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Mathematical Analysis — *On the regularity of the roots of a polynomial depending on one real parameter*, by Ferruccio Colombini, Nicola Orrù and Ludovico Pernazza, communicated on June 15, 2017.

Alla memoria di Ennio De Giorgi.

Abstract. — We investigate the regularity of functions τ of one variable such that $P(t, \tau(t)) = 0$, where $P(t, x)$ is a given polynomial of degree m in x whose coefficients are functions of class $C^{m^2!}$ of one real parameter. We show that if a root is chosen with a continuous dependence on the parameter, this function is indeed absolutely continuous. From this and a theorem of Kato one deduces that such polynomials have complete systems of roots that are absolutely continuous functions.

Key words: Square roots, bounded variation, absolute continuity

Mathematics Subject Classification: 26A15, 26A27

Introduction

The problem of finding a parametrization of the roots of a polynomial whose coefficients depend on a (real) parameter t has a long story. In the case of symmetric operators it has been analyzed by Rellich [10, Par. I.5]; in the general case of continuous dependence on the parameter the problem was solved by Kato [5, Theorems II.5.1, II.5.2]:

Theorem 0.1 (Kato). *If $P(t, x)$ is a polynomial of degree m in one indeterminate, whose coefficients depend continuously on the* (*real or complex*) *parameter t, the unordered m-ple of roots depends continuously on t* (*in the metric topology of unordered m-ples, i.e. the minimum distance of all their possible orderings*)*. If the roots are always real, or the parameter varies on a real interval, it is also possible to enumerate the roots for every value of t in a continuous way by means of m continuous functions $\sigma_i(t)$ $(i = 1, \dots, m)$.*

This theorem cannot be improved in the following sense: it is easy to see that a continuous enumeration of the roots is not possible in the case of the parameter varying in a complex domain (e.g., $P(x, t) = x^2 - t$). It is also easy to define polynomials that do not admit Lipschitz continuous enumerations of their roots, even in the case of coefficients depending smoothly (i.e., of class C^∞) on the parameters. It is possible, though, to have Lipschitz continuity of the square root of a non-negative function of class C^2, as proved by Glaeser [4]; and to have $\frac{\alpha}{m}$-Hölder continuity if the coefficients are α-Hölder continuous, as proved by Malgrange [7, IV.2.2].

Better results are possible if all the roots are real, i.e. for hyperbolic polynomials. With sufficient regularity of the coefficients, Brohnstein showed in [2] that Lipschitz continuous roots can be chosen; improved results, with additional regularity conditions, were given by Mandai in [8], Alekseevski, Kriegl, Losik and Michor in [1] and [6] and the authors in [3].

Nevertheless, the question of what can be the best possible regularity for any enumeration of the roots, or even of just one root, is still open. In this paper we show that if sufficient regularity is asked on the coefficients of the polynomial, any continuous root (and therefore, for example, any function in a Kato enumeration) is indeed absolutely continuous (for polynomials of degree 2 and 3 Spagnolo proved a slightly more precise result in [12]).

The proof is divided in two parts: in the first part, by far the longer one, after some preliminary reductions, we prove that any such root has bounded variation (Theorem 1.6); in the second part, by a simple argument, we deduce that it is indeed absolutely continuous.

A very similar result has been proved using completely different techniques by Parusiński and Rainer in [9]. An improved result has since been announced by the same authors (but it is unpublished as yet).

In the last section we include two examples, one for polynomials of degree 3 and one for m-th roots. The proof in these cases is simpler, but in the same vein of the general case: in this way the reader, if necessary, can follow the proof of Section 2 considering these cases as a guide.

1. Statements and proofs

We prove the following

Theorem 1.1. *Let $P(t, \tau)$ be a monic polynomial of degree m in τ whose coefficients are functions of class $C^{m^2!}$ in $t \in [0, T]$. Then the continuous roots of $P(t, \tau)$ are absolutely continuous.*

It is maybe worth noting that by e.g. Theorem 0.1 such roots always exist.

Lemma 1.2. *In the hypotheses of Theorem 1.1, there exists a monic polynomial $Q(t, \tau)$ of degree m^2 in τ, with real coefficients of class $C^{m^2!}$ in t, some roots of which are the real parts $\operatorname{Re}\tau_j(t)$ of the roots of $P(t, \tau)$ (resp. the imaginary parts $\operatorname{Im}\tau_j(t)$ of the roots of $P(t, \tau)$).*

Proof. It is sufficient to note that the coefficients of the polynomial

$$\prod_{i,j}\left(\tau - \frac{\tau_i + \bar{\tau}_j}{2}\right)$$

are separately symmetric functions of the τ_i's and of the $\bar{\tau}_j$'s and are therefore polynomials in the coefficients of P. □

We fix some terminology and notation that we will use consistently.

DEFINITION 1.3. If n is a positive integer and A is a set, an *unordered n-ple of elements of A* is an element of the symmetric n-product of A,

$$SP^n(A) = A^n/S_n,$$

that is, an equivalence class of the action of the symmetric group on standard n-ples of elements of A. We will denote such an unordered n-ple by $[a_1, \dots, a_n]$.

It is clear that the complex roots of a (real or complex) polynomial of degree n are well represented by unordered n-ples of complex numbers.

DEFINITION 1.4. If $\bar{a} = [a_1, \dots, a_n]$ is an unordered n-ple of elements of A and $\bar{b} = (b_1, \dots, b_k)$ is a k-ple of elements of A, we say that $\bar{b}$ is a *partial enumeration, or a subsystem, of $\bar{a}$* if there is an element $(a_1, \dots, a_n)$ in the class of $\bar{a}$ such that $b_i = a_i$ for every $i = 1, \dots, k$.

In a maybe more traditional way, if $\bar{a}$ is the unordered n-ple of complex roots of a polynomial P of degree n and $\bar{b}$ is a partial enumeration of $\bar{a}$, we will also say that $\bar{b}$ are *k roots of P, when counted with multiplicities.*

If the polynomial P depends on a parameter the same terminology will be used for unordered n-ples of roots depending on the parameter, if the condition holds for every value of the parameter, possibly restricting its possible values.

DEFINITION 1.5. Let $\rho : [a, b] \to \mathbb{R}$ be a real piecewise linear function defined on a real interval $[a, b]$, let $A \in \mathbb{R}$ and let $S \subset \mathbb{R}^2$. The number of times that ρ "crosses" the value A, that is, attains the value A at some point t_0, being smaller than A on one side of t_0 and bigger than A on the other side of t_0, but with the point $(t_0, A) \notin S$, will be called the *number of oscillations of ρ (across A) outside S* (horizontal segments are counted as single points). If ρ never attains a local maximum or minimum at A and is not identically A on any subinterval, this is in fact $|((\rho^{-1}(A) \times \{A\}) \cap ([a, b] \times \mathbb{R})) \backslash S|$.

If we make no mention of S it is to be understood that $S = \emptyset$, and if we omit to mention the value A it is to be understood that $A = 0$.

We are now ready to prove that $\operatorname{Re} \tau_j(t)$ and $\operatorname{Im} \tau_j(t)$ are of bounded variation.

PROPOSITION 1.6. *Let*

$$P(t, \tau) = \tau^m + a_1(t)\tau^{m-1} + a_2(t)\tau^{m-2} + \cdots + a_m(t)$$

be a polynomial of degree m whose coefficients are real functions $a_j(t)$ of class $C^{m!}$ in t. Let $\sigma(t)$ be a real and continuous root of P. Then $\sigma(t)$ is of bounded variation on $[0, T]$.

PROOF. We will roughly proceed as follows: given any subdivision of $[0, T]$, we will define a suitable set of "tubular domains" in $\mathbb{R} \times \mathbb{C}$ such that for all values of t all the roots belong to the union of these domains, and that they behave in a tame way inside them (i.e., the number of their oscillations inside them is bounded).

To do this, we apply induction, and indeed most of the proof is devoted to show how to make the inductive step.

More precisely, at step i we will assume that we already defined "tubular domains of level $i-1$" containing at least $i-1$ roots for every value of t (when counted with multiplicities). We then choose a suitable subsequence of "points of level i", $\mathscr{T}_i$, among those of level $i-1$ and showing that, if we modify some of the tubular domains defined at level $i-1$, and add to them new tubular domains (associated to each interval defined by the points in $\mathscr{T}_i$), all these domains together contain at least i roots for every value of t (when counted with multiplicities).

Finally, as the radii of the domains (as functions of the lengths of the intervals) will be sublinear, we will be able to find a bound independent from the subdivision.

Let then

$$0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_n = T$$

be a subdivision $\mathscr{T}_0$ of $[0, T]$, and let $\rho(t)$ be the (continuous) piecewise linear function connecting by segments the points $(t_j, \sigma(t_j))$ for $j = 0, \dots, n$.

Clearly, in order to prove that the total variation of $\rho(t)$ is bounded by a constant independent from the subdivision, we may suppose without loss of generality that $T \leq 1$ (if $T > 1$ it is sufficient to divide $[0, T]$ into many subintervals of length smaller than 1 and repeat the argument for each of them).

Let Φ_i (for $i = 1, \dots, m$) be the following statement:

"for every $h = 1 \dots, i$ there exist positive integers n_h and q_h, a subset

$$\mathscr{T}_h = \{t_{k_{h,0}} = 0, t_{k_{h,1}}, \dots, t_{k_{h,n_h}}\} \subset \mathscr{T}_0$$

of the points of the subdivision (defining intervals $L_j^h = [t_{k_{h,j-1}}, t_{k_{h,j}}]$ for $j = 1, \dots, n_h$), values $A_{h,l,j} \in \mathbb{R}$ where $l = 1, \dots, h$, $j = 1, \dots, n_h$ and a positive real constant C_h such that setting $t_{k_{h,-1}} = 0$ and $t_{k_{h,n_h+1}} = t_{k_{h,n_h}}$ and defining for $h = 1, \dots, i-1$

$$\begin{aligned}\mathscr{A}_h = \{(t, \rho) \in [0, T] \times \mathbb{C} \mid &\text{for some } l \in \{1, \dots, h\} \text{ and } j \in \{1, \dots, n_h\}, \text{ we have} \\ &t_{k_{h,j-2}} \leq t \leq t_{k_{h,j+1}} \text{ and } |\rho - A_{h,l,j}| < C_h (t_{k_{h,j+1}} - t_{k_{h,j-2}})^{(m-h)!}\},\end{aligned}$$

we have

(1) $\mathscr{T}_h \subset \mathscr{T}_{h-1}$ for all h,
(2) for every $A \in \mathbb{R}$, ρ has less than $2q_h$ oscillations across A in $[t_{k_{h,j-1}}, t_{k_{h,j}}]$ (resp. less than q_h oscillations in $[t_{k_{h,n_h}}, t_{k_{h-1,n_{h-1}}}]$) outside $\mathscr{A}_1 \cup \cdots \cup \mathscr{A}_{h-1}$;

(3) for every t and j there exist i roots of $P(t,\cdot)$ (when counted with multiplicity) $\tau_1(t),\dots,\tau_i(t)$, where the enumeration may depend on j, with

$$|\tau_l(t) - A_{i,l,j}| < C_i(t_{k_{i,j}} - t_{k_{i,j-1}})^{(m-i)!},$$

if $t \in [t_{k_{i,j-1}}, t_{k_{i,j}}]$,

$$|\tau_l(t) - A_{i,l,j}| < C_i(t_{k_{i,j}} - t)^{(m-i)!}$$

if $t < t_{k_{i,j-1}}$, and

$$|\tau_l(t) - A_{i,l,j}| < C_i(t - t_{k_{i,j-1}})^{(m-i)!}$$

if $t > t_{k_{i,j}}$, $l = 1,\dots,i''$.

Remark 1.7. The constants C_h are bounded by a quantity depending only on h, on the degree of the polynomial, on the $C^{m!}$-norm of the coefficients and on T. The integers q_h depend only on m and h.

We prove that we can carry on our induction until $i = m$. The final part of the proof after the base and inductive steps can be found on page 18.

Remark 1.8. If for some h the graph of ρ is contained in the union $\mathscr{A}_1 \cup \cdots \cup \mathscr{A}_h$, in fact we could do without the rest of the sets $\mathscr{A}_i$ with $i > h$ (however, we can also continue the proof as in the general case, since no contradiction arises).

Lemma 1.9. *Φ_1 is true.*

Proof of Lemma. Let $q_1 = m!$; there exist $\mathscr{T}_1 = \{t_{k_{1,j}}\} \subset \{t_j\}$ and $A_{1,1,j} \in \mathbb{R}$ such that

(1) $\rho(t)$ has at least q_1 oscillations across $A_{1,1,j}$ in $[t_{k_{1,j-1}}, t_{k_{1,j}}]$
(2) in the same interval $\rho(t)$ has at most q_1 oscillations across A for all $A \in \mathbb{R}$, while
(3) in the last interval $[t_{k_{1,n_1}}, t_n]$, $\rho(t)$ has at most q_1 oscillations across A, for all $A \in \mathbb{R}$:

it is sufficient to consider larger and larger intervals with endpoints on the points of the subdivision and stop the first time that there is a value in the image of ρ across which ρ oscillates more than q_1 times; we then choose $A_{1,1,j}$ as one of such values. Indeed, by our definition it is clear that adding a subinterval $[t_{j-1}, t_j]$ can increase the number of oscillations at most by 1.

But then, by continuity, $\sigma(t)$ also attains the value $A_{1,1,j}$ at least q_1 times in the same interval. We consider an interval $t_{k_{1,j-1}} \le t \le t_{k_{1,j}}$ and call α_μ, for $\mu = 1,\dots,q_1$, q_1 points in this interval where $\sigma(\alpha_\mu) = A_{1,1,j}$.

For $0 \le t \le T$, let $\tau_1(t), \dots, \tau_m(t)$ be the (maybe dicontinuous) roots of $P(t, \tau)$, ordered in such a way that

$$|A_{1,1,j} - \tau_1(t)| \le |A_{1,1,j} - \tau_2(t)| \le \cdots \le |A_{1,1,j} - \tau_m(t)|.$$

We see that

$$(A_{1,1,j} - \tau_1(t))(A_{1,1,j} - \tau_2(t)) \dots (A_{1,1,j} - \tau_m(t)) = P(t, A_{1,1,j})$$

is a function of class $C^{m!}$ defined in $[0, T]$ and vanishing $q_1 = m!$ times in $[t_{k_{1,j-1}}, t_{k_{1,j}}]$. By Lagrange's theorem it follows that for a suitable positive constant C depending only on the derivatives of the a_j's

$$|P(t, A_{1,1,j})| \le C(t_{k_{1,j}} - t_{k_{1,j-1}})^{m!}$$

if $t_{k_{1,j-1}} \le t \le t_{k_{1,j}}$,

$$|P(t, A_{1,1,j})| \le C(t_{k_{1,j}} - t)^{m!}$$

if $0 \le t < t_{k_{1,j-1}}$, and

$$|P(t, A_{1,1,j})| \le C(t - t_{k_{1,j-1}})^{m!}$$

if $t_{k_{1,j}} < t \le T$.

As a consequence, we have that the requested inequalities hold:

$$|\tau_1(t) - A_{1,1,j}| \le \sqrt[m]{C}(t_{k_{1,j}} - t_{k_{1,j-1}})^{(m-1)!}$$

if $t \in [t_{k_{1,j-1}}, t_{k_{1,j}}]$,

$$|\tau_1(t) - A_{1,1,j}| \le \sqrt[m]{C}(t_{k_{1,j}} - t)^{(m-1)!}$$

if $t < t_{k_{1,j-1}}$, and

$$|\tau_1(t) - A_{1,1,j}| \le \sqrt[m]{C}(t - t_{k_{1,j-1}})^{(m-1)!}$$

if $t > t_{k_{1,j}}$.

We now set $C_1 = 2\sqrt[m]{C}$ and the proof of Φ_1 is finished. □

Let us now show that if Φ_{i-1} holds and $i \le m$, also Φ_i holds.

The sufficiently large value of the positive integer q_i will be determined later (see page 757); in particular, we ask that $q_i \ge 6q_{i-1}$.

We first prove that there exist points $\mathscr{T}_i = \{t_{k_{i,j}}, j = 0, \dots, n_i\} \subset \mathscr{T}_{i-1} = \{t_{k_{i-1,j}}, j = 0, \dots, n_{i-1}\}$ and numbers $A_{i,i,j} \in \mathbb{R}$ (these will define the "tubular domains" added at level i) such that

a) $\rho(t)$ has at least q_i oscillations across $A_{i,i,j}$ in $[t_{k_{i,j-1}}, t_{k_{i,j}}]$ outside $\mathscr{A}_1 \cup \cdots \cup \mathscr{A}_{i-1}$
b) for all $A \in \mathbb{R}$, $\rho(t)$ has at most $2q_i$ oscillations across A in $[t_{k_{i,j-1}}, t_{k_{i,j}}]$ outside $\mathscr{A}_1 \cup \cdots \cup \mathscr{A}_{i-1}$
c) for all $A \in \mathbb{R}$, in $[t_{k_{i,n_i}}, t_{k_{i-1,n_{i-1}}}]$ (the last interval of level i) there are at most q_i oscillations of $\rho(t)$ across A outside $\mathscr{A}_1 \cup \cdots \cup \mathscr{A}_{i-1}$.

This is proved in a similar way as for the case $i = 1$, but taking care that now we have to choose only among the points in $\mathscr{T}_{i-1}$. This means that each new interval can add at most $2q_{i-1} < q_i$ more oscillations: adding one interval at a time and stopping every time the number of oscillations becomes larger than q_i defines a $\mathscr{T}_i$ that fulfils a), b) and c) and, as a consequence, conditions (1) and (2) of Φ_i.

By continuity and since the $\mathscr{A}_h$'s are unions of cylinders it follows that $\sigma(t)$ attains at least q_i times the value $A_{i,i,j}$ in the interval $[t_{k_{i,j-1}}, t_{k_{i,j}}]$, outside the set $\mathscr{A}_1 \cup \cdots \cup \mathscr{A}_{i-1}$.

We take t in one of the intervals $[t_{k_{i,j-1}}, t_{k_{i,j}}]$ and points $\alpha_1, \ldots \alpha_{q_i}$ (numbered in increasing order) in which $\sigma(t) = A_{i,i,j}$.

For any integer r_i if q_i is large enough we can find $2ir_i$ subintervals $J_{2ir_i} \supset J_{2ir_i-1} \supset \cdots \supset J_1$ and points $\beta_{j'} \in J_{j'}$, chosen among the α_h's, such that

(1) $d(\beta_{j'}, J_{j'-1}) \geq \frac{1}{2} l(J_{j'})$, for $j' = 2, \ldots, 2ir_i$, where d is the Euclidean distance and $l(J_{j'})$ denotes the length of the interval $J_{j'}$;
(2) $\beta_{j'}$ is an endpoint of $J_{j'}$;
(3) in J_1 there are more than $4q_{i-1}$ points α_h;
(4) in the interval between $\beta_{j'}$ and $J_{j'-1}$ there are more than $4q_{i-1}$ points α_h.

This is done inductively (we will suppose that q_i is a multiple of $2^{2ir_i}4q_{i-1}$; see also Fig. 1). First, we define $J_{2ir_i} = [\alpha_1, \alpha_{q_i}]$. We then consider the two subintervals of $[\alpha_1, \alpha_{\frac{q_i}{2}}]$ and $[\alpha_{\frac{q_i}{2}+1}, \alpha_{q_i}]$ (note that their union is not the whole of J_{2ir_i}); now, β_{2ir_i} will be the endpoint of J_{2ir_i} belonging to the longest of the two subintervals, while the shortest of the two will be J_{2ir_i-1}. By our hypothesis on q_i we can iterate the construction $2ir_i$ times, fulfilling in this way conditions (1)–(4).

Since for each $\beta_{j'}$ all the β's chosen after it (that is, those with index *smaller* than j') are on the same side, that is, they are all bigger or all smaller then $\beta_{j'}$ itself (they are indeed all in $J_{j'-1}$), discarding less than half of the $\beta_{j'}$'s and renumbering them, we can also assume that either $\beta_1 < \beta_2 < \cdots < \beta_{ir_i}$ or

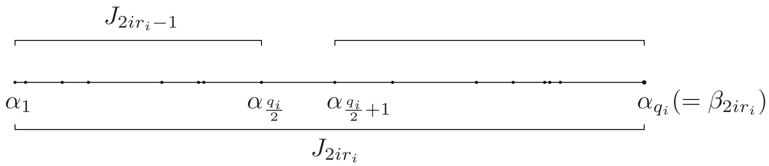

Figure 1. The choice of J_{2ir_i}, J_{2ir_i-1} and β_{2ir_i}: for every j', $J_{j'}$ contains more than $2^{j'} \cdot 4q_{i-1}$ of the points α_h.

$\beta_1 > \beta_2 > \cdots > \beta_{ir_i}$ (to do this we might have to substitute β_1, the last point, with the other endpoint of J_1).

The proof is now specular in the two cases; we will assume that $\beta_1 < \beta_2 < \cdots < \beta_{ir_i}$.

Note that between two of the β_j's there are always at least $4q_{i-1}$ of the α_h's, by construction; but in no interval of level $i-1$ there can be more then $2q_{i-1}$ points α_h (remember that also by construction, no matter what A in $\mathbb{R}$ we choose, in any interval of level $i-1$ there will never be more than $2q_{i-1}$ points where ρ has value A outside $A_1 \cup \cdots \cup A_{i-2}$; this now applies to our constant $A_{i,i,j}$): if we take the left endpoints $t_{k_{i-1,h_{j'}}}$ of the intervals of level $i-1$ containing the points $\beta_{h_{j'}}$, we have that

$$
\begin{aligned}
(1.1)\qquad t_{k_{i-1,h_1-1}} &\le t_{k_{i-1,h_1}} \le \beta_1 \le t_{k_{i-1,h_1+1}}\\
&\le t_{k_{i-1,h_2-1}} \le t_{k_{i-1,h_2}} \le \beta_2 \le t_{k_{i-1,h_2+1}}\\
&\le \cdots \le t_{k_{i-1,h_{ir_i}-1}} \le t_{k_{i-1,h_{ir_i}}} \le \beta_{ir_i} \le t_{k_{i-1,h_{ir_i}+1}}
\end{aligned}
$$

and also

$$[t_{k_{i-1,h_1-1}}, \beta_1] \subset J_1.$$

We recall that all these intervals $[t_{k_{i-1,h_{j'}}}, t_{k_{i-1,h_{j'}+1}}]$ of level $i-1$ containing the points $\beta_{j'}$ are subintervals of the same interval of level i.

We compare now the values $A_{i-1,l,h_{j'}}$ for $l = 1, \ldots, i-1$, already defined on these ir_i intervals of level $i-1$, with our new value $A_{i,i,j}$ of level i; if we group the intervals according to the number of indices l such that the constants A_{i-1,l,h_j} are below $A_{i,i,j}$ (or above it, since the l's are always $i-1$ in total) there will be i groups, therefore at least one group will gather a fraction of at least $\frac{1}{i}$ of the total number of intervals. Discarding the other intervals and points and renumbering them again, we can then assume that in all the r_i intervals the number of indices l such that the constants $A_{i-1,l,h_{j'}}$ are below $A_{i,i,j}$ is the same; we call this number u.

To avoid some very clumsy notation in the following lemma, we reorder the values $A_{i-1,l,h_{j'}}$ $(j' = 1, \ldots, r_i)$ with respect to $l = 1, \ldots, i-1$ so that

$$A_{i-1,1,h_{j'}} \le A_{i-1,2,h_{j'}} \le \cdots \le A_{i-1,u,h_{j'}} < A_{i,i,j}$$

and

$$A_{i-1,u+1,h_{j'}} \ge A_{i-1,u+2,h_{j'}} \ge \cdots \ge A_{i-1,i-1,h_{j'}} > A_{i,i,j}.$$

As a consequence of the use of Lemma 1.10 below, repeated $i-1$ times (with $K = 2C_{i-1}$), we can now prove that there are indices

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_{i-1} < \mu_1 < \mu_2 < \cdots < \mu_{(m-i+1)!+1}$$

between 1 and r_i and a constant $\tilde{C}$ depending on C_{i-1}, m and i (that we could take as $2C_{i-1}r_i^{i-1}$) such that

$$(\square)\qquad
\begin{array}{l}
\bullet \text{ for } l = 1, \dots, u \\
A_{i-1,l,h_{\lambda_p}} \geq A_{i-1,l,h_{\lambda_l}} - \tilde{C}(\beta_{\lambda_p} - t_{k_{i-1,h_{\lambda_l}-1}})^{(m-i+1)!}, \\
\text{when } p = l, l+1, \dots, i-1, \text{ and} \\
A_{i-1,l,h_{\mu_q}} \geq A_{i-1,l,h_{\lambda_l}} - \tilde{C}(\beta_{\mu_q} - t_{k_{i-1,h_{\lambda_l}-1}})^{(m-i+1)!}, \\
\text{when } q = 1, \dots, (m-i+1)! + 1; \\
\bullet \text{ for } l = u+1, \dots, i-1 \\
A_{i-1,l,h_{\lambda_p}} \leq A_{i-1,l,h_{\lambda_l}} + \tilde{C}(\beta_{\lambda_p} - t_{k_{i-1,h_{\lambda_l}-1}})^{(m-i+1)!} \\
\text{when } p = l, l+1, \dots, i-1, \text{ and} \\
A_{i-1,l,h_{\mu_q}} \leq A_{i-1,l,h_{\lambda_l}} + \tilde{C}(\beta_{\mu_q} - t_{k_{i-1,h_{\lambda_l}-1}})^{(m-i+1)!} \\
\text{when } q = 1, \dots, (m-i+1)! + 1.
\end{array}$$

LEMMA 1.10. *Suppose given integers r, s with $r = 4^m s$, points $\beta_1 < \cdots < \beta_r$ such that*

$$\begin{aligned}
t_{k_{i-1,h_1-1}} &\leq t_{k_{i-1,h_1}} \leq \beta_1 \leq t_{k_{i-1,h_1+1}} \\
&\leq t_{k_{i-1,h_2-1}} \leq t_{k_{i-1,h_2}} \leq \beta_2 \leq t_{k_{i-1,h_2+1}} \\
&\leq \cdots \leq t_{k_{i-1,h_r-1}} \leq t_{k_{i-1,h_r}} \leq \beta_r \leq t_{k_{i-1,h_r+1}}
\end{aligned}$$

and constants $A_j \in \mathbb{R}$ for $j \in \{1, \dots, r\}$; suppose also that there exists a positive constant K and a root $\tau(t)$ of $P(t, \tau)$ such that

$$|\tau(t) - A_j| < \frac{K}{2}(t_{k_{i-1,h_j}} - t_{k_{i-1,h_j-1}})^{(m-i+1)!}$$

if $t \in [t_{k_{i-1,h_j-1}}, t_{k_{i-1,h_j}}]$,

$$|\tau(t) - A_j| < \frac{K}{2}(t_{k_{i-1,h_j}} - t)^{(m-i+1)!}$$

if $t < t_{k_{i-1,h_j-1}}$, and

$$|\tau(t) - A_j| < \frac{K}{2}(t - t_{k_{i-1,h_j-1}})^{(m-i+1)!}$$

if $t > t_{k_{i-1,h_j}}$.

Then there exist indices $\lambda < \mu_1 < \cdots < \mu_s$ between 1 and r such that for any $v \in \{1, \dots, s\}$

$$A_{\mu_v} \geq A_\lambda - \tilde{K}(\beta_{\mu_v} - t_{k_{i-1,h_\lambda-1}})^{(m-i+1)!}$$

(*where $\tilde{K}$ depends only on K, r, m*).

PROOF. Consider $A_1, \dots, A_r$ and divide them in halves: $A_1, \dots A_{r/2}$ and $A_{r/2+1}, \dots, A_r$. We have that either

$$A_j > A_1 - rK(\beta_j - t_{k_{i-1,h_1-1}})^{(m-i+1)!}$$

for $j = r/2+1, \dots, r$, or there is $\tilde{q}_1 > r/2$ such that

$$A_{\tilde{q}_1} \le A_1 - rK(\beta_{\tilde{q}_1} - t_{k_{i-1,h_1-1}})^{(m-i+1)!}.$$

In the first case we define $\lambda = 1$, $\mu_v = r/2 + v$ for $v = 1, \dots, s$ and the proof is complete. In the second case we set $\tilde{p}_1 = 1$, reorder $A_1, \dots, A_{r/2}, A_{\tilde{q}_1}$ in an increasing sequence and choose j_1, l_1 (that have to exist in this case) such that A_{j_1}, A_{l_1} are consecutive and

$$A_{j_1} - A_{l_1} > K(\beta_{\tilde{q}_1} - t_{k_{i-1,h_1-1}})^{(m-i+1)!}.$$

Now, at least one half of $A_1, \dots, A_{r/2}, A_{\tilde{q}_1}$ are above A_{j_1} or below A_{l_1}: let us choose the larger of these two sets, and repeat the procedure.

If we stop before having repeated it m times we define λ and the μ_v's analogously as above, and the proof is completed; otherwise, we have $4m$ indices

$$j_1, l_1, \tilde{p}_1, \tilde{q}_1, \dots, j_m, l_m, \tilde{p}_m, \tilde{q}_m$$

such that

(1) $[\tilde{p}_1, \tilde{q}_1] \supset [\tilde{p}_2, \tilde{q}_2] \supset \cdots \supset [\tilde{p}_m, \tilde{q}_m]$,

and, for $\mu = 1, \dots, m$,

(2) $j_\mu, l_\mu \in [\tilde{p}_\mu, \tilde{q}_\mu]$, and
(3) the constants A_{j_μ} and A_{l_μ} satisfy

$$A_{j_\mu} - A_{l_\mu} > K(\beta_{\tilde{q}_\mu} - t_{k_{i-1,h_{\tilde{p}_\mu}-1}})^{(m-i+1)!}. \tag{1.2}$$

If μ' is fixed and $\mu > \mu'$, then the numbers A_{j_μ}, A_{l_μ} are either all $\ge A_{j_{\mu'}}$ or all $\le A_{l_{\mu'}}$.

We show that in this case we could find, for a certain t, $m+1$ distinct roots of $P(t, \tau)$, and this is absurd. We argue by induction: if $v = 1$ we have for $t = \beta_{\tilde{p}_1}$ two distinct roots, one near A_{j_1} and the other near A_{l_1} (we say that τ_1 is near A_{l_1} in t, and write $\tau_1 \sim A_{l_1}$, to mean that

$$|\tau_1(t) - A_{l_1}| < \frac{K}{2}(\beta_{\tilde{q}_1} - t_{k_{i-1,h_{\tilde{p}_1}-1}})^{(m-i+1)!}).$$

Suppose that for $v = k$ we have $k+1$ distinct roots $\tau_1(t), \dots, \tau_{k+1}(t)$ for $t = \beta_{\tilde{p}_k}$ and that for $\mu = 1, \dots, k$

$$\min(|\tau_\mu(\beta_{\tilde{p}_k}) - A_{j_\mu}|, |\tau_\mu(\beta_{\tilde{p}_k}) - A_{l_\mu}|) < \frac{K}{2}(\beta_{\tilde{q}_\mu} - t_{k_{i-1,h_{\tilde{p}_\mu}-1}})^{(m-i+1)!}$$

and for $\mu = k+1$

$$\min(|\tau_{k+1}(\beta_{\tilde{p}_k}) - A_{j_k}|, |\tau_{k+1}(\beta_{\tilde{p}_k}) - A_{l_k}|) < \frac{K}{2}(\beta_{\tilde{q}_k} - t_{k_{i-1,h_{\tilde{p}_k}-1}})^{(m-i+1)!}.$$

For $v = k+1$ and for $t = \beta_{\tilde{p}_{k+1}}$ we have $k+1$ distinct roots $\tau_2, \dots, \tau_{k+2}$. We find $\tau_1 \sim A_{l_1}$ in t if $A_{j_\mu}, A_{l_\mu} \geq A_{j_1}$ (for $\mu = 2, \dots, k+2$), or $\tau_1 \sim A_{j_1}$ in t if $A_{j_\mu}, A_{l_\mu} \leq A_{l_1}$ (again, for $\mu = 2, \dots, k+2$). Let us consider the first case.

We note that τ_1 is different from $\tau_2, \dots, \tau_{k+2}$; otherwise we would have

$$|\tau_1(t) - A_{l_1}| < \frac{K}{2}(\beta_{\tilde{q}_1} - t_{k_{i-1,h_{\tilde{p}_1}-1}})^{(m-i+1)!}$$

and for $|\tau_1(t) - A_{j_\mu}|$ (or for $|\tau_1(t) - A_{l_\mu}|$) we would have

$$|\tau_1(t) - A_{j_\mu}| < \frac{K}{2}(\beta_{\tilde{q}_\mu} - t_{k_{i-1,h_{\tilde{p}_\mu}-1}})^{(m-i+1)!} \leq \frac{K}{2}(\beta_{\tilde{q}_1} - t_{k_{i-1,h_{\tilde{p}_1}-1}})^{(m-i+1)!},$$

so that

$$|A_{j_1} - A_{l_1}| \leq \min(|A_{l_\mu} - A_{l_1}|, |A_{j_\mu} - A_{l_1}|) < K(\beta_{\tilde{q}_1} - t_{k_{i-1,h_{\tilde{p}_1}-1}})^{(m-i+1)!},$$

which is against inequality (1.2).

A similar argument applies to the case when $\tau_1 \sim A_{j_1}$ in t and $A_{j_\mu}, A_{l_\mu} \leq A_{l_1}$ ($\mu > 1$).

The roots $\tau_1, \tau_2, \dots, \tau_{k+2}$ are then pairwise distinct. When $v = m$ we have a contradiction. $\square$

We now fix the value of q_i choosing

$$q_i = 2^{2i4^{m(i-1)}(m-i+1)!} \cdot 4 \cdot 6 q_{i-1}, \tag{1.3}$$

which is enough to do all the constructions up to now.

Lemma 1.11. *Let $t \in [0, T]$ and consider the real numbers $A_{i-1,l,h_{\lambda_l}}$ (for $l = 1, \dots, i-1$). We can associate to them complex numbers $\tau_1(t), \dots, \tau_{i-1}(t)$ which are $i-1$ roots of $P(t, \tau)$ (when counted with multiplicities), such that setting $\tilde{C}_{i-1} = 4(i-1)\tilde{C}$ for $l = 1, \dots, i-1$ we have*

$$|\tau_l(t) - A_{i-1,l,h_{\lambda_l}}| < \tilde{C}_{i-1}(t_{k_{i-1,h_{\lambda_{i-1}}+1}} - t_{k_{i-1,h_{\lambda_1}-1}})^{(m-i+1)!}, \tag{1.4}$$

if $t_{k_{i-1,h_{\lambda_1}-1}} \leq t \leq t_{k_{i-1,h_{\lambda_{i-1}}+1}}$,

$$|\tau_l(t) - A_{i-1,l,h_{\lambda_l}}| < \tilde{C}_{i-1}(t_{k_{i-1,h_{\lambda_{i-1}}+1}} - t)^{(m-i+1)!}, \tag{1.5}$$

if $0 \leq t \leq t_{k_{i-1,h_{\lambda_1}-1}}$, and

$$|\tau_l(t) - A_{i-1,l,h_{\lambda_l}}| < \tilde{C}_{i-1}(t - t_{k_{i-1,h_{\lambda_1}-1}})^{(m-i+1)!}, \tag{1.6}$$

if $t_{k_{i-1,h_{\lambda_{i-1}}+1}} \leq t \leq T$.

Proof. Let us put

$$A_{j,k} = A_{i-1,j,h_{\lambda_k}}, \quad j,k = 1,\dots,i-1$$

and

$$R = C(t_{k_{i-1,h_{\lambda_{i-1}}+1}} - t_{k_{i-1,h_{\lambda_1}-1}})^{(m-i+1)!}$$

if $t_{k_{i-1,h_{\lambda_1}-1}} \le t \le t_{k_{i-1,h_{\lambda_{i-1}}+1}}$,

$$R = C(t_{k_{i-1,h_{\lambda_{i-1}}+1}} - t)^{(m-i+1)!}$$

if $0 \le t \le t_{k_{i-1,h_{\lambda_1}-1}}$,

$$R = C(t - t_{k_{i-1,h_{\lambda_1}-1}})^{(m-i+1)!}$$

if $t_{k_{i-1,h_{\lambda_{i-1}}+1}} \le t \le T$.

We can suppose $A_{i,i,j} = 0$ and

$$\begin{aligned} A_{1,k} \le A_{2,k} \le \cdots \le A_{u,k} < 0; \\ A_{u+1,k} \ge A_{u+2,k} \ge \cdots \ge A_{i-1,k} > 0. \end{aligned}$$

On the other hand, by relations ($\square$),

$$\begin{aligned} &A_{j,k} \ge A_{j,j} - R, && \text{and so also} \\ &A_{k,k} \ge A_{j,j} - R, && \text{for } k = j+1,\dots,i-1 \text{ and } j = 1,\dots,u, \\ &A_{j,k} \le A_{j,j} + R, && \text{and so also} \\ &A_{k,k} \le A_{j,j} + R, && \text{for } k = j+1,\dots,i-1 \text{ and } j = u+1,\dots,i-1. \end{aligned} \tag{1.7}$$

Let us define $\tilde{Y}_j = \{\rho \in \mathbb{C} : |\rho - A_{j,j}| < 2R\}$, $j = 1,\dots,i-1$. We group the $\tilde{Y}_j$ into connected components: it is not difficult to use inequalities 1.7 to show that, if $\tilde{Y}_{j_1}$ and $\tilde{Y}_{j_2}$ belong to the same component $\mathscr{C}$ and $j_1 < j_2 \le u$, then every $\tilde{Y}_j$, with $j_1 < j < j_2$, belongs to $\mathscr{C}$ too, and so these connected components will have the form

$$\tilde{Y}_1 \cup \cdots \cup \tilde{Y}_{\alpha_1} \quad \tilde{Y}_{\alpha_1+1} \cup \cdots \cup \tilde{Y}_{\alpha_2} \quad \dots \quad \tilde{Y}_{\alpha_{p-1}+1} \cup \cdots \cup \tilde{Y}_{\alpha_p}$$

($\alpha_p = u$); and similarly, for $j_1 \ge j_2 \ge u+1$,

$$\tilde{Y}_{u+1} \cup \cdots \cup \tilde{Y}_{\beta_1} \quad \tilde{Y}_{\beta_1+1} \cup \cdots \cup \tilde{Y}_{\beta_2} \quad \dots \quad \tilde{Y}_{\beta_{q-1}+1} \cup \cdots \cup \tilde{Y}_{\beta_q}$$

($\beta_q = i-1$). The two last groups of the rows, $\tilde{Y}_{\alpha_{p-1}+1} \cup \cdots \cup \tilde{Y}_{\alpha_p}$ and $\tilde{Y}_{\beta_{q-1}+1} \cup \cdots \cup \tilde{Y}_{\beta_q}$, may intersect (and in this case they are put together). Now, by part (3) of the induction hypothesis Φ_{i-1}, for all $t \in [0,T]$ the set $\bigcup_{j=1}^{\alpha_1} D(A_{j,\alpha_1}, R)$ contains a subsystem of α_1 roots of the polynomial at that same point t; we

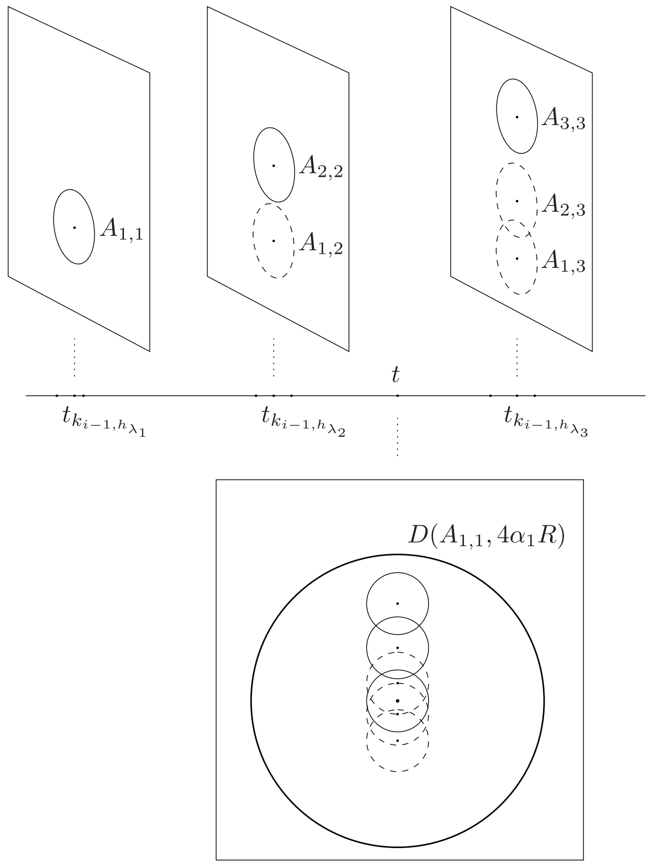

Figure 2. The disk $D(A_{1,1}, 4\alpha_1 R)$ contains the subsystem of the $\alpha_1 = 3$ roots (at point t) of its connected component. Inequalities 1.7 relate the constants $A_{j,k}$ defined in different intervals (note that in this picture the real axis is vertical).

deduce (see also Fig. 2) that $D(A_{1,1}, 4\alpha_1 R)$ also contains that subsystem, and the same is true for $D(A_{j,j}, 4\alpha_1 R)$, $j = 1, \dots, \alpha_1$. Therefore we can renumber the τ_l's so that $\tau_j \in D(A_{j,j}, 4\alpha_1 R)$ for $j = 1, \dots, \alpha_1$. The same construction allows us to find a subsystem of α_2 roots in

$$D(A_{\alpha_1+1,\alpha_1+1}, 4(\alpha_2 - \alpha_1)R),$$
$$\dots$$
$$D(A_{\alpha_2,\alpha_2}, 4(\alpha_2 - \alpha_1)R),$$

and so on. If there is a component containing discs with center both in the left and right half-plane we have that

$$D(A_{\alpha_{q-1}+1,\alpha_{q-1}+1}, 4(\alpha_p - \alpha_{p-1} + \beta_q - \beta_{q-1})R)$$

contains both the discs

$$D(A_{\alpha_{p-1}+1,i-1},R),\dots,D(A_{u,i-1},R)$$

and the discs

$$D(A_{\beta_{q-1}+1,i-1},R),\dots,D(A_{i-1,i-1},R)$$

and so $\alpha_p-\alpha_{p-1}+\beta_q-\beta_{q-1}$ roots $\tau_l(t)$ (when counted with multiplicities). So we can choose a point in each disk

$$D(A_{j,j},8(\alpha_p-\alpha_{p-1}+\beta_q-\beta_{q-1})R)$$

for $j=\alpha_{p-1}+1,\dots,u$ and $j=\beta_{q-1}+1,\dots,i-1$.

All these sets of roots are disjoint, as we now show. Indeed,

$$\tilde{Y}_1\cap\mathbb{R},\dots,\tilde{Y}_{\alpha_1}\cap\mathbb{R}\subset(a_1,b_1),$$
$$\tilde{Y}_{\alpha_1+1}\cap\mathbb{R},\dots,\tilde{Y}_{\alpha_2}\cap\mathbb{R}\subset(a_2,b_2),$$

and $b_1\le a_2$; moreover,

$$A_{j,\alpha_1}\ge A_{1,1}-R\quad\text{and so}\quad A_{j,\alpha_1}-R\ge A_{1,1}-2R\ge a_1$$

and

$$A_{j,\alpha_1}\le A_{\alpha_1,\alpha_1}\quad\text{and so}\quad A_{j,\alpha_1}+R\le A_{\alpha_1,\alpha_1}+R\le b_1$$

for $j=1,\dots,\alpha_1$. So $D(A_{j,\alpha_1},R)\cap\mathbb{R}\subset(a_1,b_1)$, and

$$\bigcup_{j=1}^{\alpha_1}(D(A_{j,\alpha_1},R)\cap\mathbb{R})\subset(a_1,b_1),$$

$$\bigcup_{j=\alpha_1+1}^{\alpha_2}(D(A_{j,\alpha_2},R)\cap\mathbb{R})\subset(a_2,b_2).$$

Hence the two subsystems of roots are disjoint. The other cases are similar.

We can then put together these roots and obtain a subsystem $(\tau_1(t),\dots,\tau_{i-1}(t))$ of $i-1$ roots of $P(t,\tau)$. $\square$

As a consequence of the two geometric-combinatorial lemmas 1.12 and 1.13 below, we may suppose also that the following property $(\diamond)$ holds:

$$(\diamond)\quad\begin{aligned}&(1)\ \text{if } l\in\{1,\dots,u\}\text{, and therefore } A_{i-1,l,h_{\lambda_l}}<A_{i,i,j}\text{, then}\\&\qquad \operatorname{Re}\tau_l(t)<A_{i,i,j}\text{ for } t=\beta_{\mu_h}\ (h=1,\dots,(m-i+1)!+1),\\&(2)\ \text{if } l\in\{u+1,\dots,i-1\}\text{, and therefore } A_{i-1,l,h_{\lambda_l}}>A_{i,i,j},\\&\qquad \text{then } \operatorname{Re}\tau_l(t)>A_{i,i,j}\text{ for } t=\beta_{\mu_h}\ (h=1,\dots,(m-i+1)!+1).\end{aligned}$$

LEMMA 1.12. *Let* $[\tau_1, \ldots, \tau_m]$ *be an unordered m-ple of real numbers, let i be a number* $\leq m$ *and* S^-, S^+ *a partition of* $\{1, \ldots, i\}$. *Suppose given intervals* $X_k = [a_k, b_k]$ *and values* $x_k \in X_k$, *where* $k = 1, \ldots, i$, *which form a subsystem of* $[\tau_1, \ldots, \tau_m]$; *and other intervals* $D_k = [c_k, d_k]$ *such that*

(1) *if* $k \in S^-$ *then* $a_k \leq c_k < d_k < 0$,
(2) *if* $k \in S^+$ *then* $0 < c_k < d_k \leq b_k$,

and other points $y_k \in D_k$, *where* $k = 1, \ldots, i$, *also forming a subsystem of* $[\tau_1, \ldots, \tau_m]$.

Then there exists a subsystem $(z_1, \ldots, z_i)$ *of* $[\tau_1, \ldots, \tau_m]$ *such that* $z_k \in X_k$ *for all indices* $k = 1, \ldots, i$ *and* $z_k < 0$ *if* $k \in S^-$, $z_k > 0$ *if* $k \in S^+$.

PROOF. Consider first the indices $k \in S^-$. If $x_k < 0$ for all such indices we define $z_k = x_k$ and pass to the indices in S^+; otherwise, we will have to find how to "substitute" the $x_k \geq 0$ with some well-chosen y_k to define the z_k's. To do that, say $x_{i_1} \geq 0$ and consider y_{i_1}: if there is no x_k coinciding with y_{i_1}, we define $z_{i_1} = y_{i_1}$; otherwise, we will find a chain of values $y_{i_1} = x_{i_2}$, $y_{i_2} = x_{i_3}, \ldots,$ until we find $y_{i_{\mu_1}} \neq x_j$, $\forall j \in S^- \setminus \{i_1, \ldots, i_{\mu_1}\}$. Define now

$$\nu_1 = \max\{j \in \{1, \ldots, \mu_1\} \mid y_{i_j} = \max\{y_{i_1}, \ldots, y_{i_{\mu_1}}\}\}$$

and set $z_{i_2} = x_{i_2}, \ldots, z_{i_{\nu_1}} = x_{i_{\nu_1}}$, $z_{i_1} = y_{i_{\nu_1}}$. All the z_{i_j} are trivially in X_{i_j} and are negative, if $j \neq i_1$; on the other hand, note that we have

$$a_{i_1} \leq c_{i_1} \leq y_{i_1} \leq y_{i_{\nu_1}} \leq d_{i_{\nu_1}} < 0 \leq x_{i_1} \leq b_{i_1},$$

so that $z_{i_1} = y_{i_{\nu_1}}$ also satisfies the required conditions. Then, if $\nu_1 < \mu_1$, define

$$\nu_2 = \max\{j \in \{\nu_1 + 1, \ldots, \mu_1\} \mid y_{i_j} = \max\{y_{i_{\nu_1+1}}, \ldots, y_{i_{\mu_1}}\}\}$$

and, analogously as above, set $z_{i_{\nu_1+2}} = x_{i_{\nu_1+2}}, \ldots, z_{i_{\nu_2}} = x_{i_{\nu_2}}$, $z_{i_{\nu_1+1}} = y_{i_{\nu_2}}$. Again, note that we have

$$a_{i_{\nu_1+1}} \leq y_{i_{\nu_1+1}} \leq y_{i_{\nu_2}} = z_{i_{\nu_1+1}} \leq y_{i_{\nu_1}} = x_{i_{\nu_1+1}} \leq b_{i_{\nu_1+1}},$$

that is $z_{i_{\nu_1+1}} \in X_{i_{\nu_1+1}}$, and $z_{i_{\nu_1+1}} = y_{i_{\nu_2}} < 0$. We repeat these steps until we reach $y_{i_{\mu_1}}$.

We then check if there are other $x_k \geq 0$ with $k \in S^- \setminus \{i_1, \ldots, i_{\mu_1}\}$. If there are, let $x_{i_{\mu_1+1}} \geq 0$ be one of them; again, we can define new chains of points $y_{i_{\mu_1+1}} = x_{i_{\mu_1+2}}$, $y_{i_{\mu_1+2}} = x_{i_{\mu_1+3}}, \ldots, y_{i_{\mu_2}}$ where $y_{i_{\mu_2}} \neq x_j$, $\forall j \in S^- \setminus \{i_1, \ldots, i_{\mu_2}\}$, and repeat the previous argument.

If, after z_k's have been defined for all the $x_k \geq 0$, there still are indices in S^- for which $x_k < 0$ and z_k is not defined, we put $z_k = x_k$ for such k's. In this way we exhaust the $k \in S^-$ (and we argue symmetrically for the $k \in S^+$).

We claim that the points z_k, $k \in S^-$, are a subsystem of $[\tau_1, \ldots, \tau_m]$. Indeed, all the values z_k are among the τ_j's; and apart from the $z_j = y_{i_{\mu_\alpha}}$'s, they coincide with a part of the subsystem x_k. On the other hand, the values $y_{i_{\mu_\alpha}}$ have (among the

z_k's) multiplicities bounded by those of the subsystem y_k. In this way we show that the z_k's with $k \in S^-$ are a subsystem; the same is true for the z_k, $k \in S^+$. But the former are <0 and the latter are >0, so that their union is also necessarily a subsystem, and the proof is completed. □

LEMMA 1.13. *Let $[\tau_1, \dots, \tau_m]$ be an unordered m-ple of complex numbers, i be a number $\leq m$ and S^-, S^+ be a partition of $\{1, \dots, i\}$. Let also*

$$X_k = \{\rho : |\rho - A_k| < R_k\},$$

where $A_k < 0$ if $k \in S^-$, whereas $A_k > 0$ if $k \in S^+$, for some positive constants R_k, and let $x_k \in X_k$ form a subsystem of $[\tau_1, \dots, \tau_m]$. Similarly, let

$$D_k = \{\rho : |\rho - \tilde{A}_k| < \tilde{R}\},$$

with $\tilde{A}_k < 0$ if $k \in S^-$, whereas $\tilde{A}_k > 0$ if $k \in S^+$ and $0 < \tilde{R} \leq R_k$ $\forall k$. Let $y_k \in D_k$ be a subsystem of $[\tau_1, \dots, \tau_m]$. We suppose that

(1) $D_k \cap \mathbb{R} \subset (A_k - R_k, 0)$ *if* $k \in S^-$,
(2) $D_k \cap \mathbb{R} \subset (0, A_k + R_k)$ *if* $k \in S^+$.

Then there exists a subsystem $(z_1, \dots, z_i)$ of $[\tau_1, \dots, \tau_m]$, such that

$$z_k \in \tilde{X}_k = \{\rho : |\rho - A_k| < 2R_k\},$$

$\operatorname{Re} z_k < 0$ *if* $k \in S^-$ *and* $\operatorname{Re} z_k > 0$ *if* $k \in S^+$.

PROOF. Let's consider first the k's in S^-. If $\operatorname{Re} x_k < 0$ for all these indices, we just set $z_k = x_k$, $k \in S^-$. Otherwise, let's choose an x_{i_1} with $\operatorname{Re} x_{i_1} \geq 0$ (see also Fig. 3). We build a chain of values

$$y_{i_1} = x_{i_2}, y_{i_2} = x_{i_3}, \dots, y_{i_{\mu_1 - 1}} = x_{i_{\mu_1}}, y_{i_{\mu_1}}$$

such that $i_1, i_2, \dots, i_{\mu_1} \in S^-$ and $y_{i_{\mu_1}} \neq x_j$ $\forall j \in S^- \backslash \{i_1, \dots, i_{\mu_1}\}$.

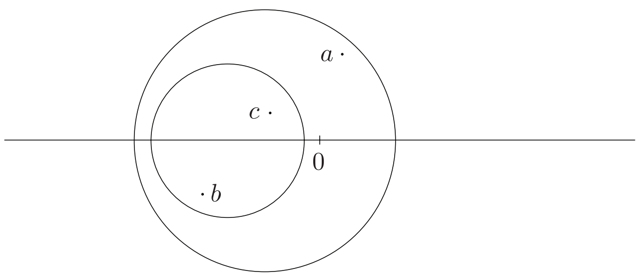

Figure 3. The definition of the z_k's in a simple case. Here $(x_1, \dots, x_6) = (c, b, a, b, b, b)$ and $(y_1, \dots, y_6) = (b, b, b, c, c, c)$; $\operatorname{Re} x_3 > 0$, so the first chain of indices could be $3, 2, 4, 1, 6$ $(= \mu_1)$, and y_4 has maximal real part among them, i.e. $v_1 = 4$, $v_2 = 6$. We would then set $z_2 = b$, $z_4 = b$, $z_3 = c$, then $z_6 = b$, $z_1 = c$, and finally $z_5 = x_5 = b$.

As in the previous lemma, let v_1 be the last index such that

$$\operatorname{Re} y_{i_{v_1}} = \max\{\operatorname{Re} y_{i_1}, \dots, \operatorname{Re} y_{i_{\mu_1}}\},$$

and let us pose $z_{i_2} = x_{i_2}, z_{i_3} = x_{i_3}, \dots, z_{i_{v_1}} = x_{i_{v_1}}, z_{i_1} = y_{i_{v_1}}$.

We have

$$A_{i_1} - R_{i_1} \leq \tilde{A}_{i_1} - \tilde{R} \leq \operatorname{Re} y_{i_1} \leq \operatorname{Re} y_{i_{v_1}} < 0 \leq \operatorname{Re} x_{i_1} \leq A_{i_1} + R_{i_1},$$

and thus $z_{i_1} \in \tilde{X}_{i_1}$.

Then we choose v_2 such that

$$\operatorname{Re} y_{i_{v_2}} = \max\{\operatorname{Re} y_{i_{v_1+1}}, \dots, \operatorname{Re} y_{i_{\mu_1}}\},$$

and define the z_{i_j}'s for $j = v_1 + 1, \dots, v_2$, analogously as above. Again, we will have that $z_{i_{v_1+1}} = y_{i_{v_2}}$, $z_{i_{v_1+1}} \in \tilde{X}_{i_{v_1+1}}$ and $\operatorname{Re} z_{i_{v_1+1}} = \operatorname{Re} y_{i_{v_2}} < 0$. We repeat this construction until we reach $y_{i_{\mu_1}}$, check if there are x_k's left with $\operatorname{Re} x_k \geq 0$, and if this is the case, build more chains; otherwise we set $z_k = x_k$ for the remaining indices in S^- (and we similarly treat the indices in S^+).

The proof that the z_k's form a subsystem of $[\tau_1, \dots, \tau_m]$ is now completely analogous to the real case in Lemma 1.12. □

Let us now consider a fixed point $t = \beta_{\mu_p}$ in the interval $[t_{k_{i-1,h_{\mu_p}}}, t_{k_{i-1,h_{\mu_p}+1}}]$; we suppose, which can be done up to a translation, that $A_{i,i,j} = 0$. Let $[\tau_1(t), \dots, \tau_m(t)]$ be the unordered m-ple of roots of $P(t, \tau)$: define for $l = 1, \dots, i-1$

$$X_l = \{\rho \in \mathbb{C} : |\rho - A_{i-1,l,h_{\lambda_l}}| < 2\tilde{C}_{i-1}(\beta_{\mu_p} - t_{k_{i-1,h_{\lambda_1}-1}})^{(m-i+1)!} = R_l = R\}$$

and, as a partition of the set $\{1, \dots, i-1\}$, let us choose as before

$$S^- = \{1, \dots, u\} = \{l : A_{i-1,l,h_{\lambda_l}} < 0\},$$
$$S^+ = \{u+1, \dots, i-1\} = \{l : A_{i-1,l,h_{\lambda_l}} > 0\}.$$

We now see that by Lemma 1.11 there is a subsystem $(x_1, \dots, x_{i-1}) = (\tau_1, \dots, \tau_{i-1})$ of $[\tau_1(t), \dots, \tau_m(t)]$ such that $x_l \in X_l$ for $l = 1, \dots, i-1$.

We define also

$$D_l = \{\rho \in \mathbb{C} : |\rho - A_{i-1,l,h_{\mu_p}}| < C_{i-1}(\beta_{\mu_p} - t_{k_{i-1,h_{\mu_p}-1}})^{(m-i+1)!} = \tilde{R}\}$$

(where $C_{i-1} < \tilde{C}_{i-1}$; $\tilde{R} < R$): we have $A_{i-1,l,h_{\mu_p}} < 0$ if $l = 1, \dots, u$, $A_{i-1,l,h_{\mu_p}} > 0$ if $l = u+1, \dots, i-1$.

By condition (3) of the induction hypothesis Φ_{i-1}, there is another subsystem $(y_1, \dots, y_{i-1})$ of $[\tau_1(t), \dots, \tau_m(t)]$ with $y_l \in D_l$.

Notice that

$$D_l \cap \mathbb{R} = (A_{i-1,l,h_{\mu_p}} - \tilde{R}, A_{i-1,l,h_{\mu_p}} + \tilde{R}) \subset (A_{i-1,l,h_{\lambda_l}} - R_l, 0),$$

if $l \in S^-$ (and a similar relation holds if $l \in S^+$), as a consequence of inequalities ($\square$) on page 755 and of the definition of $A_{i,i,j}$, which implies that the point $(\beta_{\mu_p}, A_{i,i,j}) \notin \mathscr{A}_{i-1}$.

This means that the two subsystems x_l, y_l and the discs X_l, D_l are in the hypotheses of Lemma 1.13; therefore, there are roots $(z_1, \dots, z_{i-1})$ (a subsystem of $[\tau_1(t), \dots, \tau_m(t)]$) such that

$$z_l \in \tilde{X}_l = \{\rho : |\rho - A_{i-1,l,h_{\lambda_l}}| \le 2R\},$$

and $\operatorname{Re} z_l < 0$ if $l \in S^-$, $\operatorname{Re} z_l > 0$ if $l \in S^+$ (that is, property ($\diamondsuit$) at page 760 holds).

We define $A_{i,l,j} = A_{i-1,l,h_{\lambda_l}}$ when $l < i$, and prove that for a suitable constant C_i these values, together with $A_{i,i,j}$, satisfy condition (3) of Φ_i.

Let $\tau_i(t), \tau_{i+1}(t), \dots, \tau_m(t)$ be the other roots of $P(t,\tau)$ on L_j^i (that is, we recall, $t_{k_{i,j-1}} \le t \le t_{k_{i,j}}$), ordered in such a way that

$$|A_{i,i,j} - \tau_i(t)| \le |A_{i,i,j} - \tau_{i+1}(t)| \le \cdots \le |A_{i,i,j} - \tau_m(t)|.$$

Lemma 1.14. *There exist functions $f_{i,j}(t)$ of class $C^{m!}$, whose $C^{m!}$-norm is bounded by a constant depending only on the $C^{m!}$-norm of the coefficients of $P(t,\tau)$, and bounded functions $g_{i,l,j}(t)$, whose (common) bound M depends only on the C^0-norm of the coefficients of $P(t,\tau)$, such that*

$$\begin{aligned}(1.8)\quad &(\tau_i(t) - A_{i,i,j})(\tau_{i+1}(t) - A_{i,i,j}) \dots (\tau_m(t) - A_{i,i,j}) \\ &= f_{i,j}(t) + (\tau_1(t) - A_{i,1,j})g_{i,1,j}(t) + \cdots + (\tau_{i-1}(t) - A_{i,i-1,j})g_{i,i-1,j}(t).\end{aligned}$$

Proof. For the sake of simplicity, let's put in this lemma $A^* = A_{i,i,j}$. Let

$$S_h(\tau_1, \dots, \tau_i)$$

be the sum of the products of i numbers taken in groups of h, that is the h-th elementary symmetric function on i arguments (where we take $S_0 = 1$ if $h = 0$ and $S_h = 0$ if $h > i$).

We prove by induction that $S_h(\tau_1(t), \dots, \tau_i(t))$ can be written in the form

$$(1.9)\qquad f(t) + (\tau_1(t) - A_{i,1,j})g_1(t) + \cdots + (\tau_{i-1}(t) - A_{i,i-1,j})g_{i-1}(t),$$

where f has $m!$ derivatives and the g_l's are bounded functions.

First, we have

$$\begin{aligned}&S_1(\tau_i(t) - A^*, \dots, \tau_m(t) - A^*) \\ &\quad = S_1(\tau_1(t) - A^*, \dots, \tau_m(t) - A^*) - S_1(\tau_1(t) - A^*, \dots, \tau_{i-1}(t) - A^*) \\ &\quad = f(t) - (\tau_1(t) - A_{i,1,j}) - \cdots - (\tau_{i-1}(t) - A_{i,i-1,j}),\end{aligned}$$

where $f(t) = -a_1(t) - mA^* + \sum_{l=1}^{i-1}(A^* - A_{i,l,j})$ is of class $C^{m!}$.

Suppose now that

$$\begin{gathered} S_1(\tau_i(t)-A^*,\dots,\tau_m(t)-A^*) \\ S_2(\tau_i(t)-A^*,\dots,\tau_m(t)-A^*) \\ \dots \\ S_{h-1}(\tau_i(t)-A^*,\dots,\tau_m(t)-A^*) \end{gathered}$$

can all be expressed in the form (1.9).

It is obvious that $S_v(\tau_1(t)-A^*,\dots,\tau_{i-1}(t)-A^*)$ can also be written in the form (1.9); then we have

$$\begin{aligned} S_h(\tau_i(t)-A^*,\dots,\tau_m(t)-A^*) \\ = S_h(\tau_1(t)-A^*,\dots,\tau_m(t)-A^*) \\ - S_h(\tau_1(t)-A^*,\dots,\tau_{i-1}(t)-A^*) \\ - S_{h-1}(\tau_1(t)-A^*,\dots,\tau_{i-1}(t)-A^*) \\ \cdot S_1(\tau_i(t)-A^*,\dots,\tau_m(t)-A^*) \\ - S_{h-2}(\tau_1(t)-A^*,\dots,\tau_{i-1}(t)-A^*) \\ \cdot S_2(\tau_i(t)-A^*,\dots,\tau_m(t)-A^*) - \cdots \\ - S_1(\tau_1(t)-A^*,\dots,\tau_{i-1}(t)-A^*) \\ \cdot S_{h-1}(\tau_i(t)-A^*,\dots,\tau_m(t)-A^*) \end{aligned} \tag{1.10}$$

and since the product and the sum of functions of the form (1.9) are of the same form, we can write also (1.10) in the form (1.9). This completes the induction. The case $h = m-i+1$ is our thesis. □

We also note that the functions $f_{i,j}$ of Lemma 1.14 are real, since they can be expressed as real polynomials in the coefficients of $P(t,\tau)$ and in the constants $A_{i,l,j}$.

Now $\tau_1(t), \tau_2(t), \dots, \tau_{i-1}(t), \tau_i(t)$ are i roots (when counted with multiplicities) for $0 \le t \le T$. Note that $\tau_i(\beta_{\mu_p}) = A_{i,i,j}$ as $\sigma(\beta_{\mu_p}) = A_{i,i,j}$ and τ_i is the only root that can assume the value $A_{i,i,j}$, since $\operatorname{Re}\tau_l(\beta_{\mu_p}) \neq A_{i,i,j}$, for $l = 1,\dots,i-1$ (by property $(\diamond)$).

Let

$$Q_p(t) = (t-\beta_{\mu_1})\dots(\widehat{t-\beta_{\mu_p}})\dots(t-\beta_{\mu_{(m-i+1)!+1}})$$

(polynomial of degree $(m-i+1)!$), and

$$Q(t) = \sum_{p=1}^{(m-i+1)!+1} f_{i,j}(\beta_{\mu_p})\frac{Q_p(t)}{Q_p(\beta_{\mu_p})}$$

(Lagrange interpolation polynomial); we do this for each $f_{i,j}$ given by Lemma 1.14.

We can write

$$f_{i,j}(t) = [f_{i,j}(t) - Q(t)] + Q(t).$$

Now, $f_{i,j}(t) - Q(t)$ is small since it vanishes in $(m-i+1)!+1$ points and its derivative of order $(m-i+1)!+1$ is bounded by a uniform constant depending only on the $C^{(m-i+1)!+1}$ norm of the coefficients of $P(t,\tau)$ (and on the values $A_{i,l,j}$, that are in turn bounded by the coefficients). Thus

$$|f_{i,j}(t) - Q(t)| \le C'(t_{k_{i,j}} - t_{k_{i,j-1}})^{(m-i+1)!} \tag{1.11}$$

for $t_{k_{i,j-1}} \le t \le t_{k_{i,j}}$.

As far as $Q(t)$ is concerned, by (1.8) and (1.4), since $A_{i,l,j} = A_{i-1,l,h_{\lambda_l}}$ and $\beta_{\mu_p} \notin [t_{k_{i-1,h_{\lambda_1}}-1}, t_{k_{i-1,h_{\lambda_{i-1}}}+1}]$, we have

$$\begin{aligned} |f_{i,j}(\beta_{\mu_p})| &= |(\tau_1(\beta_{\mu_p}) - A_{i,1,j})g_{i,1,j}(\beta_{\mu_p}) + \cdots + (\tau_{i-1}(\beta_{\mu_p}) - A_{i,i-1,j})g_{i,i-1,j}(\beta_{\mu_p})| \\ &\le \tilde{C}C''(i-1)(\beta_{\mu_p} - t_{k_{i-1,h_{\lambda_1}}-1})^{(m-i+1)!}. \end{aligned}$$

Hence

$$\left|\frac{f_{i,j}(\beta_{\mu_p})}{Q_p(\beta_{\mu_p})}\right| \le \tilde{C}C''(i-1)\frac{|\beta_{\mu_p} - t_{k_{i-1,h_{\lambda_1}}-1}|^{(m-i+1)!}}{|(\beta_{\mu_p} - \beta_{\mu_1})\dots\widehat{(\beta_{\mu_p} - \beta_{\mu_p})}\dots(\beta_{\mu_p} - \beta_{\mu_{(m-i+1)!+1}})|} \le C'''$$

where $C''' = \tilde{C}C''(i-1)2^{(m-i+1)!}$. In fact

$$|\beta_{\mu_p} - \beta_{\mu_{p'}}| \ge \frac{1}{2}l(J_{\mu_p})$$

if $p \ne p'$ and

$$|\beta_{\mu_p} - t_{k_{i-1,h_{\lambda_1}}-1}| \le l(J_{\mu_p})$$

since β_{μ_p} and $t_{k_{i-1,h_{\lambda_1}}-1}$ are both in J_{μ_p} (this is easy to see, since $\lambda_1 \le \mu_p \Rightarrow \beta_{\lambda_1} \in J_{\lambda_1} \subset J_{\mu_p}$ while $t_{k_{i-1,h_1}-1} \in J_1 \subset J_{\mu_p}$ and $t_{k_{i-1,h_1}-1} \le t_{k_{i-1,h_{\lambda_1}}-1} \le \beta_{\lambda_1}$).

Moreover

$$|Q_p(t)| \le (t_{k_{i,j}} - t_{k_{i,j-1}})^{(m-i+1)!}$$

for $t \in L_j^i$.

Therefore

$$|Q(t)| \le C''((m-i+1)!+1)(t_{k_{i,j}} - t_{k_{i,j-1}})^{(m-i+1)!} \tag{1.12}$$

and similar formulae hold in $[0, t_{k_{i,j-1}}]$ and in $[t_{k_{i,j}}, T]$.

Considering (1.8), (1.4), (1.11) and (1.12) we have (for $t \in L^i_j$)

$$|\tau_i(t) - A_{i,i,j}|\,|\tau_{i+1}(t) - A_{i,i,j}| \dots |\tau_m(t) - A_{i,i,j}| \le C^*_i (t_{k_{i,j}} - t_{k_{i,j-1}})^{(m-i+1)!}$$

with e.g. $C^*_i = C' + ((m+i-1)! + 1)C'' + (i-1)M\tilde{C}_{i-1}$; as a consequence, since $\tau_i(t)$ is the nearest root to $A_{i,i,j}$,

$$|\tau_i(t) - A_{i,i,j}| < 2(\sqrt[m-i+1]{C^*_i})(t_{k_{i,j}} - t_{k_{i,j-1}})^{(m-i)!}.$$

Similar formulae hold before and after L^i_j, namely

$$|\tau_i(t) - A_{i,i,j}| < 2(\sqrt[m-i+1]{C^*_i})(t_{k_{i,j}} - t)^{(m-i)!}$$

if $t \in [0, t_{k_{i,j-1}}]$ and

$$|\tau_i(t) - A_{i,i,j}| < 2(\sqrt[m-i+1]{C^*_i})(t - t_{k_{i,j-1}})^{(m-i)!}.$$

if $t \in [t_{k_{i,j}}, T]$.

The same inequalities for the roots $\tau_l(t)$ with $l < i$ follow from Lemma 1.11; thus, choosing a constant $C_i = \max\{\sqrt[m-i+1]{C^*_i}, \tilde{C}_{i-1}\}$ we have property Φ_i and the induction is complete.

We finally define

$$\mathscr{A}_m = \{(t,\rho) \in [0,T] \times \mathbb{C} \,|\, \text{for some } l \in \{1,\dots,m\} \text{ and } j \in \{1,\dots,n_m\} \text{ we have } t_{k_{m,j-2}} \le t \le t_{k_{m,j+1}} \text{ and } |\rho - A_{m,l,j}| < C_m(t_{k_{m,j+1}} - t_{k_{m,j-2}})\}.$$

In $\mathscr{A}_i - (\mathscr{A}_1 \cup \dots \cup \mathscr{A}_{i-1})$, $t \in [t_{k_{i,j-2}}, t_{k_{i,j+1}}]$, $0 \le j-2 \le j+1 \le n_i$, $\rho(t)$ has at most $6q_i$ oscillations.

Gathering the estimates and observing that $\mathscr{A}_m$ contains all the roots of $P(t,\tau)$, we can now show that the total variation of $\rho(t)$ is bounded by a constant independent from the subdivision.

Indeed, let us more generally consider a continuous function $f : [0,T] \to \mathbb{R}$ with bounded variation and let Γ be its graph, μ the Lebesgue-Stieljes measure associated to f and $|\mu|$ its total variation measure; it is well-known that $|\mu|$ is a Borel measure (see e.g. [11]).

Then, if $S \subset [0,T] \times \mathbb{R}$ is a Borel set and $g : [0,T] \to [0,T] \times \mathbb{R}$ is defined as $g(t) = (t, f(t))$, we define the total variation of f in S as

$$TV_S f = |\mu|(g^{-1}(S)).$$

As a first remark, it is clear by the properties of $|\mu|$ that if $\Gamma \subset S$ then

$$TV_S f = TVf.$$

Moreover, if S is a finite (or countable) union of disjoint Borel subsets, say $S = \bigcup_{i=1}^N S_i \subset [0,T] \times \mathbb{R}$, then

$$TV_S f = \sum_{1=i}^{N} TV_{S_i} f.$$

Other useful properties of $TV_S f$ include the fact that if S is a segment parallel to a coordinate axis, $TV_S f = 0$; e.g., if $t_i, t_j \in [0, T]$ and $c, d \in \mathbb{R}$, $TV_{\{t_i\}\times[c,d]} f = 0$ and $TV_{[t_i,t_j]\times\{c\}} f = 0$. Therefore if $S_1, \ldots, S_N$ is a partition of $[0, T] \times \mathbb{R}$, where each S_k is a finite union of rectangles $[t_{j-1}, t_j] \times [\alpha, \beta]$ (where the segments can be open, half-open or closed and the second segment can be a half-line), denoting with $\mathrm{int}(S_k)$ the interior of S_k we get

$$TVf = \sum TV_{S_k} f = \sum TV_{int(S_k)} f,$$

and conversely, if f is continous and $\sum TV_{int(S_k)} f < +\infty$ then $TVf < +\infty$.

We now apply these considerations to our piecewise-linear function ρ with $S = \bigcup_i \mathscr{A}_i$. Since the graph of ρ is contained in S as shown above, we have that

$$\begin{aligned} TV\rho = TV_S f &= TV_{\mathscr{A}_1}\rho + TV_{\mathscr{A}_2 - \mathscr{A}_1}\rho + \cdots + TV_{\mathscr{A}_m - (\mathscr{A}_1 \cup \cdots \cup \mathscr{A}_{m-1})}\rho \\ &\leq 36(C_1 q_1 T + 2C_2 q_2 T + 3C_3 q_3 T + \cdots + mC_m q_m T \\ &\quad + D(q_1 + 2q_2 + \cdots + mq_m)), \end{aligned}$$

where $D = \max\{|\tau_l(t)| \mid t \in [0, T], l = 1, \ldots, m\}$ (and $T \leq 1$); but this is what we wanted to prove, and we conclude. $\square$

Proof of Theorem 1.1. If $m = 1$, there is nothing to prove.

We then proceed by induction on m. Suppose the theorem true for polynomials of degree smaller than m; let P be a polynomial of degree m and let $\tau(t)$ be a continuous root of P. We can suppose that the sum of the roots of P is 0 on $[0, T]$; let then

$$F = \{t \in [0, T] \mid \tau_1(t) = 0, \ldots, \tau_m(t) = 0\}$$

and let $A = [0, T] \backslash F$. A is an open set in $[0, T]$ and therefore it can be written as a union of a countable family of intervals $\{I_n\}_{n\in\mathbb{N}}$, say $I_n = (r_n, s_n)$. By Proposition 1.6, we know that $\tau(t)$ has bounded variation: then, given a positive real number ε, for N large enough we have that

$$\sum_{n>N} TV_{I_n}\tau(t) < \frac{\varepsilon}{3}. \tag{1.13}$$

Let us take the total variation measure of $\tau(t)$ and call it again $|\mu|$. Since τ is a continuous function, there will be a positive real number δ' small enough to have

$$|\mu|([r_j, r_j + \delta']) < \frac{\varepsilon}{6N} \quad \text{and} \quad |\mu|([s_j - \delta', s_j]) < \frac{\varepsilon}{6N} \tag{1.14}$$

for $j = 0, \ldots, N$.

We now consider the set $I_0 \cup I_1 \cup \cdots \cup I_N$: on it the function $\tau(t)$ is absolutely continuous by the induction hypothesis, since it can be seen as a root of a polynomial whose degree is smaller than m with coefficients of the same class $C^{(m^2)!}$. Then $\tau(t)$ is also absolutely continuous on the set

$$\bigcup_{n=0}^{N}[r_n+\delta', s_n-\delta'],$$

so we can find a positive real number δ such that if a finite union of pairwise disjoint intervals (u_i, v_i) is a subset of

$$\bigcup_{n=0}^{N}[r_n+\delta', s_n-\delta']$$

of measure smaller than δ we have that

$$\sum_i |\tau(v_i)-\tau(u_i)| < \frac{\varepsilon}{3}. \tag{1.15}$$

We take now a finite family of pairwise disjoint subintervals of $[0, T]$ (say (x_h, y_h), for $h \in H$ a finite set) such that the (Lebesgue) measure of $\bigcup_h (x_h, y_h)$ is smaller than δ. We consider three different categories of subintervals:

(1) Subintervals of type A are such that $\tau(x_h)=\tau(y_h)=0$ (and clearly give no contribution to the variation);
(2) subintervals of type B are contained in some I_n with $n > N$;
(3) subintervals of type C are contained in some I_n with $n \le N$.

We note that the other subintervals intersect F and not both their endpoints belong to it. We take their first point in F and call it a_h and their last point in F and call it b_h: now, (x_h, a_h) and (b_h, y_h) are subsets of some I_n, while (if non-empty) (a_h, b_h) is of type A: therefore these subintervals can be subdivided into two or three parts, each of type A, B or C.

We now consider the intervals of type C. They can be subdivided into three parts (some of which could be empty) intersecting them with $(r_n, r_n+\delta')$, $(r_n+\delta', s_n-\delta')$ and $(s_n-\delta', s_n)$, respectively: the first and third part will be called intervals of type C_1, and the second part an interval of type C_2.

Recalling the estimates we have proved (respectively (1.13) for intervals of type B, (1.14) for those of type C_1 and (1.15) for those of type C_2) we now conclude. □

2. Two examples

2.1. *Polynomials of degree 3 with real coefficients*

We begin with a polynomial

$$P(\tau, t) = \tau^3 + a_1(t)\tau^2 + a_2(t)\tau + a_3(t)$$

where $a_1(t)$, $a_2(t)$ and $a_3(t)$ are real functions of class C^6. As usual, to simplify a bit the calculations we make a translation to eliminate the second-order term, and so we can assume that $a_1(t) \equiv 0$ (without changing the regularity of the coefficients).

We are given a continuous function $\tau(t)$ that for any $t \in [0, T]$ is a root of P and a subdivision of $[0, T]$ by points $t_0, \dots, t_N$: we now show how to bound the total variation of τ on this subdivision. We introduce the auxiliary piecewise-linear function $\rho(t)$ coinciding with τ at points $t_0, \dots, t_N$ (whose total variation will therefore be the same as that of τ on this subdivision).

The proof is done by induction also in this case, but we will need only one step of it. The base case is in fact identical to what is done in the general case in Lemma 1.9: at the end of this step we have chosen points $t_{1,1} < \cdots < t_{1,n_1}$ dividing the interval in $n_1 + 1$ subintervals, as many constants $A_{1,1,1}, \dots, A_{1,1,n_1+1}$ as there are subintervals and one constant C_1 such that in the interval $[t_{1,j-1}, t_{1,j}]$ there always is a root $\tau_1(t)$ of P in the cylinder

$$|\tau_1(t) - A_{1,1,j}| \leq C_1(t_{1,j} - t_{1,j-1})^2$$

and that in this interval $\rho(t)$ oscillates 6 times across $A_{1,1,j}$ (but no other constant in the place of $A_{1,1,j}$ would make it oscillate more times).

To do the first (and last, in this case) step of the induction, we now set $q_2 = 2^{2 \cdot 2 \cdot 64 \cdot 3} \cdot 24$ and choose new points $t_{2,1} < \cdots < t_{2,n_2}$ among the $t_{1,j}$'s and new real constants $A_{2,2,1}, \dots, A_{2,2,n_2+1}$ so that the function $\rho(t)$ oscillates q_2 times across $A_{2,2,j}$ in the interval $[t_{2,j-1}, t_{2,j}]$ (but no other constant would make it oscillate more than $2q_2$ times) *outside* the cylinders of level 1. This is done exactly in the same way as before: we add one interval of level 1 at a time, until there is some constant across which $\rho(t)$ oscillates more than q_2 times (or until we reach the end of the interval): note that every time we add one interval we add at most $q_1 = 6$ oscillations by the definition of constants at level 1.

Our goal, now, is to choose one of the constants $A_{1,1,j'}$ corresponding to intervals $[t_{1,j'-1}, t_{1,j'}]$ contained in $[t_{2,j-1}, t_{2,j}]$ and use its value as $A_{2,1,j}$, and show that $A_{2,1,j}$ and $A_{2,2,j}$ satisfy the condition of the thesis (with a suitable new constant C_2) of level 2.

We focus on one interval $[t_{2,j-1}, t_{2,j}]$: we take q_2 points $\alpha_1, \dots, \alpha_{2^{768} \cdot 24}$ where $\sigma(\alpha_i) = A_{2,2,j}$. We consider the shortest of the two intervals $[\alpha_1, \alpha_{2^{767} \cdot 24}]$ and $[\alpha_{2^{767} \cdot 24+1}, \alpha_{2^{768} \cdot 24}]$ and call it J_{768}; we also set β_{768} equal to α_1 or $\alpha_{2^{768} \cdot 24}$, choosing the one of the two points not belonging to J_{768}. We then choose J_{767} as the shortest of the two intervals containing $2^{766} \cdot 24$ points in J_{768} and define β_{767} as the endpoint of J_{768} not belonging to J_{767}, and so on, 768 times until we get to J_1 containing 24 point (but not β_1). Now, since all the points $\beta_{j'}$ with $j' < k$ belong to J_k, it is easy to see that there will be at least 384 of the points β that are an increasing sequence, or at least 384 that are a decreasing sequence. We will assume the first, but the proof is the same in the other case.

We call $h_{j'}$ the index of the interval of level 1 containing the point $\beta_{j'}$: the intervals $[t_{1,h_1-1}, t_{1,h_1}], \dots, [t_{1,h_{384}-1}, t_{1,h_{384}}]$ are separated by at least one interval of level 1, since there were at least 24 points between two successive β's. We compare the values of $A_{1,1,j'}$ for these intervals with $A_{2,2,j}$: at least half of them will be above it, or at least half below it; let us suppose that there are at least 192 of them below $A_{2,2,j}$, and let us renumber their indices again $h_1, \dots, h_{192}$. Using Lemma 1.10 we see that we find one index λ and three indices μ_1, μ_2 and μ_3 such that the constants $A_{1,1,h_\lambda}$ and $A_{1,1,h_{\mu_\nu}}$ satisfy

$$A_{1,1,h_{\mu_\nu}} \geq A_{1,1,h_{\lambda_1}} - \tilde{C}(\beta_{\mu_\nu} - t_{1,h_{\lambda_1}-1})^{(m-i+1)!},$$

where $\nu = 1, 2, 3$. Thanks to this inequalities, we can set $A_{2,1,j} = A_{1,1,h_\lambda}$. Lemma 1.11 for our case is simply another way to express the induction hypotheses: if we fix an interval of index j and consider its constant $A_{1,1,j}$, for every t there is a root nearer (in the euclidean distance in $\mathbb{C}$) to the constant $A_{1,1,j}$ than a certain distance, depending only on the (square of the) length of the interval or (for points outside the interval) on the (squared) distance from the point to the farther endpoint of the interval.

It is also evident that at the points β_{μ_ν} the real part of this root will be smaller than $A_{2,2,j}$, since the root belongs to a disc fixed by te induction hypothesis (this for the general case is a consequence of Lemmas 1.12 and 1.13, but here there is no combinatorics to take into account when $i - 1 = 1!$).

We can also write explicitely the formula of Lemma 1.14: since $a_1(t) = -\tau_1(t) - \tau_2(t) - \tau_3(t)$ and $a_2(t) = \tau_1(t)\tau_2(t) + \tau_2(t)\tau_3(t) + \tau_3(t)\tau_1(t)$,

$$\begin{aligned}
&(\tau_2(t) - A_{2,2,j})(\tau_3(t) - A_{2,2,j}) \\
&\quad = (a_2(t) - \tau_1(t)(\tau_2(t) + \tau_3(t))) - A_{2,2,j}(\tau_2(t) + \tau_3(t)) + A_{2,2,j}^2 \\
&\quad = (a_2(t) - A_{2,1,j} - (\tau_1(t) - A_{2,1,j}))(-a_1(t) - A_{2,1,j} - (\tau_1(t) - A_{2,1,j})) \\
&\qquad - A_{2,2,j}(-a_1(t) - A_{2,1,j} - (\tau_1(t) - A_{2,1,j})) + A_{2,2,j}^2 \\
&\quad = (a_2(t) - A_{2,1,j})(-a_1(t) - A_{2,1,j}) + (\tau_1(t) - A_{2,1,j}) \\
&\qquad \times (-a_2(t) + A_{2,1,j} + a_1(t) + \tau_1(t)) \\
&\qquad + A_{2,2,j}(a_1(t) + A_{2,1,j}) + A_{2,2,j}(\tau_1(t) - A_{2,1,j}) + A_{2,2,j}^2 \\
&\quad = [(a_2(t) - A_{2,1,j})(-a_1(t) - A_{2,1,j}) + A_{2,2,j}(a_1(t) + A_{2,1,j}) + A_{2,2,j}^2] \\
&\qquad + (\tau_1(t) - A_{2,1,j})(-a_2(t) + A_{2,1,j} + a_1(t) + \tau_1(t) + A_{2,2,j})
\end{aligned}$$

Now, as in the general case, the conclusion of step 2 of the induction follows.

We remind that $\tau_1(t) + \tau_2(t) + \tau_3(t) = 0$. We can estimate $VT_{\mathscr{A}_1}\rho(t)$ and $VT_{\mathscr{A}_2-\mathscr{A}_1}\rho(t)$: it remains to estimate $VT_{[0,T]\times R-(\mathscr{A}_1\cup\mathscr{A}_2)}\rho(t)$.

The function $\rho(t) - A$ ($A \in R$) has at most q_2 oscillations in $[t_{k_{2,j-1}}, t_{k_{2,j}}] \times R - (\mathscr{A}_1 \cup \mathscr{A}_2)$ ($j = 1, \dots, n_2$), since $(t, \rho(t)) \in ([0, T] \times R) - \mathscr{A}_1$ (see statement Φ_2).

Consider an interval $[t_{l-1}, t_l] \subset [t_{k_{2,j-1}}, t_{k_{2,j}}]$ in which $\rho(t)$ is linear. The graph of $\rho(t)$ outside $\mathscr{A}_2$ is made by (at most) seven segments.

If $(t_{l-1}, \rho(t_{l-1}))$, $(t_l, \rho(t_l))$ are among the endpoints of them, then they coincide with $(t_{l-1}, \sigma(t_{l-1}))$, $(t_l, \sigma(t_l))$ respectively and being outside $\mathscr{A}_2$, $\sigma(t_{l-1}) = \tau_3(t_{l-1})$, $\sigma(t_l) = \tau_3(t_l)$. If $(\tilde{t}, \rho(\tilde{t}))$ $(t_{l-1} < \tilde{t} < t_l)$ is an endpoint which belongs to $\partial\mathscr{A}_2$, it exists (by the theorem of zeroes) $(t', \sigma(t'))$, with $\sigma(t')) = \rho(\tilde{t})$. $(t', \sigma(t')) \notin \mathscr{A}_2$, so $\sigma(t') = \tau_3(t')$. Then

$$\begin{aligned}
&\sup\{|\rho(t) + A_{2,1,j} + A_{2,2,j}| : t \in [t_{l-1}, t_l], (t, \rho(t)) \notin \mathscr{A}_2\} \\
&\quad \le \sup|\tau_3(t) + A_{2,1,j} + A_{2,2,j}| = \sup|-\tau_1(t) - \tau_2(t) + A_{2,1,j} + A_{2,2,j}| \\
&\quad \le \sup|\tau_1(t) - A_{2,1,j}| + \sup|\tau_2(t) - A_{2,2,j}| \\
&\quad \le 2C_2|t_{k_{2,j}} - t_{k_{2,j-1}}|.
\end{aligned}$$

Hence

$$VT_{[0,T]\times R-(\mathscr{A}_1\cup\mathscr{A}_2)}\rho(t) \le \sum_j 2q_2 \cdot 4C_2|t_{k_{2,j}} - t_{k_{2,j-1}}| \le 8q_2C_2T.$$

This achieves the result of Proposition 1.6 in this case; the rest of the proof follows the model of Theorem 1.1.

2.2. m-th roots

Let us suppose that our polynomial has the simple form

$$\tau^m - a(t) = 0$$

where $a : \mathbb{R} \to \mathbb{R}$ is a function of class C^m (nonnegative if m is even). Let us set $\tau(t) = \sqrt[m]{a(t)}$; as an application of our method, we show that the variation of τ on the interval $I = [0, 1]$ is bounded. We will need a bound on the m-th derivative of a: let us suppose then that for every $t \in I$ we have $|a^{(m)}(t)| \le C$.

Let $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = 1$ be a subdivision of I. Consider the piecewise linear function $\sigma : I \to \mathbb{R}$ such that $\sigma(t_j) = \tau(t_j)$ for $j = 0, \dots, N$ and $\sigma(t) = \sigma(t_j) + \frac{t-t_j}{t_{j+1}-t_j}(\sigma(t_{j+1}) - \sigma(t_j))$ if $t \in (t_j, t_{j+1})$. We choose an increasing subsequence $0 = s_0 < s_1 < \cdots < s_K = 1$ of the t_j's and real constants A_j such that

(1) for every j and every l the constant A_j is different from $\tau(t_l)$;
(2) for every $0 \le j < K$ the function $\sigma(t) - A_j$ has exactly m zeros in the interval (s_j, s_{j+1});
(3) the function $\sigma(t) - A_{K-1}$ has less than m zeros in the interval (s_{K-1}, s_K);
(4) for every interval (s_j, s_{j+1}) there is no constant A such that the function $\sigma(t) - A$ has more zeros on the interval than the function $\sigma(t) - A_j$.

This choice is made progressively, starting from t_1 and adding one interval (t_j, t_{j+1}) at a time until there exist one real constant A_0 such that the function $\sigma(t) - A_0$ has m zeros, or until we reach $t_N = 1$. The points s_j are uniquely determined (the constants A_j are not).

We now study what happens on each one of the intervals (s_j, s_{j+1}); let us call it J. Let us suppose, without loss of generality, that $A_j > 0$.

Clearly, also the (continuous) function $\tau(t) - A_j$ has at least m zeros on J; therefore $a(t) - A_j^m = 0$ at least m times in J. But then, as $|a^{(m)}(t)| \leq C$,

$$
\begin{gathered}
|a^{(m-1)}(t)| \leq C(s_{j+1} - s_j) \\
|a^{(m-2)}(t)| \leq C(s_{j+1} - s_j)^2 \\
\cdots \\
|a(t) - A_j^m| \leq C(s_{j+1} - s_j)^m.
\end{gathered}
$$

We consider two cases, according to the sign of $\tau(t)$ (for any point $t \in J$).

- $\tau(t) \geq 0$

Let ζ be a primitive m-th root of unity in $\mathbb{C}$. The m-th roots of A_j^m are $A_j, \zeta A_j, \zeta^2 A_j, \ldots, \zeta^{m-1} A_j$, and clearly for any $l = 1, \ldots, m-1$

$$|\tau(t) - A_j| \leq |\tau(t) - \zeta^l A_j|,$$

therefore

$$
\begin{aligned}
|\tau(t) - A_j|^m &\leq |\tau(t) - A_j| \cdot |\tau(t) - \zeta A_j| \ldots |\tau(t) - \zeta^{m-1} A_j| \\
&\leq |\tau(t)^m - A_j^m| \leq C(s_{j+1} - s_j)^m,
\end{aligned}
$$

from which we get that

$$|\tau(t) - A_j| \leq C^{\frac{1}{m}}(s_{j+1} - s_j).$$

(The same is obviously true for $\sigma(t)$).

- $\tau(t) < 0$

First, we show that it is impossible that

$$C(s_{j+1} - s_j)^m \leq \frac{A_j^m}{2}.$$

Indeed, if this were true, we would have

$$|\tau(t)^m - A_j^m| \leq C(s_{j+1} - s_j)^m \leq \frac{A_j^m}{2}$$

from which we get $\tau(t) > \frac{A_j^m}{2}$ and so $\tau(t) > 0$, against our hypothesis.

But then from

$$C(s_{j+1} - s_j)^m > \frac{A_j^m}{2}$$

and

$$|\tau(t)^m - A_j^m| \leq C(s_{j+1} - s_j)^m$$

we deduce that

$$|\tau(t)| \leq A_j^m + C(s_{j+1} - s_j)^m \leq 3C(s_{j+1} - s_j)^m$$

which implies

$$|\tau(t)| \leq 3^{\frac{1}{m}} C^{\frac{1}{m}} (s_{j+1} - s_j)$$

and

$$|\tau(t) - A_j| \leq A_j + 3^{\frac{1}{m}} C^{\frac{1}{m}} (s_{j+1} - s_j) \leq (2^{\frac{1}{m}} + 3^{\frac{1}{m}}) C^{\frac{1}{m}} (s_{j+1} - s_j).$$

(As above, the same is true for the function σ).

Since then in any case $|\sigma(t) - A_j| \leq 5C^{\frac{1}{m}}(s_{j+1} - s_j)$, and by our choice of A_j, we deduce that the total variation of σ on J is bounded by $10mC^{\frac{1}{m}}|s_{j+1} - s_j|$.

Finally, we sum the contributions of all the intervals and find that the total variation of σ on I, that coincides with the total variation of τ on our subdivision, is bounded by $10mC^{\frac{1}{m}}$, a constant independent from the subdivision: then the total variation of τ on I is also bounded (by the same constant). From here, as above, absolute continuity follows.

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