



**Calculus of Variations** — *On the isoperimetric profile for a mixed Euclidean–Log-convex measure*, by NICOLA FUSCO and DIEGO PALLARA, communicated on February 10, 2017.

*This paper is dedicated to the memory of Professor Ennio De Giorgi.*

ABSTRACT. — We study the isoperimetric problem in  $\mathbb{R}^h \times \mathbb{R}^k$  endowed with a mixed Euclidean–Log-convex measure  $\lambda = e^{\psi(x)} dx dy$ . We prove the existence of an isoperimetric set and we show some of its qualitative properties.

KEY WORDS: Isoperimetric problem, mixed densities, existence and properties of minimizers

MATHEMATICS SUBJECT CLASSIFICATION: 49Q20, 28A75

## INTRODUCTION

The classical isoperimetric problem consists in finding the sets with minimal boundary measure under a volume constraint or, equivalently, the sets with maximal volume and given boundary measure, that are called *isoperimetric sets*. We limit ourselves to recalling that the usual approach to this problem consists in showing existence of minimizers in the class of sets with finite perimeter and in describing the solutions via suitable symmetrization techniques. Symmetrization techniques are classical, but the setting of finite perimeter sets and the proof of the optimality of the ball in  $\mathbb{R}^n$  in this class came much later and were due to E. De Giorgi, see [14]. We refer to [17, 19] and the references therein for a complete information on the whole subject, in the classical case of Lebesgue measure on  $\mathbb{R}^n$ . More generally, the same problem can be set in general contexts, such as differentiable (sub)riemannian manifolds, currents, or Euclidean spaces with densities, see e.g. [1, 3, 5, 15, 18, 20, 21]. In the latter case, suitable notions of symmetrization have been devised and applied to the study of properties of other analytical problems, see e.g. [4, 6, 7, 8, 10, 12, 16]. As a particular case, decomposing  $\mathbb{R}^n$  as a cartesian product,  $\mathbb{R}^n = \mathbb{R}^h \times \mathbb{R}^k$ , with  $h, k \geq 1$ , *mixed densities* can be considered, i.e., measures  $\lambda$  on  $\mathbb{R}^n$  arising as product measures on the factors,  $\lambda = \mu \otimes \nu$ , with  $\mu, \nu$  measures on  $\mathbb{R}^h, \mathbb{R}^k$ , respectively. The isoperimetric

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problem in such a mixed framework has been studied e.g. in [18] with  $\nu = \mathcal{L}^k$  the Lebesgue measure on  $\mathbb{R}^k$  and  $\mu$  the standard Gaussian measure on  $\mathbb{R}^h$ , relying on a suitable notion of mixed rearrangement. In the present paper we consider a mixed density, i.e., a measure  $\lambda = \mu \otimes \mathcal{L}^k$ , where  $\mu$  is a Log-convex measure on  $\mathbb{R}^h$ . Interest in such framework comes also from the recent proof of the Log-convex density conjecture by G. R. Chambers, see [9].

Let us come to a description of the content of the present paper. We consider the Euclidean space  $\mathbb{R}^n = \mathbb{R}^h \times \mathbb{R}^k$ , with  $h, k \geq 1$ , whose points are denoted by  $z = (x, y)$ , endowed with the measure  $\lambda = e^{\psi(|x|)} dx dy$ , where  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  is a  $C^\infty$ , convex, even function, and the isoperimetric problem

$$(1) \quad \inf \left\{ \int_{\partial E} e^{\psi(|x|)} d\mathcal{H}^{n-1}(x, y) : \int_E e^{\psi(|x|)} dx dy = m \right\}, \quad m > 0.$$

As the density of  $\lambda$  with respect to the Lebesgue measure is regular, the class  $\mathcal{P}_\lambda(\mathbb{R}^n)$  of sets with locally finite perimeter with respect to  $\lambda$  is the same as the classical nonweighted one, and we may cast the isoperimetric problem (1) in this class as follows:

$$(2) \quad \inf \left\{ P_\lambda(E) : \lambda(E) = \int_E e^{\psi(|x|)} dx dy = m \right\}, \quad m > 0.$$

where

$$(3) \quad P_\lambda(E) := \sup \left\{ \int_E \operatorname{div}_\lambda F d\lambda : F \in C_c^1(\mathbb{R}^n, \mathbb{R}^n), \|F\|_\infty \leq 1 \right\}$$

and

$$\operatorname{div}_\lambda F(x, y) = \sum_{j=1}^n \partial_j F_j(x, y) + \psi'(|x|) \sum_{j=1}^h \frac{x_j}{|x|} F_j(x, y).$$

In case of regular (say, Lipschitz continuous) boundaries we have the equality

$$P_\lambda(E) = \int_{\partial E} e^{\psi(|x|)} d\mathcal{H}^{n-1}(x, y).$$

Our main results concern existence, geometric properties and uniqueness of the isoperimetric sets. After introducing the suitable weighted Steiner symmetrization in our setting and discussing the main properties, we show that a symmetric isoperimetric set exists (up to translations along the  $y$  directions). Indeed, every isoperimetric set is Steiner symmetric with respect to both coordinate spaces. Moreover, we can prove that the isoperimetric set is unique (up to translations along the  $y$  directions) provided  $\min\{h, k\} = 1$  and that if  $k = 1$  and the mass  $m$  is small enough or if  $h = 1$ , the isoperimetric set is strictly convex. In order to

prove the existence of an isoperimetric set, the (classical) idea is to replace each term of a minimizing sequence by its symmetrized and to show that the new sequence converges to a set which fulfils the volume constraint. Performing this program, in our case, relies on the standard Steiner symmetrization with respect to the subspace  $\{y = 0\}$  and on a *weighted* rearrangement with respect to the subspace  $\{x = 0\}$ , which depends on the density  $\psi$ . Symmetry properties of minimizers depend upon the stability of minimizers of the functional  $J$  in (1.5), which allows us to express the weighted perimeter, with respect to weighted symmetrization, see Theorem 1.8. As a consequence of the first variation formula (2.4) and the regularity of the density  $\psi$  we get the regularity of the whole boundary of the isoperimetric sets, see Theorem 3.2. The aforementioned geometric properties and the uniqueness of the isoperimetric profiles are proved through a careful analysis of the Euler equation (2.4) as it can be formulated in view of the symmetry properties. This is done in Section 3.

### 1. PRELIMINARIES

In the following we denote by  $B_r(z)$  the  $n$ -dimensional ball with center at  $z$  and radius  $r$ . When the center  $z$  is the origin we simply write  $B_r$  instead of  $B_r(0)$ . If  $x \in \mathbb{R}^h$  the  $h$ -dimensional ball with center at  $x$  and radius  $r$  is denoted by  $B_r^{(h)}(x)$ . As before, if  $x = 0$  we simply write  $B_r^{(h)}$ . If  $0 \leq s \leq n$  we denote by  $\mathcal{H}^s$  the  $s$ -dimensional Hausdorff measure. For every set  $E \subset \mathbb{R}^n$  and every  $x \in \mathbb{R}^h$  we define

$$(1.1) \quad E_x := \{y \in \mathbb{R}^k : (x, y) \in E\} \quad \text{and} \quad v_E(x) := \mathcal{H}^k(E_x),$$

where we recall that  $\mathcal{H}^k$  coincides with the *outer Lebesgue measure* in  $\mathbb{R}^k$ , and we set

$$\pi(E)^+ := \{x \in \mathbb{R}^h : v_E(x) > 0\}.$$

We assume that the reader is well acquainted with the theory of  $BV$  functions and sets of finite perimeter. Here we just set a few notation. Given a set of locally finite perimeter  $E$  in  $\mathbb{R}^n$  we denote by  $\partial^* E$  its *reduced boundary* and by  $\nu^E$  its *generalized outer normal*.

If  $u$  is a function in  $BV(\mathbb{R}^m)$ , we say that  $u$  has approximate limit at  $z \in \mathbb{R}^m$  if there exists  $\tilde{u}(z) \in \mathbb{R}$  such that

$$\lim_{r \rightarrow 0} \int_{B_r(z)} |u(w) - \tilde{u}(z)| \, dw = 0.$$

By the Lebesgue differentiation theorem, we know that  $u = \tilde{u}$  a.e. in  $\mathbb{R}^m$ . We denote by

$$\mathcal{C}_u := \{z \in \mathbb{R}^m : \tilde{u}(z) \text{ exists}\}$$

the (Borel) set of *points of approximate continuity* of  $u$ . If  $z \in \mathcal{C}_u$  we say that  $u$  is *approximately differentiable* at  $z$  if there exists a vector  $\nabla u(z) \in \mathbb{R}^m$  such that

$$(1.2) \quad \lim_{r \rightarrow 0} \int_{B_r(z)} \frac{|u(w) - \tilde{u}(z) - \nabla u(z) \cdot (w - z)|}{r} dw = 0.$$

The set of all points  $z \in \mathbb{R}^m$  where  $\nabla u(z)$  exists is denoted by  $\mathcal{D}_u$ . Finally we recall that the approximate gradient defined in (1.2) coincides  $\mathcal{L}^m$ -a.e. with the absolutely continuous part of the measure gradient  $Du$ . Therefore, the following decomposition formula holds

$$Du = \nabla u \mathcal{L}^m + D^s u.$$

For all the other properties of sets of finite perimeter and  $BV$  functions needed in the following we refer to the book [2]. Here, we just recall the following result, essentially due to Vol'pert, see [4, Th. 2.4], stating that  $\mathcal{H}^h$ -a.e. slice of a set of finite perimeter  $E$  in  $\mathbb{R}^n$  is a set of finite perimeter in  $\mathbb{R}^k$  and relating the reduced boundary and the exterior normal of  $E$  to the ones of its slices. In the following, if  $v$  is any vector in  $\mathbb{R}^n$  we set  $v_x := (v_1, \dots, v_h)$  and  $v_y := (v_{h+1}, \dots, v_n)$ .

**THEOREM 1.1.** *Let  $E$  be a set of finite perimeter in  $\mathbb{R}^n$ . Then for  $\mathcal{H}^h$ -a.e.  $x \in \mathbb{R}^h$ ,*

- (i)  $E_x$  is a set of finite perimeter in  $\mathbb{R}^k$ ;
- (ii)  $\mathcal{H}^{k-1}(\partial^* E_x \Delta (\partial^* E)_x) = 0$ ;
- (iii) for  $\mathcal{H}^{k-1}$ -a.e.  $y$  such that  $y \in \partial^* E_x \cap (\partial^* E)_x$  we have
  - (a)  $v_y^E(x, y) \neq 0$ ,
  - (b)  $v^{E_x}(y) = \frac{v_y^E(x, y)}{|v_y^E(x, y)|}$ .

*In particular, there exists a Borel set  $G_E \subset \pi(E)^+$  such that  $\mathcal{H}^h(\pi(E)^+ \setminus G_E) = 0$  and (i)–(iii) are satisfied for every  $x \in G_E$ .*

In view of the above theorem, if  $E$  is a set of finite perimeter we may define for  $\mathcal{H}^h$ -a.e.  $x \in \mathbb{R}^h$

$$(1.3) \quad p_E(x) := \mathcal{H}^{k-1}(\partial^* E_x).$$

It is readily checked that  $p_E$  is a Borel function.

### 1.1. Steiner symmetrization

Let us now recall the definitions and properties of the *Steiner symmetrization with respect to the subspace*  $\{y = 0\}$ . For every  $E \subset \mathbb{R}^n$  we denote by  $r(x)$  the radius

of a  $k$ -dimensional ball in  $\mathbb{R}^k$  with measure  $v_E(x)$ , see (1.1). Then the Steiner symmetral of  $E$  with respect to the subspace  $\{y = 0\}$  is defined as

$$E^S = \{(x, y) \in \mathbb{R}^n : x \in \pi(E)^+, |y| < r(x)\}.$$

By construction  $v_E = v_{E^S}$  and  $\lambda(E) = \lambda(E^S)$ . If  $E = E^S$  we say that  $E$  is Steiner symmetric with respect to the subspace  $\{y = 0\}$ .

By replacing  $E_x$  with  $E^y = \{x \in \mathbb{R}^h : (x, y) \in E\}$  the Steiner symmetral  $E_S$  of  $E$  with respect to the subspace  $\{x = 0\}$  is defined similarly.

Next result, see [4, Lemma 3.1 and Prop. 3.5] deals with the properties of the function  $v_E$  defined in (1.1).

**PROPOSITION 1.2.** *Let  $E \subset \mathbb{R}^n$  be a set of finite measure and perimeter. Then  $v_E \in BV(\mathbb{R}^h)$  and  $|Dv_E|(\mathbb{R}^h) \leq P(E)$ . Moreover*

$$v_E \in W^{1,1}(\mathbb{R}^h) \text{ if and only if } \mathcal{H}^{n-1}(\{z \in \partial^* E^S : v_y^{E^S}(z) = 0\}) = 0.$$

Roughly speaking, the above proposition states that the function measuring the vertical slices of  $E$  is  $W^{1,1}$  if and only if the boundary of  $E$  has no vertical parts.

It is well known that Steiner symmetrization decreases the perimeter. The same happens also for the mixed perimeter  $P_\lambda$  defined in (3). Indeed, this follows from a more general inequality proved in [4, Prop. 3.4]. In the statement below  $p_E$  is the function defined in (1.3).

**PROPOSITION 1.3.** *Let  $E \subset \mathbb{R}^n$  be a set of finite measure and perimeter and let  $g : \mathbb{R}^h \rightarrow [0, \infty]$  be a Borel function. Then*

$$(1.4) \quad \int_{\partial^* E} g(x) d\mathcal{H}^{n-1} \geq \int_{\mathbb{R}^h} g(x) \sqrt{p_E(x)^2 + |\nabla v_E(x)|^2} dx + \int_{\mathbb{R}^h} g(x) d|D^s v_E|$$

with the equality holding when  $E = E^S$ . In particular, for any Borel set  $B \subset \mathbb{R}^h$

$$\begin{aligned} P_\lambda(E; B \times \mathbb{R}^k) &\geq \int_B e^{\psi(|x|)} \sqrt{p_{E^S}(x)^2 + |\nabla v_E(x)|^2} dx + \int_B e^{\psi(|x|)} d|D^s v_E| \\ &= P_\lambda(E^S; B \times \mathbb{R}^k). \end{aligned}$$

Observe that if  $E = E^S$ , then  $v_E(x) = \omega_k r(x)^k$ , where  $\omega_k$  denotes the measure of the  $k$ -dimensional unit ball. Therefore in this case we have  $p_E(x) = k\omega_k^{1/k} v_E(x)^{(k-1)/k}$  and  $P_\lambda(E) = J(v_E)$ , where the functional  $J$  is defined for any function  $u \in BV(\mathbb{R}^h)$  by setting

$$(1.5) \quad J(u) = \int_{\{u>0\}} e^{\psi(|x|)} \sqrt{k^2 \omega_k^{\frac{2}{k}} u(x)^{\frac{2k-2}{k}} + |\nabla u(x)|^2} dx + \int_{\mathbb{R}^h} e^{\psi(|x|)} d|D^s u|.$$

The characterization of the equality cases in the inequality  $P(E) \geq P(E^S)$ , where  $P(\cdot)$  denotes the standard perimeter, was initiated in [10] and carried on in [4, 7], see also [8] for the case of the Gaussian perimeter.

The following result is an immediate consequence of the analogous result for the Euclidean perimeter established in [4, Th. 1.2].

**THEOREM 1.4.** *Let  $U \subset \mathbb{R}^h$  be a connected open set and let  $E$  be a set of finite perimeter in  $U \times \mathbb{R}^k$  such that  $P_\lambda(E; U \times \mathbb{R}^k) = P_\lambda(E^S; U \times \mathbb{R}^k)$ . Assume that the following two conditions hold:*

- (i)  $\mathcal{H}^{n-1}(\{z \in \partial E^S : \nu_y^{E^S}(z) = 0\} \cap (U \times \mathbb{R}^k)) = 0$ ,
- (ii)  $\tilde{v}_E(x) > 0$  for  $\mathcal{H}^{h-1}$ -a.e.  $x \in U$ .

*Then  $E \cap (U \times \mathbb{R}^k)$  is equivalent to a translate along  $\mathbb{R}^k$  of  $E^S \cap (U \times \mathbb{R}^k)$ .*

**PROOF.** Let  $B \subset U$  be any Borel set. Since  $P_\lambda(E; B \times \mathbb{R}^k) \geq P_\lambda(E^S; B \times \mathbb{R}^k)$ , from the assumption  $P_\lambda(E; U \times \mathbb{R}^k) = P_\lambda(E^S; U \times \mathbb{R}^k)$  we have that  $P_\lambda(E; B \times \mathbb{R}^k) = P_\lambda(E^S; B \times \mathbb{R}^k)$ . By assumption (i) Proposition 1.2 gives that  $v_E \in W_{loc}^{1,1}(U)$ . Thus from Proposition 1.3 it follows that for every Borel set  $B \subset U$

$$(1.6) \quad \int_{\partial^* E \cap (B \times \mathbb{R}^k)} e^{\psi(|x|)} d\mathcal{H}^{n-1} = \int_B e^{\psi(|x|)} \sqrt{p_{E^S}(x)^2 + |\nabla v_E(x)|^2} dx.$$

We now set for any Borel set  $B \subset U$

$$\mu(B) := P(E; B \times \mathbb{R}^k), \quad v(B) := P(E^S; B \times \mathbb{R}^k).$$

Observe that by Propositions 1.2 and 1.3 we have

$$v(B) = \int_B \sqrt{p_{E^S}(x)^2 + |\nabla v_E(x)|^2} dx.$$

Therefore, equation (1.6) can be rewritten as

$$\int_B e^{\psi(|x|)} d\mu = \int_B e^{\psi(|x|)} dv.$$

Since  $\mu \geq v$  (again by Proposition 1.3) and  $\psi \geq c > 0$  from the above equality we have  $\mu = v$ , hence

$$P(E; U \times \mathbb{R}^k) = P(E^S; U \times \mathbb{R}^k).$$

From this inequality and the assumptions (i), (ii), the result follows by [4, Th. 1.2]. □

1.2. *Weighted symmetric rearrangements*

In the sequel, given  $0 \leq s \leq h$ , we denote by  $\lambda_s$  the measure defined by setting for any Borel set  $B \subset \mathbb{R}^h$

$$\lambda_s(B) = \int_B e^{\psi(|x|)} d\mathcal{H}^s(x).$$

With this definition in hand we may now proceed to defining the weighted spherically symmetric decreasing rearrangement of a nonnegative function  $u : \mathbb{R}^h \rightarrow [0, \infty]$  with the property that the level set  $\{x \in \mathbb{R}^h : u(x) > t\}$  has finite  $\lambda_h$  measure for every  $t > 0$ . To this aim we introduce the function  $\mu_u : [0, \infty) \rightarrow [0, \infty]$ , defined for  $t \geq 0$  as

$$\mu_u(t) = \lambda_h(\{x \in \mathbb{R}^h : u(x) > t\}),$$

which is called the *distribution function* of  $u$ . Then the *weighted decreasing rearrangement*  $u^\#$  of  $u$  is the function from  $[0, \infty)$  to  $[0, \infty]$  given by

$$u^\#(s) = \sup\{t \geq 0 : \mu_u(t) > s\}$$

for  $0 \leq s < \lambda_h(\{u > 0\})$ ,  $u^\#(s) = 0$  otherwise. Observe that the function  $u^\#$  is decreasing and right-continuous,  $u^\#(0) = \text{ess sup } u$ , and

$$(1.7) \quad \{s \geq 0 : u^\#(s) > t\} = [0, \mu_u(t)) \quad \text{for every } 0 \leq t < \text{ess sup } u.$$

The *weighted symmetric rearrangement* of  $u$  is the function from  $\mathbb{R}^h$  into  $[0, \infty]$  defined as

$$u^*(x) = u^\#(\lambda_h(B_{|x|}^{(h)})) \quad \text{for } x \in \mathbb{R}^h.$$

See [6] for a similar definition. Note that (1.7) implies that  $\mu_u(t) = \mu_{u^*}(t)$  for every  $t > 0$  and thus for every  $\alpha > 0$

$$(1.8) \quad \int_{\mathbb{R}^h} u(x)^\alpha d\lambda_h = \int_{\mathbb{R}^h} u^*(x)^\alpha d\lambda_h.$$

Most of the properties of the standard decreasing rearrangement are true also for the weighted rearrangement, by just repeating verbatim the proofs of the standard case. We present here some useful properties of the distribution function  $\mu_u$  and of the weighted symmetric rearrangement of a  $BV$  function  $u$ . The first result, which can be proved exactly as in standard case, see [11, Lemmas 3.1 and 3.2], provides a formula for the derivative of  $\mu_u$ . To this aim, given a function  $u \in BV(\mathbb{R}^h)$  we set

$$\mathcal{D}_u^+ := \{x \in \mathbb{R}^h : \nabla u(x) \neq 0\}, \quad \mathcal{D}_u^0 := \{x \in \mathbb{R}^h : \nabla u(x) = 0\},$$

**PROPOSITION 1.5.** *Let  $u \in BV(\mathbb{R}^h)$  be a nonnegative function such that  $\lambda_h(\{u > t\}) < \infty$  for every  $t > 0$ . Then for  $t > 0$*

$$\mu_u(t) = \lambda_h(\{u > t\} \cap \mathcal{D}_u^0) + \int_t^\infty d\tau \int_{\partial^*\{u>\tau\}} \frac{\chi_{\mathcal{D}_u^+}}{|\nabla u|} d\lambda_{h-1}$$

and for a.e.  $t > 0$

$$(1.9) \quad \mu'_u(t) \leq - \int_{\partial^*\{u>t\}} \frac{\chi_{\mathcal{D}_u^+}}{|\nabla u|} d\lambda_{h-1}.$$

Moreover,

$$(1.10) \quad \mu'_u(t) = - \frac{\lambda_{h-1}(\{u^* = t\})}{|\nabla u^*|_{\{u^*=t\}}} \quad \text{for a.e. } t \in u^*(\mathcal{D}_{u^*}^+).$$

and

$$(1.11) \quad \mu'_u(t) = 0 \quad \text{for a.e. } t \in (0, \infty) \setminus u^*(\mathcal{D}_{u^*}^+).$$

In the aforementioned paper [11] several fine properties of the symmetric rearrangement of a  $BV$  functions are established. In the next result we recall a few ones that are needed below, see Lemma 2.6 (v) and Part I of Theorem 1.2 of [11].

**PROPOSITION 1.6.** *Let  $u$  be a nonnegative function in  $BV(\mathbb{R}^h)$ . Then, for almost every  $t \in u^*(\mathcal{D}_{u^*}^+)$*

$$(1.12) \quad \mathcal{H}^{h-1}(\partial^*\{u^* > t\} \Delta \{u^* = t\}) = 0$$

Moreover, for almost every  $t \in (0, \infty) \setminus u^*(\mathcal{D}_{u^*}^+)$

$$(1.13) \quad \mathcal{H}^{h-1}(\partial^*\{u^* > t\} \cap \mathcal{D}_{u^*}^+) = 0 \quad \text{and} \quad \mathcal{H}^{h-1}(\partial^*\{u > t\} \cap \mathcal{D}_u^+) = 0$$

Finally, we make use of a recent, deep result proved by Chambers in [9], known as the *Log-convex density conjecture*.

**THEOREM 1.7.** *Let  $E$  be a set of finite perimeter in  $\mathbb{R}^h$  with  $\lambda_h(E) = \lambda_h(B_r^{(h)})$ . Then*

$$(1.14) \quad \lambda_{h-1}(\partial^*E) \geq \lambda_{h-1}(\partial B_r^{(h)}).$$

Assume also that

$$(1.15) \quad \psi(r) > \psi(0) \quad \text{for every } r \neq 0.$$

Then, if the equality holds in (1.14),  $E$  is equivalent to  $B_r^{(h)}$ .

Next result is a *Pólya–Szegő principle* for the functional  $J$  defined in (1.5) and tells us that this functional decreases under weighted symmetric rearrangement.



Its proof follows the argument used in the proof of [11, Th. 1.4] with some extra complications due to the fact that the integrand in the definition of  $J$  depends also on  $u$ .

**THEOREM 1.8.** *Let  $J$  be the functional defined in (1.5). Then*

$$(1.16) \quad J(u^*) \leq J(u)$$

for any nonnegative function  $u \in BV(\mathbb{R}^h)$ . Moreover, if  $\psi$  satisfies (1.15) and equality holds in (1.16), then  $u$  agrees a.e. with  $u^*$ .

**PROOF.** *Step 1.* We start by rewriting the functional  $J$  as follows:

$$J(u) = \tilde{J}(u) + k\omega_k^{\frac{1}{k}} \int_{\{u>0\}} e^{\psi(|x|)} u(x)^{\frac{k-1}{k}} dx,$$

where we have set

$$\begin{aligned} \tilde{J}(u) := & \int_{\mathcal{D}_u} e^{\psi(|x|)} (\sqrt{k^2 \omega_k^{\frac{2}{k}} u(x)^{\frac{2k-2}{k}} + |\nabla u(x)|^2} - k\omega_k^{\frac{1}{k}} u(x)^{\frac{k-1}{k}}) dx \\ & + \int_{\mathbb{R}^h \setminus \mathcal{D}_u} e^{\psi(|x|)} d|D^s u|. \end{aligned}$$

Since from (1.8)

$$(1.17) \quad \int_{\{u>0\}} e^{\psi(|x|)} u^*(x)^{\frac{k-1}{k}} dx = \int_{\{u>0\}} e^{\psi(|x|)} u(x)^{\frac{k-1}{k}} dx < \infty,$$

in order to prove (1.16) it is enough to show that

$$(1.18) \quad \tilde{J}(u^*) \leq \tilde{J}(u)$$

To this aim we define a function  $B : (0, \infty) \times [0, \infty] \rightarrow [0, 1]$  setting for  $t > 0$ ,  $0 \leq s \leq \infty$

$$B(t, s) := \begin{cases} 0 & \text{if } s = 0, \\ \frac{\sqrt{k^2 \omega_k^{\frac{2}{k}} t^{\frac{2k-2}{k}} + s^2} - k\omega_k^{\frac{1}{k}} t^{\frac{k-1}{k}}}{s} & \text{if } 0 < s < \infty, \\ 1 & \text{if } s = \infty. \end{cases}$$

It is easily checked that for every  $t > 0$  the function  $B(t, \cdot) : [0, \infty] \rightarrow [0, 1]$  is strictly increasing. Moreover the function  $C(t, \cdot) : [0, 1] \rightarrow [0, \infty]$ , defined by setting for  $r \in [0, 1]$

$$C(t, r) := \frac{1}{[B(t, \cdot)]^{-1}(r)},$$

is strictly decreasing and strictly convex. To prove the latter claim let us calculate for  $t > 0$  and  $0 < r < 1$

$$\frac{\partial C}{\partial r}(t, r) = -\frac{1}{([B(t, \cdot)]^{-1}(r))^2} \cdot \frac{1}{\partial_s B(t, [B(t, \cdot)]^{-1}(r))}.$$

Since  $[B(t, \cdot)]^{-1}$  is strictly increasing, the claim follows by observing that the function

$$s \mapsto \frac{1}{s^2 \partial_s B(t, s)} = \frac{\sqrt{\gamma_k^2 + s^2}}{\gamma_k \sqrt{\gamma_k^2 + s^2 - \gamma_k^2}},$$

where we have set  $\gamma_k = k\omega_k^{\frac{1}{k}} t^{\frac{k-1}{k}}$ , is strictly decreasing. Let us now set, for a function  $v \in BV(\mathbb{R}^h)$ ,

$$g_v(x) := \begin{cases} |\nabla v(x)| & \text{if } x \in \mathcal{D}_v^+, \\ +\infty & \text{otherwise.} \end{cases}$$

Then, using the coarea formula for  $BV$  functions,  $\tilde{J}(u)$  can be rewritten as

$$\begin{aligned} \tilde{J}(u) &= \int_{\mathcal{D}_u} e^{\psi(|x|)} B(\tilde{u}(x), g_u(x)) d|Du| + \int_{\mathbb{R}^h \setminus \mathcal{D}_u} e^{\psi(|x|)} d|D^s u| \\ &= \int_0^\infty dt \int_{\partial^* \{u>t\} \cap \mathcal{D}_u} e^{\psi(|x|)} B(\tilde{u}(x), g_u(x)) d\mathcal{H}^{h-1} \\ &\quad + \int_0^\infty dt \int_{\partial^* \{u>t\} \setminus \mathcal{D}_u} e^{\psi(|x|)} d\mathcal{H}^{h-1}. \end{aligned}$$

Thus, since for a.e.  $t \in (0, \infty)$

$$\mathcal{H}^{h-1}((\partial^* \{u > t\} \Delta \{\tilde{u} = t\}) \cap \mathcal{C}_u) = 0,$$

recalling the definition of  $g_u$  and  $B$  we have

$$(1.19) \quad \tilde{J}(u) = \int_0^\infty dt \int_{\partial^* \{u>t\}} B(t, g_u(x)) d\lambda_{h-1}.$$

*Step 2.* From Jensen's inequality we have that for a.e.  $t > 0$

$$C\left(t, \int_{\partial^* \{u>t\}} B(t, g_u(x)) d\lambda_{h-1}\right) \leq \int_{\partial^* \{u>t\}} C(t, B(t, g_u(x))) d\lambda_{h-1},$$

hence, recalling that  $C(t, \cdot)$  is strictly decreasing, we have

$$(1.20) \quad \int_{\partial^* \{u>t\}} B(t, g_u(x)) d\lambda_{h-1} \geq C^{-1}\left(t, \int_{\partial^* \{u>t\}} C(t, B(t, g_u(x))) d\lambda_{h-1}\right).$$

Recalling the definition of  $g_u$  and  $B$  and (1.9), we have that for a.e.  $t > 0$

$$\begin{aligned} & C^{-1}\left(t, \int_{\partial^*\{u>t\}} C(t, B(t, g_u(x))) d\lambda_{h-1}\right) \\ &= C^{-1}\left(t, \int_{\partial^*\{u>t\}} \frac{1}{g_u(x)} d\lambda_{h-1}\right) \\ &= C^{-1}\left(t, \int_{\partial^*\{u>t\}} \frac{\chi_{\mathcal{D}_u^+}}{|\nabla u|} d\lambda_{h-1}\right) = B\left(t, \frac{\lambda_{h-1}(\partial^*\{u > t\})}{\int_{\partial^*\{u>t\}} \frac{\chi_{\mathcal{D}_u^+}}{|\nabla u|} d\lambda_{h-1}}\right) \\ &\geq B\left(t, \frac{\lambda_{h-1}(\partial^*\{u > t\})}{-\mu'_u(t)}\right) \geq B\left(t, \frac{\lambda_{h-1}(\partial^*\{u^* > t\})}{-\mu'_u(t)}\right), \end{aligned}$$

where the last inequality follows from the isoperimetric inequality (1.14). Thus, using the representation formula (1.10), (1.12) and the fact that  $|\nabla u^*|$  is constant  $\mathcal{H}^{h-1}$ -a.e. on  $\{u^* = t\}$ , we conclude that for a.e.  $t \in u^*(\mathcal{D}_{u^*}^+)$

$$(1.21) \quad C^{-1}\left(t, \int_{\partial^*\{u>t\}} C(t, B(t, g_u(x))) d\lambda_{h-1}\right) \geq B(t, |\nabla u^*|_{|\partial^*\{u^*>t\}}) = B(t, g_{u^*})_{|\partial^*\{u^*>t\}}.$$

If instead  $t \in (0, \infty) \setminus u^*(\mathcal{D}_{u^*}^+)$ , from the second equality in (1.13) we have that  $g_u(x) = \infty$  for  $\mathcal{H}^{h-1}$ -a.e.  $x \in \partial^*\{u > t\}$ , therefore  $C(t, B(t, g_u(x))) = 0$  for  $\mathcal{H}^{h-1}$ -a.e.  $x \in \partial^*\{u > t\}$ . Thus, recalling that  $C^{-1}(t, 0) = 1$  and using the first equality in (1.13), we have

$$(1.22) \quad C^{-1}\left(t, \int_{\partial^*\{u>t\}} C(t, B(t, g_u(x))) d\lambda_{h-1}\right) = 1 = B(t, g_{u^*})_{|\partial^*\{u^*>t\}}.$$

Thus, from (1.21) and (1.22), using (1.20) and the isoperimetric inequality (1.14) again, we get that for a.e.  $t > 0$

$$\int_{\partial^*\{u>t\}} B(t, g_u(x)) d\lambda_{h-1} \geq \int_{\partial^*\{u^*>t\}} B(t, g_{u^*}(x)) d\lambda_{h-1}.$$

From this inequality, recalling (1.19), inequality (1.18) immediately follows.

Finally, let us assume that condition (1.15) holds and that  $J(u) = J(u^*)$ . By (1.17) this equality implies that  $\tilde{J}(u) = \tilde{J}(u^*)$ . At this point, from the argument we just used to deduce inequality (1.18), it is clear that for a.e.  $t > 0$

$$\lambda_{h-1}(\partial^*\{u > t\}) = \lambda_{h-1}(\partial^*\{u^* > t\}).$$

Thus, from Theorem 1.7 we get that for a.e.  $t > 0$  the level set  $\{u > t\}$  is a ball centered at the origin. At this point it is not too hard to show that  $u$  coincides  $\mathcal{H}^h$  a.e. with  $u^*$ , see the proof of [11, Lemma 4.1] for the details.  $\square$

2. EXISTENCE AND THE FIRST VARIATION FORMULA

In this section we prove the existence of an isoperimetric set with respect to the weighted volume  $\lambda(E)$  and the weighted perimeter  $P_\lambda(E)$ .

**THEOREM 2.1.** *For every  $m > 0$  the infimum in (2) is attained.*

**PROOF.** Fix  $m > 0$  and let  $E_j$  be a minimizing sequence for the problem (2). First, we perform a Steiner symmetrization of codimension  $k$  of the sets  $E_j$  and denote by  $E_j^S$  the corresponding Steiner symmetrizations. By Proposition 1.3, we have that for every  $j$

$$(2.1) \quad \lambda(E_j) = \lambda(E_j^S) = m \quad \text{and} \quad P_\lambda(E_j^S) \leq P_\lambda(E_j).$$

Moreover, by Proposition 1.2, setting  $v_j = v_{E_j}$ , the sequence  $v_j$  is bounded in  $BV(\mathbb{R}^h)$  and by (1.8) and (1.16) we have also that

$$\int_{\mathbb{R}^h} v_j \, d\lambda_h = \int_{\mathbb{R}^h} v_j^* \, d\lambda_h, \quad J(v_j^*) \leq J(v_j),$$

where  $v_j^*(x)$  is the weighted symmetric rearrangement of  $v_j$ . From these relations, recalling (2.1) and setting for every  $j$

$$F_j = \{(x, y) \in \mathbb{R}^h \times \mathbb{R}^k : \omega_k |y|^k < v_j^*(x)\},$$

we conclude that  $F_j$  is a minimizing sequence for the problem (2). Since the sequence  $v_j^*$  is bounded in  $BV(\mathbb{R}^h)$  we may assume without loss of generality that the functions  $v_j^*$  converge in  $L^1_{loc}(\mathbb{R}^h)$  to a nonnegative function  $v \in BV(\mathbb{R}^h)$ . Therefore, by well known lower semicontinuity results, see for instance [13, Th. 1.1], we may conclude that

$$J(v) \leq \liminf_{j \rightarrow \infty} J(v_j^*).$$

In turn, setting  $F := \{(x, y) \in \mathbb{R}^h \times \mathbb{R}^k : \omega_k |y|^k < v(x)\}$ , the above inequality can be rewritten as

$$P_\lambda(F) \leq \liminf_{j \rightarrow \infty} P_\lambda(F_j).$$

Therefore to conclude that  $F$  is a minimizer of (2) we need only to show that  $F$  satisfies the mass constraint  $\lambda(F) = m$ . Since the functions  $v_j^*$  converge to  $v$  in  $L^1_{loc}(\mathbb{R}^h)$  this equality follows if we show that there is no loss of mass at infinity along the minimizing sequence  $F_j$ , i.e., for every  $\varepsilon > 0$  there exist  $R_\varepsilon > 0$  and a positive integer  $j_\varepsilon$  such that

$$\int_{\mathbb{R}^h \setminus B_{R_\varepsilon}^{(h)}} e^{\psi(|x|)} v_j^*(x) \, dx \leq \varepsilon \quad \text{for every } j \geq j_\varepsilon.$$

To prove this we argue by contradiction assuming that there exists  $\varepsilon_0 > 0$  such that for every  $R > 0$

$$(2.2) \quad \int_{\mathbb{R}^h \setminus B_R^{(h)}} e^{\psi(|x|)} v_j^*(x) \, dx \geq \varepsilon_0 \quad \text{for infinitely many } j.$$

Observe that given  $R$ , since  $v_j^*(x) = v_j^\#(\lambda_h(B_{|x|}^{(h)}))$  and  $v_j^\#$  is decreasing, we have for every  $j$

$$m = \int_{\mathbb{R}^h} e^{\psi(|x|)} v_j^*(x) \, dx \geq e^{\psi(0)} \omega_h R^h v_j^\#(\lambda_h(B_R^{(h)})).$$

Thus, from this inequality, for every  $j$  for which (2.2) holds we get

$$\begin{aligned} P_\lambda(F_j) &= J(v_j^*) \geq k \omega_k^{\frac{1}{k}} \int_{\{v_j^* > 0\}} e^{\psi(|x|)} v_j^*(x)^{\frac{k-1}{k}} \, dx \\ &\geq k \omega_k^{\frac{1}{k}} \int_{\{v_j^* > 0\} \setminus B_R^{(h)}} e^{\psi(|x|)} v_j^*(x) (v_j^*(x))^{-\frac{1}{k}} \, dx \\ &\geq \frac{k \omega_k^{\frac{1}{k}} \varepsilon_0}{v_j^\#(\lambda_h(B_R^{(h)}))^{\frac{1}{k}}} \geq k \omega_k^{\frac{1}{k}} \varepsilon_0 \left( \frac{e^{\psi(0)} \omega_h R^h}{m} \right)^{\frac{1}{k}}. \end{aligned}$$

Since the perimeters  $P_\lambda(F_j)$  are bounded, this inequality is clearly impossible if  $R$  is sufficiently large. This contradiction concludes the proof.  $\square$

**REMARK 2.2.** Note that in the proof of Theorem 2.1 we have shown that for every  $m > 0$  there exists an isoperimetric set  $S$  which is Steiner symmetric with respect to the subspace  $\{y = 0\}$  and such that

$$(2.3) \quad S = \{(x, y) \in \mathbb{R}^h \times \mathbb{R}^k : \omega_k |y|^k < v_S^*(x)\}.$$

Observe that  $S$  is also Steiner symmetric with respect to the subspace  $\{x = 0\}$ .

If the boundary of the isoperimetric set  $E$  minimizing (2) is a manifold of class  $C^2$ , then a standard argument, see for instance [19, Sec. 17.5], shows that there exists  $\Lambda \in \mathbb{R}$  such that

$$(2.4) \quad H_{\partial E} + \nabla \psi \cdot \nu_x^E = \Lambda,$$

where  $H_{\partial E}$  denotes the *mean curvature* of  $E$ , i.e., the sum of the principal curvatures of  $\partial E$ . Note that, as  $\psi$  depends only upon  $x$ , the inner product is in the horizontal space  $\mathbb{R}^h$ .

On the other hand, the regularity of the isoperimetric sets for the mixed perimeter can be deduced from De Giorgi’s theory of minimal sets of finite perimeter.

The precise result is given in the theorem below. Note that in the following, when dealing with a set of finite perimeter  $E$ , we always tacitly assume that  $E$  is a Borel set such that its topological boundary  $\partial E$  coincides with the support of the perimeter measure, i.e.,

$$(2.5) \quad \partial E = \{z \in \mathbb{R}^n : 0 < \mathcal{L}^n(E \cap B_r(z)) < \omega_n r^n \text{ for every } r > 0\}.$$

The fact that a set of finite perimeter has always a Borel representative satisfying (2.5) is a well known fact, see for instance [19, Prop. 12.19]. In the following, given any Borel set  $B \subset \mathbb{R}^n$  we denote by  $\dim_{\mathcal{H}}(B)$  its *Haudorff dimension*.

**THEOREM 2.3.** *Let  $E$  be a minimizer of the isoperimetric problem (2). Then its reduced boundary  $\partial^* E$  is a manifold of class  $C^\infty$ . Moreover  $\dim_{\mathcal{H}}(\partial E \setminus \partial^* E) \leq n - 8$ .*

**PROOF.** Let  $G$  be a set of finite perimeter and fix  $R > 0$  such that  $\mathcal{H}^{n-1}(\partial^* G \cap B_R) > 0$ . Observe that there exist two constants  $\sigma_0$  and  $C_0$  depending only on  $R$  and  $\psi(R)$  such that for every  $\sigma \in (-\sigma_0, \sigma_0)$  we can find a set of finite perimeter  $F$  such that  $G \Delta F \subset\subset B_R$  and

$$\lambda(F) = \lambda(G) + \sigma, \quad |P_\lambda(G; B_R) - P_\lambda(F; B_R)| \leq C_0 |\sigma|.$$

Indeed, this fact can be proved arguing exactly as in the case of standard volume and perimeter, with the obvious modifications, see for instance the proof of [19, Lemma 17.21].

Then, arguing again as for the standard perimeter, see [19, Example 21.3], it is not too difficult to show that if  $E$  is a minimizer of the constrained problem (2) and  $B_R$  is a ball as above, there exists a constant  $M$  depending only on  $R$  and  $\psi(R)$  such that if  $B_r(z) \subset B_R$  and  $F$  is a set of finite perimeter such that  $E \Delta F \subset\subset B_r(z)$  one has

$$(2.6) \quad P_\lambda(E; B_r(z)) \leq P_\lambda(F; B_r(z)) + M |\lambda(E) - \lambda(F)|.$$

In turn, given a ball  $B_r(z) \subset B_R$ , taking  $F = E \setminus B_\varrho(z)$ , with  $0 < \varrho < r$  and letting  $\varrho \rightarrow r$ , from (2.6) one easily gets that for any  $B_r(z) \subset B_R$

$$(2.7) \quad P_\lambda(E; B_r(z)) \leq C_1 r^{n-1}$$

for some constant  $C_1$  depending, as before, only on  $R$  and  $\psi(R)$ .

Let us now consider a set  $F$  of finite perimeter such that  $E \Delta F \subset\subset B_r(z)$ . Denote by  $m(r, z)$  and  $M(r, z)$  the minimum and the maximum, respectively, of the function  $e^{\psi(|x|)}$  on  $B_r(z)$ . From (2.6) we have

$$m(r, z) P(E; B_r(z)) \leq M(r, z) P(F; B_r(z)) + C_2 r^n,$$

for some positive constant  $C_2$  depending only on  $R$  and  $\psi(R)$ . Therefore, recalling (2.7), we have

$$\begin{aligned}
 P(E; B_r(z)) &\leq P(F; B_r(z)) + \frac{M(r, z) - m(r, z)}{M(r, z)} P(E; B_r(z)) + C_2 r^n \\
 &\leq P(F; B_r(z)) + CrP(E; B_r(z)) + C_2 r^n \leq P(F; B_r(z)) + \gamma r^n,
 \end{aligned}$$

where also  $\gamma$  depends only on  $R, \psi(R)$  and  $\psi'(R)$ . In conclusion, we have proved that  $E$  is a  $\gamma$ -almost minimizer for the perimeter in  $B_R$ , that is, for any ball  $B_r(z) \subset B_R$  and for any set of finite perimeter  $F$  with  $E \Delta F \subset\subset B_r(z)$  the inequality

$$P(E; B_r(z)) \leq P(F; B_r(z)) + \gamma r^n$$

holds. From this minimality property we may conclude, see [22, Th. 1.9] or also [19, Th. 21.8 and 28.1], that  $\partial^*E$  is a  $C^{1,\alpha}$  manifold for every  $0 < \alpha < 1/2$  and that  $\dim_{\mathcal{H}}(\partial E \setminus \partial^*E) \leq n - 8$ . Moreover  $\partial^*E$  satisfies (2.4) in a distributional sense, i.e.,

$$\operatorname{div} v^E + \nabla \psi \cdot \nu_x^E = \Lambda \quad \text{on } \partial^*E$$

for some Lagrange multiplier  $\Lambda \in \mathbb{R}$ . Thus, standard elliptic regularity results imply that indeed  $\partial^*E$  is a  $C^{2,\alpha}$  manifold. Then, another standard bootstrap argument yields that  $\partial^*E$  is of class  $C^\infty$ .  $\square$

If the minimizer is Steiner symmetric with respect to both subspaces  $\{x = 0\}$  and  $\{y = 0\}$  the above regularity result can be improved as follows.

**COROLLARY 2.4.** *Let  $E$  be a minimizer of the isoperimetric problem (2). Assume that  $E$  is Steiner symmetric with respect to both subspaces  $\{x = 0\}$  and  $\{y = 0\}$ . Then*

$$(2.8) \quad (\partial E \setminus \partial^*E) \cap \{(x, y) : x \neq 0 \text{ and } y \neq 0\} = \emptyset.$$

Moreover, if  $h, k \leq 6$ , then  $\partial E$  is a  $C^\infty$  manifold.

**PROOF.** We argue by contradiction assuming that there exists  $(x_0, y_0) \in \partial E \setminus \partial^*E$  with both  $x_0$  and  $y_0$  not zero. Since  $E$  is Steiner symmetric with respect to the subspace  $\{y = 0\}$  all points  $(x_0, y)$  with  $|y| = |y_0|$  belong to the singular set  $\partial E \setminus \partial^*E$ . In turn, since  $E$  is also Steiner symmetric with respect to  $\{x = 0\}$ , the set  $\{(x, y) : |x| = |x_0|, |y| = |y_0|\}$  is contained in  $\partial E \setminus \partial^*E$ . By Theorem 2.3 this is impossible since the Hausdorff dimension of this set is  $(h - 1) + (k - 1) = n - 2$ . This contradiction proves (2.8).

Assume now that  $h, k \leq 6$  and that the singular set is not empty. Then  $n \geq 8$  and from (2.8) it follows that  $\partial E \setminus \partial^*E$  contains only points of the type  $(0, y_0)$  with  $y_0 \neq 0$  or  $(x_0, 0)$  with  $x_0 \neq 0$ . So let us assume that  $(0, y_0)$  is a singular point for some  $y_0 \neq 0$ . Then the set  $\{(0, y) : |y| = |y_0|\}$  is also contained in the singular set. But this set has dimension  $k - 1$  and since  $h \leq 6, k - 1 > n - 8$ , which is impossible by Theorem 2.3. The same argument shows that also the points of the type  $(x_0, 0)$  cannot be singular.  $\square$

## 3. PROPERTIES OF THE ISOPERIMETRIC SET

In this section we assume that *the function  $\psi$  satisfies the assumption (1.15)*. With this assumption in mind we investigate the properties and the uniqueness of the isoperimetric sets, that are the sets minimizing (2). To this end we start with the two-dimensional case, that is  $h = k = 1$ , where the arguments are similar, but simpler than in the general case  $n \geq 3$ .

**THEOREM 3.1.** *Let  $h = k = 1$  and  $m > 0$ . Up to a vertical translation, every isoperimetric set  $E$  with  $\lambda(E) = m$  is a  $C^\infty$ , bounded, strictly convex set, Steiner symmetric with respect to both coordinate axes. Moreover  $E$  is unique.*

**PROOF.** Given  $m > 0$ , let  $S$  be an isoperimetric set with  $\lambda(S) = m$  Steiner symmetric with respect to both coordinate axes as in (2.3). By Theorem 2.3 the boundary of  $S$  is a  $C^\infty$  manifold. Since  $S$  is Steiner symmetric with respect to both axes, it is a connected set and since its Euclidean perimeter is finite,  $S$  is also bounded.

Therefore, there exist two even  $BV$  functions,  $f : (-a, a) \rightarrow (0, \infty)$  and  $g : (-b, b) \rightarrow [0, \infty)$ , such that for every  $x \in (-a, a)$  and  $y \in (-b, b)$

$$S_x = (-f(x), f(x)), \quad S^y = (-g(y), g(y)).$$

Note that the functions  $f$  and  $g$  are both decreasing when restricted to the intervals  $(0, a)$  and  $(0, b)$ , respectively. Moreover, from the first variation formula (2.4) we deduce that these functions satisfy the equations

$$(3.1) \quad -f'' - \psi'(x)f'(x)(1 + f'(x)^2) = \Lambda(1 + f'(x)^2)^{3/2}$$

and

$$(3.2) \quad -g'' + \psi'(g(y))(1 + g'(y)^2) = \Lambda(1 + g'(y)^2)^{3/2},$$

respectively, on any interval where they are smooth. Observe also that if the normal at a point  $(x_0, y_0)$  of the boundary of  $S$  is not horizontal, then  $f(x_0) = y_0$  and  $f$  is  $C^\infty$  in a neighborhood of  $x_0$ . Similarly, if the normal at  $(x_0, y_0)$  is not vertical, then  $g(y_0) = x_0$  and  $g$  is  $C^\infty$  in a neighborhood of  $y_0$ .

Since  $S$  is smooth the exterior normal at the point  $(g(0), 0)$  is  $(1, 0)$ . Therefore,  $g$  is smooth in a neighborhood of 0 and since 0 is a maximum point for  $g$  we have  $g'(0) = 0$ ,  $g''(0) \leq 0$ . Thus, from (3.2) we have

$$\psi'(g(0)) - \Lambda = g''(0) \leq 0.$$

Hence, thanks to assumption (1.15) we have that  $\Lambda \geq \psi'(g(0)) > 0$ .

Let us now assume that at a point  $(x_0, y_0) \in \partial S$ , with  $x_0 \in (0, a)$ , the exterior normal is vertical. Then  $f(x_0) = y_0$ ,  $f$  is smooth in a neighborhood of  $x_0$  and  $f'(x_0) = 0$ . Therefore, from (3.1) we obtain  $f''(x_0) = -\Lambda < 0$ . Thus  $x_0$  is a local strict maximum and this is impossible since  $f$  is decreasing. This shows that, except for the points  $(0, \pm b)$ , the normal to the boundary of  $S$  is never vertical. In turn, as observed before this yields that  $g \in C^\infty(-b, b)$ .



Moreover  $g' \neq 0$  in  $(0, b)$ . In fact if there were  $y_0 \in (0, b)$  such that  $g'(y_0) = 0$ , then also  $g''(y_0) = 0$ , otherwise  $y_0$  would be a strict local minimum or maximum and this is impossible since  $g$  is decreasing. Then from (3.2) we would get that

$$\psi'(g(y_0)) = \Lambda$$

and thus by the uniqueness of solutions to the equation (3.2) we would conclude that  $g$  is constant, hence  $S$  is a rectangle. But this is impossible since  $\partial S$  is smooth. So  $g'$  never vanishes in  $(0, b)$ , hence  $f$  is  $C^\infty$  in  $(-a, a)$ .

Finally, observe that using again (3.2) and recalling that  $\Lambda > 0$ , we get that for every  $y \neq 0$

$$(3.3) \quad \begin{aligned} g''(y) &= \psi'(g(y))(1 + g'(y)^2) - \Lambda(1 + g'(y)^2)^{3/2} \\ &< (\psi'(g(0)) - \Lambda)(1 + g'(y)^2) \leq 0. \end{aligned}$$

Thus  $g$  is strictly concave, hence  $S$  is strictly convex.

Let us now show that  $S$  is the only isoperimetric set Steiner symmetric with respect to the  $x$  axis satisfying (2.3). We argue by contradiction assuming that there exist two solutions  $f_2$  and  $f_1$  of (3.1) defined in two intervals  $(-a_2, a_2)$ ,  $(-a_1, a_1)$ , respectively, and corresponding to two isoperimetric sets with the same mass. From the regularity of the isoperimetric sets we have that  $f_2'(0) = f_1'(0) = 0$  and thus from the uniqueness of solutions of (3.1) we deduce also that  $f_2'(x) = f_1'(x)$  for every  $x$  such that  $|x| < \min\{a_1, a_2\}$ . Therefore, since  $f_1(a_1) = f_2(a_2) = 0$  and

$$\int_0^{a_1} e^{\psi(x)} \sqrt{1 + f_1'(x)^2} dx = \int_0^{a_2} e^{\psi(x)} \sqrt{1 + f_2'(x)^2} dx,$$

we conclude immediately that the two functions  $f_1$  and  $f_2$  coincide.

Let us conclude the proof by showing that  $S$  is the unique isoperimetric set up to vertical translations. Indeed, if  $E$  is another isoperimetric set with the same mass as  $S$ , arguing as in the proof of Theorem 2.1 we first consider the Steiner symmetrization  $E^S$  of  $E$  with respect to the  $x$  axis. Then,

$$E^S = \{(x, y) : 2|y| < v_E(x)\},$$

where  $v_E$  is defined as in (1.1). Replacing  $v_E$  by its weighted symmetric rearrangement  $v_E^*$ , we set

$$(3.4) \quad F := \{(x, y) : 2|y| < v_E^*(x)\}.$$

Since  $\lambda(E) = \lambda(E^S) = \lambda(F)$  and, by Proposition 1.3 and (1.16),

$$P_\lambda(E) \geq P_\lambda(E^S) \geq P(F),$$

we conclude that indeed all the previous inequalities are in fact equalities. Moreover, since  $F$  is Steiner symmetric by construction,  $F$  coincides with  $S$ . Note that from the equality  $P_\lambda(E^S) = P_\lambda(F)$  we have by Theorem 1.8 that

$E^S = F = S$ . Finally,  $P_\lambda(E^S) = P_\lambda(E)$  and both assumptions (i) and (ii) of Theorem 1.4 are satisfied, since  $v_y^{E^S} = 0$  only at  $(\pm a, 0)$  and  $v_E(x) = 2f(x) > 0$  for each  $x \in (-a, a)$ . Thus,  $E$  is a vertical translation of  $E^S$ , hence a vertical translation of  $S$ . □

We now consider the general case  $n \geq 3$ . In this case our result reads as follows.

**THEOREM 3.2.** *Let  $n \geq 3$  and  $m > 0$ . Up to a translation in the  $y$  direction, every isoperimetric set  $E$  with  $\lambda(E) = m$  is  $C^\infty$ , bounded and Steiner symmetric with respect to both coordinate axes. If  $h = 1$  or  $k = 1$ ,  $E$  is also unique. Moreover, if  $k = 1$  and  $m < m_0$ , for some  $m_0 > 0$  depending only on  $n$  and  $\psi$ , or if  $h = 1$ ,  $E$  is strictly convex.*

**PROOF.** *Step 1.* Given  $m > 0$  let  $S$  be an isoperimetric set with  $\lambda(S) = m$ , Steiner symmetric with respect to the subspace  $\{y = 0\}$  and satisfying (2.3). By Corollary 2.4

$$(3.5) \quad \mathcal{M} := \partial S \setminus ((\mathbb{R}^h \times \{0\}) \cup (\{0\} \times \mathbb{R}^k)) \text{ is a } C^\infty \text{ manifold.}$$

Let us now consider the function  $v_S = v_S^*$ . The support of  $v_S$  is either a closed ball of radius  $a$  or the whole  $\mathbb{R}^h$  and from Proposition 1.2 we have  $v_S \in BV(\mathbb{R}^h)$ . Moreover, since  $v_S = v_S^*$  the function  $v_S$  depends only on  $|x|$ . Therefore, there exists a function  $r : (0, a) \rightarrow (0, \infty)$ , with  $a = \infty$  if the support of  $v_S$  is  $\mathbb{R}^h$ , such that  $v_S(x) = \omega_k r(|x|)^k$  for every  $x \in \mathbb{R}^h$ ,  $0 < |x| < a$ . Note that  $r \in BV_{loc}((0, a))$  and that  $r$  is decreasing, since  $v_S = v_S^*$ . Moreover, since the manifold  $\mathcal{M}$  in (3.5) is smooth the extended graph of  $r$  over the interval  $(0, a)$  is a  $C^\infty$  curve. Let us denote it by  $\Gamma_r$ .

If  $(x_0, y_0) \in \partial S$ ,  $0 < |x_0| < a$ , and  $v_y^S(x_0, y_0) \neq 0$ , then  $|y_0| = r(|x_0|)$  and  $r$  is  $C^\infty$  in a neighborhood of  $|x_0|$ . Moreover, the coordinates of the points  $(x, y) \in \mathcal{M}$  verify the equation  $|y|^2 - r^2(|x|) = 0$  in a neighborhood  $U$  of  $(x_0, y_0)$ . Therefore the exterior normal vector field to  $\partial S \cap U$  is given by

$$v^S(x, y) = \frac{\left(-\frac{x}{|x|}r(|x|)r'(|x|), y\right)}{\left(r(|x|)^2r'(|x|)^2 + |y|^2\right)^{1/2}}.$$

Since the mean curvature  $H_{\partial S}$  is equal to  $\operatorname{div} v^S$ , taking the divergence of the right hand side of the above equality and using the fact that  $|y|^2 = r^2(|x|)$  on  $\partial S \cap U$ , the first variation equation (2.4) becomes

$$(3.6) \quad -\frac{r''(\varrho)}{(1+r'(\varrho)^2)^{3/2}} - \frac{(h-1)r'(\varrho)}{\varrho(1+r'(\varrho)^2)^{1/2}} + \frac{(k-1)}{r(\varrho)(1+r'(\varrho)^2)^{1/2}} - \frac{\psi'(\varrho)r'(\varrho)}{(1+r'(\varrho)^2)^{1/2}} = \Lambda,$$

where we have set  $\varrho = |x|$ .

Let us now assume that at a point  $(\varrho_0, \sigma_0) \in \Gamma_r$  the normal to  $\Gamma_r$  is vertical. Then  $r(\varrho_0) = \sigma_0$ ,  $r$  is smooth in a neighborhood of  $\varrho_0$  and  $r'(\varrho_0) = 0$ ,  $r''(\varrho_0) \leq 0$ . However, it cannot be  $r''(\varrho_0) < 0$  because in this case  $r$  would have a strict local maximum and this is impossible because  $r$  is decreasing. Hence,  $r''(\varrho_0) = 0$  and from (3.6) we have

$$(3.7) \quad \frac{k-1}{r(\varrho_0)} = \Lambda.$$

Therefore, if  $k > 1$ , from the local uniqueness of solutions of equation (3.6) we conclude that  $r$  is constant in the interval  $(0, a)$ . Similarly, if  $k = 1$ , from (3.7) we have that  $\Lambda = 0$  and thus, again from the local uniqueness of solutions of the equations of (3.6), it follows that  $r$  is constant in the interval  $(0, a)$  and thus  $a$  must be finite. But if  $r$  is constant the points  $(x, y)$  with  $|x| = a$ ,  $|y| = r(a)$  are singular points of  $\partial S$  and this is impossible since these points form a set of Hausdorff dimension  $n - 2$ . This proves that the normal vector to  $\Gamma_r$  is never vertical. Thus  $\Gamma_r$  coincides with the graph of a smooth decreasing function  $g : (0, b) \rightarrow (0, a)$ , with  $b$  possibly equal to  $+\infty$ , such that  $g(r(\varrho)) = \varrho$  for each  $\varrho \in (0, a)$ . Moreover, arguing as in the proof of (3.6) we get that  $g$  satisfies the equation

$$(3.8) \quad -\frac{g''(\sigma)}{(1 + g'(\sigma)^2)^{3/2}} + \frac{(h-1)}{g(\sigma)(1 + g'(\sigma)^2)^{1/2}} - \frac{(k-1)g'(\sigma)}{\sigma(1 + g'(\sigma)^2)^{1/2}} + \frac{\psi'(g(\sigma))}{(1 + g'(\sigma)^2)^{1/2}} = \Lambda.$$

Observe that if  $\sigma_0 \in (0, b)$  then  $g'(\sigma_0) < 0$ . To prove this we argue as before observing that if  $g'(\sigma_0) = 0$  then necessarily also  $g''(\sigma_0) = 0$  and thus from (3.8) we obtain

$$\frac{(h-1)}{g(\sigma_0)} + \psi'(g(\sigma_0)) = \Lambda.$$

Therefore, by the local uniqueness of solutions of equation (3.8) it follows that  $g$  is constant in the interval  $(0, b)$  and thus  $b$  must be finite. But if  $g$  is constant the points  $(x, y)$  with  $|x| = g(b)$ ,  $|y| = b$  are singular points of  $\partial S$  and again this is impossible since these points form a set of Hausdorff dimension  $n - 2$ . This shows that  $g' < 0$  in  $(0, b)$  and thus the normal to  $\Gamma_r$  is never horizontal. In turn, this implies that  $r$  is  $C^\infty$  in the interval  $(0, a)$ .

Finally, let  $E$  be any isoperimetric set with the  $\lambda(E) = m$ . Arguing as in the final part of the proof of Theorem 3.1 we first construct the Steiner symmetrization  $E^S$  of  $E$  with respect to  $\{x = 0\}$  and then the set  $F$  defined as in (3.4). Again, since  $P_\lambda(F) = P_\lambda(E^S) = P_\lambda(E)$ , from the first equality we get that  $E^S = F$ . Then, we consider the function  $r$  such that  $v_F(x) = \omega_k r(|x|)^k$  for every  $x \in \mathbb{R}^h$ .

Since by the previous analysis  $r$  is  $C^\infty$  in  $(0, a)$  we have

$$\mathcal{H}^{n-1}(\{(x, y) \in \partial^* F : |x| < a, v_y^F(x, y) = 0\}) = 0.$$

Thus, by Theorem 1.4 we conclude that  $E$  is a vertical translation of  $F$ .

*Step 2.* Let us now prove that  $S$  is bounded. This follows if we show that both intervals  $(0, a)$  and  $(0, b)$  are bounded.

To prove that  $a$  is finite, let us first consider the case  $k = 1$ . In this case, if  $a = +\infty$ , then from Proposition 1.3 we get

$$P_\lambda(S) = 2 \int_{\mathbb{R}^h} e^{\psi(|x|)} \sqrt{1 + r'(|x|)^2} dx = \infty,$$

which is impossible. So, let us assume that  $k > 1$ . If  $a = +\infty$ , then  $r(\varrho) \rightarrow 0$  as  $\varrho \rightarrow +\infty$ , since otherwise, again by Proposition 1.3, we would get  $P_\lambda(S) = \infty$ . Now, from (3.6) we have

$$(3.9) \quad \frac{r''(\varrho)}{(1 + r'(\varrho)^2)^{3/2}} > \frac{(k - 1)}{r(\varrho)(1 + r'(\varrho)^2)^{1/2}} - \Lambda.$$

From this inequality it follows that there exists  $\varrho_0 > 0$  such that

$$(3.10) \quad \varrho > \varrho_0 \quad \text{and} \quad r'(\varrho) > -1 \quad \Rightarrow \quad r''(\varrho) > 1.$$

In fact this implication follows from (3.9) by observing that if  $\varrho > \varrho_0$  and  $0 > r'(\varrho) > -1$  then

$$\frac{r''(\varrho)}{(1 + r'(\varrho)^2)^{3/2}} > \frac{(k - 1)}{\sqrt{2}r(\varrho)} - \Lambda > 1$$

if we choose  $\varrho_0$  sufficiently large. Observe now that if there were  $\varrho_1 > \varrho_0$  such that  $r'(\varrho_1) > -1$ , then from (3.10) we would get that  $r''(\varrho_1) > 1$  and thus, using again (3.10) in a right neighborhood of  $\varrho_1$ , that  $r'(\varrho_2) = 0$  for some  $\varrho_2 \in (\varrho_1, \varrho_1 + 1)$ . And this is impossible. Therefore we must conclude that  $r'(\varrho) \leq -1$  for every  $\varrho > \varrho_0$  and this inequality immediately yields that  $a$  is finite.

Let us now show that also  $b$  is finite. If  $h = 1$  this is trivially true. In fact, if  $b = +\infty$  then the projection of  $\mathcal{M}$  over  $\mathbb{R}^{n-1}$  would be  $\mathbb{R}^{n-1} \setminus \{0\}$  and thus would have infinite  $\mathcal{H}^{n-1}$  measure. But then also  $\mathcal{M}$  would have infinite  $\mathcal{H}^{n-1}$  measure, hence  $P_\lambda(S)$  would be infinite, which is impossible.

Assume now that  $h \geq 2$  and that  $b = +\infty$ . Then  $g(\sigma) \rightarrow 0$  as  $\sigma \rightarrow +\infty$ , otherwise  $\lambda(S) = \infty$ . If also  $g'(\sigma) \rightarrow 0$  as  $\sigma \rightarrow +\infty$ , passing to the limit in (3.8) we get  $g''(\sigma) \rightarrow +\infty$ , which is clearly impossible. On the other hand, if there exists  $\varepsilon > 0$  such that  $g'(\sigma) < -\varepsilon$  for  $\sigma$  large, then we have that  $g(\sigma) \rightarrow -\infty$  which is also impossible. Therefore, since no one of the two previous instances may occur, we

conclude that there exist a sequence  $\sigma_i$ , with  $\sigma_i \rightarrow +\infty$  such that each  $\sigma_i$  is a local maximum for  $g'$  and  $g'(\sigma_i) \rightarrow 0$ . Then, from equation (3.8) we get

$$\frac{(h-1)}{g(\sigma_i)(1+g'(\sigma_i)^2)^{1/2}} - \frac{(k-1)g'(\sigma_i)}{\sigma_i(1+g'(\sigma_i)^2)^{1/2}} + \frac{\psi'(g(\sigma_i))}{(1+g'(\sigma_i)^2)^{1/2}} \leq \Lambda.$$

But this is impossible since the left hand side of this equation tends to  $+\infty$  as  $i \rightarrow \infty$ . This final contradiction shows that also  $b < \infty$ .

*Step 3.* To prove that  $S$  is smooth by Corollary 2.4 it is enough to show that all the points in  $\partial S \cap (\{x=0\} \cup \{y=0\})$  are regular points for  $\partial S$ .

We first show that  $\partial S \cap \{y=0\}$  has no singular point. If  $k=1$  this follows immediately by observing that if  $(x_0, 0) \in \partial S$  is a singular point then all the points in the sphere  $\{|x|=|x_0|\}$  are singular. By Theorem 2.3 this is impossible since this sphere has dimension  $n-2$ . So, let us assume that  $k \geq 2$ . Observe that to show that all the points in  $\partial S \cap \{y=0\}$  are regular it is enough to prove that

$$(3.11) \quad \lim_{\sigma \rightarrow 0^+} g'(\sigma) = 0.$$

Indeed if this is true then  $g'(0) = 0$  and the normal vector field is continuous at all the points of  $\partial S \cap \{y=0\}$ . In turn, the continuity of the normal, by the almost minimality of  $S$  established in the proof of Theorem 2.3, implies that the normal vector field is  $C^{1,\alpha}$  in a neighborhood of  $\partial S \cap \{y=0\}$  for some  $\alpha > 0$ , see [19, Th. 26.3]. Hence, arguing again as in the proof of Theorem 2.3, we conclude that  $\partial S$  is  $C^\infty$  in a neighborhood of  $\partial S \cap \{y=0\}$ .

To show (3.11) we multiply both sides of (3.8) by  $\sigma^{k-1}$ , thus getting

$$\frac{d}{d\sigma} \left( \frac{-\sigma^{k-1}g'(\sigma)}{(1+g'(\sigma)^2)^{1/2}} \right) = \Lambda\sigma^{k-1} - \frac{(h-1)\sigma^{k-1}}{g(\sigma)(1+g'(\sigma)^2)^{1/2}} - \frac{\sigma^{k-1}\psi'(g(\sigma))}{(1+g'(\sigma)^2)^{1/2}}.$$

Since  $k \geq 2$ ,  $\sigma^{k-1}g'(\sigma)/(1+g'(\sigma)^2)^{1/2}$  converges to zero as  $\sigma \rightarrow 0^+$ , therefore, integrating the previous equality from 0 to  $\sigma$ , we get

$$\begin{aligned} 0 &\leq \frac{-g'(\sigma)}{(1+g'(\sigma)^2)^{1/2}} \\ &= \frac{1}{\sigma^{k-1}} \int_0^\sigma \left( \Lambda - \frac{(h-1)}{g(\tau)(1+g'(\tau)^2)^{1/2}} - \frac{\psi'(g(\tau))}{(1+g'(\tau)^2)^{1/2}} \right) \tau^{k-1} d\tau \leq \frac{\Lambda\sigma}{k}, \end{aligned}$$

thus showing (3.11).

To prove that  $\partial S \cap \{x=0\}$  has no singular points, arguing as before it is enough to assume that  $h \geq 2$  and to show that

$$\lim_{\varrho \rightarrow 0^+} r'(\varrho) = 0.$$

To this aim, we multiply equation (3.6) by  $\varrho^{h-1}$  so to get

$$\frac{d}{d\varrho} \left( \frac{-\varrho^{h-1}r'(\varrho)}{(1+r'(\varrho)^2)^{1/2}} \right) = \Lambda\varrho^{h-1} - \frac{(k-1)\varrho^{h-1}}{r(\varrho)(1+r'(\varrho)^2)^{1/2}} + \frac{\varrho^{h-1}\psi'(g(\varrho))}{(1+g'(\varrho)^2)^{1/2}}.$$

Now, the conclusion follows as in the previous case.

*Step 4.* Let us show that if  $h = 1$  or  $k = 1$  then  $S$  is unique. To this aim, let us first assume that  $h = 1$ . Assume that there exist two different solutions  $g_2$  and  $g_1$  of (3.8) such that the corresponding profiles have both the same mass  $m$ , with  $g_2 > 0$  in  $(-b_2, b_2)$  and  $g_1 > 0$  in  $(-b_1, b_1)$  for some  $b_1 \geq b_2$ .

**CLAIM 1.** *There exist  $\sigma_0, \delta$ , with  $0 \leq \sigma_0 < \sigma_0 + \delta \leq b_2$ , such that either*

$$(3.12) \quad \begin{aligned} g'_1(\sigma_0) = g'_2(\sigma_0), \quad g'_2(\sigma) < g'_1(\sigma) < 0, \quad g_2(\sigma) > g_1(\sigma) \\ \text{for every } \sigma \in (\sigma_0, \sigma_0 + \delta) \end{aligned}$$

*or the same inequalities as in (3.12) hold with  $g_1$  exchanged with  $g_2$ .*

In order to prove the claim observe that  $\max\{g_1(\sigma) - g_2(\sigma) : \sigma \in [0, b_2]\} > 0$ . In fact, otherwise  $g_1 \leq g_2$  and since the isoperimetric profiles corresponding to  $g_1$  and  $g_2$  have the same mass we easily conclude that  $g_1 = g_2$ . Therefore, we may assume that there exists  $\bar{\sigma} \in [0, b_2]$  such that

$$g_1(\sigma) - g_2(\sigma) \leq g_1(\bar{\sigma}) - g_2(\bar{\sigma}) \quad \text{for every } \sigma \in [0, b_2].$$

Now, two cases may occur.

First, let us assume that  $g_1(\sigma) > g_2(\sigma)$  for every  $\sigma \in [\bar{\sigma}, b_2]$  or that  $\bar{\sigma} = b_2$ . In this case there must be some point in  $(0, \bar{\sigma})$  where  $g_1$  is strictly smaller than  $g_2$ , since otherwise the mass of the isoperimetric profile corresponding to  $g_1$  would be strictly bigger than the one of the profile corresponding to  $g_2$ . Thus, let us denote by  $\sigma'$  the greatest  $\sigma \in (0, \bar{\sigma})$  such that  $g_1(\sigma') = g_2(\sigma')$ . By minimality, we have that  $g'_2(\sigma') \leq g'_1(\sigma')$ . In fact the stronger inequality  $g'_2(\sigma') < g'_1(\sigma')$  holds, because if  $g'_1(\sigma') = g'_2(\sigma')$  then by the local uniqueness of solutions of the equation (3.8) we would conclude that  $g_1 = g_2$ . Observe that in a left neighborhood of  $\sigma'$  we have that  $g_1 < g_2$  and  $g'_2 < g'_1$ . Then, we denote by  $\sigma_0$  the largest point in  $[0, \sigma']$  such that  $g'_1(\sigma_0) = g'_2(\sigma_0)$ . Note that since  $g'_2(0) = g'_1(0)$  such a point always exists. Finally, observe that by construction  $g'_2(\sigma) < g'_1(\sigma)$  and  $g_2(\sigma) > g_1(\sigma)$  for every  $\sigma \in (\sigma_0, \sigma')$ , thus proving Claim 1 in this case.

Let us now prove the claim when there exists a point  $\sigma \in [\bar{\sigma}, b_2]$  such that  $g_1(\sigma) = g_2(\sigma)$ . Denoting by  $\sigma'$  the first one of such points and arguing as before we have that  $g'_1(\sigma') < g'_2(\sigma')$ . Then, denoting by  $\sigma_0$  the largest point in  $[\bar{\sigma}, \sigma']$  such that  $g'_1(\sigma_0) = g'_2(\sigma_0)$  we conclude as above that  $g'_1(\sigma) < g'_2(\sigma)$  and  $g_1(\sigma) > g_2(\sigma)$  for every  $\sigma \in (\sigma_0, \sigma')$ , thus proving (3.12) with  $g_1$  exchanged with  $g_2$ .

Let us show that Claim 1 yields the uniqueness of the isoperimetric profile when  $h = 1$ . To this aim, for any  $\sigma$  in the interval  $(\sigma_0, \sigma_0 + \delta)$  where (3.12) holds, from (3.8) we have, using the fact that  $\psi'$  is increasing,

$$\begin{aligned} g_1''(\sigma) - g_2''(\sigma) &= \frac{k-1}{\sigma} (g_2'(\sigma)(1 + g_2'(\sigma)^2) - g_1'(\sigma)(1 + g_1'(\sigma)^2)) \\ &\quad + \psi'(g_1(\sigma))(1 + g_1'(\sigma)^2) - \psi'(g_2(\sigma))(1 + g_2'(\sigma)^2) \\ &\quad + \Lambda((1 + g_2'(\sigma)^2)^{3/2} - (1 + g_1'(\sigma)^2)^{3/2}) \\ &\leq \psi'(g_2(\sigma))(1 + g_1'(\sigma)^2) - \psi'(g_2(\sigma))(1 + g_2'(\sigma)^2) \\ &\quad + \Lambda((1 + g_2'(\sigma)^2)^{3/2} - (1 + g_1'(\sigma)^2)^{3/2}) \\ &\leq \Lambda((1 + g_2'(\sigma)^2)^{3/2} - (1 + g_1'(\sigma)^2)^{3/2}). \end{aligned}$$

Setting  $M = \|g_2'\|_{L^\infty(\sigma_0, \sigma_0 + \delta)}$  and integrating this inequality from  $\sigma_0$  to  $\sigma$  we then get

$$g_1'(\sigma) - g_2'(\sigma) \leq c\Lambda(1 + M^2) \int_{\sigma_0}^{\sigma} (g_1'(\tau) - g_2'(\tau)) d\tau$$

for some positive absolute constant  $c$  independent of  $g_1$  and  $g_2$ . In turn, this inequality implies that for every  $\sigma \in (\sigma_0, \sigma_0 + \delta)$

$$\max_{[\sigma_0, \sigma]}(g_1' - g_2') \leq c\Lambda(1 + M^2)(\sigma - \sigma_0) \max_{[\sigma_0, \sigma]}(g_1' - g_2').$$

But, this inequality is clearly impossible if we choose  $\sigma$  such that  $c\Lambda(1 + M^2)(\sigma - \sigma_0) < 1$ . This contradiction concludes the proof of uniqueness in this case.

We now assume  $k = 1$ . In this case we are going to study equation (3.6). Again, we argue by contradiction, supposing that there exist two solutions  $r_2$  and  $r_1$  of (3.6) whose corresponding isoperimetric sets have the same mass. By the regularity of the boundary of the isoperimetric profile we have that  $r_2'(0) = r_1'(0)$ . However, we cannot conclude that  $r_2'$  coincides with  $r_1'$  since equation (3.6) degenerates at 0. In any case, passing to the limit as  $\varrho \rightarrow 0^+$  we have  $\Lambda = -hr_2''(0) = -hr_1''(0) \geq 0$ . On the other hand if  $r_2'(\varrho) = r_1'(\varrho) < 0$  for some  $\varrho > 0$ , then by uniqueness it follows that  $r_2'$  coincides with  $r_1'$  and this immediately implies that  $r_1$  and  $r_2$  also coincide since the corresponding profiles have the same mass. Therefore, without loss of generality, we may assume that there exists an interval  $(0, \varrho_0)$  such that  $0 > r_1'(\varrho) > r_2'(\varrho)$  for every  $\varrho \in (0, \varrho_0]$ . Let us then set  $M = \|r_2'\|_{L^\infty(0, \varrho_0)}$ . Then, from (3.6) we have for every  $\varrho \in (0, \varrho_0]$

$$\begin{aligned} r_1''(\varrho) - r_2''(\varrho) &= \left(\frac{h-1}{\varrho} + \psi'(\varrho)\right)[r_2'(\varrho)(1 + r_2'(\varrho)^2) - r_1'(\varrho)(1 + r_1'(\varrho)^2)] \\ &\quad + \Lambda[(1 + r_2'(\varrho)^2)^{3/2} - (1 + r_1'(\varrho)^2)^{3/2}] \\ &< \Lambda[(1 + r_2'(\varrho)^2)^{3/2} - (1 + r_1'(\varrho)^2)^{3/2}]. \end{aligned}$$

Integrating this equation we then get that for every  $\varrho \in (0, \varrho_0]$

$$0 < r'_1(\varrho) - r'_2(\varrho) \leq c\Lambda(1 + M^2) \int_0^\varrho (r'_1(\tau) - r'_2(\tau)) d\tau,$$

for some positive absolute constant  $c$  independent of  $r_1$  and  $r_2$ . In turn, from this inequality we get that for every  $\varrho \in (0, \varrho_0]$

$$\max_{[0, \varrho]}(r'_1 - r'_2) \leq c\Lambda(1 + M^2)\varrho \max_{[0, \varrho]}(r'_1 - r'_2).$$

But, this inequality is clearly impossible if we choose  $\varrho$  such that  $c\Lambda(1 + M^2)\varrho < 1$ . This contradiction concludes the proof of uniqueness also in this case.

*Step 5.* Let us now show that  $S$  is strictly convex when  $h = 1$ . To this aim we prove the following

**CLAIM 2.1.** *There exists no interval  $(\sigma_0, \sigma_1)$ , with  $0 \leq \sigma_0 < \sigma_1 < b$ , such that  $g''(\sigma) < 0$  for every  $\sigma \in (\sigma_0, \sigma_1)$  and  $g''(\sigma_1) = 0$ .*

In order to prove this claim we argue by contradiction assuming that an interval as above exists and setting  $d(\sigma) = g'(\sigma)/(1 + g'(\sigma)^2)^{1/2}$  for every  $\sigma \in (\sigma_0, \sigma_1)$ . Then, we rewrite (3.8) as

$$-d'(\sigma) - \frac{(k - 1)d(\sigma)}{\sigma} + \frac{\psi'(g(\sigma))}{(1 + g'(\sigma)^2)^{1/2}} = \Lambda.$$

Differentiating this equation in the interval  $(\sigma_0, \sigma_1)$  we get

$$-d''(\sigma) - \frac{(k - 1)d'(\sigma)}{\sigma} + \frac{(k - 1)d(\sigma)}{\sigma^2} + \psi''(g(\sigma))d(\sigma) - \psi'(g(\sigma))g'(\sigma)d'(\sigma) = 0.$$

Observe that in the interval  $(\sigma_0, \sigma_1)$  we have  $d'(\sigma) < 0$ , while  $d'(\sigma_1) = 0$ . Therefore  $d''(\sigma_1) \geq 0$ . Recalling that  $k > 1$  and that  $g'(\sigma_1) < 0$ , hence  $d(\sigma_1) < 0$ , from the above equation we obtain

$$0 \leq d''(\sigma_1) = d(\sigma_1) \left( \frac{k - 1}{\sigma_1^2} + \psi''(g(\sigma_1)) \right) < 0.$$

This contradiction proves Claim 2.1.

Let us now set  $A = \{\sigma \in (0, b) : g''(\sigma) < 0\}$ . Observe that  $A$  is not empty since otherwise  $g''(\sigma) \geq 0$  for every  $\sigma \in (0, b)$ . Then, since  $g'(0) = 0$  and  $g'(\sigma) \neq 0$  for  $\sigma \in (0, b)$ , we would have  $g'(\sigma) > 0$  for every  $\sigma \in (0, b)$  and this would imply that  $g$  is strictly increasing, which is impossible.

Observe that the claim above implies that  $A$  has only one connected component  $(\sigma_0, b)$  for some  $0 \leq \sigma_0 < b$ . Moreover  $\sigma_0 = 0$ , otherwise  $g''(\sigma) \geq 0$  in  $(0, \sigma_0)$



and the same argument as above would imply that  $g$  is strictly increasing in  $(0, \sigma_0)$ . In conclusion, we have proved that  $g''(\sigma) < 0$  for every  $\sigma \in (0, b)$ , hence  $S$  is strictly convex.

Assume now that  $k = 1$ . In this case we start first by observing that if  $m \rightarrow 0$  then also  $a \rightarrow 0$ . To see this it is enough to check that if  $B_{r(m)}$  is the ball such that  $\lambda(B_{r(m)}) = m$ , then  $P_\lambda(B_{r(m)}) \rightarrow 0$  as  $m \rightarrow 0$  and to estimate

$$P_\lambda(B_{r(m)}) \geq P_\lambda(S) = \int_{B_a} e^{\psi(|x|)} \sqrt{4 + |\nabla v_S|^2} dx \geq 2e^{\psi(0)} \omega_{n-1} a^{n-1}.$$

Then, we have the following

**CLAIM 2.2.** *There exists  $m_0 > 0$  such that if  $0 < m < m_0$  there exists no interval  $(\varrho_0, \varrho_1)$  with  $0 \leq \varrho_0 < \varrho_1 < a$  such that  $r''(\varrho) < 0$  in  $(\varrho_0, \varrho_1)$  and  $r''(\varrho_1) = 0$ .*

In order to prove the claim we argue by contradiction assuming that an interval as above exists. Similarly to the previous case, we set  $d(\varrho) = r'(\varrho)/(1 + r'(\varrho)^2)^{1/2}$  and we rewrite (3.6) as

$$-d'(\varrho) - \frac{(h-1)d(\varrho)}{\varrho} - \psi'(\varrho)d(\varrho) = \Lambda.$$

Differentiating this equation in the interval  $(\varrho_0, \varrho_1)$  we get that

$$-d''(\varrho) - \frac{(h-1)d'(\varrho)}{\varrho} + \frac{(h-1)d(\varrho)}{\varrho^2} - \psi''(\varrho)d(\varrho) - \psi'(\varrho)d'(\varrho) = 0.$$

Observe that in the interval  $(\varrho_0, \varrho_1)$  we have  $d(\varrho) < 0$ ,  $d'(\varrho) < 0$ , while  $d'(\varrho_1) = 0$ . Therefore  $d''(\varrho_1) \geq 0$ . But from the above equation we obtain

$$(3.13) \quad 0 \leq d''(\varrho_1) = d(\varrho_1) \left( \frac{h-1}{\varrho_1^2} - \psi''(\varrho_1) \right).$$

Note that

$$\frac{h-1}{\varrho_1^2} - \psi''(\varrho_1) \geq \frac{h-1}{a^2} - \max_{[0,a]} \psi''(\varrho) > 0$$

provided that  $0 < a < a_0$  for a sufficiently small  $a_0$  depending only on  $h$  and  $\psi$ , hence  $m < m_0$  for some  $m_0$  depending only on  $h$  and  $\psi$ . Thus, we get a contradiction since the right hand side of (3.13) is strictly negative and this contradiction proves Claim 2.

From Claim 2, arguing as in the case  $h = 1$  we conclude that  $r''(\varrho) < 0$  for each  $\varrho \in (0, a)$ , thus proving that  $S$  is strictly convex. □

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