\tau = \hat{\Omega}_\varepsilon \times \hat{T}_\tau.$$

Here and in the definitions below $[r]$ denotes the integer part of $r \in \boldsymbol{R}$.

For $x \in \boldsymbol{R}^N$ and $t \in [0, +\infty)$ we define

$$\left[\frac{x}{\varepsilon}\right]_Y = \left(\left[\frac{x_1}{\varepsilon}\right], \dots, \left[\frac{x_N}{\varepsilon}\right]\right),$$

and also denote

$$x = \varepsilon\left(\left[\frac{x}{\varepsilon}\right]_Y + \left\{\frac{x}{\varepsilon}\right\}_Y\right), \quad t = \tau\left(\left[\frac{t}{\tau}\right] + \left\{\frac{t}{\tau}\right\}\right).$$

Then we introduce the space and the space-time cell containing (x, t) as

$$Y_\varepsilon(x) = \varepsilon\left(\left[\frac{x}{\varepsilon}\right]_Y + Y\right), \quad Q_\tau(x,t) = \varepsilon\left(\left[\frac{x}{\varepsilon}\right]_Y + Y\right) \times \tau\left(\left[\frac{t}{\tau}\right] + \Sigma\right).$$

The following operator is a space-time version of the space unfolding operator introduced in [9].

Definition 2.1 (Time-periodic unfolding operator). For w Lebesgue-measurable on Ω_T the time-periodic unfolding operator $\mathscr{T}_\tau$ is defined as

$$\mathscr{T}_\tau(w)(x, y, t, s) = \begin{cases} w\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y, \tau\left[\frac{t}{\tau}\right] + \tau s\right), & (x, y, t, s) \in \Lambda_\tau \times Q, \\ 0, & \text{otherwise.} \end{cases} \qquad \square$$

Clearly for w_1, w_2 as in Definition 2.1

$$\mathscr{T}_\tau(w_1 w_2) = \mathscr{T}_\tau(w_1)\mathscr{T}_\tau(w_2). \tag{2.1}$$

We need both an average operator in space-time and one in space only.

Definition 2.2 (Local average operators). Let w be integrable in Ω_T. The space-time average operator is defined by

$$\mathscr{M}_\tau(w)(x, t) = \begin{cases} \dfrac{1}{\varepsilon^N \tau} \displaystyle\int_{Q_\tau(x,t)} w(\zeta, \theta)\, \mathrm{d}\zeta\, \mathrm{d}\theta, & \text{if } (x, t) \in \Lambda_\tau, \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

For $\bar{t} = \tau\left(\left[\frac{t}{\tau}\right] + s\right)$ we define the space average operator as

$$\tilde{\mathscr{M}}_\tau(w)(x, t, s) = \begin{cases} \dfrac{1}{\varepsilon^N} \displaystyle\int_{Y_\varepsilon(x)} w(\zeta, \bar{t})\, \mathrm{d}\zeta, & \text{if } (x, t, s) \in \Lambda_\tau \times \Sigma, \\ 0, & \text{otherwise.} \end{cases} \tag{2.3} \qquad \square$$

Remark 2.3. From our definitions it follows

$$\mathscr{M}_\tau(w)(x, t) = \iint_Q \mathscr{T}_\tau(w)(x, t, y, s)\, \mathrm{d}y\, \mathrm{d}s = \mathscr{M}_Q(\mathscr{T}_\tau(w))(x, t), \tag{2.4}$$

where in general $\mathscr{M}_I$ denotes the integral average on the set I. We also have

$$\tilde{\mathscr{M}}_\tau(w)(x,t,s) = \int_Y \mathscr{T}_\tau(w)(x,y,t,s)\,\mathrm{d}y = \mathscr{M}_Y(\mathscr{T}_\tau(w))(x,t,s). \tag{2.5}$$
□

In practice the average operators will be mostly used in connection with the oscillation operators which we define presently.

Definition 2.4. Let w be as in Definition 2.2. The space-time oscillation operator is defined as

$$\mathscr{Z}_\tau(w)(x,y,t,s) = [\mathscr{T}_\tau(w) - \mathscr{M}_\tau(w)](x,y,t,s), \tag{2.6}$$

and the space oscillation operator is defined as

$$\tilde{\mathscr{Z}}_\tau(w)(x,y,t,s) = [\mathscr{T}_\tau(w) - \tilde{\mathscr{M}}_\tau(w)](x,y,t,s). \tag{2.7}$$
□

Notice that

$$\tilde{\mathscr{Z}}_\tau(w) = \mathscr{Z}_\tau(w) - \mathscr{M}_Y(\mathscr{Z}_\tau(w)). \tag{2.8}$$

2.3. *Basic properties of the operator $\mathscr{T}_\tau$*

In this Subsection we collect some properties of the operators defined in Subsection 2.2. First we state a list of results for the sake of further reference; their proofs can be given essentially as in [10] and are therefore mostly omitted. Indeed in them the time variable does not play any special role.

In the following $p \in [1,\infty)$ unless otherwise noted. Also, functions depending only on the microscopic variables (y,s), or only on (x,t), are often considered trivially extended to $\Omega_T \times Q$.

Proposition 2.5. *The operator $\mathscr{T}_\tau : L^p(\Omega_T) \to L^p(\Omega_T \times Q)$ is linear and continuous.*

In addition for all $w \in L^p(\Omega_T)$ we have

$$\|\mathscr{T}_\tau(w)\|_{L^p(\Omega_T\times Q)} \le \|w\|_{L^p(\Omega_T)}, \tag{2.9}$$

and

$$\left|\int_{\Omega_T} w\,\mathrm{d}x\,\mathrm{d}t - \iint_{\Omega_T\times Q} \mathscr{T}_\tau(w)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\right| \le \int_{\Lambda_\tau} |w|\,\mathrm{d}x\,\mathrm{d}t. \tag{2.10}$$

Lemma 2.6. *Let $\phi \in W^{1,1}(\Omega_T \times Q)$, and define*

$$\phi^\tau(x,t) = \phi\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\tau}\Big), \quad (x,t) \in \Omega_T, \tag{2.11}$$

where ϕ has been extended by Q-periodicity to $\Omega_T \times \boldsymbol{R}^{N+1}$. Then in $\Omega_T \times Q$

$$\frac{\partial}{\partial s}\mathscr{T}_\tau(\phi^\tau) = \tau\mathscr{T}_\tau\Big(\frac{\partial\phi}{\partial t}\Big) + \mathscr{T}_\tau\Big(\frac{\partial\phi}{\partial s}\Big), \tag{2.12}$$

and

$$\nabla_y\mathscr{T}_\tau(\phi^\tau) = \varepsilon\mathscr{T}_\tau(\nabla_x\phi) + \mathscr{T}_\tau(\nabla_y\phi). \tag{2.13}$$

Proof. To prove (2.12) we note

$$\begin{aligned}
\frac{\partial}{\partial s}\mathscr{T}_\tau(\phi^\tau)(x,t,y,s) &= \frac{\partial}{\partial s}\left[\phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y, \tau\left[\frac{t}{\tau}\right] + \tau s, \left[\frac{x}{\varepsilon}\right]_Y + y, \left[\frac{t}{\tau}\right] + s\right)\right]\chi_{\Lambda_\tau} \\
&= \frac{\partial}{\partial s}\left[\phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y, \tau\left[\frac{t}{\tau}\right] + \tau s, y, s\right)\right]\chi_{\Lambda_\tau} \\
&= \tau\frac{\partial\phi}{\partial t}\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y, \tau\left[\frac{t}{\tau}\right] + \tau s, y, s\right)\chi_{\Lambda_\tau} \\
&\quad + \frac{\partial\phi}{\partial s}\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y, \tau\left[\frac{t}{\tau}\right] + \tau s, y, s\right)\chi_{\Lambda_\tau} \\
&= \tau\mathscr{T}_\tau\Big(\frac{\partial\phi}{\partial t}\Big) + \mathscr{T}_\tau\Big(\frac{\partial\phi}{\partial s}\Big).
\end{aligned}$$

Equation (2.13) can be proved similarly. □

Proposition 2.7. *For ϕ measurable on Q, extended by Q-periodicity to the whole of $\boldsymbol{R}^N \times \boldsymbol{R}$, define the sequence*

$$\phi^\tau(x,t) = \phi\Big(\frac{x}{\varepsilon}, \frac{t}{\tau}\Big), \quad (x,t) \in \boldsymbol{R}^N \times \boldsymbol{R}.$$

Then

$$\mathscr{T}_\tau(\phi^\tau)(x,y,t,s) = \begin{cases} \phi(y,s), & (x,y,t,s) \in \Lambda_\tau, \\ 0, & \text{otherwise}. \end{cases} \tag{2.14}$$

Moreover, if $\phi \in L^p(Q)$ as $\varepsilon, \tau \to 0$

$$\mathscr{T}_\tau(\phi^\tau) \to \phi, \quad \text{strongly in } L^p(\Omega_T \times Q). \tag{2.15}$$

If there exist $\nabla_y\phi, \frac{\partial\phi}{\partial s} \in L^p(Q)$ then

$$\nabla_y(\mathscr{T}_\tau(\phi^\tau)) \to \nabla_y\phi, \quad \text{strongly in } L^p(\Omega_T \times Q), \tag{2.16}$$

$$\frac{\partial}{\partial s}(\mathscr{T}_\tau(\phi^\tau)) \to \frac{\partial}{\partial s}\phi, \quad \text{strongly in } L^p(\Omega_T \times Q). \tag{2.17}$$

PROPOSITION 2.8 (Convergences). *Let $\{w_\tau\}$ be a sequence of functions in $L^p(\Omega_T)$.*

If $w_\tau \to w$ strongly in $L^p(\Omega_T)$ as $\varepsilon, \tau \to 0$, then

$$\mathscr{T}_\tau(w_\tau) \to w, \quad \textit{strongly in } L^p(\Omega_T \times Q). \tag{2.18}$$

If we only assume that (2.18) *holds true and that $w_\tau \geq C > 0$, then we have*

$$\mathscr{T}_\tau(w_\tau^{-1}) \to w^{-1}, \quad \textit{strongly in } L^p(\Omega_T \times Q). \tag{2.19}$$

If w_τ is a bounded sequence of functions in $L^p(\Omega_T)$, $p > 1$, then up to a subsequence

$$\mathscr{T}_\tau(w_\tau) \rightharpoonup \hat{w}, \quad \textit{weakly in } L^p(\Omega_T \times Q), \tag{2.20}$$

and

$$w_\tau \rightharpoonup \mathscr{M}_Q(\hat{w}), \quad \textit{weakly in } L^p(\Omega_T). \tag{2.21}$$

REMARK 2.9. We apply (2.19) to the case $w_\tau = \phi^\tau$, ϕ^τ as in (2.11). Actually the only classes for which (2.18) is known to hold in this context, are sums of the following cases: $\phi = f_1(x,t) f_2(y,s)$, $\phi \in L^p(Y \times \Sigma; C(\Omega_T))$, $\phi \in L^p(\Omega_T; C(Y \times \Sigma))$. In all such cases $\mathscr{T}_\tau(\phi^\tau) \to \phi$ strongly in $L^p(\Omega_T \times Q)$ (see [1, 9, 10]). □

The following result may give a fairly precise picture of the compactness of unfolded sequences of functions.

PROPOSITION 2.10. *Let $w \in L^p(\Omega_T)$. Assume that if $h \in \boldsymbol{R}^N$, $z \in \boldsymbol{R}$, $E \subset \Omega_T$ with $|h| + |z| + |E| \leq \delta$ then*

$$\int_{\Omega_T} |w(x+h, t+z) - w(x,t)|^p \,\mathrm{d}x\,\mathrm{d}t + \int_E |w(x,t)|^p \,\mathrm{d}x\,\mathrm{d}t \leq \omega(\delta), \tag{2.22}$$

where $\omega : [0, +\infty) \to [0, +\infty)$ is an increasing function with $\omega(0) = 0$. In (2.22) *w is extended to* 0 *out of Ω_T. Then if $|h_1| + |h_2| + |z_1| + |z_2| \leq \delta$,*

$$\begin{aligned} \int_{\boldsymbol{R}^{2N+2}} &|\mathscr{T}_\tau(w)(x+h_1, t+z_1, y+h_2, s+z_2) - \mathscr{T}_\tau(u_\tau)(x,t,y,s)| \\ &\leq \gamma\omega(\gamma(\delta + \varepsilon + \tau)). \end{aligned} \tag{2.23}$$

PROOF. We give the details of the proof for translations in the space variables; the general case is similar. Let us denote here for all $v : \boldsymbol{R}^{2N} \to \boldsymbol{R}$, $h \in \boldsymbol{R}^N$, $\delta > 0$

$$\begin{gathered} v^h(x,y) = v(x+h, y), \quad v_h(x,y) = v(x, y+h); \\ \Omega_T(\delta) = \{(x,t) \in \Omega_T \mid \mathrm{dist}((x,t), \partial\Omega_T) < \delta\}; \\ \Lambda_\tau - h = \{(x,t) \in \boldsymbol{R}^{N+1} \mid (x+h, t) \in \Lambda_\tau\}. \end{gathered}$$

Then we compute

$$\begin{aligned}(2.24)\qquad &\int_{\mathbf{R}^{2N+2}} |\mathscr{T}_\tau(w)^h - \mathscr{T}_\tau(w)|^p \\ &= \int_{(\Lambda_\tau - h)\setminus\Lambda_\tau} \int_Q |\mathscr{T}_\tau(w)^h|^p + \int_{\Lambda_\tau\setminus(\Lambda_\tau - h)} \int_Q |\mathscr{T}_\tau(w)|^p \\ &+ \int_{(\Lambda_\tau - h)\cap\Lambda_\tau} \int_Q \left| w\left(\varepsilon\left[\frac{x+h}{\varepsilon}\right]_Y + \varepsilon y, \tau\left[\frac{t}{\tau}\right] + \tau s\right)\right. \\ &\left. - w\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y, \tau\left[\frac{t}{\tau}\right] + \tau s\right)\right|^p.\end{aligned}$$

Notice that for a suitable $\gamma = \gamma(N)$

$$Q_\tau(x,t) \cap [\Lambda_\tau\setminus(\Lambda_\tau - h)] \neq \emptyset \quad \text{implies} \quad Q_\tau(x,t) \subset \Omega_T(\gamma(|h| + \varepsilon));$$

then by means of a standard change of variable (see [10, Proposition 2.5]) the second integral on the right hand side of (2.24) is bounded by

$$\int_{\Omega_T(\gamma(|h|+\varepsilon))} |w|^p \,\mathrm{d}x\,\mathrm{d}t \leq \omega(\gamma(|h| + \varepsilon)).$$

The first integral there can be majorized in the same way.

As to the last integral in (2.24) we recall that for any two real numbers r_1, r_2 we have

$$[r_1 + r_2] = [r_1] + [r_2] + j, \quad j \in \{0, 1\}.$$

Thus

$$\varepsilon\left[\frac{x+h}{\varepsilon}\right]_Y + \varepsilon y = \varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon\left[\frac{h}{\varepsilon}\right]_Y + \varepsilon\xi + \varepsilon y,$$

for a $\xi = \xi(\varepsilon, x, h) \in \{0,1\}^N$. Then we have in any case

$$\begin{aligned}(2.25)\qquad &\left| w\left(\varepsilon\left[\frac{x+h}{\varepsilon}\right]_Y + \varepsilon y, \tau\left[\frac{t}{\tau}\right] + \tau s\right) - w\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y, \tau\left[\frac{t}{\tau}\right] + \tau s\right)\right|^p \\ &\leq \sum_{i=1}^{2^N} \left| w\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + k_i + \varepsilon y, \tau\left[\frac{t}{\tau}\right] + \tau s\right)\right. \\ &\left. - w\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y, \tau\left[\frac{t}{\tau}\right] + \tau s\right)\right|^p,\end{aligned}$$

where denoting $\{\xi_i\} = \{0,1\}^N$ we have set

$$k_i = \varepsilon\left[\frac{h}{\varepsilon}\right]_Y + \varepsilon\xi_i, \quad |k_i| \le \gamma(N)(|h| + \varepsilon).$$

On the right hand side of (2.25) w is defined as 0 if its arguments is outside of Ω_T. With this convention, the integral of each summand on the right hand side of (2.25) can be bounded above by

$$\int_{\Lambda_\tau} |w(x + k_i, t) - w(x,t)|^p \,\mathrm{d}x\,\mathrm{d}t \le \omega(|k_i|) \le \omega(\gamma(|h| + \varepsilon)). \tag{2.26}$$

Next we consider translations in the microscopic space variables; we have

$$\begin{aligned}\int_{\boldsymbol{R}^{2N+2}} |\mathscr{T}_\tau(w)_h - \mathscr{T}_\tau(w)|^p = &\int_{\Lambda_\tau\times[Q\setminus Q_h]} \left| w\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y, \tau\left[\frac{t}{\tau}\right] + \tau s\right)\right|^p \\ &+ \int_{\Lambda_\tau\times Q_h} \left| w\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y + \varepsilon h, \tau\left[\frac{t}{\tau}\right] + \tau s\right)\right. \\ &\left. - w\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y, \tau\left[\frac{t}{\tau}\right] + \tau s\right)\right|^p,\end{aligned} \tag{2.27}$$

where

$$Q_h = Y_h \times \Sigma, \quad Y_h = \{y \in Y \mid y + h \in Y\}.$$

The last integral in (2.27) can be bounded essentially as in (2.26). The first integral on the right hand side in (2.27) equals after a change of variable

$$\int_{\Lambda_\tau^h} |w(x,t)|^p \,\mathrm{d}x\,\mathrm{d}t, \quad \text{where } \Lambda_\tau^h = \left\{(x,t) \in \Lambda_\tau \;\middle|\; \left\{\frac{x}{\varepsilon}\right\}_Y \in Y\setminus Y_h\right\}.$$

We conclude by observing that (the sum below is extended to all cells contained in Λ_τ)

$$|\Lambda_\tau^h| \le \sum_i |Q_\tau(x_i, t_i) \cap \Lambda_\tau^h| \le \gamma \sum_i \varepsilon^N \tau |Q\setminus Q_h| \le \gamma |h|. \qquad \square$$

Notice that as a consequence of Definition 2.2 and of Lemma 2.6, if $w \in W^{1,p}(\Omega_T)$

$$\nabla_y \mathscr{Z}_\tau(w) = \nabla_y \tilde{\mathscr{Z}}_\tau(w) = \varepsilon \mathscr{T}_\tau(\nabla_x w), \tag{2.28}$$

$$\frac{\partial}{\partial s}\mathscr{Z}_\tau(w) = \tau \mathscr{T}_\tau(w_t). \tag{2.29}$$

THEOREM 2.11. *Let* $\{w_\tau\}$ *be a sequence converging strongly to* w *in* $L^p(0,T; W^{1,p}(\Omega))$, *as* $\varepsilon, \tau \to 0$, *then*

$$\mathscr{T}_\tau(\nabla w_\tau) \to \nabla w, \qquad \text{strongly in } L^p(\Omega_T \times Q), \tag{2.30}$$

$$\frac{1}{\varepsilon}\tilde{\mathscr{Z}}_\tau(w_\tau) \to y^c \cdot \nabla w, \quad \text{strongly in } L^p(\Omega_T \times \Sigma; W^{1,p}(Y)), \tag{2.31}$$

where

$$y^c = \Big(y_1 - \frac{1}{2}, y_2 - \frac{1}{2}, \dots, y_N - \frac{1}{2}\Big).$$

Let $p > 1$ *and let* $\{w_\tau\}$ *be a sequence converging weakly to* w *in* $L^p(0,T, W^{1,p}(\Omega))$. *Then, up to a subsequence, there exists* $\tilde{w} = \tilde{w}(x,y,t,s) \in L^p(\Omega_T \times \Sigma; W^{1,p}_{per}(Y))$, $\mathscr{M}_Y(\tilde{w}) = 0$, *such that as* $\varepsilon, \tau \to 0$

$$\mathscr{T}_\tau(\nabla w_\tau) \rightharpoonup \nabla w + \nabla_y \tilde{w}, \qquad \text{weakly in } L^p(\Omega_T \times Q), \tag{2.32}$$

$$\frac{1}{\varepsilon}\tilde{\mathscr{Z}}_\tau(w_\tau) \rightharpoonup y^c \cdot \nabla w + \tilde{w}, \quad \text{weakly in } L^p(\Omega_T \times \Sigma; W^{1,p}(Y)). \tag{2.33}$$

PROOF. The limit in (2.30) follows from the strong convergence of ∇w_τ and Proposition 2.8. To prove (2.31) we note that, applying the Poincaré–Wirtinger inequality in Y to the function $\frac{1}{\varepsilon}\tilde{\mathscr{Z}}_\tau(w_\tau) - y^c \cdot \nabla w$ and (2.30), we get

$$\begin{aligned} &\Big\| \frac{1}{\varepsilon}\tilde{\mathscr{Z}}_\tau(w_\tau) - y^c \cdot \nabla w \Big\|_{L^p(\Omega_T \times Q)} \\ &\quad \le \gamma \Big\| \nabla_y\Big(\frac{1}{\varepsilon}\tilde{\mathscr{Z}}_\tau(w_\tau)\Big) - \nabla w \Big\|_{L^p(\Omega_T \times Q)} \to 0. \end{aligned} \tag{2.34}$$

Now we turn to the proof of (2.32) and (2.33). Since $\nabla_y\big(\frac{1}{\varepsilon}\tilde{\mathscr{Z}}_\tau(w_\tau)\big) = \mathscr{T}_\tau(\nabla w_\tau)$, the limit relation (2.33) implies (2.32).

Then noting that $\nabla_y\big(\frac{1}{\varepsilon}\tilde{\mathscr{Z}}_\tau(w_\tau)\big)$ is bounded in $L^p(\Omega_T \times Q)$ we have

$$\Big\| \frac{1}{\varepsilon}\tilde{\mathscr{Z}}_\tau(w_\tau) - y^c \cdot \nabla w \Big\|_{L^p(\Omega_T \times Q)} \le \gamma \Big\| \nabla_y\Big(\frac{1}{\varepsilon}\tilde{\mathscr{Z}}_\tau(w_\tau)\Big) - \nabla w \Big\|_{L^p(\Omega_T \times Q)} \le K, \tag{2.35}$$

where K is a positive constant independent of ε and τ. Then there exists $\tilde{w}(x,y,t,s) \in L^p(\Omega_T \times \Sigma; W^{1,p}(Y))$ such that, up to a subsequence

$$\frac{1}{\varepsilon}\tilde{\mathscr{Z}}_\tau(w_\tau) - y^c \cdot \nabla w \rightharpoonup \tilde{w}, \quad \text{weakly in } L^p(\Omega_T \times \Sigma; W^{1,p}(Y)). \tag{2.36}$$

It is easy to show that $\mathscr{M}_Y\big(\frac{1}{\varepsilon}\tilde{\mathscr{Z}}_\tau(w_\tau) - y^c \cdot \nabla w\big) = 0$, so that $\mathscr{M}_Y(\tilde{w}) = 0$. The Y periodicity of $\tilde{w}$ can be proven following the lines of the proof in Theorem 3.5 of [10]. □

Next we deal with results connected with scalings specific to parabolic problems. For example the following proposition should be compared with Proposition 3.1 of [10], where a different scaling appears.

PROPOSITION 2.12. *Let $p > 1$ and let $\{w_\tau\}$ be a sequence converging weakly in $L^p(0,T;W^{1,p}(\Omega))$ to w, and also satisfying the estimate*

$$\left\| \tau^m \frac{\partial w_\tau}{\partial t} \right\|_{L^p(\Omega_T)} \leq \gamma, \tag{2.37}$$

with $0 \leq m < 1$. Then

$$\mathscr{T}_\tau(w_\tau) \rightharpoonup w, \quad \text{weakly in } L^p(\Omega_T; W^{1,p}(Q)). \tag{2.38}$$

PROOF. Using (2.9), (2.12) and (2.13) owing to the stated weak convergence of $\{w_\tau\}$ we have the estimates

$$\|\mathscr{T}_\tau(w_\tau)\|_{L^p(\Omega_T\times Q)} \leq \|w_\tau\|_{L^p(\Omega_T)} \leq \gamma, \tag{2.39}$$

$$\|\nabla_y \mathscr{T}_\tau(w_\tau)\|_{L^p(\Omega_T\times Q)} \leq \varepsilon \|\nabla w_\tau\|_{L^p(\Omega_T)} \leq \gamma\varepsilon, \tag{2.40}$$

$$\left\| \frac{\partial}{\partial s} \mathscr{T}_\tau(w_\tau) \right\|_{L^p(\Omega_T\times Q)} \leq \tau^{1-m} \left\| \tau^m \frac{\partial w_\tau}{\partial t} \right\|_{L^p(\Omega_T)} \leq \gamma\tau^{1-m}, \tag{2.41}$$

so that there exist a subsequence and $\hat{w} \in L^p(\Omega_T; W^{1,p}(Q))$ such that

$$\mathscr{T}_\tau(w_\tau) \rightharpoonup \hat{w}, \quad \text{weakly in } L^p(\Omega; W^{1,p}(Q)), \tag{2.42}$$

$$\nabla_y \mathscr{T}_\tau(w_\tau) \to 0, \quad \text{strongly in } L^p(\Omega_T \times Q), \tag{2.43}$$

$$\frac{\partial}{\partial s} \mathscr{T}_\tau(w_\tau) \to 0, \quad \text{strongly in } L^p(\Omega_T \times Q), \tag{2.44}$$

and $\nabla_y \hat{w} = \frac{\partial \hat{w}}{\partial s} = 0$, so that $\hat{w}$ does not depend on y and s. Then from (2.21) we have

$$w(x,t) = \mathscr{M}_Q(\hat{w})(x,t) = \hat{w}(x,t). \qquad \square$$

Next we prove the following

LEMMA 2.13. *If* (2.37) *is in force with $p > 1$, $0 \leq m \leq 1$, then*

$$\int_{\Omega_T} \int_Q |\mathscr{M}_\tau(w)(x,t) - \tilde{\mathscr{M}}_\tau(w)(x,t,s)|^p \,\mathrm{d}x \,\mathrm{d}t \,\mathrm{d}y \,\mathrm{d}s \leq \gamma\tau^{p(1-m)}. \tag{2.45}$$

PROOF. For $(x,t) \in \Lambda_\tau$ and recalling that $\bar{t} = \tau\left(\left[\frac{t}{\tau}\right] + s\right)$, we have on applying twice Hölder inequality

$$\begin{aligned}(2.46)\qquad &\mathcal{M}_\tau(w)(x,t) - \tilde{\mathcal{M}}_\tau(w)(x,t,s)\\ &= \frac{1}{\varepsilon^N \tau}\int_{Q_\tau(x,t)} [w(\zeta,\theta) - w(\zeta,\bar{t})]\,\mathrm{d}\zeta\,\mathrm{d}\theta\\ &\le \frac{\tau^{1-\frac{1}{p}}}{\varepsilon^N\tau}\left(\int_{Q_\tau(x,t)}\int_{\bar{t}}^{\theta}\left|\frac{\partial w}{\partial\lambda}(\zeta,\lambda)\right|^p \mathrm{d}\lambda\,\mathrm{d}\zeta\,\mathrm{d}\theta\right)^{\frac{1}{p}}(\varepsilon^N\tau)^{1-\frac{1}{p}}\\ &\le \frac{\tau^{1-\frac{1}{p}}}{\varepsilon^{\frac{N}{p}}}\left(\int_{Q_\tau(x,t)}\left|\frac{\partial w}{\partial\lambda}(\zeta,\lambda)\right|^p \mathrm{d}\lambda\,\mathrm{d}\zeta\right)^{\frac{1}{p}}.\end{aligned}$$

Then after integrating over $\Omega_T \times Q$ and changing variables

$$z = \frac{\zeta - \varepsilon\left[\frac{x}{\varepsilon}\right]_Y}{\varepsilon}, \quad \sigma = \frac{\lambda - \tau\left[\frac{t}{\tau}\right]}{\tau}$$

we find

$$\begin{aligned}(2.47)\qquad &\int_{\Omega_T}\int_Q |\mathcal{M}_\tau(w)(x,t) - \tilde{\mathcal{M}}_\tau(w)(x,t,s)|^p\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\\ &\le \int_{\Omega_T}\int_Q \tau^p \int_Q\left|\frac{\partial w}{\partial\lambda}\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + z\varepsilon, \tau\left[\frac{t}{\tau}\right] + \sigma\tau\right)\right|^p \mathrm{d}\sigma\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\\ &= \tau^p\int_{\Omega_T}\int_Q \mathcal{T}_\tau\left(\left|\frac{\partial w}{\partial t}\right|^p\right)(x,z,t,\sigma)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}\sigma \le \gamma\tau^{p(1-m)},\end{aligned}$$

where we have made use of (2.9) and of (2.37). □

2.4. *Fast oscillations in time*

We assume here that

$$(2.48)\qquad \frac{\tau^{1-m}}{\varepsilon} \le C \in (0,+\infty),$$

where $0 \le m < 1$. Actually there are different subcases which are treated in the following results.

PROPOSITION 2.14. *Let $\{w_\tau\}$ be a sequence converging strongly to w in $L^p(0,T; W^{1,p}(\Omega))$ and satisfying the estimate* (2.37) *with* $0 \le m < 1$. *If*

$$(2.49)\qquad \lim_{\varepsilon,\tau\to 0}\frac{\tau^{1-m}}{\varepsilon} = 0,$$

then

$$\frac{1}{\varepsilon}\mathscr{Z}_\tau(w_\tau) \to y^c \cdot \nabla w \quad \text{strongly in } L^p(\Omega_T; W^{1,p}(Q)). \tag{2.50}$$

PROOF. To prove (2.50) we first note that

$$\nabla_y\Big(\frac{1}{\varepsilon}\mathscr{Z}_\tau(w_\tau)\Big) = \nabla_y\Big(\frac{1}{\varepsilon}\mathscr{T}_\tau(w_\tau)\Big) = \mathscr{T}_\tau(\nabla_x w_\tau) \to \nabla_x w, \tag{2.51}$$

in $L^p(\Omega_T \times Q)$ where we used property (2.13) and (2.18) applied to $\nabla_x w$. From (2.9), (2.12) and assumptions (2.37), (2.49) we obtain

$$\begin{aligned}\left\|\frac{\partial}{\partial s}\Big(\frac{1}{\varepsilon}\mathscr{Z}_\tau(w_\tau)\Big)\right\|_{L^p(\Omega_T\times Q)} &= \left\|\frac{\partial}{\partial s}\Big(\frac{1}{\varepsilon}\mathscr{T}_\tau(w_\tau)\Big)\right\|_{L^p(\Omega_T\times Q)} \\ &= \frac{\tau}{\varepsilon}\left\|\mathscr{T}_\tau\Big(\frac{\partial w_\tau}{\partial t}\Big)\right\|_{L^p(\Omega_T\times Q)} \\ &\le \frac{\tau}{\varepsilon}\left\|\frac{\partial w_\tau}{\partial t}\right\|_{L^p(\Omega_T)} \le \gamma\frac{\tau^{1-m}}{\varepsilon} \to 0,\end{aligned} \tag{2.52}$$

as $\varepsilon, \tau \to 0$. Having disposed of the convergence of the derivatives, we turn to the sequence itself. We may apply the Poincaré–Wirtinger inequality in Q to the function $\mathscr{Z}_\tau(w_\tau)/\varepsilon - y^c \cdot \nabla w$, since its mean value in Q vanishes. We obtain that

$$\begin{aligned}&\left\|\frac{1}{\varepsilon}\mathscr{Z}_\tau(w_\tau) - y^c\cdot\nabla w\right\|_{L^p(\Omega_T\times Q)} \\ &\quad\le \gamma\left\|\nabla_y\Big(\frac{1}{\varepsilon}\mathscr{Z}_\tau(w_\tau)\Big) - \nabla w\right\|_{L^p(\Omega_T\times Q)} + \gamma\left\|\frac{\partial}{\partial s}\Big(\frac{1}{\varepsilon}\mathscr{Z}_\tau(w_\tau)\Big)\right\|_{L^p(\Omega_T\times Q)}\end{aligned} \tag{2.53}$$

goes to 0 as $\varepsilon, \tau \to 0$ as a consequence of (2.51) and (2.52). □

PROPOSITION 2.15. *Let* $\{w_\tau\}$ *be a sequence converging strongly to* w *in* $W^{1,p}(\Omega_T)$.

If $\frac{\tau}{\varepsilon} \to \ell$ *as* $\varepsilon, \tau \to 0$, *then*

$$\mathscr{T}_\tau\Big(\frac{\partial w_\tau}{\partial t}\Big) \to \frac{\partial w}{\partial t}, \qquad \text{strongly in } L^p(\Omega_T \times Q), \tag{2.54}$$

$$\frac{1}{\varepsilon}\mathscr{Z}_\tau(w_\tau) \to y^c\cdot\nabla w + \ell\Big(s - \frac{1}{2}\Big)\frac{\partial w}{\partial t} \quad \text{strongly in } L^p(\Omega_T; W^{1,p}(Q)). \tag{2.55}$$

PROOF. Convergence (2.54) follows from the assumed strong convergence and from Proposition 2.8. Equation (2.55) can be proven reasoning as in the proof of Proposition 2.14. □

THEOREM 2.16. *Let $p > 1$ and let $\{w_\tau\}$ be a sequence converging weakly to w in $L^p(0,T;W^{1,p}(\Omega))$ and satisfying the estimate (2.37) with $0 \le m < 1$. If (2.49) is in force, then up to a subsequence there exists $\hat{w} \in L^p(\Omega_T; W^{1,p}_{per}(Q))$ such that as $\varepsilon, \tau \to 0$*

$$\mathcal{T}_\tau(\nabla w_\tau) \rightharpoonup \nabla w + \nabla_y \hat{w}, \qquad \textit{weakly in } L^p(\Omega_T \times Q), \tag{2.56}$$

$$\frac{1}{\varepsilon}\mathcal{Z}_\tau(w_\tau) \rightharpoonup y^c \cdot \nabla w + \hat{w}, \quad \textit{weakly in } L^p(\Omega_T; W^{1,p}(Q)). \tag{2.57}$$

Actually $\mathcal{M}_Q(\hat{w}) = 0$ and

$$\frac{\partial \hat{w}}{\partial s} = 0, \tag{2.58}$$

so that $\hat{w} = \hat{w}(x,t,y)$.

PROOF. In order to prove (2.57) we appeal to Poincaré–Wirtinger inequality as in the proof of Proposition 2.14. Indeed we have

$$\begin{aligned}\left\| \frac{1}{\varepsilon}\mathcal{Z}_\tau(w_\tau) - y^c \cdot \nabla w \right\|_{L^p(\Omega_T\times Q)} &\le \gamma \left\| \nabla_y\left(\frac{1}{\varepsilon}\mathcal{Z}_\tau(w_\tau)\right) - \nabla w \right\|_{L^p(\Omega_T\times Q)} \\ &\quad + \gamma \left\| \frac{\partial}{\partial s}\left(\frac{1}{\varepsilon}\mathcal{Z}_\tau(w_\tau)\right) \right\|_{L^p(\Omega_T\times Q)}.\end{aligned} \tag{2.59}$$

Recalling (2.28) and the stated weak convergence, the first term on the right hand side of (2.59) is uniformly bounded on ε, τ. When we recall also (2.29), we have

$$\left\| \frac{\partial}{\partial s}\left(\frac{1}{\varepsilon}\mathcal{Z}_\tau(w_\tau)\right) \right\|_{L^p(\Omega_T\times Q)} = \left\| \frac{\tau}{\varepsilon}\mathcal{T}_\tau\left(\frac{\partial w_\tau}{\partial t}\right) \right\|_{L^p(\Omega_T\times Q)} \le \gamma \frac{\tau^{1-m}}{\varepsilon}. \tag{2.60}$$

Then the whole right hand side of (2.59) is uniformly bounded on ε, τ and there exists $\hat{w} \in L^p(\Omega_T; W^{1,p}(Q))$ such that, up to a subsequence

$$\frac{1}{\varepsilon}\mathcal{Z}_\tau(w_\tau) - y^c \cdot \nabla w \rightharpoonup \hat{w}, \quad \text{weakly in } L^p(\Omega_T; W^{1,p}(Q)), \tag{2.61}$$

that is (2.57).

Since by construction $\mathcal{M}_Q\big(\frac{1}{\varepsilon}\mathcal{Z}_\tau(w_\tau) - y^c \cdot \nabla w\big) = 0$, then $\mathcal{M}_Q(\hat{w}) = 0$.

Taking into account (2.28) again we see that the limit relation (2.57) implies (2.56).

On invoking (2.49), we see that (2.60) and (2.57) imply (2.58).

It remains to prove the Y-periodicity of $\hat{w}$, which can be done following [10]. □

REMARK 2.17. Under the assumptions of Theorem 2.16, we can of course apply also Theorem 2.11. The two functions $\hat{w}$ and $\tilde{w}$ so determined however coincide, since we may apply Lemma 2.13 in

$$\frac{1}{\varepsilon}\tilde{\mathscr{Z}}_\tau(w_\tau) - \frac{1}{\varepsilon}\mathscr{Z}_\tau(w_\tau) = \frac{\tau^{1-m}}{\varepsilon}\frac{[\mathscr{M}_\tau(w_\tau) - \tilde{\mathscr{M}}_\tau(w_\tau)]}{\tau^{1-m}}. \qquad \square$$

THEOREM 2.18. *Let $p > 1$ and let $\{w_\tau\}$ be a sequence converging weakly to w in $L^p(0, T; W^{1,p}(\Omega))$ and satisfying the estimate* (2.37) *with* $0 < m < 1$. *If* (2.48) *is in force, then up to a subsequence, there exists $\hat{w} \in L^p(\Omega_T; W^{1,p}_{per}(Q))$ such that as $\varepsilon, \tau \to 0$* (2.56) *and* (2.57) *hold true and $\mathscr{M}_Q(\hat{w}) = 0$. Moreover*

$$\frac{\tau}{\varepsilon}\mathscr{T}_\tau\Big(\frac{\partial w_\tau}{\partial t}\Big) \to \frac{\partial \hat{w}}{\partial s}, \quad \textit{weakly in } L^p(\Omega_T \times Q). \tag{2.62}$$

PROOF. The proof stays essentially unchanged from the one of Theorem 2.16. Indeed the only difference is that the rightmost hand side in (2.60) does not tend to 0. Actually this was used only to prove (2.58) which is not relevant here.

However, by the same token, we have to provide an argument to prove the Σ-periodicity of $\hat{w}$. We introduce a test function $\psi \in \mathscr{C}^\infty_c(\Omega_T \times Y)$, and compute

$$\begin{aligned}
&\int_{\Omega_T}\int_Y \frac{1}{\varepsilon}[\mathscr{Z}_\tau(x, y, t, 1) - \mathscr{Z}_\tau(x, y, t, 0)]\psi(x, y, t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&= \frac{1}{\varepsilon}\int_{\Omega_T}\int_Y \Big[w_\tau\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big]_Y + \varepsilon y, \tau\Big[\frac{t}{\tau}\Big] + \tau\Big) \\
&\quad - w_\tau\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big]_Y + \varepsilon y, \tau\Big[\frac{t}{\tau}\Big]\Big)\Big]\psi(x, y, t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&= \frac{1}{\varepsilon}\int_{\Omega_T}\int_Y w_\tau\Big(\varepsilon\Big[\frac{x}{\varepsilon}\Big]_Y + \varepsilon y, \tau\Big[\frac{t}{\tau}\Big]\Big)[\psi(x, y, t-\tau) - \psi(x, y, t)]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&= \frac{1}{\varepsilon}\int_{\Omega_T}\int_Y \mathscr{T}_\tau(w_\tau)(x, y, t, 0)[\psi(x, y, t-\tau) - \psi(x, y, t)]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&= \frac{\tau}{\varepsilon}\int_{\Omega_T}\int_Y \mathscr{T}_\tau(w_\tau)(x, y, t, 0)\frac{\psi(x, y, t-\tau) - \psi(x, y, t)}{\tau}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.
\end{aligned} \tag{2.63}$$

Next we observe the trace inequality

$$\begin{aligned}
&\|\mathscr{T}_\tau(w_\tau)(\cdot, \cdot, \cdot, 0)\|_{L^p(\Omega_T \times Y)} \\
&\quad \le \gamma\|\mathscr{T}_\tau(w_\tau)\|_{L^p(\Omega_T \times Q)} + \gamma\Big\|\frac{\partial}{\partial s}\mathscr{T}_\tau(w_\tau)\Big\|_{L^p(\Omega_T \times Q)} \\
&\quad \le \gamma\|w_\tau\|_{L^p(\Omega_T)} + \gamma\tau^{1-m}\Big\|\tau^m\frac{\partial w_\tau}{\partial t}\Big\|_{L^p(\Omega_T)} \le K.
\end{aligned} \tag{2.64}$$

Then using Hölder inequality we get

$$\left| \int_{\Omega_T} \int_Y \frac{1}{\varepsilon} [\mathscr{Z}_\tau(x, y, t, 1) - \mathscr{Z}_\tau(x, y, t, 0)] \psi(x, y, t) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t \right| \leq K \frac{\tau}{\varepsilon} \to 0, \tag{2.65}$$

as $\varepsilon, \tau \to 0$. Combining this with (2.57), considering that $y^c \cdot \nabla w$ does not depend on s, and that $\frac{\partial \hat{w}}{\partial s} \in L^p(\Omega_T \times Q)$, we get

$$\int_{\Omega_T} \int_Y [\hat{w}(x, y, t, 1) - \hat{w}(x, y, t, 0)] \psi(x, y, t) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t = 0, \tag{2.66}$$

implying that $\hat{w}$ is Σ-periodic. □

Remark 2.19. The convergence in (2.62) allows us to avoid the unloading of the time derivative onto the test function in the homogenization process, see (4.17), (4.18). In turn this avoids the appearance of non-local terms as in [12], see Remark 4.3 below. □

In the case $m = 0$ Theorem 2.18 is replaced with the following stronger formulation.

Theorem 2.20. *Let $p > 1$ and let $\{w_\tau\}$ be a sequence converging weakly to w in $W^{1,p}(\Omega_T)$, and satisfying estimate* (2.37) *with $m = 0$. If*

$$\lim_{\varepsilon, \tau \to 0} \frac{\tau}{\varepsilon} = \ell \in (0, +\infty), \tag{2.67}$$

then there exists $\mathring{w} \in W^{1,p}(\Omega_T \times Q)$ such that

$$\mathscr{T}_\tau(\nabla w_\tau) \rightharpoonup \nabla w + \nabla_y \mathring{w}, \qquad \text{weakly in } L^p(\Omega_T \times Q), \tag{2.68}$$

$$\mathscr{T}_\tau\left(\frac{\partial w_\tau}{\partial t}\right) \rightharpoonup w_t + \frac{\mathring{w}_s}{\ell}, \qquad \text{weakly in } L^p(\Omega_T \times Q), \tag{2.69}$$

$$\frac{1}{\varepsilon} \mathscr{Z}_\tau(w_\tau) \rightharpoonup y^c \cdot \nabla w + \ell\left(s - \frac{1}{2}\right) w_t + \mathring{w}, \qquad \text{weakly in } L^p(\Omega_T; W^{1,p}(Q)). \tag{2.70}$$

Here $\mathring{w}$ is periodic in Q and is such that $\mathscr{M}_Q(\mathring{w}) = 0$.

The proof of Theorem 2.20 can be easily given along the lines of the proof of Theorem 2.16. The periodicity of $\mathring{w}$ follows reasoning as in Theorem 3.5 in [10], since in this case the time derivative is controlled as the space gradient.

Remark 2.21. In fact even under the assumptions of Theorem 2.20, Theorem 2.18 is valid excepting the periodicity of $\hat{w}$. In fact in Theorem 2.20 $\mathring{w} = \hat{w} - \ell(s - 1/2) w_t$. □

2.5. *Slow oscillations in time*

We assume here that

$$\lim_{\varepsilon,\tau\to 0}\frac{\varepsilon}{\tau^{1-m}}=0, \tag{2.71}$$

for a given $0 \le m < 1$.

PROPOSITION 2.22. *Let* $\{w_\tau\}$ *be a bounded sequence in* $L^p(0,T;W^{1,p}(\Omega))$, *satisfying* (2.37) *and* (2.71). *Then*

$$\left(\frac{\varepsilon}{\tau^r}\right)^{1+\alpha}\frac{1}{\varepsilon}\mathscr{Z}_\tau(w_\tau)\to 0,\quad \textit{strongly in } L^p(\Omega_T;W^{1,p}(Q)), \tag{2.72}$$

for all $\alpha > 0$, $0 < r \le 1-m$. *We can take* $\alpha = 0$ *if* $r < 1-m$.

PROOF. Since $\mathscr{Z}_\tau(w_\tau)$ has zero average in Q, we may apply Poincaré–Wirtinger inequality to it, obtaining

$$\begin{aligned}\left\|\frac{1}{\varepsilon}\mathscr{Z}_\tau(w_\tau)\right\|_{L^p(\Omega_T\times Q)} \\ \le \gamma\left\|\nabla_y\left(\frac{1}{\varepsilon}\mathscr{Z}_\tau(w_\tau)\right)\right\|_{L^p(\Omega_T\times Q)}+\gamma\left\|\frac{\partial}{\partial s}\left(\frac{1}{\varepsilon}\mathscr{Z}_\tau(w_\tau)\right)\right\|_{L^p(\Omega_T\times Q)},\end{aligned} \tag{2.73}$$

which yields, when invoking the properties of $\mathscr{T}_\tau$ and (2.37),

$$\begin{aligned}\left\|\frac{1}{\varepsilon}\mathscr{Z}_\tau(w_\tau)\right\|_{L^p(\Omega_T\times Q)} \\ \le \gamma\|\mathscr{T}_\tau(\nabla w_\tau)\|_{L^p(\Omega_T\times Q)}+\gamma\frac{\tau}{\varepsilon}\left\|\mathscr{T}_\tau\left(\frac{\partial w_\tau}{\partial t}\right)\right\|_{L^p(\Omega_T\times Q)}\le K\left(1+\frac{\tau^{1-m}}{\varepsilon}\right).\end{aligned} \tag{2.74}$$

We multiply (2.74) by $(\varepsilon/\tau^r)^{1+\alpha}$, r, α as in the statement, and infer the sought after limiting relation. □

REMARK 2.23. We single out for future reference (proof of Theorem 5.2) the following immediate consequences of the results in this Section: if $w \in W^{1,p}(\Omega_T)$ and $\varepsilon/\tau^r \to 0$, then for any given $\alpha > 0$, $0 < r \le 1$ as ε, $\tau \to 0$

$$\mathscr{T}_\tau\left(\frac{\partial w}{\partial t}\right)\to\frac{\partial w}{\partial t},\qquad \text{strongly in } L^p(\Omega_T\times Q), \tag{2.75}$$

$$\mathscr{T}_\tau(\nabla w)\to\nabla w,\qquad \text{strongly in } L^p(\Omega_T\times Q), \tag{2.76}$$

$$\frac{1}{\varepsilon}\tilde{\mathscr{Z}}_\tau(w)\to y^c\cdot\nabla w,\quad \text{strongly in } L^p(\Omega_T\times\Sigma;W^{1,p}(Y)), \tag{2.77}$$

$$\left(\frac{\varepsilon}{\tau^r}\right)^{1+\alpha}\frac{1}{\varepsilon}\mathscr{Z}_\tau(w)\to 0,\qquad \text{strongly in } L^p(\Omega_T;W^{1,p}(Q)). \tag{2.78}$$

□

3. A parabolic homogenization problem

3.1. Assumptions

Let $a : \Omega_T \times Q \to \boldsymbol{R}$, $a_1 : \Omega_T \times Y \to \boldsymbol{R}$, $a_2 : \Omega_T \times \Sigma \to \boldsymbol{R}$ be measurable functions. We assume that they satisfy the uniform estimates

$$0 < C^{-1} \le a, a_1, a_2 \le C < \infty, \tag{3.1}$$

for some $C > 1$. Let then

$$A^\tau(x,t) = \mathscr{A}^\tau\Big(x, t, \frac{x}{\varepsilon}, \frac{t}{\tau}\Big), \quad \mathscr{A}^\tau = \mathscr{A}^\tau(x,t,y,s),$$

be a sequence of $N \times N$ matrices such that for all $\tau > 0$

$$\|\mathscr{A}^\tau_{ij}\|_\infty \le C, \quad i, j = 1, \dots, N; \quad \mathscr{A}^\tau \xi \cdot \xi \ge C^{-1}|\xi|^2, \quad \xi \in \boldsymbol{R}^N. \tag{3.2}$$

We also assume that a_1 [respectively a_2, $\mathscr{A}^\tau$] are Lipschitz continuos in t [respectively in x, (t,s)] and that

$$\left|\frac{\partial a_1}{\partial t}\right|, \left|\frac{\partial a_2}{\partial x_i}\right|, \left|\frac{\partial \mathscr{A}^\tau}{\partial t}\right|, \left|\frac{\partial \mathscr{A}^\tau}{\partial s}\right| \le C, \tag{3.3}$$

for all $i = 1, \dots, N$ and for all relevant arguments. We denote

$$a^\tau(x,t) = a\Big(x, t, \frac{x}{\varepsilon}, \frac{t}{\tau}\Big), \quad a_1^\tau(x,t) = a_1\Big(x, t, \frac{x}{\varepsilon}\Big), \quad a_2^\tau(x,t) = a_2\Big(x, t, \frac{t}{\tau}\Big).$$

We always assume that A^τ, a^τ, a_1^τ, a_2^τ are measurable in Ω_T. This is known to be the case for functions in the classes of Remark 2.9.

Let $f \in L^2(\Omega_T)$ be the source term in the diffusion equation (see (3.5)). In fact all our results in this paper are still valid if we more generally allow f to depend on the unknown, i.e., if we let $f : \Omega_T \times \boldsymbol{R} \to \boldsymbol{R}$ be measurable and

$$|f(x,t,u)| \le g(x,t) + C|u|, \quad (x,t,u) \in \Omega_T \times \boldsymbol{R}, \tag{3.4}$$

where $g \in L^2(\Omega_T)$. Here we must assume f to be Lipschitz continuous in the variable u uniformly with respect to $(x,t) \in \Omega_T$. Essentially this greater generality is possible owing to the strong convergence result of Corollary 3.4. We have chosen to present the proofs in the slightly simpler case $f = f(x,t)$ in order to achieve a more compact presentation.

3.2. Estimates

Consider the parabolic problem

$$a_1^\tau a_2^\tau \frac{\partial u_\tau}{\partial t} - \operatorname{div}(A^\tau \nabla u_\tau) = f, \qquad (x,t) \in \Omega_T, \tag{3.5}$$

$$u_\tau(x,t) = 0, \qquad (x,t) \in \partial\Omega \times [0,T], \tag{3.6}$$

$$u_\tau(x,0) = u_0^\tau(x), \quad x \in \Omega. \tag{3.7}$$

Here for a given initial data $u_0 \in L^2(\Omega)$, we let $\{u_0^\tau\}$ be a sequence in $W_0^{1,2}(\Omega)$ such that $u_0^\tau \to u_0$ strongly in $L^2(\Omega)$, and $\|u_0^\tau\|_{W^{1,2}(\Omega)} \le C/\sqrt{\tau}$, where C depends on $\|u_0\|_{L^2(\Omega)}$.

In the following propositions we assume all the needed smoothness of the solution u_τ, whose existence is classical under standard regularity assumptions (see [16]). This can be done by means of an approximation procedure of the data and coefficients in the equation.

Proposition 3.1. *Let u_τ be the solution to problem* (3.5)–(3.7). *We have the standard energy estimate*

$$\max_{0 \le t \le T} \int_\Omega u_\tau^2 \,\mathrm{d}x + \int_0^T \int_\Omega |\nabla u_\tau|^2 \,\mathrm{d}x \,\mathrm{d}t \le \gamma, \tag{3.8}$$

where γ is a constant independent of τ.

Proof. Choose u_τ / a_2^τ as a test function in (3.5) and integrate by parts in Ω_T, to get for all $\bar{t} \in (0,T)$

$$\frac{1}{2}\int_\Omega a_1^\tau u_\tau^2(\bar{t}) - \frac{1}{2}\int_\Omega \int_0^{\bar{t}} (a_1^\tau)_t u_\tau^2 + \int_\Omega \int_0^{\bar{t}} \frac{A^\tau}{a_2^\tau} \nabla u_\tau \cdot \nabla u_\tau - \int_\Omega \int_0^{\bar{t}} u_\tau \frac{A^\tau}{|a_2^\tau|^2} \nabla u_\tau \cdot \nabla a_2^\tau$$

$$= \int_\Omega \int_0^{\bar{t}} f \frac{u_\tau}{a_2^\tau} + \frac{1}{2}\int_\Omega a_1^\tau (u_0^\tau)^2.$$

Thus by means of Cauchy–Schwarz inequality and of our structural assumptions we infer

$$\int_\Omega |u_\tau|^2(\bar{t}) + \int_\Omega \int_0^{\bar{t}} |\nabla u_\tau|^2 \le \gamma \left[\int_\Omega \int_0^{\bar{t}} |u_\tau|^2 + \int_\Omega \int_0^{\bar{t}} |f|^2 + \int_\Omega (u_0^\tau)^2 \right]. \tag{3.9}$$

Next Gronwall's inequality yields

$$\int_\Omega |u_\tau|^2(\bar{t}) \le \gamma, \quad 0 < \bar{t} < T.$$

Finally on letting $\bar{t} \to T$ in (3.9) we obtain (3.8). □

PROPOSITION 3.2. *For all* $\tau > 0$

$$\tau \int_0^T \int_\Omega \left(\frac{\partial u_\tau}{\partial t}\right)^2 \mathrm{d}x\,\mathrm{d}t + \tau \max_{0 \le t \le T} \int_\Omega |\nabla u_\tau|^2(t)\,\mathrm{d}x \le \gamma, \tag{3.10}$$

where γ *is a constant independent of* τ.

PROOF. We select $\frac{\partial u_\tau}{\partial t}$ as a testing function in (3.5) so that on integrating by parts in the space variables we get

$$\int_0^T \int_\Omega a_1^\tau a_2^\tau \left(\frac{\partial u_\tau}{\partial t}\right)^2 + \int_0^T \int_\Omega A^\tau \nabla u_\tau \cdot \nabla \frac{\partial u_\tau}{\partial t} = \int_0^T \int_\Omega f \frac{\partial u_\tau}{\partial t}. \tag{3.11}$$

Hence

$$\begin{aligned} &\int_0^T \int_\Omega a_1^\tau a_2^\tau \left(\frac{\partial u_\tau}{\partial t}\right)^2 + \frac{1}{2}\int_0^T \int_\Omega \frac{\partial}{\partial t}(A^\tau \nabla u_\tau \cdot \nabla u_\tau) \\ &\qquad - \frac{1}{2}\int_0^T \int_\Omega \frac{\partial \mathscr{A}^\tau}{\partial t} \nabla u_\tau \cdot \nabla u_\tau - \frac{1}{2}\int_0^T \int_\Omega \frac{\partial \mathscr{A}^\tau}{\partial s} \frac{1}{\tau} \nabla u_\tau \cdot \nabla u_\tau \\ &\quad = \int_0^T \int_\Omega f \frac{\partial u_\tau}{\partial t}. \end{aligned} \tag{3.12}$$

The second integral on the left hand side of (3.12) can be evaluated exactly. After an application of Cauchy–Schwarz inequality we are led to

$$\begin{aligned} &\int_0^T \int_\Omega \left(\frac{\partial u_\tau}{\partial t}\right)^2 + \int_\Omega |\nabla u_\tau|^2(T) \\ &\qquad \le \frac{\gamma}{\tau} \int_0^T \int_\Omega |\nabla u_\tau|^2 + \gamma \int_0^T \int_\Omega f^2 + \gamma \int_\Omega |\nabla u_0^\tau|^2, \end{aligned} \tag{3.13}$$

whence (3.10) follows by taking (3.8) and our assumption $\|\nabla u_0^\tau\|_{L^2(\Omega)} \le C/\sqrt{\tau}$ into account, and making the obvious remark that T in the proof can be replaced with any $t \in (0, T)$. □

PROPOSITION 3.3 (Time compactness). *If* $0 < \sigma < T/2$ *there exists* $\gamma = \gamma(\sigma) > 0$ *such that for any* $0 < h < \sigma/2$ *we have*

$$\begin{aligned} &\int_\sigma^{T-\sigma} \int_\Omega |u_\tau(x, t+h) - u_\tau(x, t)|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad \le \gamma(1 + \|u_\tau\|^2_{L^2(\Omega_T)} + \|\nabla u_\tau\|^2_{L^2(\Omega_T)})\sqrt{h}. \end{aligned} \tag{3.14}$$

PROOF. Let $\sigma \in (0, T/2)$, $0 < h < \sigma/2$, and define

$$\varphi_h(x, t) = -\zeta(t) \int_t^{t+h} u_\tau(x, s)\,\mathrm{d}s,$$

where $\zeta \in C_0^1(\sigma/2, T - \sigma/2)$ is a nonnegative function such that $\zeta = 1$ in $(\sigma, T - \sigma)$ and $|\zeta'| \leq \gamma/\sigma$. In this proof for any $v = v(x, t)$ we denote $\tilde{v}(x, t) = v(x, t + h)$.

Testing equation (3.5) written at times t and respectively $t + h$ with φ_h / a_2^τ and respectively $\varphi_h / \widetilde{a_2^\tau}$ we get on subtracting the two integral formulations

$$
\begin{aligned}
(3.15)\qquad & -\int_0^T \int_\Omega [\widetilde{a_1^\tau}\widetilde{u_\tau} - a_1^\tau u_\tau] \frac{\partial \varphi_h}{\partial t} - \int_0^T \int_\Omega [\widetilde{a_{1t}^\tau}\widetilde{u_\tau} - a_{1t}^\tau u_\tau] \varphi_h \\
& + \int_0^T \int_\Omega \widetilde{A^\tau} \nabla \widetilde{u_\tau} \cdot \nabla \Big(\frac{\varphi_h}{\widetilde{a_2^\tau}} \Big) - \int_0^T \int_\Omega A^\tau \nabla u_\tau \cdot \nabla \Big(\frac{\varphi_h}{a_2^\tau} \Big) \\
& = \int_0^T \int_\Omega \left[\frac{\tilde{f}}{\widetilde{a_2^\tau}} - \frac{f}{a_2^\tau} \right] \varphi_h.
\end{aligned}
$$

The first integral on the left hand side of (3.15) equals

$$
\begin{aligned}
(3.16)\qquad & \int_0^T \int_\Omega [\widetilde{a_1^\tau}\widetilde{u_\tau} - a_1^\tau u_\tau] \Big\{ \zeta' \int_t^{t+h} u_\tau(x, s)\, \mathrm{d}s + \zeta[\widetilde{u_\tau} - u_\tau] \Big\} \\
& = \int_0^T \int_\Omega [\widetilde{u_\tau} - u_\tau]^2 \zeta a_1^\tau + \int_0^T \int_\Omega [\widetilde{a_1^\tau}\widetilde{u_\tau} - a_1^\tau u_\tau] \zeta' \int_t^{t+h} u_\tau(x, s)\, \mathrm{d}s \\
& + \int_0^T \int_\Omega \widetilde{u_\tau}[\widetilde{u_\tau} - u_\tau] \zeta [\widetilde{a_1^\tau} - a_1^\tau].
\end{aligned}
$$

Clearly the first term on the right hand side of (3.16) essentially is the one estimated in the statement. The second integral there can be majorized by means of Hölder inequality by

$$
\begin{aligned}
(3.17)\qquad & \gamma \|\zeta'\|_\infty \Big(\int_\Omega \int_{\sigma/2}^{T-\sigma/2} |\widetilde{u_\tau}|^2 + |u_\tau|^2 \Big)^{\frac{1}{2}} \left(\int_\Omega \int_{\sigma/2}^{T-\sigma/2} \left| \int_t^{t+h} u_\tau(x, s)\, \mathrm{d}s \right|^2 \right)^{\frac{1}{2}} \\
& \leq \gamma \|u_\tau\|_{L^2(\Omega_T)}^2 \sqrt{h}.
\end{aligned}
$$

The third term on the right hand side of (3.16) is estimated, invoking the time regularity of a_1, by

$$
(3.18)\qquad \gamma \int_\Omega \int_{\sigma/2}^{T-\sigma/2} |\widetilde{u_\tau}|(|\widetilde{u_\tau}| + |u_\tau|)|\widetilde{a_1^\tau} - a_1| \leq \gamma \|u_\tau\|_{L^2(\Omega_T)}^2 h.
$$

We turn to estimating the other terms in (3.15). The second integral there can be treated as in (3.17). The third and fourth integrals in (3.15) can be bounded in the same way, that is

$$\left|\int_0^T \int_\Omega \widetilde{A^\tau} \nabla \widetilde{u_\tau} \cdot \nabla\left(\frac{\varphi_h}{\widetilde{a_2^\tau}}\right)\right| \tag{3.19}$$
$$\leq \gamma \int_\Omega \int_{\sigma/2}^{T-\sigma/2} \left|\frac{\nabla \widetilde{u_\tau}}{\widetilde{a_2^\tau}}\right| \left|\int_t^{t+h} \nabla u_\tau(x,s)\,\mathrm{d}s\right|$$
$$+ \gamma \int_\Omega \int_{\sigma/2}^{T-\sigma/2} \frac{|\nabla \widetilde{u_\tau}|\,|\nabla \widetilde{a_2^\tau}|}{\widetilde{a_2^\tau}^2} \left|\int_t^{t+h} u_\tau(x,s)\,\mathrm{d}s\right|$$
$$\leq \gamma \|\nabla u_\tau\|_{L^2(\Omega_T)} \left(\int_\Omega \int_{\sigma/2}^{T-\sigma/2} \left|\int_t^{t+h} \nabla u_\tau(x,s)\,\mathrm{d}s\right|^2 + \left|\int_t^{t+h} u_\tau(x,s)\,\mathrm{d}s\right|^2\right)^{\frac{1}{2}}$$
$$\leq \gamma \|\nabla u_\tau\|_{L^2(\Omega_T)} (\|\nabla u_\tau\|_{L^2(\Omega_T)} + \|u_\tau\|_{L^2(\Omega_T)}) \sqrt{h}.$$

Finally the integral on the right hand side of (3.15) is dealt with by a similar and even simpler argument, contributing a quantity

$$\gamma \|f\|_{L^2(\Omega_T)} \|u_\tau\|_{L^2(\Omega_T)} \sqrt{h}. \tag{3.20}$$

Collecting estimates (3.15)–(3.20) we prove at once (3.14). □

COROLLARY 3.4. *By extracting a subsequence if needed we may assume*

$$u_\tau \to u, \qquad \textit{strongly in } L^2(\Omega_T) \textit{ and weakly in } C(0,T;L^2(\Omega)); \tag{3.21}$$
$$\nabla u_\tau \to \nabla u, \quad \textit{weakly in } L^2(\Omega_T). \tag{3.22}$$

PROOF. Both claims follow straightforwardly from Propositions 3.1 and 3.3, and from classical results. □

3.3. A special case

Let us look also at the following problem

$$a^\tau(x,t)\frac{\partial u_\tau}{\partial t} - \operatorname{div}\left(A^\tau\left(x,\frac{x}{\varepsilon}\right)\nabla u_\tau\right) = f, \qquad (x,t) \in \Omega_T, \tag{3.23}$$
$$u_\tau(x,t) = 0, \qquad (x,t) \in \partial\Omega \times [0,T], \tag{3.24}$$
$$u_\tau(x,0) = u_0(x), \quad x \in \Omega. \tag{3.25}$$

Here we assume $u_0 \in W_0^{1,2}(\Omega)$.

PROPOSITION 3.5. *If* $u_0 \in W_0^{1,2}(\Omega)$ *then the solution to* (3.23)–(3.25) *satisfies*

$$\int_0^T \int_\Omega \left(\frac{\partial u_\tau}{\partial t}\right)^2 \mathrm{d}x\,\mathrm{d}t + \max_{0\le t\le T} \int_\Omega (u_\tau(x,t)^2 + |\nabla u_\tau(x,t)|^2)\,\mathrm{d}x \le \gamma, \tag{3.26}$$

where γ *is independent of* τ.

PROOF. We multiply (3.23) against $\frac{\partial u_\tau}{\partial t}$ and integrate by parts in the space variables, reasoning as in the proof of Proposition 3.2. In the simpler case at hand we immediately get

$$\begin{aligned}\int_0^T \int_\Omega a^\tau \left(\frac{\partial u_\tau}{\partial t}\right)^2 + \frac{1}{2}\int_\Omega A^\tau \nabla u_\tau(T)\cdot\nabla u_\tau(T) \\ \le \frac{1}{2}\int_\Omega A^\tau \nabla u_0 \cdot \nabla u_0 + \int_0^T \int_\Omega f\frac{\partial u_\tau}{\partial t}.\end{aligned} \tag{3.27}$$

The estimates of $\frac{\partial u_\tau}{\partial t}$ and ∇u_τ follow upon an application of Cauchy–Schwarz inequality. Finally the estimate of u_τ is simply a consequence of the standard formula

$$u_\tau(x,t) = u_0(x) + \int_0^t \frac{\partial u_\tau}{\partial z}(x,z)\,\mathrm{d}z. \qquad \square$$

REMARK 3.6. Clearly Corollary 3.4 is still in force under the assumptions of Proposition 3.5. $\square$

4. THE LIMIT PROBLEM IN THE CASE OF FAST OSCILLATIONS IN TIME

We look here at the case

$$\tau \le \gamma\varepsilon^2, \tag{4.1}$$

and assume throughout that there exist bounded functions $B : \Omega_T \times Q \to \boldsymbol{R}^{N^2}$, $b_1 : \Omega_T \times Y \to \boldsymbol{R}$ and $b_2 : \Omega_T \times \Sigma \to \boldsymbol{R}$ such that

$$\mathscr{T}_\tau(A^\tau) \to B, \quad \text{strongly in } L^1(\Omega_T \times Q), \tag{4.2}$$

$$\mathscr{T}_\tau(a_1^\tau) \to b_1, \quad \text{strongly in } L^1(\Omega_T \times Y), \tag{4.3}$$

$$\mathscr{T}_\tau(a_2^\tau) \to b_2, \quad \text{strongly in } L^1(\Omega_T \times \Sigma). \tag{4.4}$$

We also need assume

$$\mathscr{T}_\tau(a_{1t}^\tau) \rightharpoonup b_{1t}, \quad \text{weakly in } L^2(\Omega_T \times Y), \tag{4.5}$$

$$\mathscr{T}_\tau(\nabla a_2^\tau) \to \nabla b_2, \quad \text{strongly in } L^1(\Omega_T \times \Sigma), \tag{4.6}$$

and the convergence at time $t = 0$

$$\mathscr{T}_\varepsilon(a_1^\tau(0)) \rightharpoonup b_1(0), \quad \text{weakly in } L^2(\Omega \times Y). \tag{4.7}$$

About the known cases of convergences of the type above see Remark 2.9.

Proposition 4.1. *Let* (4.1) *be in force and let* u_τ *be the solution of problem* (3.5)–(3.7). *Then there exist* $u \in L^2(0, T, W^{1,2}(\Omega))$ *and* $\hat{u} \in L^2(\Omega_T;\ W^{1,2}_{per}(Q))$ *such that* $\mathscr{M}_Q(\hat{u}) = 0$ *and up to a subsequence*

$$u_\tau \rightharpoonup u, \qquad \text{weakly in } L^2(0, T; W^{1,2}(\Omega)), \tag{4.8}$$

$$\mathscr{T}_\tau(u_\tau) \rightharpoonup u, \qquad \text{weakly in } L^2(\Omega_T; W^{1,2}(Q)), \tag{4.9}$$

$$\mathscr{T}_\tau(\nabla u_\tau) \rightharpoonup \nabla u + \nabla_y \hat{u}, \quad \text{weakly in } L^2(\Omega_T \times Q), \tag{4.10}$$

$$\frac{\tau}{\varepsilon}\mathscr{T}_\tau\left(\frac{\partial u_\tau}{\partial t}\right) \rightharpoonup \frac{\partial \hat{u}}{\partial s}, \qquad \text{weakly in } L^2(\Omega_T \times Q). \tag{4.11}$$

The convergence $u_\tau \to u$ *is in fact strong in* $L^2(\Omega_T)$, *so that from Proposition* 2.8 *it follows* $\mathscr{T}_\tau(u_\tau) \to u$ *strongly in* $L^2(\Omega_T \times Q)$.

Proof. The claim readily follows from Proposition 2.12, Theorem 2.18 and Corollary 3.4, by taking into account the estimates proved in Section 3. □

Theorem 4.2. *Let* (4.1)–(4.7) *be in force, and assume*

$$\lim_{\varepsilon, \tau \to 0} \frac{\tau}{\varepsilon^2} = \ell \in (0, \infty). \tag{4.12}$$

Then the pair $(u, \hat{u})$ *as in Proposition* 4.1 *is the unique solution* (*in the class specified in the Proposition*) *of*

$$\begin{aligned} \int_{\Omega_T} \int_Q \Big\{ &-u(b_1\phi)_t + \ell^{-1} b_1 \hat{u}_s \Psi + B[\nabla_x u \\ &+ \nabla_y \hat{u}]\left[\nabla_x\left(\frac{\phi}{b_2}\right) + \frac{1}{b_2}\nabla_y \Psi\right]\Big\} \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s \\ &= \int_{\Omega_T} \int_\Sigma \frac{f}{b_2}\phi \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s + \int_\Omega \int_Y u_0(x)\phi(x,0)b_1(x,0,y)\,\mathrm{d}x\,\mathrm{d}y, \end{aligned} \tag{4.13}$$

for all $\phi \in W^{1,2}(\Omega_T)$ *with* $\phi(x, t) = 0$ *on* $\partial\Omega \times [0, T]$ *and* $\phi(x, T) = 0$, *and* $\Psi \in L^2(\Omega_T \times \Sigma; W^{1,2}_{per}(Y))$.

Proof. First we prove the macroscopic part of (4.13), i.e., the equality itself with $\Psi = 0$. To this end we do not need (4.12).

We use ϕ/a_2^τ as a test function for equation (3.5), where $\phi \in \mathscr{C}^\infty(\Omega_T)$ with $\phi = 0$ on $\partial\Omega \times [0, T]$ and $\phi(x, T) = 0$. Integrating by parts in Ω_T we get

$$(4.14)\qquad -\int_{\Omega_T} u_\tau[a_1^\tau\phi_t + a_{1t}^\tau\phi]\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} A^\tau\nabla u_\tau \cdot \left[\frac{\nabla\phi}{a_2^\tau} - \phi\frac{\nabla a_2^\tau}{(a_2^\tau)^2}\right]\mathrm{d}x\,\mathrm{d}t$$
$$= \int_{\Omega_T} f\frac{\phi}{a_2^\tau}\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega u_0^\tau(x)\phi(x,0)a_1^\tau(x,0)\,\mathrm{d}x.$$

Unfolding the equation above we get

$$(4.15)\qquad -\int_{\Omega_T}\int_Q \{\mathscr{T}_\tau(a_1^\tau)\mathscr{T}_\tau(u_\tau)\mathscr{T}_\tau(\phi_t) + \mathscr{T}_\tau(a_{1t}^\tau)\mathscr{T}_\tau(u_\tau)\mathscr{T}_\tau(\phi)\}$$
$$+\int_{\Omega_T}\int_Q \mathscr{T}_\tau(A^\tau)\mathscr{T}_\tau(\nabla u_\tau)\mathscr{T}_\tau(\nabla\phi)\mathscr{T}_\tau\Big(\frac{1}{a_2^\tau}\Big)$$
$$-\int_{\Omega_T}\int_Q \mathscr{T}_\tau(A^\tau)\mathscr{T}_\tau(\nabla u_\tau)\mathscr{T}_\tau(\phi)\mathscr{T}_\tau(\nabla a_2^\tau)\Big(\mathscr{T}_\tau\Big(\frac{1}{a_2^\tau}\Big)\Big)^2$$
$$= \int_{\Omega_T}\int_Q \mathscr{T}_\tau(f)\mathscr{T}_\tau(\phi)\mathscr{T}_\tau\Big(\frac{1}{a_2^\tau}\Big)$$
$$+\int_\Omega\int_Y \mathscr{T}_\tau(u_0^\tau)\mathscr{T}_\tau(\phi(0))\mathscr{T}_\tau(a_1^\tau(0))\,\mathrm{d}x\,\mathrm{d}y + R^\tau,$$

where $R^\tau = o(1)$, as $\varepsilon, \tau \to 0$. Then taking the limit $\varepsilon, \tau \to 0$ and recalling Proposition 4.1 as well as (2.18), (2.19), (4.2)–(4.7), we get

$$(4.16)\qquad \int_{\Omega_T}\int_Q \Big\{-b_1u\phi_t - b_{1t}u\phi + B(\nabla_x u + \nabla_y\hat{u})\Big(\frac{\nabla\phi}{b_2} - \phi\frac{\nabla b_2}{b_2^2}\Big)\Big\}\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s$$
$$= \int_{\Omega_T}\int_\Sigma \frac{f}{b_2}\phi\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s + \int_\Omega\int_Y u_0\phi(x,0)b_1(x,0,y)\,\mathrm{d}x\,\mathrm{d}y,$$

amounting to the differential equation in the macroscopic variables.

Next we turn to the proof of the equation in the microscopic quantities, where we first appeal to (4.12). We use a test function

$$\Phi = \frac{\tau}{\varepsilon}\phi(x,t)\psi\Big(\frac{x}{\varepsilon},\frac{t}{\tau}\Big),$$

where ϕ is the same as above and $\psi \in W_{per}^{1,2}(Q)$ is extended periodically both in y and s to the whole $\boldsymbol{R}^{N+1}$. So testing (3.5) with Φ and then integrating by parts gives

$$(4.17)\qquad \int_{\Omega_T}\Big\{\frac{\tau}{\varepsilon}a_1^\tau a_2^\tau\frac{\partial u_\tau}{\partial t}\phi\psi + \frac{\tau}{\varepsilon}A^\tau\nabla u_\tau\cdot(\nabla\phi)\psi + \frac{\tau}{\varepsilon^2}A^\tau\nabla u_\tau\cdot(\nabla_y\psi)\phi\Big\}\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_{\Omega_T}\frac{\tau}{\varepsilon}f\phi\psi\,\mathrm{d}x\,\mathrm{d}t.$$

Then unfolding we get

$$\int_{\Omega_T}\int_Q \frac{\tau}{\varepsilon}\mathscr{T}_\tau(a_1^\tau)\mathscr{T}_\tau(a_2^\tau)\mathscr{T}_\tau\Big(\frac{\partial u_\tau}{\partial t}\Big)\mathscr{T}_\tau(\phi)\mathscr{T}_\tau(\psi)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s \tag{4.18}$$
$$+\int_{\Omega_T}\int_Q \frac{\tau}{\varepsilon^2}\mathscr{T}_\tau(A^\tau)\mathscr{T}_\tau(\nabla u_\tau)\mathscr{T}_\tau(\phi)\mathscr{T}_\tau(\nabla_y\psi)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s$$
$$= -\int_{\Omega_T}\int_Q \frac{\tau}{\varepsilon}\mathscr{T}_\tau(A^\tau)\mathscr{T}_\tau(\nabla u_\tau)\mathscr{T}_\tau(\nabla\phi)\mathscr{T}_\tau(\psi)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s$$
$$+\int_{\Omega_T}\frac{\tau}{\varepsilon}f\phi\psi\,\mathrm{d}x\,\mathrm{d}t + R^\tau.$$

In the limit as $\varepsilon,\tau\to 0$ the right hand side vanishes. The left hand side, by virtue of Proposition 4.1, converges to

$$\int_{\Omega_T}\int_Q\{\hat{u}_s b_1 b_2\phi\psi + \ell B(\nabla u+\nabla_y\hat{u})\phi\nabla_y\psi\}\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s. \tag{4.19}$$

By the density of the tensor product $\mathscr{C}^\infty(\Omega_T)\otimes W^{1,2}_{per}(Q)$ in $L^2(\Omega_T\times\Sigma; W^{1,2}_{per}(Y))$ these results hold for every $\Psi\in L^2(\Omega_T\times\Sigma; W^{1,2}_{per}(Y))$. If we select for the sake of formal symmetry a test function Φ/a_2^τ, minor modifications to the argument above yield (4.13).

The uniqueness is proved in a rather standard fashion, exploiting the linearity of the problem. If $(u_i,\hat{u}_i)$, $i=1,2$, are two solutions of the problem, we essentially take as testing functions $\phi=u_1-u_2$, $\Psi=\hat{u}_1-\hat{u}_2$, after unloading the time derivative on u_i again, via a Steklov averaging procedure. Here we use also Gronwall's theorem and thus, if f is allowed to depend on u, its Lipschitz continuity in u. □

Remark 4.3. The microscopic part of our homogenized equation (4.13) does not contain any non-local term, as found instead in [12, formula (3.5)]. Indeed [12, Remark 3.4] stated that such a term is zero if, in our notation, $\frac{\partial u_\tau}{\partial t}$ is uniformly bounded in $L^2(\Omega_T)$. Therefore, here we have shown that the non-local term vanishes also under the weaker estimate (2.37), $m=1/2$. In our approach this bound follows from the regularity in (3.3). However the regularity of the matrix $\mathscr{A}^\tau$ is used only to estimate $\frac{\partial u_\tau}{\partial t}$ and is irrelevant in the homogenization process.

Finally, we remark that the function u_1 in the notation of [12] can be written as

$$u_1(x,t,y,s)=\hat{u}(x,y,t,s)-\mathscr{M}_Y(\hat{u})(x,t,s),$$

for $\hat{u}$ as in Theorem 4.2. □

THEOREM 4.4. *Let* (4.1)–(4.4) *be in force, and assume*

$$\lim_{\varepsilon,\tau\to 0}\frac{\tau}{\varepsilon^2}=0. \tag{4.20}$$

Let the pair $(u,\hat{u})$ *be as in Proposition* 4.1. *Then* $\hat{u}=\hat{u}(x,t,y)$, *i.e.*, $\hat{u}_s=0$, *and* $(u,\hat{u})$ *is the solution of*

$$\int_{\Omega_T}\int_Q\left\{-u(b_1\phi)_t+B[\nabla_x u+\nabla_y\hat{u}]\left[\nabla_x\left(\frac{\phi}{b_2}\right)+\frac{1}{b_2}\nabla_y\Psi\right]\right\}\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s$$
$$=\int_{\Omega_T}\int_\Sigma\frac{f}{b_2}\phi\,\mathrm{d}x\,\mathrm{d}t+\int_\Omega\int_Y u_0(x)\phi(x,0)b_1(x,0,y)\,\mathrm{d}x\,\mathrm{d}y, \tag{4.21}$$

for all $\phi\in W^{1,2}(\Omega_T)$ *with* $\phi(x,t)=0$ *on* $\partial\Omega\times[0,T]$ *and* $\phi(x,T)=0$, *and* $\Psi\in L^2(\Omega_T;W^{1,2}_{per}(Y))$.

PROOF. The proof of the macroscopic differential equation is the same as in Theorem 4.2.

Concerning the microscopic equation, we remark that when (4.20) is in force, then $\hat{u}$ does not depend on s (see Theorem 2.16). Then we test the equation (3.5) with a function

$$\Phi=\varepsilon\frac{\phi(x,t)}{a_2^\tau(x,t)}\psi\left(\frac{x}{\varepsilon}\right),$$

with $\phi\in\mathscr{C}^\infty(\Omega_T)$, $\phi(x,T)=0$ and $\psi\in W^{1,2}_{per}(Y)$ extended periodically to the whole $\boldsymbol{R}^N$, obtaining

$$\int_{\Omega_T}A^\tau\nabla u_\tau(\nabla_y\psi)\frac{\phi}{a_2^\tau}$$
$$=\varepsilon\int_{\Omega_T}\left\{a_1^\tau u_\tau\phi_t\psi+a_{1t}^\tau u_\tau\phi\psi-A^\tau\nabla u_\tau\cdot\left(\frac{\nabla\phi}{a_2^\tau}-\frac{\phi\nabla a_2^\tau}{(a_2^\tau)^2}\right)\psi+f\phi\frac{\psi}{a_2}\right\}\mathrm{d}x\,\mathrm{d}t$$
$$+\varepsilon\int_\Omega u_0^\tau(x)a_1^\tau(x,0)\phi(x,0)\psi\left(\frac{x}{\varepsilon}\right)\mathrm{d}x. \tag{4.22}$$

Clearly the right hand side of (4.22) vanishes as $\varepsilon,\tau\to 0$. When we unfold the left hand side we obtain in the limit

$$\int_{\Omega_T}\int_Q\mathscr{T}_\tau(A^\tau)\mathscr{T}_\tau(\nabla u_\tau)\mathscr{T}_\tau(\phi)\mathscr{T}_\tau(\nabla_y\psi)\mathscr{T}_\tau\left(\frac{1}{a_2^\tau}\right)$$
$$\to\int_{\Omega_T}\int_Q\frac{B}{b_2}(\nabla u+\nabla_y\hat{u})\phi\nabla_y\psi=0. \tag{4.23}$$

Owing to the density of the tensor product $\mathscr{C}^\infty(\Omega_T)\otimes W^{1,2}_{per}(Y)$ in $L^2(\Omega_T; W^{1,2}_{per}(Y))$, from (4.23) we obtain (4.21) for every $\Psi\in L^2(\Omega_T; W^{1,2}_{per}(Y))$, concluding the proof.

The uniqueness of solutions follows as in Theorem 4.2. □

5. The limit problem in the case of slow oscillations in time

We consider here the case

$$\lim_{\varepsilon,\tau\to 0}\frac{\varepsilon^2}{\tau}=0. \tag{5.1}$$

Proposition 5.1. *Let* (5.1) *be in force and let u_τ be the solution of problem* (3.5)–(3.7). *Then there exist $u\in L^2(0,T,W^{1,2}(\Omega))$ and $\tilde u\in L^2(\Omega_T\times\Sigma; W^{1,2}_{per}(Y))$ such that $\mathscr{M}_Y(\tilde u)=0$ and up to a subsequence*

$$u_\tau \rightharpoonup u, \qquad \textit{weakly in } L^2(0,T;W^{1,2}(\Omega)), \tag{5.2}$$

$$\mathscr{T}_\tau(u_\tau)\rightharpoonup u, \qquad \textit{weakly in } L^2(\Omega_T;W^{1,2}(Q)), \tag{5.3}$$

$$\mathscr{T}_\tau(\nabla u_\tau)\rightharpoonup \nabla u+\nabla_y\tilde u, \qquad \textit{weakly in } L^2(\Omega_T\times Q). \tag{5.4}$$

The convergence $u_\tau\to u$ is in fact strong in $L^2(\Omega_T)$, so that from Proposition 2.8 *follows $\mathscr{T}_\tau(u_\tau)\to u$ strongly in $L^2(\Omega_T\times Q)$.*

Proof. The claim follows at once from Theorem 2.11, Proposition 2.12 and Corollary 3.4, on invoking the estimates of Section 3. □

Theorem 5.2. *Let* (4.2)–(4.7) *be in force, and also assume* (5.1). *Then the pair $(u,\tilde u)$ as in Proposition* 5.1 *is the unique solution of the problem*

$$\int_{\Omega_T}\int_Q\left\{-u(b_1\phi)_t+B[\nabla_x u+\nabla_y\tilde u]\left[\nabla_x\left(\frac{\phi}{b_2}\right)+\frac{1}{b_2}\nabla_y\Psi\right]\right\}\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s$$
$$=\int_{\Omega_T}\int_\Sigma\frac{f}{b_2}\phi\,\mathrm{d}x\,\mathrm{d}t+\int_\Omega\int_Y u_0(x)\phi(x,0)b_1(x,0,y)\,\mathrm{d}x\,\mathrm{d}y, \tag{5.5}$$

for all $\phi\in W^{1,2}(\Omega_T)$ with $\phi=0$ on $\partial\Omega\times[0,T]$ and $\phi(x,T)=0$, and $\Psi\in L^2(\Omega_T\times\Sigma; W^{1,2}_{per}(Y))$.

Proof. The macroscopic differential equation (4.16) can be proved as in Theorem 4.2.

Next we introduce a test function

$$\Phi=\varepsilon\phi(x,t)\psi\left(\frac{x}{\varepsilon},\frac{t}{\tau}\right)a_2^\tau(x,t)^{-1},$$

where $\phi \in \mathscr{C}^\infty(\Omega_T)$ with $\phi = 0$ on $\partial\Omega \times [0, T]$ and $\phi(x, T) = 0$, and $\psi \in W^{1,2}_{per}(Q)$, with $\psi(y, 0) = 0$, $\psi(y, 1) = 0$. We understand ψ to be extended periodically both in y and s to the whole $\boldsymbol{R}^{N+1}$. Then testing (3.5) with Φ and integrating by parts we get

$$\int_{\Omega_T} \left\{ -\frac{\varepsilon}{\tau} a_1^\tau u_\tau \phi \psi_s + A^\tau \nabla u_\tau \cdot (\nabla_y \psi) \frac{\phi}{a_2^\tau} \right\} \mathrm{d}x\, \mathrm{d}t = \varepsilon \int_{\Omega_T} \left\{ u_\tau (a_1^\tau \phi)_t \psi - A^\tau \nabla u_\tau \cdot \left(\nabla \frac{\phi}{a_2^\tau} \right) \psi + \frac{f \phi \psi}{a_2^\tau} \right\} \mathrm{d}x\, \mathrm{d}t. \tag{5.6}$$

The right hand side of (5.6) goes to zero as $\varepsilon, \tau \to 0$. Unfolding the left hand side we see that it equals

$$\begin{aligned} &\int_{\Omega_T} \int_Q \left\{ -\frac{\varepsilon}{\tau} \mathscr{T}_\tau(a_1^\tau) \mathscr{T}_\tau(u_\tau) \mathscr{T}_\tau(\phi) \mathscr{T}_\tau(\psi_s) \right\} \\ &\quad + \int_{\Omega_T} \int_Q \left\{ \mathscr{T}_\tau(A^\tau) \mathscr{T}_\tau(\nabla u_\tau) \mathscr{T}_\tau(\phi) \mathscr{T}_\tau(\nabla_y \psi) \mathscr{T}_\tau\left(\frac{1}{a_2^\tau}\right) \right\} + R^\tau \\ &=: J_1 + J_2 + R^\tau. \end{aligned} \tag{5.7}$$

As a consequence of Theorem 2.11 we have as $\varepsilon, \tau \to 0$

$$J_2 \to \int_{\Omega_T} \int_Q B(\nabla u + \nabla_y \tilde{u}) \cdot (\nabla_y \psi) \frac{\phi}{b_2}. \tag{5.8}$$

Next we show that the term J_1 is vanishing in the limit. By recalling Definitions 2.2 and 2.4 we find

$$\begin{aligned} J_1 = &-\frac{\varepsilon^2}{\tau} \int_{\Omega_T} \int_Q \frac{1}{\varepsilon} \mathscr{Z}_\tau(u_\tau) \mathscr{T}_\tau(a_1^\tau) \mathscr{T}_\tau(\phi) \mathscr{T}_\tau(\psi_s) \\ &-\frac{\varepsilon^2}{\tau} \int_{\Omega_T} \int_Q \frac{1}{\varepsilon} \mathscr{M}_\tau(u_\tau) \mathscr{T}_\tau(a_1^\tau) \mathscr{T}_\tau(\phi) \mathscr{T}_\tau(\psi_s) =: J_{11} + J_{12}. \end{aligned} \tag{5.9}$$

By taking into account Proposition 2.22 with $m = r = 1/2$ and $\alpha = 1$, we see that $J_{11} \to 0$ as $\varepsilon, \tau \to 0$. We split J_{12} again, as in

$$\begin{aligned} J_{12} = &-\frac{\varepsilon^2}{\tau} \int_{\Omega_T} \int_Q \frac{1}{\varepsilon} \mathscr{M}_\tau(u_\tau) \mathscr{M}_\tau(\phi) \mathscr{T}_\tau(a_1^\tau) \mathscr{T}_\tau(\psi_s) \\ &-\frac{\varepsilon^2}{\tau} \int_{\Omega_T} \int_Q \frac{1}{\varepsilon} \mathscr{Z}_\tau(\phi) \mathscr{M}_\tau(u_\tau) \mathscr{T}_\tau(a_1^\tau) \mathscr{T}_\tau(\psi_s) =: J_{121} + J_{122}. \end{aligned} \tag{5.10}$$

Again we have $J_{122} \to 0$ as $\varepsilon, \tau \to 0$ by virtue of Remark 2.23 (with $m = r = 1/2$ and $\alpha = 1$).

Next we invoke Lemma 2.6 and specifically (2.12) to write

$$J_{121} = -\frac{\varepsilon^2}{\tau}\int_{\Omega_T}\int_Q \frac{1}{\varepsilon}\mathcal{M}_\tau(u_\tau)\mathcal{M}_\tau(\phi)\mathcal{T}_\tau(a_1^\tau)\frac{\partial}{\partial s}\mathcal{T}_\tau(\psi). \tag{5.11}$$

Recalling that ψ is zero both at $s = 0$ and at $s = 1$ we have after integrating by parts and using (2.12)

$$\begin{aligned} J_{121} &= \frac{\varepsilon^2}{\tau}\int_{\Omega_T}\int_Q \frac{1}{\varepsilon}\mathcal{M}_\tau(u_\tau)\mathcal{M}_\tau(\phi)\Big(\frac{\partial}{\partial s}\mathcal{T}_\tau(a_1^\tau)\Big)\mathcal{T}_\tau(\psi) \\ &= \varepsilon\int_{\Omega_T}\int_Q \mathcal{M}_\tau(u_\tau)\mathcal{M}_\tau(\phi)\mathcal{T}_\tau(a_{1t}^\tau)\mathcal{T}_\tau(\psi) \to 0. \end{aligned} \tag{5.12}$$

By a routine density argument, we see that (5.5) is in force for all test functions as claimed in the statement.

The uniqueness of solutions follows as in Theorem 4.2. □

6. The limit problem in the case $m = 0$, $\tau \sim \varepsilon$

Here we find a homogenized formulation for problem (3.23)–(3.25) in the special case where (2.67) holds true. We assume throughout that the requirements in Subsection 3.1 are fulfilled. We are also going to require that there exist bounded functions $B : \Omega \times Y \to \boldsymbol{R}^{N^2}$, $b : Q \to \boldsymbol{R}$ such that

$$\mathcal{T}_\tau(A^\tau) \to B, \quad \text{strongly in } L^1(\Omega \times Y), \tag{6.1}$$

$$\mathcal{T}_\tau(a^\tau) \to b, \quad \text{strongly in } L^1(\Omega_T \times Q). \tag{6.2}$$

Owing to estimate (3.26), in the notation of Section 2 we may take $m = 0$.

Proposition 6.1. *Let* (2.67) *be in force and let* u_τ *be the solution of problem* (3.23)–(3.25). *Then there exist* $u \in L^2(0, T; W^{1,2}(\Omega))$ *and* $\mathring{u} \in L^2(\Omega_T; W^{1,2}_{per}(Q))$ *such that* $\mathcal{M}_Q(\mathring{u}) = 0$ *and up to a subsequence*

$$u_\tau \rightharpoonup u, \qquad \textit{weakly in } W^{1,2}(\Omega_T), \tag{6.3}$$

$$\mathcal{T}_\tau(u_\tau) \rightharpoonup u, \qquad \textit{weakly in } L^2(\Omega_T; W^{1,2}(Q)), \tag{6.4}$$

$$\mathcal{T}_\tau(\nabla u_\tau) \rightharpoonup \nabla u + \nabla_y \mathring{u}, \qquad \textit{weakly in } L^2(\Omega_T \times Q), \tag{6.5}$$

$$\mathcal{T}_\tau\Big(\frac{\partial u_\tau}{\partial t}\Big) \rightharpoonup \frac{\partial u}{\partial t} + \ell^{-1}\frac{\partial \mathring{u}}{\partial s}, \qquad \textit{weakly in } L^2(\Omega_T \times Q). \tag{6.6}$$

The convergence $u_\tau \to u$ *is in fact strong in* $L^2(\Omega_T)$, *so that from Proposition* 2.8 *it follows* $\mathcal{T}_\tau(u_\tau) \to u$ *strongly in* $L^2(\Omega_T \times Q)$.

PROOF. The claim follows from Proposition 2.12, Theorem 2.20 and Remark 3.6, by taking into account the estimates proved in Subsection 3.3. □

THEOREM 6.2. *Let* (6.1)–(6.2) *be in force. Then the pair* $(u, \mathring{u})$ *as in Proposition* 6.1 *solves*

$$\int_{\Omega_T}\int_Q \{(u_t + \ell^{-1}\mathring{u}_s)b\phi + B[\nabla_x u + \nabla_y \mathring{u}][\nabla_x \phi + \nabla_y \Psi]\}\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s = \int_{\Omega_T}\int_\Sigma f\phi\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s, \tag{6.7}$$

for all $\phi \in L^2((0,T)\times\Sigma; W^{1,2}(\Omega))$ *with* $\phi = 0$ *on* $\partial\Omega \times [0,T]$ *for a.e.* $s \in \Sigma$, *and* $\Psi \in L^2(\Omega_T \times \Sigma; W^{1,2}_{per}(Y))$.

If $\nabla_x b \in L^\infty(\Omega_T \times Q)$ *such solution is unique.*

PROOF. To prove the macroscopic part of equation (6.7), we test (3.23) with the time-oscillating function

$$\phi^\tau(x,t) = \phi\Big(x, t, \frac{t}{\tau}\Big),$$

for $\phi \in L^2(\Sigma; \mathscr{C}^\infty(\Omega_T))$, and $\phi = 0$ for $(x,t) \in \partial\Omega \times [0,T]$, obtaining

$$\int_{\Omega_T} \frac{\partial u_\tau}{\partial t} a^\tau \phi^\tau + \int_{\Omega_T} A^\tau \nabla u_\tau \cdot \nabla\phi^\tau = \int_{\Omega_T} f\phi^\tau. \tag{6.8}$$

On unfolding we are led to

$$\int_{\Omega_T}\int_Q \mathscr{T}_\tau\Big(\frac{\partial u_\tau}{\partial t}\Big)\mathscr{T}_\tau(a^\tau)\mathscr{T}_\tau(\phi^\tau) + \int_{\Omega_T}\int_Q \mathscr{T}_\tau(A^\tau)\mathscr{T}_\tau(\nabla u_\tau)\mathscr{T}_\tau(\nabla\phi^\tau) = \int_{\Omega_T}\int_Q \mathscr{T}_\tau(f)\mathscr{T}_\tau(\phi^\tau) + R^\tau. \tag{6.9}$$

Then taking the limit $\varepsilon, \tau \to 0$, recalling (6.1), (6.2), Proposition 6.1 and Remark 2.9 we get

$$\int_{\Omega_T}\int_Q (u_t + \ell^{-1}\mathring{u}_s)b\phi + \int_{\Omega_T}\int_Q B(\nabla_x u + \nabla_y \mathring{u})\nabla_x\phi = \int_{\Omega_T}\int_\Sigma f\phi. \tag{6.10}$$

Next we prove the microscopic part of equation (6.7). For this purpose we test the equation (3.23) with a function

$$\varepsilon\varphi(x,t)\psi\Big(\frac{x}{\varepsilon}, \frac{t}{\tau}\Big)$$

where $\varphi \in \mathscr{C}^{\infty}(\Omega_T)$ with $\varphi = 0$ on $\partial\Omega \times [0, T]$, and $\psi \in W^{1,2}_{per}(Q)$ is extended periodically both in y and s to the whole $\boldsymbol{R}^{N+1}$, obtaining

$$\int_{\Omega_T} A^{\tau}\nabla u_{\tau} \cdot (\nabla_y \psi)\varphi = -\varepsilon \int_{\Omega_T} a^{\tau} \frac{\partial u_{\tau}}{\partial t} \varphi\psi - \varepsilon \int_{\Omega_T} A^{\tau}\nabla u_{\tau} \cdot (\nabla\varphi)\psi + \varepsilon \int_{\Omega_T} f\varphi\psi. \tag{6.11}$$

The right hand side of equation (6.11) goes to zero as $\varepsilon, \tau \to 0$. Unfolding the left hand side we see that it equals

$$\int_{\Omega_T}\int_{Q} \mathscr{T}_{\tau}(A^{\tau})\mathscr{T}_{\tau}(\nabla u_{\tau})\mathscr{T}_{\tau}(\nabla_y \psi)\mathscr{T}_{\tau}(\varphi) + R^{\tau}. \tag{6.12}$$

Recalling Proposition 6.1, as $\varepsilon, \tau \to 0$ we get from (6.11)–(6.12)

$$\int_{\Omega_T}\int_{Q} B(\nabla_x u + \nabla_y \mathring{u})(\nabla_y \psi)\varphi = 0. \tag{6.13}$$

By a routine density argument we see that (6.7) is in force for all test functions as claimed in the statement.

In order to prove uniqueness of solutions, we preliminarily remark that by virtue of (6.7) we may write for any solution $(u, \mathring{u})$

$$\mathring{u}(x, t, y, s) = u^{*}(x, t, y) + \bar{u}(x, t, s), \tag{6.14}$$

where u^{*} is the unique solution of

$$\operatorname{div}_y(B(x, y)[\nabla_x u + \nabla_y u^{*}]) = 0$$

such that $\mathscr{M}_Y(u^{*}) = 0$ and u^{*} is Y-periodic. Then $\mathscr{M}_{\Sigma}(\bar{u}) = 0$ and

$$\frac{\partial \mathring{u}}{\partial s} = \frac{\partial \bar{u}}{\partial s} \tag{6.15}$$

does not depend on y. Next we invoke again the integral equation (6.7). Take there

$$\phi(x, t, s) = \frac{\varphi(x, t)}{\mathscr{M}_Y(b)(x, t, s)}, \tag{6.16}$$

for φ satisfying the requirements in the statement, and obtain

$$\int_{\Omega_T}\int_{Q} \left\{ (u_t + \ell^{-1}\mathring{u}_s)\varphi + B[\nabla_x u + \nabla_y \mathring{u}]\left[\nabla_x\left(\frac{\varphi}{\mathscr{M}_Y(b)}\right) + \nabla_y \Psi\right]\right\} \mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s = \int_{\Omega_T}\int_{\Sigma} f\frac{\varphi}{\mathscr{M}_Y(b)} \mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s. \tag{6.17}$$

The contribution of the term $\mathring{u}_s\varphi$ vanishes by periodicity. Then we are back to a formulation similar e.g., to (4.21), and can proceed accordingly. That is to say, given two solutions $(u_1, \mathring{u}_1)$, $(u_2, \mathring{u}_2)$ we may conclude $u_1 = u_2$. Thus $u_1^* = u_2^*$ by the definition of u_i^*. We make use a last time of (6.7), and of (6.15), to infer

$$\int_{\Omega_T}\int_{\Sigma} \ell^{-1}(\bar{u}_{1s} - \bar{u}_{2s})\mathscr{M}_Y(b)(x,t,s)\phi(x,t,s)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s = 0,$$

whence

$$\bar{u}_{1s} - \bar{u}_{2s} = 0.$$

It follows $\bar{u}_1 = \bar{u}_2$ since $\mathscr{M}_\Sigma(\bar{u}_1) = \mathscr{M}_\Sigma(\bar{u}_2) = 0$. □

REMARK 6.3. One can see that as a difference from the other macroscopic equations obtained in the homogenization limit in this paper, the macroscopic part of (6.7) contains a residual microscopic time derivative $\mathring{u}_s$. In contrast, the term $\hat{u}_s$ in (4.13) belongs to the microscopic equation. See also Remark 7.6. □

REMARK 6.4. The case investigated in this Section is actually covered by Theorem 5.2, if we take $a^\tau = a_1^\tau a_2^\tau$, so that $b = b_1 b_2$. Here we show how to reconcile equations (5.5) and (6.7). In the latter take

$$\phi(x,t,s) = \frac{\varphi(x,t)}{b_2(x,t,s)},$$

where φ is admissible as in Theorem 5.2. Observe that in the resulting equation, owing to the periodicity of $\mathring{u}$ in s, we have

$$\int_{\Omega_T}\int_{Q} \ell^{-1}\mathring{u}_s(x,t,y,s)b_1(x,t,y)\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s = 0.$$

After integrating by parts the term $u_t b_1 \varphi$ we recover (5.5), since Ψ/b_2 is admissible whenever Ψ is. □

7. REDUCTION TO THE MACROSCOPIC SCALE

In order to reduce the homogenized problems for the three different scalings of the parameters ε, τ to macroscopic formulations, we first introduce the cell functions χ_i. Let us denote the elements of the limit matrix in (4.2) by

$$B(x,t,y,s) = (b_{i,j}(x,t,y,s))_{1\le i,j\le N}.$$

DEFINITION 7.1. If (4.12) is in force, for $1 \le i \le N$ the functions $\chi_i(x,t,y,s)$ satisfy $\mathscr{M}_Q(\chi_i) = 0$ and are the Q-periodic solutions of the problem

$$\ell^{-1}b_1(x,y,t)\chi_{is} - \sum_{j,k=1}^{N} \frac{\partial}{\partial y_j}\Big(\frac{b_{j,k}(x,t,y,s)}{b_2(x,t,s)}\frac{\partial(\chi_i - y_i)}{\partial y_k}\Big) = 0 \quad \text{in } \Omega_T \times Q. \tag{7.1}$$

If (4.20) is in force, for $1 \le i \le N$ the functions $\chi_i(x, t, y)$ satisfy $\mathscr{M}_Y(\chi_i) = 0$ and are the Y-periodic solutions of the problem

$$\sum_{j,k=1}^{N} \frac{\partial}{\partial y_j} \left(\mathscr{M}_\Sigma \left(\frac{b_{j,k}}{b_2} \right)(x, t, y) \frac{\partial(\chi_i - y_i)}{\partial y_k} \right) = 0 \quad \text{in } \Omega_T \times Y. \tag{7.2}$$

If (5.1) is in force, for $1 \le i \le N$ the functions $\chi_i(x, t, y, s)$ satisfy $\mathscr{M}_Y(\chi_i) = 0$ and are the Y-periodic solutions of the problem

$$\sum_{j,k=1}^{N} \frac{\partial}{\partial y_j} \left(\frac{b_{j,k}(x, t, y, s)}{b_2(x, t, s)} \frac{\partial(\chi_i - y_i)}{\partial y_k} \right) = 0 \quad \text{in } \Omega_T \times Q. \tag{7.3}$$ □

Next we prove

Theorem 7.2. *Let u denote the limit of the sequence $\{u_\tau\}$ of solutions to problems* (3.5)–(3.7), *obtained in Theorems* 4.2, 4.4 *and* 5.2.

Then u is the solution of the following homogenized problem

$$a^{hom} u_t - \operatorname{div}(A^{hom} \nabla u) - E^{hom} \cdot \nabla u = F^{hom} f, \quad (x, y) \in \Omega_T, \tag{7.4}$$

$$u(x, t) = 0, \qquad (x, t) \in \partial\Omega \times (0, T), \tag{7.5}$$

$$u(x, 0) = u_0(x), \qquad x \in \Omega, \tag{7.6}$$

where

$$a^{hom}(x, t) = \mathscr{M}_Y(b_1), \tag{7.7}$$

$$A^{hom}(x, t) = \mathscr{M}_Q \left(\frac{B}{b_2} (I - [\nabla_y \chi_1 | \dots | \nabla_y \chi_N]) \right), \tag{7.8}$$

$$E^{hom}(x, t) = \mathscr{M}_Q \left(B(I - [\nabla_y \chi_1 | \dots | \nabla_y \chi_N]) \frac{\nabla_x b_2}{|b_2|^2} \right), \tag{7.9}$$

$$F^{hom}(x, t) = \mathscr{M}_\Sigma \left(\frac{1}{b_2} \right), \tag{7.10}$$

and the χ_i have been introduced in Definition 7.1.

Proof. If (4.12) is in force, we factorize as

$$\hat{u}(x, t, y, s) = -\nabla_x u(x, t) \cdot \sum_{i=1}^{N} \chi_i(x, t, y, s) e_i, \quad (x, t, y, s) \in \Omega_T \times Q, \tag{7.11}$$

where χ_i is defined by (7.1). By using (7.11) in (4.13) with $\phi = 0$ we obtain the problem (7.1) in the microscopic space-time cell, which is satisfied thanks to our definition of χ_i. Then considering (4.13) with $\Psi = 0$ and using again (7.11), we get equation (7.4).

The formulations in the other cases are obtained in a similar way. Namely if (4.20) is in force, we use the factorization

$$\hat{u}(x,t,y) = -\nabla_x u(x,t)\cdot\sum_{i=1}^{N}\chi_i(x,t,y)e_i, \quad (x,t,y)\in\Omega_T\times Y, \tag{7.12}$$

where χ_i is defined by (7.2). If instead (5.1) is in force we write

$$\tilde{u}(x,t,y,s) = -\nabla_x u(x,t)\cdot\sum_{i=1}^{N}\chi_i(x,t,y,s)e_i, \quad (x,y,t,s)\in\Omega_T\times Q, \tag{7.13}$$

where χ_i is defined by (7.3). □

7.1. The case $m = 0$, $\tau \sim \varepsilon$

In this case the elements $b_{i,j}$ of the limit matrix in (6.1) depend only on (x, y).

Definition 7.3. If (2.67) is in force, for $1 \le i \le N$ the functions $\chi_i(x, y)$ satisfy $\mathscr{M}_Y(\chi_i) = 0$ and are the Y-periodic solutions of the problem

$$\sum_{j,k}^{N}\frac{\partial}{\partial y_j}\Big(b_{j,k}(x,y)\frac{\partial(\chi_i - y_i)}{\partial y_k}\Big) = 0, \quad (x,y)\in\Omega\times Y. \tag{7.14}$$ □

Theorem 7.4. *Let u denote the limit of the sequence $\{u_\tau\}$ of solutions to problems* (3.23)–(3.25)*, obtained in Theorem* 6.2.

Then u is the solution of the homogenized problem (7.4)–(7.6)*, where*

$$a^{hom}(x,t) = 1, \tag{7.15}$$

$$A^{hom}(x,t) = \mathscr{M}_Q\Big(\frac{B}{\mathscr{M}_Y(b)}(I - [\nabla_y\chi_1|\dots|\nabla_y\chi_N])\Big), \tag{7.16}$$

$$E^{hom}(x,t) = \mathscr{M}_Q\Big(B(I - [\nabla_y\chi_1|\dots|\nabla_y\chi_N])\nabla_x\Big(\frac{1}{\mathscr{M}_Y(b)}\Big)\Big), \tag{7.17}$$

$$F^{hom}(x,t) = \mathscr{M}_\Sigma\Big(\frac{1}{\mathscr{M}_Y(b)}\Big), \tag{7.18}$$

and χ_i are as in Definition 7.3.

Proof. In (6.7) we split $\mathring{u}(x,t,y,s)$ as in (6.14), and factorize as

$$\mathring{u}(x,t,y,s) = -\nabla_x u(x,t)\cdot\sum_{i=1}^{N}\chi_i(x,y)e_i + \bar{u}(x,t,s), \tag{7.19}$$

$$(x,t,y,s)\in\Omega_T\times Q,$$

where χ_i is defined by (7.14). By using (7.19) in (6.7) with $\phi = 0$ we obtain the problem (7.14) in the microscopic space cell, which is satisfied thanks to our definition of χ_i. Then considering (6.7) with $\Psi = 0$ and recalling that $\phi = \phi(x, t, s)$, we get in the distribution sense

$$\int_Y u_t(x,t)b(x,t,y,s)\,\mathrm{d}y + \int_Y \ell^{-1}\mathring{u}_s(x,t,y,s)b(x,t,y,s)\,\mathrm{d}y - \int_Y \mathrm{div}_x(B(\nabla_x u(x,t) + \nabla_y \mathring{u})) = f(x,t). \tag{7.20}$$

Then using the factorization (7.19) in (7.20) we obtain

$$\mathscr{M}_Y(b)u_t + \mathscr{M}_Y(b)\ell^{-1}\bar{u}_s - \mathrm{div}_x(\mathscr{M}_Y(B)\nabla_x u) + \mathrm{div}_x(\mathscr{M}_Y(B\nabla_y\chi_i)\nabla_x u) = f. \tag{7.21}$$

Next, on dividing by $\mathscr{M}_Y(b)$ and integrating in Σ we get

$$u_t + \int_\Sigma \ell^{-1}\bar{u}_s\,\mathrm{d}s + \mathscr{M}_Q\Big(B(I - [\nabla_y\chi_1|\dots|\nabla_y\chi_N])\nabla_x\Big(\frac{1}{\mathscr{M}_Y(b)}\Big)\Big)\nabla_x u - \mathrm{div}_x\Big[\mathscr{M}_Q\Big(\frac{B}{\mathscr{M}_Y(b)}(I - [\nabla_y\chi_1|\dots|\nabla_y\chi_N])\Big)\nabla_x u\Big] = \mathscr{M}_\Sigma\Big(\frac{f}{\mathscr{M}_Y(b)}\Big). \tag{7.22}$$

The thesis follows once we note that the second term on the left hand side of (7.22) vanishes since $\bar{u}$ is Σ-periodic. □

REMARK 7.5. It is worthwhile remarking the following characterization of $\bar{u}$. From equation (7.21) we get the differential equation in the variable s

$$\bar{u}_s = \ell\Big[-u_t + \frac{1}{\mathscr{M}_Y(b)}\,\mathrm{div}_x(\mathscr{M}_Y(B(I - [\nabla_y\chi_1|\dots|\nabla_y\chi_N]))\nabla_x u) + \frac{f}{\mathscr{M}_Y(b)}\Big].$$

Of course this equation should be understood in the suitable weak sense of (6.7), and complemented with the information that $\bar{u}$ is Σ-periodic and $\mathscr{M}_\Sigma(\bar{u}) = 0$.

REMARK 7.6. Though $\mathring{u}_s$ disappears from the single scale formulation of (6.7), actually its presence forces one to divide by $\mathscr{M}_Y(b)$ as in (7.22), therefore implying the structure in (7.15)–(7.18). □

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