



Calculus of Variations — *H-measures and propagation effects*, by LUC TARTAR, communicated on February 10, 2017.

In memory of Ennio De Giorgi.

ABSTRACT. — The notion of H-measures, introduced by the author and applied to propagation effects for wave equations is recalled, and extended to similar questions for linearized elasticity.

KEY WORDS: H-measures, homogenization

MATHEMATICS SUBJECT CLASSIFICATION: 74Q10, 35B27

A little less than twenty years ago, I wrote (for *Annali della Scuola Normale Superiore di Pisa*) an article [Ta98] in memory of Ennio De Giorgi, entitled *Homogenization and Hyperbolicity*, where I discussed about finite propagation speeds in homogenization, for the wave equation, the Maxwell–Heaviside system in electromagnetism, and the Cauchy–Lamé equation in linearized elasticity. [For his unification of electricity and magnetism, Maxwell used ideas about æther which led him to an enormous system of 26 equations, and it was not an easy task for Heaviside to deduce the system which one uses now, hence it is fair to both of them to call it the Maxwell–Heaviside system. Cauchy wrote the elasticity equation $\rho \hat{\partial}_i^2 u_i - \sum_j \partial_j \sigma_{ij} = f_i$ for density ρ , displacement u , force f , and his stress tensor σ . Using a linearization for a 2-dimensional analogue of a 1-dimensional computation by D. Bernoulli for a vibrating string, he obtained a constitutive law $\sigma = 2\mu\varepsilon + \mu \operatorname{div}(u)I$, with $2\varepsilon_{ij} = \partial_i u_j + \partial_j u_i$. Lamé noticed that the general law for an isotropic material is $\sigma = 2\mu\varepsilon + \lambda \operatorname{div}(u)I$, so that it is fair to both of them to call the elasticity equation with two parameters the Cauchy–Lamé equation.]

I have written many times how sad I find that so many study problems of “fake mechanics” using Γ -convergence, which is not a way to honour the memory of Ennio De Giorgi: those who consider themselves his followers should learn that *minimization of potential energy is not physics*, since *total energy is conserved*. *If energy hides at mesoscopic levels*, one should learn where it goes. For elasticity, one should work with the hyperbolic system governing the *evolution*, and not only with *stationary solutions*, of course.

One *flaw of thermodynamics* is that it gives no intuition of what internal energy is, and how it moves around at various mesoscopic levels and in various directions. As a way to analyse this situation, I want to check here what the

transport theorem of H-measures says for *linearized elasticity*, although I have pointed out its defects, but seismologists seem to use only linearized elasticity; however, they also seem to use only isotropic materials, but I want to address the question in a possibly anisotropic material.

I had introduced H-measures for the question of *small amplitude homogenization*: they describe at quadratic order the oscillations and concentration effects in a weakly converging sequence. I then found how to use them for studying propagation for sequences of solutions to some hyperbolic systems (like those mentioned above), but when I wrote the results in [Ta90], I showed the results in opposite order, because I found the propagation result more important: it gave at last a definition of *what a curved beam of light is* (the fake scalar light for the wave equation or the real polarized light for the Maxwell–Heaviside system).

My feeling is that nobody cared, maybe because one thought that it was already known.

The classical computations of geometrical optics are for the scalar wave equation (with smooth coefficients), and not for more general hyperbolic systems, and they concern solutions which look like *distorted plane waves* (and not like curved beams of light out of which nothing may happen), and in this framework one shows that *there exists* an asymptotic expansion for large frequency ν for which an *amplitude* satisfies a transport equation which needs the gradient of a *phase*, itself a solution of an Hamilton–Jacobi equation *valid only away from caustics*.

I proved that *for all* sequences of solutions of a scalar wave equation (with smooth coefficients) converging weakly (i.e. in the limit of infinite frequency) an H-measure satisfies a first order partial differential equation, and this equation tells *where energy and momentum go*, but they do not see a phase, since they use no characteristic length; there is no difficulty having many curved beams going through the same point (x, t) but with different directions. My method also has the advantage to extend to a class of hyperbolic systems.

Lars Hörmander studied *propagation of microlocal regularity*, which takes place *out of a wave front set*, which is a no-man’s land, but a curved beam of light should not be considered a no-man’s land, since one wants to know how much energy or momentum it carries; computations of particular solutions showing explicit forms of non-regularity (called singularities) are shown, and it may be propaganda to speak of propagation of singularities: why look at such details when one does not see the essential, energy?

1. BASICS OF H-MEASURES

I worked in an open set $\Omega \subset \mathbb{R}^N$, and I defined a kind of “pseudo-differential calculus” on $L^2(\Omega)$, but *modulo compact operators*. [Patrick Gérard, who introduced H-measures [Gé91] and variants in a slightly different way than me and for different reasons, wrote that one cannot define H-measures (which he called microlocal defect measures, a not so good name) on manifolds, but I have mentioned that one should use *a manifold with a volume form*.]

For $b \in L^\infty(\Omega)$, the multiplication operator M_b is defined by

$$(1.1) \quad M_b u = bu \quad \text{for } u \in L^2(\Omega) : \|M_b\|_{\mathcal{L}(L^2(\Omega); L^2(\Omega))} = \|b\|_{L^\infty(\Omega)}.$$

For $a \in L^\infty(\mathbb{R}^N)$, the operator P_a is defined by

$$(1.2) \quad \mathcal{F} P_a u = a \overline{\mathcal{F} u} \quad \text{for } u \in L^2(\mathbb{R}^N) : \|P_a\|_{\mathcal{L}(L^2(\mathbb{R}^N); L^2(\mathbb{R}^N))} = \|a\|_{L^\infty(\mathbb{R}^N)},$$

where \mathcal{F} is the Fourier transform, with inverse $\overline{\mathcal{F}}$, given (using the notation of Laurent Schwartz) by

$$(1.3) \quad \begin{aligned} \mathcal{F} u(\xi) &= \int_{\mathbb{R}^N} e^{-2i\pi(x, \xi)} u(x) dx, \quad \text{for } u \in L^1(\mathbb{R}^N), \\ \overline{\mathcal{F}} v(x) &= \int_{\mathbb{R}^N} e^{+2i\pi(x, \xi)} v(\xi) d\xi, \quad \text{for } v \in L^1(\mathbb{R}^N), \end{aligned}$$

which extend into isometries on $L^2(\mathbb{R}^N)$. For defining H-measures I restricted my attention to

$$(1.4) \quad a \in C(\mathbb{S}^{N-1}), \quad \text{extended as } a\left(\frac{\xi}{|\xi|}\right) \text{ on } \mathbb{R}^N \setminus \{0\},$$

but variants are useful, for which I refer to my book [Ta10]. Technically, a is defined on the space of half-lines, i.e. the quotient of $\mathbb{R}^N \setminus \{0\}$ by the equivalence relation that ξ is equivalent to $r\xi$ for all $r > 0$.

LEMMA 1.1 (first commutation lemma). *If a satisfies (1.4) and $b \in C_0(\mathbb{R}^N)$, then the commutator $[P_a, M_b] = P_a M_b - M_b P_a$ is a compact operator on $L^2(\mathbb{R}^N)$.*

PROOF. One approaches uniformly b by a sequence $b_n \in \mathcal{S}(\mathbb{R}^N)$ such that $\mathcal{F} b_n$ has compact support, in $|\xi| \leq \rho_n$. Since a limit in norm of compact operators is compact, and since $[P_a, M_b]$ is the limit in norm of $[P_a, M_{b_n}]$, it suffices to show that each $[P_a, M_{b_n}]$ (or $\mathcal{F}[P_a, M_{b_n}]$) is a compact operator. One then has

$$(1.5) \quad \mathcal{F}[P_a, M_{b_n}]v(\xi) = \int_{\mathbb{R}^N} \mathcal{F} b_n(\xi - \eta)(a(\xi) - a(\eta))\mathcal{F}v(\eta) d\eta,$$

and one wants to decompose the kernel $K(\xi, \eta) = \mathcal{F} b_n(\xi - \eta)(a(\xi) - a(\eta))$ into two pieces, one bounded by $\varepsilon|\mathcal{F} b_n(\xi - \eta)|$ and one in $L^2(\mathbb{R}^N \times \mathbb{R}^N)$; the first piece gives an operator bounded by a convolution operator, of norm $\leq \varepsilon\|\mathcal{F} b_n\|_{L^1(\mathbb{R}^N)}$, the second one is an Hilbert–Schmidt operator, hence compact; if one can do this for all $\varepsilon > 0$, then $\mathcal{F}[P_a, M_{b_n}]$ is a limit in norm (as $\varepsilon \rightarrow 0$) of compact operators, and is then compact.

Using the uniform continuity of a on the unit sphere \mathbb{S}^{N-1} , one has

$$(1.6) \quad |a(\xi) - a(\eta)| \leq \varepsilon \quad \text{if} \quad \left| \frac{\xi}{|\xi|} - \frac{\eta}{|\eta|} \right| \leq \delta(\varepsilon),$$

and since one only needs to consider pairs with $|\xi - \eta| \leq \rho_n$, one finds that

$$(1.7) \quad |\xi - \eta| \leq \rho_n \quad \text{and} \quad \min\{|\xi|, |\eta|\} \geq \frac{\rho_n}{\delta(\varepsilon)} \quad \text{imply} \quad \left| \frac{\xi}{|\xi|} - \frac{\eta}{|\eta|} \right| \leq \delta(\varepsilon);$$

on the other piece $\max\{|\xi|, |\eta|\} \leq \frac{\rho_n}{\delta(\varepsilon)} + \rho_n$, and the restriction of K is in $L^2(\mathbb{R}^N \times \mathbb{R}^N)$. [Thanks to a result of Raphaël Coifman, Rochberg, and Weiss [C-R-W76], the conclusion of Lemma 1.1 is true if $b \in L^\infty(\mathbb{R}^N) \cap VMO(\mathbb{R}^N)$.]

I then introduced the class of *symbols* of the form

$$(1.8) \quad s(x, \xi) = \sum_m a_m(\xi) b_m(x), \quad \text{with} \quad \sum_m \|a_m\|_{C(\mathbb{S}^{N-1})} \|b_m\|_{C_0(\Omega)} = \kappa < +\infty,$$

and I associated the *standard operator* S_s defined by

$$(1.9) \quad S_s = \sum_m P_{a_m} M_{b_m} \in \mathcal{L}(L^2(\Omega); L^2(\mathbb{R}^N)), \quad \text{with} \quad \|S_s\|_{\mathcal{L}(L^2(\Omega); L^2(\mathbb{R}^N))} \leq \kappa,$$

and S_s does not depend upon the decomposition (1.8) since one has

$$(1.10) \quad \mathcal{F} S_s u(\xi) = \int_{\Omega} e^{-2i\pi(x, \xi)} s\left(x, \frac{\xi}{|\xi|}\right) u(x) dx$$

a.e. $\xi \in \mathbb{R}^N$ for $u \in L^1(\Omega) \cap L^2(\Omega)$.

I defined an *operator of symbol* s to be any operator from $L^2(\Omega)$ to $L^2(\mathbb{R}^N)$ which differs from S_s by a compact operator. In the case $\Omega = \mathbb{R}^N$, using the first commutation lemma (Lemma 1.1), one example is

$$(1.11) \quad L_s = \sum_m M_{b_m} P_{a_m} \in \mathcal{L}(L^2(\mathbb{R}^N); L^2(\mathbb{R}^N)), \quad \text{with} \quad \|L_s\|_{\mathcal{L}(L^2(\mathbb{R}^N); L^2(\mathbb{R}^N))} \leq \kappa,$$

and L_s does not depend upon the decomposition (1.8) since one has

$$(1.12) \quad L_s u(x) = \int_{\mathbb{R}^N} e^{+2i\pi(x, \xi)} s\left(x, \frac{\xi}{|\xi|}\right) \mathcal{F} u(\xi) d\xi, \quad \text{for } u \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N).$$

Then I deduced the existence of H-measures:

THEOREM 1.1 (existence of H-measures). *If U^n converges to 0 in $L^2_{loc}(\Omega; \mathbb{R}^p)$ weak, then there exists a subsequence U^m and an H-measure μ , which is an Hermitian symmetric non-negative $p \times p$ matrix of Radon measures on $\Omega \times \mathbb{S}^{N-1}$ such that: for every $j, k \in \{1, \dots, p\}$, for every $\varphi_1, \varphi_2 \in C_c(\Omega)$, for every operators $L_{s_1}, L_{s_2} \in \mathcal{L}(L^2(\Omega); L^2(\mathbb{R}^N))$ with symbols s_1, s_2 , one has*

$$(1.13) \quad \int_{\mathbb{R}^N} L_{s_1}(\varphi_1 U_j^m) \overline{L_{s_2}(\varphi_2 U_k^m)} d\xi \rightarrow \langle \mu_{jk}, s_1 \varphi_1 \overline{s_2 \varphi_2} \rangle.$$

Sketch of proof. It suffices to show (1.13) for operators of the form $P_a M_b$ and since the b can be included in φ_1 or φ_2 , one just needs to extract (by a Cantor diagonal argument) a subsequence such that for a countable dense set of $\psi \in C(\mathbb{S}^{N-1})$ and for φ_1, φ_2 in a countable set, union of countable dense sets of $C_{K_\ell}(\Omega)$ (where K_ℓ is an increasing sequence of compact sets with union Ω),

$$(1.14) \quad \lim_{m \rightarrow \infty} \int_{\mathbb{R}^N} \mathcal{F}(\varphi_1 U_j^m) \overline{\mathcal{F}(\varphi_2 U_k^m)} \psi \left(\frac{\xi}{|\xi|} \right) d\xi = L(\varphi_1, \varphi_2, \psi) \text{ exists,}$$

and using the first commutation lemma (Lemma 1.1) one deduces then that

$$(1.15) \quad L(\varphi_1, \varphi_2, \psi) \text{ only depends upon } \varphi_1 \overline{\varphi_2} \text{ and } \psi.$$

This limit then defines a linear continuous mapping from $C_c(\Omega)$ into the dual of $C(\mathbb{S}^{N-1})$, i.e. the (Banach) space of Radon measures on \mathbb{S}^{N-1} , denoted $\mathcal{M}(\mathbb{S}^{N-1})$. Using the kernel theorem of Laurent Schwartz, this operator has a kernel belonging to $\mathcal{D}'(\Omega \times \mathbb{S}^{N-1})$, and that the distribution kernel is actually a Radon measure results from non-negativity, since for $\varphi_1 = \varphi_2$ and $\psi \geq 0$ the limit is ≥ 0 . Since Jacques-Louis Lions had told me that he had taught with Lars Gårding an explicit construction of the kernel theorem (and I had looked at their simplified proof [G-L59], which uses Fourier transform and Sobolev spaces), I decided to avoid the kernel theorem in [Ta90] and my proof just uses Functional Analysis, in particular Hilbert–Schmidt operators. When I wrote my book [Ta10], I noticed that the referee from Math Reviews for [G-L59] attributed the idea to Leon Ehrenpreis [Eh56] (which I did not read).

Besides noticing that $\mu = 0$ is equivalent to U^m converging strongly to 0 in $L^2_{loc}(\Omega; \mathbb{R}^p)$, I observed that *linear differential constraints* on U^n imply *pointwise algebraic constraints* on μ :

LEMMA 1.2 (localization principle). *If*

$$(1.16) \quad \sum_{j=1}^p \frac{\partial}{\partial x_j} \left(\sum_{k=1}^N A_{jk} U_k^n \right) \rightarrow 0 \text{ in } H^{-1}_{loc}(\Omega) \text{ strong,}$$

with all $A_{jk} \in C(\Omega)$, then the H-measure μ (for any subsequence) satisfies

$$(1.17) \quad \sum_{j=1}^p \sum_{k=1}^N \xi_j A_{jk}(x) \mu_{k\ell} = 0 \text{ in } \Omega \times \mathbb{S}^{N-1}, \text{ for } \ell = 1, \dots, p.$$

PROOF. Multiplying by $\varphi \in C^1_c(\Omega)$ gives the same property for φU^n since a sequence which converges weakly to 0 in $L^2(\Omega)$ converges strongly to 0 in $H^{-1}_{loc}(\Omega)$. Applying $(-\Delta)^{-1/2}$, and noticing that all the terms have compact support (because of the truncation by φ), (1.16) implies

$$(1.18) \quad \sum_{j=1}^p R_j \left(\sum_{k=1}^N A_{jk} U_k^n \right) \rightarrow 0 \text{ in } L^2(\mathbb{R}^N) \text{ strong,}$$

where the *Riesz operator* R_j has symbol $i \frac{\xi_j}{|\xi|}$. One then applies the defining property of H-measures (in Theorem 1.1) to the subsequence U^m and one deduces (1.17) multiplied by φ , which one then lets vary.

EXAMPLE 1.1 (scalar case). *Let $u^n \rightharpoonup 0$ in $L^2_{loc}(\Omega)$ weak and correspond to a scalar H-measure $\mu (\geq 0)$;*

$$(1.19) \quad \text{if } \sum_j b_j \frac{\partial u^n}{\partial x_j} \rightarrow 0 \text{ in } H^{-1}_{loc}(\Omega) \text{ strong, with } b_j \in C^1(\Omega), j = 1, \dots, N,$$

$$\text{then } P\mu = 0 \text{ in } \Omega \times \mathbb{S}^{N-1}, \text{ with } P(x, \xi) = \sum_j b_j(x) \xi_j \text{ in } \Omega \times \mathbb{R}^N,$$

so that the support of μ is included in the zero set of P , hence the name I chose for the effect. If the zero set of P is empty, then $\mu = 0$.

EXAMPLE 1.2 (gradients). *Let $U^n \rightharpoonup 0$ in $L^2_{loc}(\Omega; \mathbb{R}^N)$ weak and correspond to an H-measure μ ;*

$$(1.20) \quad \text{if } \frac{\partial U_j^n}{\partial x_k} - \frac{\partial U_k^n}{\partial x_j} \rightarrow 0 \text{ in } H^{-1}_{loc}(\Omega) \text{ strong, } j, k = 1, \dots, N,$$

$$\text{then } \mu_{jk} = \xi_j \xi_k \pi, j, k = 1, \dots, N \text{ in } \Omega \times \mathbb{S}^{N-1},$$

$$\text{for a scalar non-negative Radon measure } \pi \text{ in } \Omega \times \mathbb{S}^{N-1}.$$

Similarly, in the vector-valued case

$$(1.21) \quad \text{if } u_i^n \rightharpoonup 0 \text{ in } H^1_{loc}(\Omega) \text{ weak, } i = 1, \dots, p,$$

$$\text{and } V_{ij}^n = \frac{\partial u_i^n}{\partial x_j}, i = 1, \dots, p, j = 1, \dots, N,$$

corresponds to an H-measure μ ,

then $\mu_{ij;lm} = \xi_j \xi_m \pi_{il}$, $i, l = 1, \dots, p$, $j, m = 1, \dots, N$, in $\Omega \times \mathbb{S}^{N-1}$,

for a $p \times p$ non-negative Hermitian symmetric matrix

of Radon measures π in $\Omega \times \mathbb{S}^{N-1}$.

Indeed, from $\frac{\partial V_{ij}^n}{\partial x_k} - \frac{\partial V_{ik}^n}{\partial x_j} = 0$, one deduces that $\xi_k \mu_{ij;lm} = \xi_j \mu_{ik;lm}$, and multiplying by ξ_k and summing in k gives $\mu_{ij;lm} = \xi_j v_{i;lm}$ (with $v_{i;lm} = \sum_k \xi_k \mu_{ik;lm}$); Hermitian symmetry then gives $\xi_j v_{i;lm} = \overline{\mu_{lm;ij}} = \xi_m \overline{v_{l;ij}}$ and multiplying by ξ_j and summing in j gives $v_{i;lm} = \xi_m \pi_{il}$ (with $\pi_{il} = \sum_m \xi_j \overline{v_{l;ij}}$).

EXAMPLE 1.3 (wave equation). *Let $u^n \rightharpoonup 0$ in $H^1_{loc}((0, T) \times \Omega)$ weak and (using $x_0 = t$) $\text{grad}_{t,x} u^n$ correspond to an H-measure μ , satisfying (by Example 1.2)*

$\mu_{jk} = \xi_j \xi_k \pi$ for $j, k = 0, \dots, N$ in $(0, T) \times \Omega \times \mathbb{S}^N$. If u^n satisfies a “wave-like” equation

$$(1.22) \quad \frac{\partial}{\partial t} \left(\rho \frac{\partial u^n}{\partial t} \right) - \sum_{j,k=1}^N \frac{\partial}{\partial x_j} \left(C_{jk} \frac{\partial u^n}{\partial x_k} \right) \rightarrow 0 \text{ in } H_{loc}^{-1}((0, T) \times \Omega) \text{ strong,}$$

with $\rho, C_{jk} \in C((0, T) \times \Omega)$, $j, k = 1, \dots, N$,

then π satisfies $Q\pi = 0$ in $(0, T) \times \Omega \times \mathbb{S}^N$,

$$\text{with } Q = \rho(t, x) \xi_0^2 - \sum_{j,k=1}^N C_{jk}(t, x) \xi_j \xi_k \text{ in } (0, T) \times \Omega \times \mathbb{R}^{N+1}.$$

[It is a wave equation if $\rho > 0$ and C is symmetric positive definite, but it is elliptic if $\rho > 0$ and C is negative definite, in which case the zero set of Q is empty, hence $\pi = 0$.]

Indeed, the localization principle (Lemma 1.2) gives $\xi_\ell Q\pi = 0$ for every ℓ , and multiplying by ξ_ℓ and summing in ℓ gives (1.22).

EXAMPLE 1.4 (linearized elasticity). For $N \geq 2$, let $u_i^n \rightarrow 0$ in $H_{loc}^1((0, T) \times \Omega)$ weak for $i = 1, \dots, N$, and (using $x_0 = t$) $V_{ij}^n = \frac{\partial u_i^n}{\partial x_j}$ (for $j = 0, \dots, N$) correspond to an H -measure μ , satisfying (by Example 1.2) $\mu_{ij;lm} = \xi_j \xi_m \pi_{il}$ in $(0, T) \times \Omega \times \mathbb{S}^N$ for $i, l = 1, \dots, N$, and $j, m = 0, \dots, N$. Let u^n satisfy a “Cauchy–Lamé-like” system

$$(1.23) \quad \frac{\partial}{\partial t} \left(\rho \frac{\partial u_i^n}{\partial t} \right) - \sum_{j=1}^N \frac{\partial \sigma_{ij}^n}{\partial x_j} \rightarrow 0 \text{ in } H_{loc}^{-1}((0, T) \times \Omega) \text{ strong, } i = 1, \dots, N,$$

$$\text{with } \sigma_{ij}^n = \sum_{k,l=1}^N C_{ij;kl} \varepsilon_{kl}^n, \quad \varepsilon_{ij}^n = \frac{1}{2} \left(\frac{\partial u_i^n}{\partial x_j} + \frac{\partial u_j^n}{\partial x_i} \right), \quad i, j = 1, \dots, N,$$

$$\text{and } \rho, C_{ij;kl} = C_{ji;kl} = C_{ij;lk} \in C((0, T) \times \Omega), \quad i, j, k, l = 1, \dots, N.$$

[It is a Cauchy–Lamé system if $\rho > 0$ and if C satisfies some symmetry and positivity requirement.]

Then π satisfies

$$(1.24) \quad \xi_0^2 \rho \pi_{i,r} - \sum_{k=1}^N A_{ik}(e) \pi_{kr} = 0 \quad \text{in } (0, T) \times \Omega \times \mathbb{S}^N, \quad i, r = 1, \dots, N,$$

with $e = (\xi_1, \dots, \xi_N)$, where $A(e)$ is the acoustic tensor:

$$(1.25) \quad A_{ik}(e) = \sum_{j,l=1}^N C_{ij;kl} e_j e_l, \quad i, k = 1, \dots, N \text{ in } (0, T) \times \Omega \times \mathbb{R}^N.$$

Indeed, the localization principle (Lemma 1.2) gives for each i an equation valid for all pairs of indices (r, s) , and it appears to be the i th equation in (1.24) multiplied by ξ_s : since $\xi \in \mathbb{S}^N$, not all ξ_s may vanish, and one obtains (1.24). If at a point $(t_0, x_0) \in (0, T) \times \Omega$ and $(\xi_0, e) \in \mathbb{S}^N$ the matrix $A(e) - \xi_0^2 \rho I$ is invertible, then it is true in an open neighbourhood (since the coefficients are continuous), and on this set (1.24) implies that $\pi = 0$, and one deduces that

$$(1.26) \quad \text{the support of } \pi \text{ is included in the set where } \det(A(e) - \xi_0^2 \rho I) = 0.$$

The acoustic tensor was introduced for the study of plane-waves for the *constant coefficient case*: taking $\eta \in \mathbb{S}^{N-1}$, one looks for a plane-wave solution of the form $u_i(x, t) = f_i((x, \eta) - vt)$, $i = 1, \dots, N$, i.e. having (phase) velocity v ; one finds that f'' is an eigenvector of $A(\eta)$ for the eigenvalue ρv^2 (because $f'' = 0$ corresponds to constant $\frac{\partial u_i}{\partial x_j}$, so that nothing changes, and it is not called a wave). [It is a *phase velocity*, and not a *group velocity*, but I have shown in [Ta98] that, under an hypothesis of very strong ellipticity, and for coefficients only depending upon x , that the maximum phase velocity in a direction serves also as maximum group velocity in that direction.]

The computations shown for Example 1.4 are about having sequences of solutions which converge weakly but not strongly, without imposing them any particular form, like plane-waves (again a difference between *there exists* and *for all*).

In hyperbolic cases, one may be able to obtain some partial differential equations on H-measures, which express *in some way* how the oscillations and concentration effects (which the H-measures take into account) propagate, but the proof uses smoothness hypotheses.

LEMMA 1.3 (second commutation lemma). *For $\Omega = \mathbb{R}^N$, if $a \in \text{Lip}(\mathbb{S}^{N-1})$ (extended as $a(\frac{\xi}{|\xi|})$ in $\mathbb{R}^N \setminus \{0\}$),*

$$(1.27) \quad \text{if } b \in X^1(\mathbb{R}^N) = \left\{ b \in \mathcal{F}L^1(\mathbb{R}^N) \mid \frac{\partial b}{\partial x_j} \in \mathcal{F}L^1(\mathbb{R}^N), j = 1, \dots, N \right\}, \text{ then}$$

the commutator $[P_a, M_b]$ maps $L^2(\mathbb{R}^N)$ into $H^1(\mathbb{R}^N)$;

if also $a \in C^1(\mathbb{S}^{N-1})$, then

$$\frac{\partial}{\partial x_j} [P_a, M_b] \text{ has symbol } \xi_j \sum_{k=1}^N \frac{\partial a}{\partial \xi_k} \frac{\partial b}{\partial x_k}, \text{ for } j = 1, \dots, N.$$

[In [Ta90], I used a result of Alberto Calderón for improving the regularity hypotheses, but I shall only use here my initial approach.]

PROOF. One must bound the norm in $L^2(\mathbb{R}^N)$ of

$$(1.28) \quad |\xi| \mathcal{F}[P_a, M_b]v(\xi) = |\xi| \int_{\mathbb{R}^N} \mathcal{F}b(\xi - \eta)(a(\xi) - a(\eta))\mathcal{F}v(\eta) d\eta,$$

in terms of the norm in $L^2(\mathbb{R}^N)$ of $\mathcal{F}v$. Let κ be the Lipschitz constant of a on \mathbb{S}^{N-1} , so that

$$(1.29) \quad |a(\xi) - a(\eta)| \leq \kappa \left| \frac{\xi}{|\xi|} - \frac{\eta}{|\eta|} \right| \leq \frac{\kappa |\xi - \eta|}{\min\{|\xi|, |\eta|\}}, \quad \xi, \eta \in \mathbb{R}^N \setminus \{0\},$$

then one bounds $|\xi| |a(\xi) - a(\eta)|$ in two different ways:

$$(1.30) \quad \begin{aligned} \text{if } |\xi| \leq |\eta|, \text{ then } |\xi| |a(\xi) - a(\eta)| &\leq \kappa |\xi - \eta|, \\ \text{if } |\xi| \geq |\eta|, \text{ then } |\xi| |a(\xi) - a(\eta)| &\leq |\xi - \eta| |a(\xi) - a(\eta)| + |\eta| |a(\xi) - a(\eta)| \\ &\leq |\xi - \eta| 2 \|a\|_{L^\infty(\mathbb{R}^N)} + \kappa |\xi - \eta|, \end{aligned}$$

from which the first part of (1.27) follows.

For proving the second part of (1.27), one approaches (in $X^1(\mathbb{R}^N)$ norm) b by a sequence $b_n \in \mathcal{S}(\mathbb{R}^N)$ such that $\mathcal{F}b_n$ has compact support, in $|\xi| \leq \rho_n$. Since

$$(1.31) \quad \mathcal{F} \left(\frac{\partial}{\partial x_j} [P_a, M_{b_n}] \right) v(\xi) = \int_{\mathbb{R}^N} \mathcal{F}b_n(\xi - \eta) 2i\pi \xi_j (a(\xi) - a(\eta)) \mathcal{F}v(\eta) d\eta,$$

and one only considers $|\xi - \eta| \leq \rho_n$, one improves (1.29) by considering a Taylor expansion at order 1,

$$(1.32) \quad a \left(\frac{\xi}{|\xi|} \right) - a \left(\frac{\eta}{|\eta|} \right) = \nabla a \left(\frac{\xi}{|\xi|} \right) \cdot \left(\frac{\xi}{|\xi|} - \frac{\eta}{|\eta|} \right) + \text{error},$$

assuming that $\min\{|\xi|, |\eta|\} \geq r$, with r large, hence $\left| \frac{\xi}{|\xi|} - \frac{\eta}{|\eta|} \right| \leq \frac{|\xi - \eta|}{\min\{|\xi|, |\eta|\}} \leq \frac{\rho_n}{r}$ is small; the error in (1.32) is bounded using the modulus of uniform continuity of ∇a on \mathbb{S}^{N-1} , so that it is $\leq \varepsilon \left| \frac{\xi}{|\xi|} - \frac{\eta}{|\eta|} \right| \leq \frac{\varepsilon |\xi - \eta|}{\min\{|\xi|, |\eta|\}}$ with $\varepsilon \left(\frac{\rho_n}{r} \right)$ tending to 0 as r tends to ∞ ; in bounding the corresponding term in (1.31), there is a multiplying factor $2\pi |\xi_j|$, taken care of by using $|\xi| \leq \min\{|\xi|, |\eta|\} + \rho_n$, and this term then corresponds to an operator of norm $\leq C\varepsilon \|b_n\|_{X^1(\mathbb{R}^N)} \|a\|_{Lip(\mathbb{R}^N)}$, with C tending to 2π as r tends to ∞ . In order to make the standard operator of symbol $s_n(x, \xi) = \xi_j \sum_{k=1}^N \frac{\partial a}{\partial \xi_k} \frac{\partial b_n}{\partial x_k}$ appear, one introduces a second small term by replacing in the right side of (1.32) $\frac{\eta}{|\eta|}$ by $\frac{\eta}{|\xi|}$, and the difference is $\leq \frac{||\xi| - |\eta||}{|\xi| |\eta|} \leq \frac{|\xi - \eta|}{|\xi| |\eta|} \leq \frac{|\xi - \eta|}{r |\xi|}$, and this correction corresponds to an operator of norm $\leq \frac{2\pi}{r} \|b_n\|_{X^1(\mathbb{R}^N)} \|a\|_{Lip(\mathbb{R}^N)}$. Of course, one puts together all the corresponding integrals over the set $\min\{|\xi|, |\eta|\} < r$, which is included in the set $\max\{|\xi|, |\eta|\} < r + \rho_n$, and such terms correspond to Hilbert–Schmidt operators, which are compact; this shows that the difference between $\mathcal{F} \left(\frac{\partial}{\partial x_j} [P_a, M_{b_n}] \right)$ and $\mathcal{F}S_n$ is a compact operator, since it is a limit in norm (as r tends to ∞) of compact operators.

Then, as n tends to ∞ , one notices that $\mathcal{F}\left(\frac{\partial}{\partial x_j}[P_a, M_{b_n}]\right)$ converges in norm to $\mathcal{F}\left(\frac{\partial}{\partial x_j}[P_a, M_b]\right)$, that $\mathcal{F}S_{s_n}$ converges in norm to $\mathcal{F}S_s$, so that the difference of these two limits is the limit in norm of compact operators, hence is compact.

COROLLARY 1.1. *For $\Omega = \mathbb{R}^N$, if $a \in C^1(\mathbb{S}^{N-1})$ (extended as $a\left(\frac{\xi}{|\xi|}\right)$ in $\mathbb{R}^N \setminus \{0\}$), and $b \in X^1(\mathbb{R}^N)$*

$$(1.33) \quad P_a b \frac{\partial u}{\partial x_j} = b \frac{\partial(P_a u)}{\partial x_j} + Lu, \text{ for } u \in L^2(\mathbb{R}^N),$$

with L having symbol $\xi_j\{a, b\}$, $j = 1, \dots, N$,

where the Poisson bracket is defined for two functions on $\mathbb{R}^N \times \mathbb{R}^N$ by

$$(1.34) \quad \{f, g\} = \sum_{j=1}^N \left(\frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial \xi_j} \right).$$

[For an Hamiltonian system $\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}$, $\frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j}$ (the first example of which was written by Lagrange for a question of perturbation of an elliptic orbit in celestial mechanics), the sign chosen for the Poisson bracket is such that $\frac{dF(q, p)}{dt} = \{H, F\} = \sum_{j=1}^N \frac{\partial H}{\partial p_j} \frac{\partial F}{\partial q_j} - \frac{\partial H}{\partial q_j} \frac{\partial F}{\partial p_j}$ for all smooth functions F .]

PROOF. Indeed, $P_a b \frac{\partial u}{\partial x_j} = P_a \frac{\partial(bu)}{\partial x_j} - P_a \frac{\partial b}{\partial x_j} u$, and the first term is $\frac{\partial}{\partial x_j} P_a b u = \frac{\partial}{\partial x_j} [P_a, M_b]u + \frac{\partial}{\partial x_j} b P_a u$, i.e. $Lu + \frac{\partial}{\partial x_j} b P_a u$ by the second commutation lemma (Lemma 1.3), while the second term is $-\frac{\partial b}{\partial x_j} P_a u + Ku$ for a compact operator K by the first commutation lemma (Lemma 1.1), and $\frac{\partial}{\partial x_j} b P_a u - \frac{\partial b}{\partial x_j} P_a u = b \frac{\partial}{\partial x_j} P_a u$.

For generalizing the second commutation lemma (Lemma 1.3) to a larger class of operators, it is the standard operators which should be used, with a natural change of definition for the class of symbols:

$$(1.35) \quad s(x, \xi) = \sum_m a_m(\xi) b_m(x) \quad \text{with} \quad \sum_m \|a_m\|_{C^1(\mathbb{S}^{N-1})} \|b_m\|_{X^1(\mathbb{R}^N)} < +\infty.$$

LEMMA 1.4. *If S_1, S_2 are the standard operators with symbols s_1, s_2 satisfying (1.35), then*

$$(1.36) \quad \frac{\partial}{\partial x_j} [S_1, S_2] \text{ has symbol } \xi_j\{s_1, s_2\}.$$

PROOF. It follows from the case $S_1 = P_a M_b$, $S_2 = P_c M_d$: let $A = [M_b, P_c]$ and $B = [M_d, P_a]$, then, using the fact that $\frac{\partial}{\partial x_j}$, P_a , and P_c commute, and that M_b and M_d commute, one has

$$\begin{aligned}
 (1.37) \quad & \frac{\partial}{\partial x_j} (P_a M_b P_c M_d - P_c M_d P_a M_b) \\
 &= P_a \frac{\partial}{\partial x_j} ((A + P_c M_b) M_d) - P_c \frac{\partial}{\partial x_j} ((B + P_a M_d) M_b) \\
 &= P_a \frac{\partial}{\partial x_j} A M_d + P_a P_c \frac{\partial}{\partial x_j} M_b M_d - P_c \frac{\partial}{\partial x_j} B M_d M_b - P_c P_a \frac{\partial}{\partial x_j} M_d M_b \\
 &= P_a \frac{\partial}{\partial x_j} A M_d - P_c \frac{\partial}{\partial x_j} B M_d M_b, \quad \text{which has symbol} \\
 & a(\xi_j \{b, c\})d - c(\xi_j \{d, a\})b = \xi_j(a\{b, c\}d + b\{a, d\}c) = \xi_j\{ab, cd\}.
 \end{aligned}$$

2. TRANSPORT EFFECTS

In order to work with partial differential equations in an open set $\Omega \subset \mathbb{R}^N$, I need a local version of the space $X^1(\mathbb{R}^N)$ used in (1.27), and since $X^1(\mathbb{R}^N) \subset C_0^1(\mathbb{R}^N)$, because $\mathcal{FL}^1(\mathbb{R}^N) \subset C_0(\mathbb{R}^N)$, I choose

$$(2.1) \quad X_{loc}^1(\Omega) = \{b \in C^1(\Omega) \mid \varphi b \in X^1(\mathbb{R}^N) \text{ for all } \varphi \in C_c^\infty(\Omega)\}.$$

[For $1 \leq p \leq 2$, one has $W^{s,p}(\mathbb{R}^N) \subset \mathcal{FL}^1(\mathbb{R}^N)$ for $s > \frac{N}{p}$, hence $W^{s,p}(\mathbb{R}^N) \subset X^1(\mathbb{R}^N)$ for $s > 1 + \frac{N}{p}$, and one may replace $C_c^\infty(\Omega)$ in (2.1) by the Hölder space $C_c^{m,\alpha}(\Omega)$ if $m + \alpha > 1 + \frac{N}{2}$.]

In [Ta90], I studied the transport property when u_n satisfies a first order scalar equation (like in Example 1.1), and I used a right side f_n not necessarily linked to u_n : it was useful as a first step toward creating a theory for semi-linear equations (which is not yet done), but in such a case one needs to consider the H-measure for a subsequence (u_m, f_m) , and the transport equation for μ_{11} (i.e. the H-measure for u_m) has a source term involving μ_{12} ; using the localization principle (Lemma 1.2) and the non-negative Hermitian symmetry of H-measures (Theorem 1.1), both the supports of μ_{11} and μ_{12} are included in the zero set of P (defined in 1.19), although the support of μ_{22} may not be included in it.

Here, I just want to consider the case where $f_n = S_s u_n$, for the *standard operator* S_s with symbol s .

THEOREM 2.1. *If u_n converges weakly to 0 in $L^2(\Omega)$ and corresponds to an H-measure μ and if*

$$(2.2) \quad \sum_{j=1}^N b_j \frac{\partial u_n}{\partial x_j} + S_s u_n \rightarrow 0 \quad \text{in } L_{loc}^2(\Omega) \text{ strong,}$$

with b_1, \dots, b_N real and belonging to $X_{loc}^1(\Omega)$, then μ satisfies

$$(2.3) \quad P(x, \xi)\mu = 0 \text{ in } \Omega \times \mathbb{S}^{N-1}, \text{ with } P(x, \xi) = \sum_{j=1}^N b_j(x)\xi_j \text{ in } \Omega \times \mathbb{R}^N,$$

$$\langle \mu, \{\Phi, P\} + (2\Re s - \operatorname{div}(b))\Phi \rangle = 0 \text{ for all } \Phi \in C_c^1(\Omega \times \mathbb{S}^{N-1}).$$

PROOF. The first part of (2.3) was seen in (1.19). For $\varphi \in C_c^1(\Omega)$, and $\psi \in C_c^\infty(\Omega)$ real and equal to 1 on support(φ) (so that $\psi\varphi = \varphi$), φu_n satisfies

$$(2.4) \quad \sum_{j=1}^N \psi b_j \frac{\partial(\varphi u_n)}{\partial x_j} - \left(\sum_{j=1}^N b_j \frac{\partial\varphi}{\partial x_j} \right) u_n + S_s \varphi u_n \rightarrow 0 \text{ in } L^2(\mathbb{R}^N) \text{ strong,}$$

since the commutator $[S_s, \varphi]$ is compact on $L^2(\mathbb{R}^N)$ by the first commutation lemma (Lemma 1.1). Since $\psi b_j \in X^1(\mathbb{R}^N)$ by the definition (2.1), one may apply P_a to (2.4) with $a \in C^1(\mathbb{S}^{N-1})$, and use Corollary 1.1 of the second commutation lemma (Lemma 1.3), and obtain

$$(2.5) \quad \sum_{j=1}^N \psi b_j \frac{\partial(P_a \varphi u_n)}{\partial x_j} + L \varphi u_n - P_a \left(\sum_{j=1}^N b_j \frac{\partial\varphi}{\partial x_j} \right) u_n + P_a S_s \varphi u_n \rightarrow 0$$

$$\text{in } L^2(\mathbb{R}^N) \text{ strong, with } L \text{ having symbol } \sum_{j=1}^N \xi_j \{a, \psi b_j\}.$$

The next step is to multiply (2.5) by $\overline{\varphi u_n}$ and add the complex conjugate of (2.4) multiplied by $P_a \varphi u_n$, and it is here that the hypothesis that the b_j and ψ are real is used:

(2.6) the quantity

$$\begin{aligned} & \sum_{j=1}^N \psi b_j \frac{\partial(P_a \varphi u_n \overline{\varphi u_n})}{\partial x_j} + \left(L \varphi u_n - P_a \left(\sum_{j=1}^N b_j \frac{\partial\varphi}{\partial x_j} \right) u_n + P_a S_s \varphi u_n \right) \overline{\varphi u_n} \\ & + P_a \varphi u_n \overline{\left(S_s \varphi u_n - \left(\sum_{j=1}^N b_j \frac{\partial\varphi}{\partial x_j} \right) u_n \right)} \end{aligned}$$

then tends to 0 in $L^1(\mathbb{R}^N)$ strong. One then integrates (2.6) against a test function $w \in C_c^\infty(\Omega)$ and one takes the limit as n tends to ∞ , which makes the H-measure μ appear:

$$(2.7) \quad \left\langle \mu, -a\varphi\bar{\varphi} \sum_j \frac{\partial(\psi b_j w)}{\partial x_j} + \left(\sum_j \xi_j \{a, \psi b_j\} \varphi - a \left(\sum_j b_j \frac{\partial\varphi}{\partial x_j} \right) + a s \varphi \right) \bar{\varphi} w \right. \\ \left. + a\varphi w \left(\overline{s\varphi} - \sum_j b_j \frac{\partial\bar{\varphi}}{\partial x_j} \right) \right\rangle = 0.$$

Since each term has a φ or $\bar{\varphi}$, one may replace ψ by 1. One then observes that the terms $-a\varphi\bar{\varphi}\sum_{j=1}^N\frac{\partial(b_jw)}{\partial x_j}$, $-a(\sum_{j=1}^N b_j\frac{\partial\varphi}{\partial x_j})\bar{\varphi}w$, and $-a\varphi w\sum_{j=1}^N b_j\frac{\partial\bar{\varphi}}{\partial x_j}$ add up to $-a\sum_{j=1}^N\frac{\partial(b_j\varphi\bar{\varphi}w)}{\partial x_j}$, and the remaining terms also use $\varphi\bar{\varphi}w$, but one may simply say that one may choose w and then take φ equal to 1 on the support of w , so that (2.7) becomes

$$(2.8) \quad \left\langle \mu, -a\sum_{j=1}^N\frac{\partial(b_jw)}{\partial x_j} + \left(\sum_{j=1}^N \xi_j\{a, b_j\}\right)w + asw + a\bar{s}w \right\rangle = 0$$

for all $w \in C_c^\infty(\Omega)$, $a \in C^1(\mathbb{S}^{N-1})$.

Then one observes that

$$(2.9) \quad \text{for } \Phi = aw, \{\Phi, P\} = \sum_{k=1}^N\frac{\partial a}{\partial \xi_k}w\left(\sum_{j=1}^N \xi_j\frac{\partial b_j}{\partial x_k}\right) - \sum_{k=1}^N a\frac{\partial w}{\partial k}b_k$$

$$= \left(\sum_{j=1}^N \xi_j\{a, b_j\}\right)w - \sum_{j=1}^N a\frac{\partial w}{\partial_j}b_j,$$

so that in (2.8) μ is applied to $\{\Phi, P\} + (2\Re s - \text{div}(b))\Phi$, proving (2.3) in the particular case $\Phi = aw$. One deduces (2.3) by an argument of density of linear combinations of tensor products.

Equation (2.3) is a first order partial differential equation in (x, ξ) for μ , written in weak form so that the partial derivatives appear on the test function Φ ; the *characteristic curves* for this equation are given by

$$(2.10) \quad \frac{dx_j}{d\tau} = \frac{\partial P}{\partial \xi_j}, \quad \frac{d\xi_j}{d\tau} = -\frac{\partial P}{\partial x_j}, \quad j = 1, \dots, N,$$

which imply that P is constant, and one only uses those curves corresponding to $P = 0$ since the support of μ is included there. However, it is useful to *forget the constraint* $\xi \in \mathbb{S}^{N-1}$, and observe that one uses the quotient space of $\mathbb{R}^N \setminus \{0\}$ by the equivalence relation that ξ is equivalent to $r\xi$ for all $r > 0$: using \mathbb{S}^{N-1} is just a way of picking an element in each equivalence class (which is a half-ray).

If one chooses two initial data for (2.10), $(x(\tau_0), \xi(\tau_0))$ and $(x'(\tau_0), \xi'(\tau_0))$ with $x'(\tau_0) = x(\tau_0)$ but $\xi'(\tau_0) = \lambda\xi(\tau_0)$, then the solution has $x'(\tau) = x(\tau)$ and $\xi'(\tau) = \lambda\xi(\tau)$ for all τ , i.e. (2.10) induces an evolution equation for half-rays, explaining why (2.10) is called an equation for *bicharacteristic rays*.

In (2.10), the equation for x is independent of ξ , and once x is known the equation for ξ is linear, so that existence and uniqueness of solutions of (2.10) holds if the coefficients b_j are locally Lipschitz continuous.

A localization procedure is needed on open sets Ω , since I stated the second commutation lemma (Lemma 1.3) and its Corollary 1.1 on \mathbb{R}^N , but any improvement of the regularity hypothesis for b there will permit to improve the regularity hypothesis in Theorem 2.1 and in other applications.

A key point is that the equation (or system) considered must have a sesquilinear conservation law for *complex* solutions, even though one may be only interested in real solutions, since the localization in ξ results from using “pseudo-differential” operators, which may map real functions into complex functions.

LEMMA 2.1. *Let $\rho, C_{jk}, j, k = 1, \dots, N \in C^2((0, T) \times \Omega)$ be real with C symmetric, and satisfy $\rho \geq \alpha, C \geq \alpha I$ for some $\alpha > 0$. Assume that $v \in C^0(0, T; H^1(\Omega)) \cap C^1(0, T; L^2(\Omega))$ satisfies*

$$(2.11) \quad \frac{\partial}{\partial t} \left(\rho \frac{\partial v}{\partial t} \right) - \sum_{j,k=1}^N \frac{\partial}{\partial x_j} \left(C_{jk} \frac{\partial v}{\partial x_k} \right) = f \in L^2((0, T) \times \Omega),$$

then

$$(2.12) \quad \begin{aligned} f \frac{\partial \bar{v}}{\partial t} + \bar{f} \frac{\partial v}{\partial t} &= \frac{\partial}{\partial t} \left(\rho \frac{\partial v}{\partial t} \frac{\partial \bar{v}}{\partial t} + \sum_{j,k=1}^N C_{jk} \frac{\partial v}{\partial x_j} \frac{\partial \bar{v}}{\partial x_k} \right) \\ &\quad - \sum_{j,k=1}^N \frac{\partial}{\partial x_j} \left(C_{jk} \frac{\partial v}{\partial x_k} \frac{\partial \bar{v}}{\partial t} + C_{jk} \frac{\partial \bar{v}}{\partial x_k} \frac{\partial v}{\partial t} \right) \\ &\quad + \frac{\partial \rho}{\partial t} \frac{\partial v}{\partial t} \frac{\partial \bar{v}}{\partial t} - \sum_{j,k=1}^N \frac{\partial C_{jk}}{\partial t} \frac{\partial v}{\partial x_j} \frac{\partial \bar{v}}{\partial x_k} \end{aligned}$$

in the sense of distributions in $(0, T) \times \Omega$.

Sketch of proof. Formally, (2.11) implies (2.12) by developing the derivatives of products and regrouping the terms, but second order derivatives of v appear in the computation, hence it is not a proof.

A proof relies on the well-posedness of the wave equation with appropriate boundary conditions.

Since (2.12) is local, one just needs to show it for φv with $\varphi \in C_c^\infty((0, T) \times \Omega)$, equal to 1 on an open set, where (2.12) is then proved. If the support of φ is included in $(0, T) \times \omega$ for an open set ω , then φv solves the wave equation with Dirichlet conditions on the boundary of ω (and 0 initial data), and (2.12) is just the non-integrated form of the *identity of energy* (one may separate the real part and the imaginary part of (2.11) since the coefficients are real), applied to test functions $w(x)g(t)$ for example. However, although one derivative in t on the coefficients serves for constructing (weak) solutions with $u \in L^\infty(0, T; H_0^1(\omega))$ and $\frac{\partial u}{\partial t} \in L^\infty(0, T; L^2(\omega))$, the proof which I taught in 1974/75 at University of Wisconsin (in a graduate course whose lecture notes, written by graduate students of John Nohel, were gathered in [Ta78]) was for ρ constant and C independent of t , and here (and later for the linearized elasticity system) I use a more general framework (already used by Jacques-Louis Lions in his first book [Li61]), and the analogous proof seems to require two derivatives in t (see Appendix): first

one proves regularity in t , then one shows uniqueness of weak solutions, which implies that $u \in C^0([0, T]; H_0^1(\omega)) \cap C^1([0, T]; L^2(\omega))$, and finally one shows that u satisfies an identity of energy.

THEOREM 2.2. *Let $u^n \in C^0(0, T; H^1(\Omega)) \cap C^1(0, T; L^2(\Omega))$ be a sequence of solutions of a wave equation*

$$(2.13) \quad \frac{\partial}{\partial t} \left(\rho \frac{\partial u^n}{\partial t} \right) - \sum_{j,k=1}^N \frac{\partial}{\partial x_j} \left(C_{jk} \frac{\partial u^n}{\partial x_k} \right) + \sum_{j=0}^N S^j \frac{\partial u^n}{\partial x_j} \rightarrow 0 \quad \text{in } L_{loc}^2((0, T) \times \Omega) \text{ strong,}$$

with $\rho, C_{jk} \in X_{loc}^1((0, T) \times \Omega) \cap C^2((0, T) \times \Omega)$, $j, k = 1, \dots, N$, ρ being real > 0 and the matrix C being real symmetric positive definite (but since the coefficients are smooth, one may replace C by its symmetric part, and absorb the lower order terms that this creates into the operators S^1, \dots, S^N), and S^0, \dots, S^N being the standard operators with symbols s^0, \dots, s^N . Assume that $u^n \rightarrow 0$ in $H_{loc}^1((0, T) \times \Omega)$ weak and (using $x_0 = t$) that $\text{grad}_{t,x} u^n$ corresponds to an H -measure μ , satisfying (by Example 1.2) $\mu_{jk} = \xi_j \xi_k \pi$ for $j, k = 0, \dots, N$; then π satisfies

$$(2.14) \quad Q\pi = 0 \text{ in } (0, T) \times \Omega \times \mathbb{S}^N, \text{ with } Q = \rho(t, x)\xi_0^2 - \sum_{j,k=1}^N C_{jk}(t, x)\xi_j \xi_k$$

in $(0, T) \times \Omega \times \mathbb{R}^{N+1}$, and

$$\left\langle \pi, \{\Psi, Q\} + \sum_{j=0}^N (\xi_j s^j + \xi_j \bar{s}^j) \Psi \right\rangle = 0, \text{ for all } \Psi \in C_c^1((0, T) \times \Omega \times \mathbb{S}^N).$$

PROOF. The first part of (2.14) was seen in (1.22). For $\varphi \in C_c^2((0, T) \times \Omega)$, and $\psi \in C_c^\infty((0, T) \times \Omega)$ real and equal to 1 on support(φ), φu^n satisfies

$$(2.15) \quad \frac{\partial}{\partial t} \left(\psi \rho \frac{\partial(\varphi u^n)}{\partial t} \right) - \sum_{j,k=1}^N \frac{\partial}{\partial x_j} \left(\psi C_{jk} \frac{\partial(\varphi u^n)}{\partial x_k} \right) + \sum_{j=0}^N S^j \frac{\partial(\varphi u^n)}{\partial x_j} + A^n \rightarrow 0$$

in $L^2(\mathbb{R}^{N+1})$ strong,

since the commutator of S^j and M_φ is compact and applied to a weakly converging sequence gives a strongly converging sequence, and the terms which are linear in u^n (with no derivatives) converge to 0 in $L^2(\mathbb{R}^{N+1})$ strong (by the compact embedding of H^1 into L_{loc}^2); finally, the term A^n is linear in the derivatives $\frac{\partial u^n}{\partial x_k}$, $k = 0, \dots, N$, with coefficients containing one of the derivatives $\frac{\partial \varphi}{\partial x_l}$, $l = 0, \dots, N$, and these terms will disappear later, by taking φ equal to 1 on the support of a test function $w \in C_c^2((0, T) \times \Omega)$.

Since by the definition (2.1) $\psi\rho, \psi C_{jk} \in X^1(\mathbb{R}^N)$, $j, k = 1, \dots, N$, one may apply P_a to (2.15) with $a \in C^1(\mathbb{S}^{N-1})$, and use Corollary 1.1 of the second commutation lemma (Lemma 1.3), and obtain

$$(2.16) \quad \frac{\partial}{\partial t} \left(\psi\rho \frac{\partial(P_a\varphi u^n)}{\partial t} \right) - \sum_{j,k=1}^N \frac{\partial}{\partial x_j} \left(\psi C_{jk} \frac{\partial(P_a\varphi u^n)}{\partial x_k} \right) + P_a \sum_{j=0}^N S^j \frac{\partial(\varphi u^n)}{\partial x_j} \\ + P_a A^n + \sum_{k=0}^N L^k \frac{\partial(\varphi u^n)}{\partial x_k} \rightarrow 0 \quad \text{in } L^2(\mathbb{R}^{N+1}) \text{ strong,}$$

$$L^0, L^k \text{ having symbol } \xi_0\{a, \psi\rho\} \text{ and } \sum_{j=1}^N \xi_j\{a, \psi C_{jk}\}, \quad k = 1, \dots, N.$$

One then wants to apply Lemma 2.1 to $v = P_a\varphi u^n$, and apply a test function $w \in C_c^2((0, T) \times \Omega)$. Choosing φ (hence ψ) equal to 1 on the support of w , the term involving A^n disappears, and a few limits get simpler.

The term similar to $\frac{\partial}{\partial t} \left(\rho \frac{\partial v}{\partial t} \frac{\partial \bar{v}}{\partial t} + \sum_{j,k=1}^N C_{jk} \frac{\partial v}{\partial x_j} \frac{\partial \bar{v}}{\partial x_k} \right)$ applied to w gives at the limit

$$(2.17) \quad \lim_n \int_{\mathbb{R} \times \mathbb{R}^N} - \frac{\partial w}{\partial t} \left(\rho \frac{\partial(P_a\varphi u^n)}{\partial t} \frac{\partial \overline{(P_a\varphi u^n)}}{\partial t} \right. \\ \left. + \sum_{j,k} C_{jk} \frac{\partial(P_a\varphi u^n)}{\partial x_j} \frac{\partial \overline{(P_a\varphi u^n)}}{\partial x_k} \right) dt dx \\ = - \left\langle \frac{\partial w}{\partial t} \left(\rho\mu_{00} + \sum_{j,k} C_{jk}\mu_{jk} \right), a\bar{a} \right\rangle \\ = - \left\langle \pi, \left(\rho\xi_0^2 + \sum_{j,k} C_{jk}\xi_j\xi_k \right) \frac{\partial w}{\partial t} a\bar{a} \right\rangle.$$

The term similar to $-\sum_{j,k=1}^N \frac{\partial}{\partial x_j} \left(C_{jk} \frac{\partial v}{\partial x_k} \frac{\partial \bar{v}}{\partial t} + C_{jk} \frac{\partial \bar{v}}{\partial x_k} \frac{\partial v}{\partial t} \right)$ applied to w gives at the limit

$$(2.18) \quad \lim_n \int_{\mathbb{R} \times \mathbb{R}^N} \sum_{j,k} \frac{\partial w}{\partial x_j} \left(C_{jk} \frac{\partial(P_a\varphi u^n)}{\partial x_k} \frac{\partial \overline{(P_a\varphi u^n)}}{\partial t} \right. \\ \left. + C_{jk} \frac{\partial \overline{(P_a\varphi u^n)}}{\partial x_k} \frac{\partial(P_a\varphi u^n)}{\partial t} \right) dt dx \\ = \sum_{j,k=1}^N \left\langle \frac{\partial w}{\partial x_j} C_{jk} 2\mu_{k0}, a\bar{a} \right\rangle = \left\langle \pi, 2\xi_0 \sum_{j=1}^N \left(\sum_{k=1}^N C_{jk}\xi_k \right) \frac{\partial w}{\partial x_j} a\bar{a} \right\rangle.$$

The term similar to $\frac{\partial \rho}{\partial t} \frac{\partial v}{\partial t} \frac{\partial \bar{v}}{\partial t} - \sum_{j,k=1}^N \frac{\partial C_{jk}}{\partial t} \frac{\partial v}{\partial x_j} \frac{\partial \bar{v}}{\partial x_k}$ applied to w gives at the limit

$$\begin{aligned}
 (2.19) \quad & \lim_n \int_{\mathbb{R} \times \mathbb{R}^N} w \left(\frac{\partial \rho}{\partial t} \frac{\partial (P_a \varphi u^n)}{\partial t} \frac{\partial \overline{(P_a \varphi u^n)}}{\partial t} \right. \\
 & \quad \left. - \sum_{j,k} \frac{\partial C_{jk}}{\partial t} \frac{\partial (P_a \varphi u^n)}{\partial x_j} \frac{\partial \overline{(P_a \varphi u^n)}}{\partial x_k} \right) dt dx \\
 & = \left\langle w \left(\frac{\partial \rho}{\partial t} \mu_{00} - \sum_{j,k} \frac{\partial C_{jk}}{\partial t} \mu_{jk} \right), a\bar{a} \right\rangle \\
 & = \left\langle \pi, \left(\frac{\partial \rho}{\partial t} \xi_0^2 - \sum_{j,k} \frac{\partial C_{jk}}{\partial t} \xi_j \xi_k \right) wa\bar{a} \right\rangle.
 \end{aligned}$$

The term similar to $f \frac{\partial \bar{v}}{\partial t}$ applied to w gives at the limit

$$\begin{aligned}
 (2.20) \quad & \lim_n \int_{\mathbb{R} \times \mathbb{R}^N} -w \left(P_a \sum_{j=0}^N S^j \frac{\partial (\varphi u^n)}{\partial x_j} + \sum_{k=0}^N L^k \frac{\partial (\varphi u^n)}{\partial x_k} \right) \frac{\partial \overline{(P_a \varphi u^n)}}{\partial t} dt dx \\
 & = - \left\langle w \left(\sum_{j=0}^N s^j \mu_{j0} \right), a\bar{a} \right\rangle - \left\langle w \left(\xi_0 \{a, \rho\} \mu_{00} + \sum_{j,k=1}^N \xi_j \{a, C_{jk}\} \mu_{k0} \right), \bar{a} \right\rangle \\
 & = - \left\langle \pi, \xi_0 \left(\sum_{j=0}^N s^j \xi_j \right) wa\bar{a} + \xi_0 \left(\xi_0^2 \{a, \rho\} + \sum_{j,k=1}^N \xi_j \xi_k \{a, C_{jk}\} \right) w\bar{a} \right\rangle.
 \end{aligned}$$

The term similar to $\bar{f} \frac{\partial v}{\partial t}$ applied to w gives at the limit

$$(2.21) \quad - \left\langle \pi, \xi_0 \left(\sum_{j=0}^N \bar{s}^j \bar{\xi}_j \right) wa\bar{a} + \xi_0 \left(\xi_0^2 \{\bar{a}, \rho\} + \sum_{j,k=1}^N \bar{\xi}_j \bar{\xi}_k \{\bar{a}, C_{jk}\} \right) wa \right\rangle.$$

One must then write that the sum of (2.17), (2.18), and (2.19) is equal to the sum of (2.20) and (2.21), but a good way to gather the terms is to use the function $\Phi = wa\bar{a}$, and to make the Poisson bracket $\{\xi_0 \Phi, Q\}$ appear (with $Q = \rho \xi_0^2 - \sum_{j,k=1}^N C_{jk} \xi_j \xi_k$).

In (2.17), one notices that $\rho \xi_0^2 - \sum_{j,k=1}^N C_{jk} \xi_j \xi_k = 2\rho \xi_0^2$ since $Q = 0$ on the support of π , hence (2.17) is π applied to $-\xi_0 \frac{\partial Q}{\partial \xi_0} \frac{\partial \Phi}{\partial t}$.

(2.18) is π applied to $-\xi_0 \sum_j \frac{\partial Q}{\partial \xi_j} \frac{\partial \Phi}{\partial x_j}$.

(2.19) is π applied to $\frac{\partial Q}{\partial t} \Phi$, which is only a part of the expected term $\frac{\partial Q}{\partial t} \frac{\partial (\xi_0 \Phi)}{\partial \xi_0}$.

The sum of (2.20) and (2.21) has π applied to $-\xi_0 \Phi \sum_j (s^j \xi_j + \bar{s}^j \bar{\xi}_j)$, and then π applied to $-\xi_0 (\xi_0^2 \{a, \rho\} + \sum_{j,k=1}^N \xi_j \xi_k \{a, C_{jk}\}) w\bar{a}$ and π applied to

$-\xi_0(\xi_0^2\{\bar{a}, \rho\} + \sum_{j,k=1}^N \xi_j \xi_k \{\bar{a}, C_{jk}\})wa$, and by developing the Poisson brackets in these last two terms, one finds exactly the missing part of the Poisson bracket $\{\xi_0 \Phi, Q\}$, and the equation obtained is (2.14) with $\Psi = \xi_0 \Phi$.

Since ξ_0 does not vanish on the support of π (due to the sign condition on ρ), and since linear combinations (with complex coefficients) of terms of the form $wa\bar{a}$ can approach any smooth function in (x, ξ) , one deduces that (2.14) holds.

Equation (2.14) is a first order partial differential equation in (x, ξ) for π , written in weak form so that the partial derivatives appear on the test function Ψ ; the *characteristic curves* for this equation are given by

$$(2.22) \quad \frac{dx_j}{d\tau} = \frac{\partial Q}{\partial \xi_j}, \quad \frac{d\xi_j}{d\tau} = -\frac{\partial Q}{\partial x_j}, \quad j = 1, \dots, N,$$

which imply that Q is constant, and one only uses those curves corresponding to $Q = 0$ since the support of π is included there. As for (2.10), it is useful to *forget the constraint* $\xi \in \mathbb{S}^N$, and use the quotient space of $\mathbb{R}^{N+1} \setminus \{0\}$ by the equivalence relation that ξ is equivalent to $r\xi$ for all $r > 0$: if one chooses two different initial data for (2.22), $(x(\tau_0), \xi(\tau_0))$ and $(x'(\tau_0), \xi'(\tau_0))$ with $x'(\tau_0) = x(\tau_0)$ but $\xi'(\tau_0) = \lambda \xi(\tau_0)$, then the solution has $x'(\tau) = x(\lambda\tau)$ and $\xi'(\tau) = \lambda \xi(\lambda\tau)$ for all τ , i.e. (2.22) induces an evolution equation for half-rays, and (2.22) is an equation for *bicharacteristic rays*.

Since $\frac{\partial Q}{\partial x_j}$ is quadratic in ξ , the solution without the constraint $\xi \in \mathbb{S}^N$ could blow up, and it then seems better to impose the constraint and consider instead

$$(2.23) \quad \frac{dx_j}{d\tau} = \frac{\partial Q}{\partial \xi_j}, \quad \frac{d\xi_j}{d\tau} = -\frac{\partial Q}{\partial x_j} + \left(\sum_{k=1}^N \xi_k \frac{\partial Q}{\partial x_k} \right) \frac{\xi_j}{|\xi|^2}, \quad j = 0, \dots, N.$$

Since (2.23) for $j = 0$ contains $\frac{dt}{d\tau} = 2\rho\xi_0$, which does not vanish (because on $Q = 0$ one has $\xi_0 \neq 0$), one may use t for parametrizing the bicharacteristic rays: for $j = 1, \dots, N$ one has $\frac{dx_j}{d\tau} = -2(C\xi)_j$, so that

$$(2.24) \quad \frac{dx}{dt} = -\frac{C\xi}{\rho\xi_0}.$$

which may be interpreted as a *local group velocity*, that at which the energy propagates for high frequencies corresponding to the Fourier direction (ξ_0, ξ) . The *phase velocity* is only defined for plane waves: if for a unit spatial vector $\eta \in \mathbb{S}^{N-1}$ one considers functions of the form $f((x, \eta) - vt)$, then the phase velocity (in the direction η) is v ; if one considers highly oscillatory f then their Fourier transform use points going to infinity in the direction of $(-v, \eta,)$ in \mathbb{R}^{N+1} , so that (if the sequences converges weakly to 0 in $L^2_{loc}(\mathbb{R}^{N+1})$) the H-measure will be Dirac masses in the two directions $\pm(\xi_0, \xi)$ with $v = -\frac{\xi_0}{|\xi|}$ and $\eta = \frac{\xi}{|\xi|}$.

Without the not so natural constraint $(\xi_0, \xi) \in \mathbb{S}^N$, ξ_0 has dimension T^{-1} and ξ has dimension L^{-1} , while $\frac{C}{\rho}$ has dimension $L^2 T^{-2}$, so that both $\frac{C\xi}{\rho\xi_0}$ and $\frac{\xi_0}{|\xi|}$ have dimension LT^{-1} .

As pointed out by Patrick Gérard (after [Ta90], where I only considered the case of coefficients depending upon x), the usual hypothesis for ensuring uniqueness of solutions of (2.22) or (2.23) is local Lipschitz continuity, which uses second order derivatives of the coefficients (ρ and C_{ij} , $i, j = 1, \dots, N$), which I assume bounded here, but the abstract framework shown in the Appendix only requires second derivatives in t .

If $v \in L^2(\mathbb{R}^N)$ (or $L^2(\mathbb{R}^{N+1})$ here), then $\mathcal{F}v(-\xi) = \overline{\mathcal{F}v(\xi)}$, so that the H-measure corresponding to a sequence of real functions charges as much a direction $-\xi$ than the direction ξ , and one cannot send a “beam of light” in a direction without sending also a “beam of light” in the opposite direction: for headlights of a car, the source of light is near the focus of a (piece of a) parabolic mirror, and (in the approximation of geometrical optics) all the reflected rays are parallel to the direction of its axis.

In order to show the analogue of Lemma 2.1 for linearized elasticity ($N \geq 2$), one adds hypotheses (to those stated in (1.23) for Example 1.4), of symmetry, positivity, and regularity. For symmetry, one adds

$$(2.25) \quad C_{ij,kl} = C_{kl,ij}, \quad i, j, k, l = 1, \dots, N \text{ in } (0, T) \times \Omega,$$

so that the acoustic tensor $A(e)$ defined in (1.25) is symmetric. For positivity, one adds

$$(2.26) \quad \text{there exists } \alpha > 0, \quad \sum_{i,j,k,l=1}^N C_{ij,kl} M_{ij} M_{kl} \geq \alpha |M|^2$$

for all symmetric M , a.e. in $(0, T) \times \Omega$,

which is the *very strong ellipticity* condition, more constraining than the *strong ellipticity* condition

$$(2.27) \quad \text{there exists } \alpha > 0, \quad A(e) \geq \alpha |e|^2 I \text{ for all } e \in \mathbb{R}^N, \text{ a.e. in } (0, T) \times \Omega,$$

(or strong Legendre–Hadamard condition) which is the condition (2.26) only for matrices $M = a \otimes e + e \otimes a$. [I wrote that I do not know physical reasons for imposing (2.26), but (2.27) expresses the positivity of speeds for plane waves. Georges Verchery pointed out a derivation of (2.26) based on argument of thermodynamics, but my point is that thermodynamics is a faulty theory, which pretends to describe macroscopic properties without paying attention to meso-structures, and I advocate developing a better physical theory, incorporating information about transport at mesoscopic levels, describing heat as a sum of various modes propagating energy in various directions, using variants of H-measures, as I am trying here.]

LEMMA 2.2. Let $\rho, C_{ij;kl}, i, j, k, l = 1, \dots, N \in C^2((0, T) \times \Omega)$ be real, with $\rho \geq \alpha > 0$, and C satisfying the symmetries in (1.23), (2.25), and the very strong ellipticity condition (2.26). Let $v_1, \dots, v_N \in C^0(0, T; H^1(\Omega)) \cap C^1(0, T; L^2(\Omega))$ satisfy

$$(2.28) \quad \frac{\partial}{\partial t} \left(\rho \frac{\partial v_i}{\partial t} \right) - \sum_{j,k,l=1}^N \frac{\partial}{\partial x_j} \left(C_{ij;kl} \frac{\partial v_k}{\partial x_l} \right) = f_i \in L^2((0, T) \times \Omega), \quad i = 1, \dots, N,$$

then, in the sense of distributions in $(0, T) \times \Omega$

$$(2.29) \quad \begin{aligned} & \frac{\partial}{\partial t} \left(\sum_{i=1}^N \rho \frac{\partial v_i}{\partial t} \frac{\partial \bar{v}_i}{\partial t} + \sum_{i,j,k,l=1}^N C_{ij;kl} \frac{\partial v_k}{\partial x_l} \frac{\partial \bar{v}_i}{\partial x_j} \right) \\ & - \sum_{i,j,k,l=1}^N \frac{\partial}{\partial x_j} \left(C_{ij;kl} \frac{\partial v_k}{\partial x_l} \frac{\partial \bar{v}_i}{\partial t} + C_{ij;kl} \frac{\partial \bar{v}_k}{\partial x_l} \frac{\partial v_i}{\partial t} \right) \\ & = \sum_{i=1}^N \frac{\partial \rho}{\partial t} \frac{\partial v_i}{\partial t} \frac{\partial \bar{v}_i}{\partial t} + \sum_{i,j,k,l=1}^N \frac{\partial C_{ij;kl}}{\partial t} \frac{\partial v_k}{\partial x_l} \frac{\partial \bar{v}_i}{\partial x_j} + \sum_{i=1}^N \left(f_i \frac{\partial \bar{v}_i}{\partial t} + \bar{f}_i \frac{\partial v_i}{\partial t} \right). \end{aligned}$$

Sketch of proof. Formally, (2.28) implies (2.29) by developing the derivatives of products and regrouping the terms, but second order derivatives of v_1, \dots, v_N appear in the computation, hence it is not a proof, and a proof relies on the well-posedness of the linearized elasticity system with appropriate boundary conditions.

Since the coefficients $(\rho, C_{ij;kl}, i, j, k, l = 1, \dots, N)$ are real, one may separate the real part and the imaginary part of (2.28), and at the end add the results for the real part and for the imaginary part. Since (2.29) is local, one just needs to show it for $\varphi v_1, \dots, \varphi v_N$ with $\varphi \in C_c^\infty((0, T) \times \Omega)$, equal to 1 on an open set, where (2.29) is then proved. If the support of φ is included in $(0, T) \times \omega$ for an open set ω , these functions solve the linearized elasticity system with Dirichlet conditions on the boundary of ω (and 0 initial data), and (2.29) is just the non-integrated form of the *identity of energy*. Like for the proof of Lemma 2.1, one uses an abstract result shown in Appendix: first one proves regularity in t , then one shows uniqueness of weak solutions, which implies that $v_i \in C^0([0, T]; H_0^1(\omega)) \cap C^1([0, T]; L^2(\omega))$, $i = 1, \dots, N$, and finally one shows that they satisfy an identity of energy.

In order to avoid technicalities, I now want to check what the method gives concerning transport of H-measures for the homogeneous linearized elasticity system with *constant coefficients*.

THEOREM 2.3. Let $u_i^n \in C^0(0, T; H^1(\Omega)) \cap C^1(0, T; L^2(\Omega))$, $i = 1, \dots, N$ be a sequence of solutions of a linearized elasticity system

$$(2.30) \quad \frac{\partial}{\partial t} \left(\rho \frac{\partial u_i^n}{\partial t} \right) - \sum_{j,k,l=1}^N \frac{\partial}{\partial x_j} \left(C_{ij;kl} \frac{\partial u_k^n}{\partial x_l} \right) = 0 \quad \text{in } (0, T) \times \Omega, \quad i = 1, \dots, N,$$

with constant $\rho > 0$, $C_{ij;kl}$, $i, j, k, l = 1; \dots, N$ satisfying the symmetry conditions (1.23) and (2.25), and the very strong ellipticity condition (2.26). Assume that $u_i^n \rightarrow 0$ in $H_{loc}^1((0, T) \times \Omega)$ weak for $i = 1, \dots, N$, and (using $x_0 = t$) $V_{ij}^n = \frac{\partial u_i^n}{\partial x_j}$ (for $j = 0, \dots, N$) correspond to an H-measure μ , satisfying (by Example 1.2) $\mu_{ij;lm} = \xi_j \xi_m \pi_{il}$ for $i, l = 1, \dots, N$, and $j, m = 0, \dots, N$. By (1.26) the support of π is included in the set where $\det(A(e) - \xi_0^2 \rho I) = 0$, with $\xi = (\xi_0, e)$ and the acoustic tensor $A(e)$ defined in (1.25), and π satisfies

$$(2.31) \quad \frac{\partial}{\partial t} \left(\left(\rho \xi_0^2 \sum_i \pi_{ii} \right) + \left(\sum_{i,k} A_{ik}(e) \pi_{ki} \right) \right) - \sum_{i,j,k} \frac{\partial}{\partial x_j} \left(\xi_0 \frac{\partial A_{ik}(e)}{\partial \xi_j} \frac{\pi_{ki} + \pi_{ik}}{2} \right) = 0 \quad \text{in } \Omega \times \mathbb{S}^N.$$

PROOF. For $\varphi \in C_c^2((0, T) \times \Omega)$, φu_i^n , $i = 1, \dots, N$ solves

$$(2.32) \quad \frac{\partial}{\partial t} \left(\rho \frac{\partial(\varphi u_i^n)}{\partial t} \right) - \sum_{j,k,l=1}^N \frac{\partial}{\partial x_j} \left(C_{ij;kl} \frac{\partial(\varphi u_k^n)}{\partial x_l} \right) = g_i^n + h_i^n \quad \text{in } \mathbb{R} \times \mathbb{R}^N, \quad i = 1, \dots, N,$$

where

$$(2.33) \quad g_i^n = 2 \frac{\partial \varphi}{\partial t} \rho \frac{\partial u_i^n}{\partial t} - \sum_{j,k,l=1}^N 2 \frac{\partial \varphi}{\partial x_j} C_{ij;kl} \frac{\partial u_k^n}{\partial x_l} \quad \text{in } \mathbb{R} \times \mathbb{R}^N, \quad i = 1, \dots, N,$$

h_i^n tends strongly to 0 in $L^2(\mathbb{R} \times \mathbb{R}^N)$.

By taking $\varphi = 1$ on the support of a test function w , the contributions of the g_i^n , $i = 1, \dots, N$ to the equation for the H-measure disappear, since they contain the derivatives of φ , which are 0 on the support of w .

For $a \in C^1(\mathbb{S}^{N-1})$ one may apply P_a to (2.32) and obtain

$$(2.34) \quad \frac{\partial}{\partial t} \left(\rho \frac{\partial(P_a \varphi u_i^n)}{\partial t} \right) - \sum_{j,k,l=1}^N \frac{\partial}{\partial x_j} \left(C_{ij;kl} \frac{\partial(P_a \varphi u_k^n)}{\partial x_l} \right) = P_a g_i^n + P_a h_i^n$$

in $\mathbb{R} \times \mathbb{R}^N$, $i = 1, \dots, N$,

avoiding Corollary 1.1 of the second commutation lemma (Lemma 1.3), since the coefficients are constant.

One then applies Lemma 2.2 to $v_i = P_a \varphi u_i^n$, $i = 1, \dots, N$, and one uses a test function $w \in C_c^2((0, T) \times \Omega)$, but choosing φ equal to 1 on the support of w , so that the limits are

$$\begin{aligned}
 (2.35) \quad & \lim_n \left\langle \frac{\partial}{\partial t} \left(\sum_{i=1}^N \rho \frac{\partial(P_a \varphi u_i^n)}{\partial t} \frac{\partial \overline{P_a \varphi u_i^n}}{\partial t} + \sum_{i,j,k,l=1}^N C_{ij;kl} \frac{\partial(P_a \varphi u_k^n)}{\partial x_l} \frac{\partial \overline{P_a \varphi u_i^n}}{\partial x_j} \right), w \right\rangle \\
 &= - \sum_{i=1}^N \left\langle \mu_{i0; i0}, \rho a \bar{a} \frac{\partial w}{\partial t} \right\rangle - \sum_{i,j,k,l=1}^N \left\langle \mu_{kl; ij}, C_{ij;kl} a \bar{a} \frac{\partial w}{\partial t} \right\rangle \\
 &= - \sum_{i=1}^N \left\langle \pi_{ii}, \rho \xi_0^2 a \bar{a} \frac{\partial w}{\partial t} \right\rangle - \sum_{i,k=1}^N \left\langle \pi_{ki}, \left(\sum_{j,l=1}^N C_{ij;kl} \xi_j \xi_l \right) a \bar{a} \frac{\partial w}{\partial t} \right\rangle.
 \end{aligned}$$

and

$$\begin{aligned}
 (2.36) \quad & \lim_n - \left\langle \sum_{i,j,k,l} \frac{\partial}{\partial x_j} \left(C_{ij;kl} \frac{\partial(P_a \varphi u_k^n)}{\partial x_l} \frac{\partial \overline{P_a \varphi u_i^n}}{\partial t} + C_{ij;kl} \frac{\partial \overline{P_a \varphi u_k^n}}{\partial x_l} \frac{\partial(P_a \varphi u_i^n)}{\partial t} \right), w \right\rangle \\
 &= \sum_{i,j,k,l=1}^N \left\langle \mu_{kl; i0} + \mu_{i0; kl}, C_{ij;kl} a \bar{a} \frac{\partial w}{\partial x_j} \right\rangle \\
 &= \sum_{i,k=1}^N \left\langle \pi_{ki} + \pi_{ik}, \xi_0 \left(\sum_{j,l=1}^N C_{ij;kl} \xi_l a \bar{a} \frac{\partial w}{\partial x_j} \right) \right\rangle,
 \end{aligned}$$

and the other terms tend to 0. If one denotes $\Phi = a \bar{a} w$, then one has proved

$$\begin{aligned}
 (2.37) \quad & - \left\langle \rho \xi_0^2 \left(\sum_{i=1}^N \pi_{ii} \right) + \sum_{i,k=1}^N A_{ik}(e) \pi_{ki}, \frac{\partial \Phi}{\partial t} \right\rangle \\
 &+ \sum_{i,j,k=1}^N \left\langle \frac{\pi_{ki} + \pi_{ik}}{2}, \xi_0 \frac{\partial A_{ik}(e)}{\partial \xi_j} \frac{\partial \Phi}{\partial x_j} \right\rangle = 0,
 \end{aligned}$$

recalling that e is the spatial part of ξ (of components ξ_1, \dots, ξ_N). Using linear combinations of these special Φ permits to use any smooth function, hence the partial differential equation (2.31).

One notices that (2.31) has no derivatives in ξ of the H-measure π , since they come from the variations in (t, x) of the coefficients, which here are assumed constant.

For the transport of the H-measure π , one completes (2.31) with (1.24), i.e. $\xi_0^2 \rho \pi_{i,r} = \sum_{k=1}^N A_{ik}(e) \pi_{kr}$ for $i, r = 1, \dots, N$, which implies that on the support of π an eigenvalue of $A(e)$ is $\xi_0^2 \rho$, i.e. (1.26).

The situation for linearized elasticity is then not exactly like that for a scalar first order equation or a wave equation: after using the quadratic conservation law behind the balance of (total) energy, one must still use the consequences of the localization principle, and (1.24) contains more information than (1.26).

Let us check what these equations say in the *isotropic case*, where μ is the shear modulus and λ the Lamé parameter, and the very strong ellipticity condition corresponds to $\mu > 0, 2\mu + N\lambda > 0$:

$$(2.38) \quad \sigma_{ij} = 2\mu\epsilon_{ij} + \lambda \operatorname{div}(u)\delta_{ij}, \quad i, j = 1, \dots, N, \text{ so that}$$

$$A(e) = \mu|e|^2 I + (\mu + \lambda)e \otimes e, \quad e \in \mathbb{R}^N, \text{ with eigenvalues and eigenvectors}$$

$$\lambda_P(e) = (2\mu + \lambda)|e|^2, \text{ with eigenvector proportional to } e \text{ (multiplicity 1),}$$

$$\lambda_S(e) = \mu|e|^2, \text{ with eigenvector any } f \text{ orthogonal to } e, \text{ (multiplicity } N-1),$$

the index P recalling (longitudinal) P-waves or pressure waves ($\operatorname{div}(u)$ satisfies a wave equation), and the index S recalling (transverse) S-waves or shear waves ($\operatorname{curl}(u)$ satisfies a wave equation).

Since by (1.26) the support of π is included in the set where $\det(A(e) - \xi_0^2 \rho I) = 0$, it selects two disjoint closed sets of “ \mathbb{S}^N ” (because the coefficients do not depend upon (t, x)):

$$(2.39) \quad \mathbb{S}_P^N, \text{ defined by } \xi_0^2 \rho = (2\mu + \lambda)|e|^2,$$

$$\mathbb{S}_S^N, \text{ defined by } \xi_0^2 \rho = \mu|e|^2,$$

and (1.24), which expresses the localization principle (Lemma 1.2), means that π decomposes into two parts living in disjoint open sets, π^P in a neighbourhood of $(0, T) \times \Omega \times \mathbb{S}_P^N$ with support in $(0, T) \times \Omega \times \mathbb{S}_P^N$, and π^S in a neighbourhood of $(0, T) \times \Omega \times \mathbb{S}_S^N$ with support in $(0, T) \times \Omega \times \mathbb{S}_S^N$, having the form

$$(2.40) \quad \pi_{jk}^P = e_j e_k v^P, \quad j, k = 1, \dots, N, \quad v^P \text{ a non-negative Radon measure}$$

with support in $(0, T) \times \Omega \times \mathbb{S}_P^N$,

$$\pi_{jk}^S = M_{jk} v^S, \quad j, k = 1, \dots, N, \quad v^S \text{ a non-negative Radon measure}$$

with support in $(0, T) \times \Omega \times \mathbb{S}_S^N$,

and M Hermitian symmetric ≥ 0 , v^S -integrable,

with $(Me, e) = 0$, v^S -almost everywhere.

[Without $\xi \in \mathbb{S}^N$, (2.39) is consistent with $\frac{\mu}{\rho}$ and $\frac{\lambda}{\rho}$ having dimension $L^2 T^{-2}$, but imposing $\xi \in \mathbb{S}^N$ gives $\xi_0^2 = \frac{2\mu + \lambda}{2\mu + \lambda + \rho}$ on \mathbb{S}_P^N and $\xi_0^2 = \frac{\mu}{2\mu + \lambda + \rho}$ on \mathbb{S}_S^N , where one adds quantities with different dimensions.]

By taking $r = i$ in (1.24) and summing in i one obtains

$$(2.41) \quad \sum_{i,k=1}^N A_{ik}(e)\pi_{ki} = \rho \xi_0^2 \sum_{i=1}^N \pi_{ii} \quad \text{in } (0, T) \times \Omega \times \mathbb{S}^N,$$

so that the t derivative term in (2.31) is $2 \frac{\partial}{\partial t} (\rho \xi_0^2 \sum_{i=1}^N \pi_{ii})$, and by (2.40)

$$(2.42) \quad \sum_{i=1}^N \pi_{ii} = \begin{cases} |e|^2 v^P & \text{on } (0, T) \times \Omega \times \mathbb{S}_P^N \\ \text{trace}(M) v^S & \text{on } (0, T) \times \Omega \times \mathbb{S}_S^N \end{cases}.$$

The x_j derivative term in (2.31) is $-\sum_{i,k=1}^N (\xi_0 \frac{\partial A_{ik}(e)}{\partial \xi_j} \frac{\pi_{ki} + \pi_{ik}}{2})$, and since by (2.38) $A_{ik}(e) = \mu |e|^2 \delta_{ik} + (\mu + \lambda) e_i e_k$, and remembering that e is a notation for the spatial part of ξ , one has

$$(2.43) \quad \frac{\partial A_{ik}(e)}{\partial \xi_j} = 2\mu \xi_j \delta_{ik} + (\mu + \lambda) (\xi_k \delta_{ij} + \xi_i \delta_{kj}),$$

$$i, j, k = 1, \dots, N \text{ in } (0, T) \times \Omega \times \mathbb{R}^{N+1}.$$

The sum to consider has a different form on \mathbb{S}_P^N or \mathbb{S}_S^N

$$(2.44) \quad - \sum_{i,k=1}^N \xi_0 \frac{\partial A_{ik}(e)}{\partial \xi_j} \frac{\pi_{ki} + \pi_{ik}}{2}$$

$$= \begin{cases} -2(2\mu + \lambda) \xi_0 \xi_j |e|^2 v^P & \text{on } (0, T) \times \Omega \times \mathbb{S}_P^N, \\ -2\mu \xi_0 \xi_j \text{trace}(M) v^S & \text{on } (0, T) \times \Omega \times \mathbb{S}_S^N, \end{cases}$$

since by (2.40) Me is 0 v^S -almost everywhere on $(0, T) \times \Omega \times \mathbb{S}_S^N$.

On $(0, T) \times \Omega \times \mathbb{S}_P^N$ one then has $\sum_{i=1}^N \pi_{ii} = |e|^2 v^P$ satisfying a first order transport equation, in the direction $\frac{-2(2\mu + \lambda)\xi_0}{2\rho\xi_0^2} e$, i.e. $\frac{-(2\mu + \lambda)}{\rho\xi_0} e$, which when compared to (2.24) is what one expects for the wave equation satisfied by $\text{div}(u^n)$. Similarly, on $(0, T) \times \Omega \times \mathbb{S}_S^N$ one has $\sum_{i=1}^N \pi_{ii} = \text{trace}(M) v^S$ satisfying a first order transport equation, in the direction $\frac{-2\mu\xi_0}{2\rho\xi_0^2} e$, i.e. $\frac{-\mu}{\rho\xi_0} e$, which when compared to (2.24) is what one expects for the wave equation satisfied by $\text{curl}(u^n)$.

However, if on $(0, T) \times \Omega \times \mathbb{S}_P^N$ the trace $\sum_{i=1}^N \pi_{ii}$ permits to recover what π is there, one does not recover what π is on $(0, T) \times \Omega \times \mathbb{S}_S^N$ using only $\text{trace}(M) v^S$.

There is then more work to do for the question of transport of H-measures in linearized elasticity.

Another type of question which I hope to address in the future is to clarify the question of *initial data*, or more generally of boundary effects, like *reflection effects for beams of light*, but there are also effects at internal interfaces (between different materials), like *refraction effects for beams of light*.

Using my multiscales H-measures, which I introduced in [Ta15], or new and more efficient microlocal tools, I also hope to address in the future the question of explaining the formal computations of Joe Keller's *Geometrical Theory of Diffraction* (GTD), and in particular an experimental observation which I learned

from Michel Gondran, and an interesting 1818 episode which he mentions in his book with his son [G-G14]: for a prize of Académie des Sciences in Paris, on the subject of diffraction, Fresnel had imagined an hypothetic medium called æther (which was thought to exist until the 1887 experiment of Michelson and Morley), but a member of the jury (Poisson) who disliked the wave nature of light (and preferred Newton's idea of a particle nature of light) found that Fresnel's ideas imply that there would be a bright spot in the middle of the shadow of a solid opaque sphere illuminated by a point source, which he thought nonsense; however, the president of the jury (Arago) was a partisan of the wave nature of light, and he ordered it to be checked in a careful experiment, and the spot is there, which one now calls either the *Poisson spot* or the *Arago spot*.

Since Joe Keller had mentioned to me that his computations of *light creeping into the shadow* remind of the tunneling effect in quantum mechanics, except that light does not go through the obstacle, but *around the obstacle*, there is much more than questions about light behind studying his formal computations.

3. APPENDIX

One considers a second order equation (not necessarily hyperbolic, since the x variable is not explicit in this abstract framework) in the classical framework with three separable Hilbert spaces (sometimes attributed to Gel'fand) which I learned from Jacques-Louis Lions: V (with norm $\|\cdot\|$) dense in H (with norm $|\cdot|$), identified to its dual H' , so that H is dense in V' (with dual norm $\|\cdot\|_*$); the equation is

$$(A1) \quad \frac{d}{dt} \left(M \frac{du}{dt} \right) + Au = f \quad \text{in } (0, T), \text{ with initial data } u(0) = a, \frac{du}{dt}(0) = b,$$

with

$$(A2) \quad M \in C^0([0, T]; \mathcal{L}(H, H)), \quad \frac{dM}{dt} \in L^1(0, T; \mathcal{L}(H, H)), \\ M^*(t) = M(t) \quad \text{for all } t \in [0, T], \\ \text{and there exists } \gamma > 0 \text{ such that } (Mh, h) \geq \gamma|h|^2 \text{ for all } h \in H,$$

and

$$(A3) \quad A \in C^0([0, T]; \mathcal{L}(V, V')), \quad \frac{dA}{dt} \in L^1(0, T; \mathcal{L}(V, V')), \\ A^*(t) = A(t) \quad \text{for all } t \in [0, T], \\ \text{and there exists } \alpha > 0 \text{ such that } \langle Av, v \rangle_{V', V} \geq \alpha\|v\|^2 \text{ for all } v \in V.$$

Under the hypotheses (A2)–(A3) one first proves an existence result for weak solutions:

(A4) if $f \in L^1(0, T; H)$, $u_0 \in V$, $u_1 \in H$, there exists $u \in L^\infty(0, T; V)$,
 $\frac{du}{dt} \in L^\infty(0, T; H)$, $u(0) = a$, which makes sense since $u \in \text{Lip}([0, T]; H)$,

weak solution of (A1) in the sense that

$$\begin{aligned} & \int_0^T \left(- \left(M \frac{du}{dt}, e \right) \frac{d\varphi}{dt} + \langle Au, e \rangle_{V', V} \varphi \right) dt \\ &= \int_0^T (f, e) \varphi dt + (M(0)b, e) \varphi(0), \end{aligned}$$

for all $e \in V$, $\varphi \in C^1([0, T])$, with $\varphi(T) = 0$,

so that $M \frac{du}{dt}$ is absolutely continuous in V' , with initial data $M(0)b$.

This is proved using a method attributed to Galerkin, but also to Faedo (and to Ritz), where for a ‘‘Galerkin basis’’ (linearly independent elements of V which span a dense subspace in V , hence of H) w_1, \dots one looks for an approximate solution $u_n \in V_n = \text{span}\{w_1, \dots, w_n\}$ satisfying

$$\begin{aligned} \text{(A5)} \quad & \left(\frac{d}{dt} \left(M \frac{du_n}{dt} \right), w_j \right) + \langle Au_n, w_j \rangle_{V, V'} = (f, w_j) \text{ in } (0, T), \quad j = 1, \dots, n, \text{ with} \\ & u_n(0) = a_n \in V_n \text{ (converging to } a \text{ in } V), \\ & \frac{du_n}{dt}(0) = b_n \in V_n \text{ (converging to } b \text{ in } H), \end{aligned}$$

which is an ordinary differential equation whose global existence is proved by finding bounds independent of t and convergence (of a subsequence) in corresponding weak or weak $*$ topologies is proved by having bounds independent of n : they follow from taking in (A5) the combination of w_1, \dots, w_n giving $\frac{du_n}{dt}$, since

$$\begin{aligned} \text{(A6)} \quad & \frac{d}{dt} \left(\left(M \frac{du_n}{dt}, \frac{du_n}{dt} \right) + \langle Au_n, u_n \rangle_{V, V'} \right) \\ &= 2 \left(f, \frac{du_n}{dt} \right) - \left(\frac{dM}{dt} \frac{du_n}{dt}, \frac{du_n}{dt} \right) + \left\langle \frac{dA}{dt} u_n, u_n \right\rangle_{V, V'}, \end{aligned}$$

from which uniform bounds are found (using Gronwall’s inequality). Although u_n is bounded in $C^0([0, T]; V)$, the weak $*$ limit u_∞ of a subsequence is found in $L^\infty(0, T; V)$; although $\frac{du_n}{dt}$ is bounded in $C^0([0, T]; H)$, taking another weak $*$ limit gives $\frac{du_\infty}{dt}$ in $L^\infty(0, T; H)$; also, it does not permit to pass to the limit in the quadratic (or sesqui-linear) terms in (A6). One then looks for a regularity result in t : assuming bounds on $\frac{df}{dt}$ (and natural improvements on a and b), one looks for a bound of $\frac{d^2 u_n}{dt^2}$ in H and $\frac{du_n}{dt}$ in V , but

$$\begin{aligned}
 \text{(A7)} \quad & \frac{d}{dt} \left(\left(M \frac{du_n^2}{dt^2}, \frac{d^2u_n}{dt^2} \right) + \left\langle A \frac{du_n}{dt}, \frac{du_n}{dt} \right\rangle_{V, V'} \right) \\
 &= 2 \left(f, \frac{d^2u_n}{dt^2} \right) - 3 \left(\frac{dM}{dt} \frac{d^2u_n}{dt^2}, \frac{d^2u_n}{dt^2} \right) \\
 &\quad - 2 \left(\frac{d^2M}{dt^2} \frac{du_n}{dt}, \frac{d^2u_n}{dt^2} \right) - \left\langle \frac{dA}{dt} \frac{du_n}{dt}, \frac{du_n}{dt} \right\rangle_{V, V'} \\
 &\quad - 2 \left\langle \frac{dA}{dt} u_n, \frac{d^2u_n}{dt^2} \right\rangle_{V, V'},
 \end{aligned}$$

and the last term is not under control since one has no bound for $\frac{d^2u_n}{dt^2}$ in V , so that one writes

$$\begin{aligned}
 \text{(A8)} \quad & -2 \left\langle \frac{dA}{dt} u_n, \frac{d^2u_n}{dt^2} \right\rangle_{V, V'} = -2 \frac{d}{dt} \left(\left\langle \frac{dA}{dt} u_n, \frac{du_n}{dt} \right\rangle_{V, V'} \right) \\
 &\quad + 2 \left(\left\langle \frac{dA}{dt} \frac{du_n}{dt}, \frac{du_n}{dt} \right\rangle_{V, V'} \right) \\
 &\quad + 2 \left(\left\langle \frac{d^2A}{dt^2} u_n, \frac{du_n}{dt} \right\rangle_{V, V'} \right),
 \end{aligned}$$

and everything gets under control, at the expense of assuming that $\frac{d^2A}{dt^2} \in L^1(0, T; \mathcal{L}(V, V'))$, besides assuming that $a \in D(A(0))$ and $b \in V$, but the hypothesis $\frac{d^2M}{dt^2} \in L^1(0, T; \mathcal{L}(H, H))$ is also used.

This permits to find solutions more regular than $C^0([0, T]; V) \cap C^1([0, T]; H)$. For proving uniqueness of weak solutions, i.e. show that $f = 0, a = 0, b = 0$ imply $u = 0$, one notices that (A4) in this case implies

$$\begin{aligned}
 \text{(A9)} \quad & \int_0^T \left(- \left(M \frac{du}{dt}, \frac{d\psi}{dt} \right) + \langle Au, \psi \rangle_{V', V} \right) dt = 0 \\
 & \text{if } \psi \in C^0([0, T]; V) \cap C^1([0, T]; H) \text{ with } \psi(T) = 0,
 \end{aligned}$$

and taking for ψ a solution of

$$\begin{aligned}
 \text{(A10)} \quad & \frac{d}{dt} \left(M^* \frac{d\psi}{dt} \right) + A^* \psi = 0 \text{ in } (0, T), \text{ with final data } \psi(T) = 0, \\
 & \frac{d\psi}{dt}(T) = c \in V,
 \end{aligned}$$

one deduces that

$$\text{(A11)} \quad (u(T), M^*(T)c) = 0 \quad \text{for all } c \in V, \text{ hence } u(T) = 0;$$

applying then a similar argument on $[0, S]$ with $S \in (0, T)$ gives $u = 0$.

If one approaches $f \in L^1(0, T; H)$ strongly by a sequence $f_n \in L^1(0, T; H)$ with $\frac{df_n}{dt} \in L^1(0, T; H)$, $a \in V$ strongly by a sequence $a_n \in D(A(0))$, and $b \in H$ strongly by a sequence $b_n \in V$, the sequence of solutions u_n is a Cauchy sequence in $C^0([0, T]; V) \cap C^1([0, T]; H)$, and since (A6) holds for this (new) definition of u_n , it is valid for the limit u , the unique weak solution corresponding to data f, a, b .

REFERENCES

- [C-R-W76] R. COIFMAN - R. ROCHBERG - G. WEISS, “Factorization theorems for Hardy spaces in several variables,” *Ann. of Math. (2)* 103 (1976), no. 3, 611–635.
- [Eh56] L. EHRENPREIS, “On the theory of kernels of Schwartz,” *Proc. Amer. Math. Soc.* 7 (1956), 713–718.
- [G-L59] L. GÅRDING - J.-L. LIONS, “Functional analysis,” *Nuovo Cimento* (10) 14 (1959) supplemento, 9–66.
- [Gé91] P. GÉRARD, “Microlocal defect measures,” *Comm. Partial. Diff. Eq.* 16, 1761–1794 (1991).
- [G-G14] M. GONDRAN - A. GONDRAN, *Mécanique quantique: Et si Einstein et de Broglie avaient aussi raison?*, Éditions Matériologiques, Paris, 2014.
- [Li61] J.-L. LIONS, *Équations différentielles opérationnelles et problèmes aux limites*, Springer, 1961.
- [Ta78] L. TARTAR, *Topics in nonlinear analysis*, Pub. Math. Orsay Paris-Sud, 1978.
- [Ta90] L. TARTAR, “ H -measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations,” *Proc. Royal Soc. Edinburgh*, 115A, 193–230, (1990).
- [Ta98] L. TARTAR, “Homogenization and Hyperbolicity,” *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* 25 (1997), no. 3–4, 785–805 (1998).
- [Ta10] L. TARTAR, *The General Theory of Homogenization, a Personalized Introduction*, *Lectures Notes of Unione Matematica Italiana*, vol. 7, Springer (2010).
- [Ta15] L. TARTAR, “Multi-scales H -measures,” *Discrete and Continuous Dynamical Systems*, S 8 (2015), 77–90.

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