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Partial Differential Equations — *Well-posedness for multi-dimensional junction problems with Kirchoff-type conditions*, by Pierre-Louis Lions and Panagiotis Souganidis, communicated on April 21, 2017.

Abstract. — We consider multi-dimensional junction problems for first- and second-order pde with Kirchoff-type Neumann boundary conditions and we show that their generalized viscosity solutions are unique. It follows that any viscosity-type approximation of the junction problem converges to a unique limit. The results here are the first of this kind and extend previous work by the authors for one-dimensional junctions. The proofs are based on a careful analysis of the behavior of the viscosity solutions near the junction, including a blow-up argument that reduces the general problem to a one-dimensional one. As in our previous note, no convexity assumptions and control theoretic interpretation of the solutions are needed.

Key words: Hamilton–Jacobi equations, networks, discontinuous Hamiltonians, junction problesm, stratification problems, comparison principle, viscosity solutions

Mathematics Subject Classification: 35F21, 49L25, 35B51, 49L20

1. Notation and terminology

Given $x \in \mathbb{R}^d$ we write $x = (x', x_d)$ with $x' \in \mathbb{R}^{d-1}$. For $i = 1, \dots, K$, $\Pi_i := \{(x', x_{d,i}) \in \mathbb{R}^d : x' \in \mathbb{R}^{d-1}, x_{d,i} \le 0\}$ are half-planes intersecting along the line $L := \{(x', 0) : x' \in \mathbb{R}^{d-1}\}$ and set $\Pi := \bigcup_{i=1}^K \Pi_i$. For simplicity we write x_i instead of $x_{d,i}$. Given $u \in C(\overline{\Pi}; \mathbb{R})$, if $(x', x_i) \in \overline{\Pi}_i$, we write $u_i(x', x_i) := u(0, \dots, x', x_i, 0 \dots .)$; when possible, to simplify the notation, we drop the subscript on u_i and simply write $u(x', x_i)$. In this setting $u_{x_i}(0, x_i)$ is the exterior normal derivative of $u_i : \Pi \to \mathbb{R}$ on L. We consider K-junction one dimensional problems in the domain $I := \bigcup_{i=1}^K I_i$ with junction $\{0\}$, where, for $i = 1, \dots, K$, $I_i := (-a_i, 0)$ and $a_i \in [-\infty, 0)$. We work with functions $u \in C(\overline{I}; \mathbb{R})$ and, for $x = (x_1, \dots, x_K) \in \overline{I}$, we write $u_i(x_i) = u(0, \dots, x_i, \dots, 0)$; when possible, to simplify the writing, we drop the subscript on u_i and write $u(x_i)$. We also use the notation u_{x_i} and $u_{x_i x_i}$ for the first and second derivatives of u_i in x_i. For $w \in C(\overline{I} \times [0, T])$ and $t_0 \in (0, T]$, $J^+ w(0, t_0)$ and $J^- w(0, t_0)$ denote respectively the super- and sup-jets or differentials of w at $(0, t_0)$, which may be, of course, empty. If $(p_1, \dots, p_K, a) \in J^+ w(0, t_0)$, then, for all $(x, t) \in \overline{I} \times [0, T]$, $w(x_i, t) \le w(0, t_0) + p_i x_i + a(t - t_0) + \mathrm{o}(|x| + |t - t_0|)$. If w is independent of some variables; these two definition are simplified accordingly. If $(p_1, \dots, p_K, a) \in J^- w(0, t_0)$, then $w(x_i, t) \ge w(0, t_0) + p_i x_i + a(t - t_0) + \mathrm{o}(|x| + |t - t_0|)$.

Throughout the paper we work with viscosity sub- and super-solutions. In most cases, however, we will not be using the term viscosity. Also we will not keep repeating that $i \in \{1, \dots, K\}$ but rather we will say for all i.

2. Introduction

We study the well-posedness of the generalized viscosity solutions to time dependent multi-dimensional junction problems satisfying a Kirchoff-type Neumann condition at the junction. We prove that the solutions satisfy a comparison principle and, hence, are unique. It is then immediate that viscosity approximations satisfying the same boundary condition converge to the unique solution. Our results, which are the first of this kind, are simple, self-contained and depend on elementary considerations about viscosity solutions and, we emphasize, do not require any convexity assumptions and the control theoretical interpretation of the solutions.

This work is a continuation of our previous paper (Lions and Souganidis [10]) where we introduced the notion of state constraint solution to one-dimensional junction problems, proved its well-posedness, and considered, for the first time, the limit of Kirchoff-type viscosity approximations.

We also show that the so-called flux limiter solutions introduced and studied in the references below for convex problems reduce to Kirchoff-type generalized viscosity solutions. Hence, uniqueness follows immediately by the simple arguments in this note.

Among the long list of references on this topic with convex Hamiltonians we refer to Achdou and Tchou [1], Barles, Briani and Chasseigne [2, 3], Barles, Briani, Chasseigne and Imbert [4], Barles and Chasseigne [5], Bressan and Hong [6], Imbert and Monneu [7] and Imbert and Nguen [8].

We are interested in the well-posedeness of continuous solutions $u : \overline{\Pi} \to \mathbb{R}$ to the Kirchoff-type initial boundary value problem

$$
\left\{
\begin{array}{ll}
u_{i,t} + H_i(Du_i, u_i, x, t) = 0 & \text{in } \Pi_i \times (0, T], \\
\min(\Sigma_i u_{i,x_i} - B, \min_i(u_{i,t} + H_i(Du_i, u_i, x, t))) \leq 0 & \text{on } L \times (0, T], \\
\max(\Sigma_i u_{i,x_i} - B, \max_i(u_{i,t} + H_i(Du_i, u_i, x, t))) \geq 0 & \text{on } L \times (0, T],
\end{array}
\right. \tag{1}
$$

with

$$
B \in \mathbb{R} \quad \text{and} \quad u(\cdot, 0) = u_0 \quad \text{on } \overline{\Pi}, \tag{2}
$$

where

$$
u_0 \in BUC(\overline{\Pi}). \tag{3}
$$

For each i, we assume that

$$
\left\{
\begin{array}{l}
H_i \text{ is coercive in } p \text{ uniformly on } x, t \text{ and bounded } u, \text{ Lipshitz continuous} \\
\text{in } u \text{ and } t, \text{ and uniformly continuous in } p, u, x, t \text{ for bounded } p \text{ and } u.
\end{array}
\right. \tag{4}
$$

As always for time-independent problems the Lipshitz continuity of H_i in u is replaced by

$$
H_i \text{ is strictly increasing in } u. \tag{5}
$$

We remark that, as it will be clear from the proofs, the particular choice of the Neumann condition in (1) is by no means essential. The arguments actually apply to more general boundary conditions of the form $G(u_{x_1}, \ldots, u_{x_K}, u)$, with the map $(p_1, \ldots, p_k, u) \to G(u_{x_1}, \ldots, u_{x_K}, u)$ strictly increasing with respect to all its arguments.

The main result is:

THEOREM 2.1. *Assume* (4). *If* $u, v \in BUC(\overline{\Pi} \times [0, T])$ *are respectively a sub- and super-solution to* (1) *with* $u_i(\cdot, 0) \leq v_i(\cdot, 0)$ *on* $\overline{\Pi}_i$, *then* $u \leq v$ *on* $\overline{\Pi} \times [0, T]$. *Moreover, the initial boundary value problem* (1) *has a unique solution* $u \in BUC(\overline{\Pi} \times [0, T])$.

As it will become apparent from the proof, it is possible to generalize the result to problem like

$$(6) \quad \begin{cases} u_{i,t} + H_i(x_i D^2 u_i, Du_i, u_i, x, t) = 0 \quad \text{in } \Pi_i \times (0, T], \\ \min(\Sigma_i u_{i,x_i} - B, \min_i(u_{i,t} + H_i(0, Du_i, u_i, x, t))) \leq 0 \quad \text{on } L \times (0, T], \\ \max(\Sigma_i u_{i,x_i} - B, \max_i(u_{i,t} + H_i(0, Du_i, u_i, x, t))) \geq 0 \quad \text{on } L \times (0, T], \\ u(\cdot, 0) = u_0 \quad \text{on } \overline{\Pi}, \end{cases}$$

when, in addition to (4), each H_i is degenerate elliptic with respect to the Hessian. Since the arguments are almost identical to the ones for the proof of Theorem 2.1, we do not present any details.

Next we state the result about the convergence of viscosity approximations to (1). The claim is immediate from the fact that any limit is solution to (1) and, hence, we do not write the proof. We remark that we can easily use "more complicated" second-order approximations than the one below.

For $\varepsilon > 0$ consider the initial boundary value problem

$$(7) \quad \begin{cases} u_{i,\varepsilon,t} - \varepsilon \Delta u_{i,\varepsilon} + H_i(Du_{i,\varepsilon}, u_{i,\varepsilon}, x, t) = 0 \quad \text{in } \Pi_i \times (0, T], \\ \Sigma_i u_{i,\varepsilon,x_i} = B, \quad \text{on } L \times (0, T], \\ u_{i,\varepsilon}(\cdot, 0) = u_{0,i} \quad \text{on } \overline{\Pi}_i, \end{cases}$$

which, in view of the (4), has a unique solution $u \in BUC(\overline{\Pi} \times [0, T])$.

THEOREM 2.2. *Assume* (3) *and* (4). *Then* $u = \lim_{\varepsilon \to 0} u_\varepsilon$ *exists and* u *is the unique solution to* (1).

Since the proof is an immediate consequence of well known estimates and the uniqueness result, we will not discuss it any further.

We also show that, in the context of the one-dimensional time dependent junction problems, the flux-limiter solutions put forward in [7] are actually generalized viscosity solution to (8) with appropriate choice of B in the Kirchoff condition, and, hence, are unique. This provides a simple and straightforward proof of the uniqueness without the need to consider cumbersome test functions and invoke any convexity.

Following the last remark, we emphasize that Kirchoff-type conditions appear to be the "correct" ones, that is, they are compatible with the maximum principle. This can be easily seen, for example, at the level of second-order equations by considering affine solutions in each branch.

In a forthcoming paper [11], we discuss problems with more general dependence on the Hessian both in the equations and along the junctions. We also consider "stratification"-type problems, that is junctions with branches of different dimension, and, we present results about the convergence of semi-discrete in time approximations with error estimates. Finally, we consider solutions which are not necessarily Lipschitz.

In this note, to simplify the notation and explain the ideas better, we present all the arguments in the special case $d = 1$, in which case (1) reduces to

$$\begin{cases} u_{i,t} + H_i(u_{i,x_i}, x, t) = 0 \quad \text{in } \ I \times (0, T], \\ \min(\Sigma_i u_{i,x_i} - B, \min_i(u_{i,t} + H_i(u_{i,x_i}, 0, t))) \leq 0 \quad \text{on } \ \{0\} \times (0, T], \\ \max(\Sigma_i u_{i,x_i} - B, \max_i(u_{i,t} + H_i(u_{i,x_i}, 0, t))) \geq 0 \quad \text{on } \ \{0\} \times (0, T], \\ u(\cdot, 0) = u_0 \quad \text{on } \ \bar{I}. \end{cases} \tag{8}$$

Organization of the paper.
In the next section we state and prove an elementary lemma which is the basic tool for the proof of the comparison principle which is presented in Section 4. Section 5 is about the relation with the the flux limiters.

3. A general lemma

We introduce and prove a general lemma which is the basic tool for the proof of Theorem 2.1. It applies to problems of one-dimensional junctions with Kirchoff condition and expands the class of "gradients" that can be used in the inequalities at the junction.

Lemma 3.1. *Assume that* $H_1, \ldots H_K \in C(\mathbb{R})$, $p_1, \ldots, p_K, q_1, \ldots, q_K \in \mathbb{R}$ *and* $a, b \in R$ *are such that, for all* $i \in \{1, \ldots, K\}$,

$$\begin{cases} \text{(i)} \quad p_i \geq q_i \text{ and } a + H_i(p_i) \leq 0 \leq b + H_i(q_i), \\ \text{(ii)} \quad \min(\Sigma_i p_i', \min_i(a + H_i(p_i'))) \leq 0 \text{ for all } p_i' \leq p_i, \\ \text{(iii)} \quad \max(\Sigma_i q_i', \max_i(b + H_i(q_i'))) \geq 0 \text{ for all } q_i' \geq q_i. \end{cases} \tag{9}$$

Then $a \leq b$.

Proof. We argue by contradiction and assume that $a > b$.

Modifying $p_1, \ldots, p_K, q_1, \ldots, q_K, a$ and b by small amounts and using the continuity of $H_1, \ldots, H_K$, we may assume that

$$p_i > q_i \quad \text{and} \quad a + \max_i H_i(p_i) < 0 < b + \min_i H_i(q_i).$$

If $\Sigma_i q_i \geq 0$, then letting $p_i' = q_i$ in (9)(ii) yields $\min_i(a + H_i(q_i)) < 0$, which is not possible given that it assumed that $a > b$. A similar argument yields a contradiction if $\Sigma_i p_i \leq 0$.

Next we assume that $\Sigma_i p_i > 0 > \Sigma_i q_i$, and let $c \in (b, a)$ and $r_i \in (q_i, p_i)$ be such that $H_i(r_i) + c = 0$. If $\Sigma_i r_i \geq 0$ (resp. $\Sigma_i r_i \leq 0$), we choose $p_i' = r_i$ (resp. $q_i' = r_i$) and argue as before. □

4. One-dimensional time-dependent junctions

Here we prove Theorem 2.1 for the initial value problem (8). The argument in the multi-dimensional setting is almost identical and we leave it up to the reader to fill in the details. The existence of solutions is immediate from Perron's method or Theorem 2.2.

To simplify the presentation here we take

$$B = 0.$$

Although the proof is not long, to clarify the strategy and highlight the new ideas, we present first a heuristic description of the argument assuming that $u_i, v_i \in C^1(\bar{I}_i \times [0, T])$ with possible discontinuities in the spatial derivative as i changes; note that, since it also assumed that $u, v \in C(\bar{I} \times [0, T])$, this assumption gives that $u_t(0, t)$, $v_t(0, t)$ exist for all $t \in (0, T]$.

Following the proof of the classical maximum principle, we assume that, for $\delta > 0$, the $\max_{x \in \bar{I} \times [0,T]}[(u - v)(x, t) - \delta t]$ is achieved at $(x_0, t_0) \in \bar{I} \times [0, T]$ with $t_0 > 0$. If $x_0 \neq 0$, we argue as in the classical uniqueness proof. Hence, we continue assuming that $x_0 = 0$. Let $a = u_t(0, t_0)$ and $b = v_t(0, t_0)$. It follows that $a \geq b + \delta > b$.

The functions $U(x_i) = u(x_i, t_0)$ and $V(x_i) = v(x_i, t_0)$ are smooth sub- and super-(viscosity) solutions of

$$\begin{cases} a + H_i(U_{x_i}, x_i, t_0) \leq 0 \text{ in } \bar{I}_i \quad \text{and} \\ \quad \min(\Sigma_i U_{x_i}(0^-), a + \min_i H_i(U_{x_i}(0^-), 0, t_0)) \leq 0, \\ b + H_i(V_{x_i}, x_i, t_0) \geq 0 \text{ in } \bar{I}_i \quad \text{and} \\ \quad \max(\Sigma_i V_{x_i}(0^-), b + \max_i H_i(V_{x_i}(0^-), 0, t_0)) \geq 0, \end{cases} \tag{10}$$

while $U(x_i) - V(x_i) \leq U(0) - V(0)$, which in turn implies that $U_{x_i}(0^-) \geq V_{x_i}(0^-)$.

We get a contradiction using Lemma 3.1 provided we verify that (9) holds for the obvious choices of $H_1, \ldots, H_K, p_1, \ldots, p_K, q_1, \ldots, q_K$. And this is immediate since (9)(i) is part of (10), while (9)(ii),(iii) follow from the observation that $J^+U_i(0) = (-\infty, U_{x_i}(0^-)]$ and $J^-V_i(0) = [V_{x_i}(0^-), \infty)$ and the fact that inequalities must hold in the viscosity sense.

We continue now with the actual proof which consists of making the above rigorous for $u, v \in C(\bar{I} \times [0, T])$.

Proof. Without loss of generality, we assume that $u, v \in C^{0,1}(\bar{I} \times [0, T])$ are respectively a sub- and super-solution to (8) and $u(\cdot, 0) \leq v(\cdot, 0)$.

Using the classical sup- and inf-convolutions we may assume that u and v are respectively semiconvex and semiconcave with respect to t. Of course, this means that we need to consider (8) in a smaller time interval and evaluate H at a different time. It is, however, standard that this does not alter the outcome, and, hence, we omit the details.

Suppose that, for some $\delta > 0$, the $\max_{x\in\bar{I}\times[0,T]}[(u-v)(x,t)-\delta t]$ is achieved at $(x_0,t_0)\in\bar{I}\times[0,T]$ with $t_0>0$. If $x_0\neq 0$, we argue as in the classical uniqueness proof. Hence, we continue assuming that $x_0=0$.

In view of the assumed semiconvexity and semiconcavity of the u and v respectively, both of them are differentiable with respect to t at $(0,t_0)$. Let $a=u_t(0,t_0)$ and $b=v_t(0,t_0)$. It follows that

$$a\geq b+\delta>b.$$

The next step is an observation, which, heuristically speaking, establishes a C^1-type property for the sub- and super-jets of semiconvex and semiconcave functions near points of differentiability. Since the claim may be useful in other contexts, we state it as a separate lemma.

LEMMA 4.1. *Let z be a Lipshitz continuous semiconvex in time solution of $z_t+H(u_x,u,x,t)\leq 0$ in $(c,0)\times[0,T]$, and assume that $\bar{a}=z_t(0,t_0)$ exists. If $J_t^+z(x,t)$ is the subdifferential of z with respect to t at (x,t), then*

$$\lim_{(x,t)\to 0}\ \sup_{p\in J_t^+z(x,t)}|p-\bar{a}|=0. \tag{11}$$

A similar statement is true for the subdifferential in t, if z is a Lipshitz continuous semiconcave in time supersolution which such that $z_t(0,t_0)$ exists for some $t_0>0$.

The claim follows from the classical facts that the semiconvexity implies that z is actually differentiable at every (x,t) such that $J_t^+z(x,t)\neq\emptyset$, and, for semiconvex functions, derivatives converge to derivatives. The x-dependence is dealt using the Lipshitz continuity.

Continuing the ongoing proof we observe that Lemma 4.1 yields $\eta_i^{\pm}:\bar{I}_i\times[0,T]\to\mathbb{R}$ such that $\lim_{(x_i,t)\to(0,t_0)}\eta_i^{\pm}(x_i,t)=0$ and, in the viscosity sense and in a neighborhood of $(0,t_0)$,

$$a+H_i(u_{x_i},0,t_0)\leq\eta_i^+(x_i,t)\quad\text{and}\quad b+H_i(v_{x_i},0,t_0)\geq\eta_i^-(x_i,t). \tag{12}$$

Indeed, if $(p_i,\bar{p}_i)\in J^+u(x,t)$, then $\bar{p}_i\in J_t^+u(x,t)$, and the claim follows from the previous observations and the continuity properties of H_i.

Next we use a blow up argument at $(0,t_0)$ on all branches to reduce the problem to a time independent setting to which we can apply Lemma 3.1.

For $\varepsilon>0$ let

$$u_i^\varepsilon(x_i,t) = \frac{u(\varepsilon x_i, t_0 + \varepsilon(t-t_0)) - u(0,t_0)}{\varepsilon} \quad \text{and}$$

$$v_i^\varepsilon(x_i,t) = \frac{v(\varepsilon x_i, t_0 + \varepsilon(t-t_0)) - v(0,t_0)}{\varepsilon}.$$

In view of the choice of $(0,t_0)$, the properties of u and v and the observations above, for every $i \in \{1,\dots,K\}$ and $\varepsilon > 0$, we have

$$u_i^\varepsilon \le v_i^\varepsilon + \delta(t-t_0), \tag{13}$$

and, as $\varepsilon \to 0$, along subsequences and locally uniformly in (x,t),

$$\begin{cases} u_{i,t}^\varepsilon(x_i,t) = u_t(\varepsilon x_i, t_0 + \varepsilon(t-t_0)) \to a, \\ v_{i,t}^\varepsilon(x_i,t) = v_t(\varepsilon x_i, t_0 + \varepsilon(t-t_0)) \to b, \end{cases} \tag{14}$$

and

$$u_i^\varepsilon(x_i,t) - u_i^\varepsilon(x_i,t_0) \to at \quad \text{and} \quad v_i^\varepsilon(x_i,t) - v_i^\varepsilon(x_i,t_0) \to bt. \tag{15}$$

Fix a subsequence $\varepsilon_n \to 0$ such that $u_i^{\varepsilon_n}(x_i,t) \to U_i(x_i) + a(t-t_0)$ and $v_i^{\varepsilon_n}(x_i,t) \to V_i(x_i) + b(t-t_0)$ and notice that, in view of (15), both U_i and V_i are independent of t.

It follows that

$$\begin{cases} U_i \le V_i \text{ in } (-\infty,0) \quad \text{and} \quad U_i(0) = V_i(0) = 0, \\ a + \overline{H}(U_{i,x_i}) \le 0 \quad \text{and} \quad b + \overline{H}(V_{i,x_i}) \ge 0, \\ \min(\Sigma_i U_{i,x_i}, \min_i(a + \overline{H}_i(U_{i,x_i}))) \le 0, \\ \max(\Sigma_i V_{i,x_i}, \max_i(b + \overline{H}_i(V_{i,x_i}))) \ge 0, \end{cases} \tag{16}$$

where, for notational simplicity, we write $\overline{H}_i(p)$ in place of $H_i(p,0,t_0)$, and, finally, recall that

$$a > b.$$

Next we get a contradiction using Lemma 3.1. While the choice of the H_i's is obvious, some work is necessary to identify $p_1,\dots,p_K,q_1,\dots,p_K$ such that (9) holds.

Set

$$\underline{p}_i := \liminf_{x_i\to 0} \frac{U_i(x_i)}{x_i}, \quad \bar{p}_i := \limsup_{x_i\to 0} \frac{U_i(x_i)}{x_i},$$

$$\underline{q}_i := \liminf_{x_i\to 0} \frac{V_i(x_i)}{x_i} \quad \text{and} \quad \bar{q}_i := \limsup_{x_i\to 0} \frac{V_i(x_i)}{x_i},$$

and recall that $J^+U_i(0) = (-\infty, \underline{p}_i]$ and $J^-V_i(0) = [\bar{q}_i, \infty)$.

Observe that $U_i \le V_i$ does not necessarily yield $\underline{p}_i \ge \bar{q}_i$, the latter being enough to conclude using Lemma 3.1 with $p_i = \underline{p}_i$ and $q_i = \bar{q}_i$. Notice, however, that $U_i \le V_i$ implies

$$\bar{p}_i \ge \underline{q}_i. \tag{17}$$

Although $\bar{p}_i \notin J^+U_i(0)$ and $\underline{q}_i \notin J^-V_i(0)$, unless U_i and V_i are respectively differentiable at 0, we claim that (9) holds for $p_i = \bar{p}_i$ and $q_i = \underline{q}_i$.

A classical blow-up argument (see, for example, Jensen and Souganidis [9]) shows that

$$\begin{aligned} & a + \overline{H}_i(p) \le 0 \quad \text{for all } p \in [\underline{p}_i, \bar{p}_i] \quad \text{and} \\ & b + \overline{H}_i(q) \ge 0 \quad \text{for all } q \in [\underline{q}_i, \bar{q}_i]. \end{aligned} \tag{18}$$

Moreover, if $p'_i \le \underline{p}_i$ for all i, then $p'_i \in J^+U_i(0)$ and, hence, $\min(\Sigma_i p'_i, \min_i(a + \overline{H}_i(p'_i))) \le 0$.

If, for some fixed i_{i_0}, $p'_{i_0} \in [\underline{p}_{i_0}, \bar{p}_{i_0}]$, then, in view of (18), $a + H_{i_0}(p'_{i_0}, 0, t_0) \le 0$, and again $\min(\Sigma_i p'_i, \min_i(a + \overline{H}_i(p'_i))) \le 0$. It follows that (9)(ii) holds.

A similar argument yields (9)(iii), while (9)(i) is obviously true, in view of (17) and (18).

5. Flux-limiter solutions are generalized Kirchoff solutions

We show here that the flux-limiter solutions to time-depending one dimension junction problems, which were introduced in [7], are actually generalized viscosity solutions to (8) for an appropriate choice of B in the Kirchoff-condition.

We begin recalling the notion of flux-limiter solution. Following [7], we assume that, for all $i = 1, \dots, K$,

$$\tilde{H}_i \in C(\mathbb{R}) \text{ is convex with a unique minimum at } p_i^0, \tag{19}$$

and define $\tilde{H}_i^{\pm} : \mathbb{R} \to \mathbb{R}$ by

$$\tilde{H}_i^-(p) = \begin{cases} \tilde{H}_i(p) & \text{if } p \le p_i^0, \\ \tilde{H}_i(p_i^0) & \text{if } p \ge p_i^0, \end{cases} \quad \text{and} \quad \tilde{H}_i^+(p) = \begin{cases} \tilde{H}_i(p_i^0) & \text{if } p \le p_i^0, \\ \tilde{H}_i(p) & \text{if } p \ge p_i^0; \end{cases} \tag{20}$$

note that [7] considers quasiconvex $\tilde{H}_i$'s but to keep things simple here we assume convexity. Finally, to simplify the presentation we assume that we deal with continuous solutions.

Fix $A \ge A_0 = \max_i \min_{\mathbb{R}} \tilde{H}$ and let $\tilde{I}_i = (0, \infty)$ and $\tilde{I} = \bigcup_{i=1}^K \tilde{I}_i$. Then $\tilde{u} \in BUC(\bar{\tilde{I}} \times [0, T])$ is an A-limiter solution of the junction problem if

$$\begin{cases} \tilde{u}_{i,t} + \tilde{H}_i(\tilde{u}_{i,x_i}) = 0 & \text{in } \tilde{I}_i \times (0, T], \\ \tilde{u}_t + \max(A, \max_i \tilde{H}_i^-(\tilde{u}_{i,x_i})) = 0 & \text{on } \{0\} \times (0, T]. \end{cases} \tag{21}$$

For each i, let p_i^A be the unique solution to $\tilde{H}_i(p) = A$, satisfying $p_i^A \geqq p_i^0$, which exists in view of (20).

Proposition 5.1. *If $\tilde{u}$ is an A-limiter solution, that is, it satisfies (21), then $u : \bar{I} \to R$ defined by $u(x) = \tilde{u}(-x)$ is a generalized solution to (8) for $B = -\Sigma_{i=1}^K p_i^A$ and $H_i(p) = \tilde{H}_i(-p)$.*

PROOF. The conclusion follows once we show that $\tilde{u}$ is a solution to

$$
\left\{
\begin{array}{ll}
\tilde{u}_{i,t} + \tilde{H}_i(\tilde{u}_{i,x_i}) = 0 \quad \text{in } \tilde{I}_i \times (0,T], \\
\min(-\Sigma_{i=1}^K \tilde{u}_{i,x_i} - B, \min_i(\tilde{u}_{i,t} + \tilde{H}_i(\tilde{u}_{i,x_i})) \leq 0 \quad \text{on } \{0\} \times (0,T], \\
\max(-\Sigma_{i=1}^K \tilde{u}_{i,x_i} - B, \max_i(\tilde{u}_{i,t} + \tilde{H}_i(\tilde{u}_{i,x_i})) \geq 0 \quad \text{on } \{0\} \times (0,T].
\end{array}
\right. \tag{22}
$$

Clearly we only need check the inequalities on $\{0\} \times (0,T]$. We begin with the sub-solution property and assume that, for some $t_0 \in (0,T]$ and for each i, $(p_i, a) \in J^+ u_i(0,t_0)$.

Since $\tilde{u}$ is an A-limiter solution, for all i, we have

$$
a + A = a + \tilde{H}(p_i^A) \leq 0 \quad \text{and} \quad a + \tilde{H}_i^-(p_i) \leq 0. \tag{23}
$$

Arguing by contradiction we assume that

$$
-\Sigma_{i=1}^K p_i + \Sigma_{i=1}^K p_i^A > 0. \tag{24}
$$

It then follows that there exists i_0 such $p_{i_0} < p_{i_0}^A$. Then, if $p_{i_0}^0 \leq p_{i_0}$, since we are in the increasing part of $\tilde{H}_{i_0}$, we have $\tilde{H}_{i_0}(p_{i_0}) \leq \tilde{H}_{i_0}(p_{i_0}^A) = A$, and, hence

$$
a + \tilde{H}_{i_0}(p_{i_0}) \leq 0. \tag{25}
$$

If $p_{i_0}^0 \geq p_{i_0}$, then $\tilde{H}_{i_0}(p_{i_0}) = \tilde{H}_{i_0}^-(p_{i_0})$, and again we have (25), and, hence, the sub-solution property.

For the super-solution property we assume that, for some $t_0 \in (0,T]$ and for each i, $(q_i, a) \in J^- u_i(0,t_0)$. It then follows from the definition of the A-limiter solution that

$$
a + \max\Big(A, \max_i \tilde{H}_i^-(q_i)\Big) \geq 0. \tag{26}
$$

If $\max_i \tilde{H}_i^-(q_i) \geq A$, then, since $\tilde{H}_i(q_i) \geq \tilde{H}_i^-(q_i)$,

$$
\max\Big(-\Sigma_{i=1}^K q_i - B, \max_i(\tilde{a} + \tilde{H}_i(q_i))\Big) \geq 0. \tag{27}
$$

If $A > \max_i \tilde{H}_i^-(q_i)$, then, for all i,

$$
a + \tilde{H}_{i_0}^+(p_i^A) = a + A \geq 0. \tag{28}
$$

Assume that $-\Sigma_{i=1}^K q_i + \Sigma_{i=1}^K p_i^A \leq 0$, for otherwise (27) is satisfied.

Then there must exist some i_0 such that $q_{i_0} \geq p_{i_0}^A$, which implies that $\tilde{H}_{i_0}(q_{i_0}) \geq \tilde{H}_{i_0}^+(p_i^A) = A$, and (27) holds again.

The claim then follows using that $u(x) = \tilde{u}(-x)$. □

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