Rend. Lincei Mat. Appl. 29 (2018), 31–61
DOI 10.4171/RLM/792

Continuum Mechanics — *An axiomatic framework for the mechanics of generalized continua*, by Gianpietro Del Piero, communicated on November 10, 2017.

Dedicated to the memory of Professor Giuseppe Grioli

Abstract. — The foundations of the mechanics of generalized continua are revisited in the light of the theoretical progress made in the last decades. The resulting axiomatic framework is independent of the concepts of motion and inertia, and provides a simple and unifying formulation for several classes of generalized continua.

Key words: Foundations of continuum mechanics, virtual power, generalized continua, micromorphic continua, microstructure

Mathematics Subject Classification: 74A05, 74A35, 74A60

1. Introduction

In the last half of the past century, much work has been done in the direction of the axiomatization of mechanics in the spirit of Hilbert's *sixth problem* [25]. Just to quote the milestones, in 1959 the conjecture of Cauchy on the dependence of the contact actions on the normal was proved by Noll to be a consequence of the balance law of linear momentum [32]. In 1963, the same author proved that the balance laws of Euler, generally considered as the basic axioms of mechanics, are in fact consequences of the more fundamental axiom of the indifference of power [33]. And again Noll, in 1973, proved that Newton's law of action and reaction is a consequence of the additivity of the external actions over disjoint sets [36].

Starting from Noll's results, Gurtin and Martins [23] in 1976 and Šilhavý [43] in 1985 showed that the existence of the Cauchy stress is a consequence of the assumption that the system of contact actions has both a surface and a volume density. Under this regularity assumption the contact force admits a double representation, and this is expressed by an equation which, due to its similarity with the balance equation of linear momentum, has been called a *pseudobalance equation* [11]. With this equation, the existence of the Cauchy stress tensor can be proved bypassing the law of linear momentum. This led to revolutionary consequences. Indeed, as pointed out by Noll in [38] and briefly commented in Section 3.7 below, this led to the removal of the concepts of *motion* and *inertia* from the list of the fundamental objects of mechanics.

In the recent past, I devoted a number of papers to extend these results to the mechanics of generalized continua [11, 12, 13]. The resulting axiomatic framework starts with the selection of a set of *state variables*. Their variations, the *virtual velocities*, are put in duality with a set of *external actions*. The duality relation, the *external virtual power*, has the form of a volume plus a surface integral. The assumed existence of a volume density for the contact actions allows to transform this relation into a volume integral, the *internal virtual power*, which is a duality relation between *internal actions* and *generalized strain rates*. When subjected to the restrictions due to the indifference of power, this relation takes a *reduced form*, which identifies the objects to be mutually related by *constitutive equations*, thereby determining the field equations of the *incremental equilibrium problem*.

After a brief preliminary review of the conceptual framework of classical continuum mechanics in Sect. 2, the emerging alternative axiomatics is applied to classical continua in Sect. 3 and to generalized continua in Sect. 4. Section 5 focuses on the class of *micromorphic continua*, which are generalized continua characterized by a single tensorial state variable, the microscopic deformation. Finally, some particular subclasses, obtained subjecting the state variable to internal constraints, are reviewed in Sect. 6. They include second-order continua, crystalline continua obeying the Cauchy–Born rule, unconstrained and constrained micropolar continua, and the classical theories of plates and beams.

2. The conceptual framework of classical continuum mechanics

The two axioms of classical continuum mechanics are the Euler balance laws of linear and angular momentum

$$\begin{aligned} \int_\Pi b(x)\,dV + \int_{\partial\Pi} s_{\partial\Pi}(x)\,dA &= 0, \\ \int_\Pi x\times b(x)\,dV + \int_{\partial\Pi} x\times s_{\partial\Pi}(x)\,dA &= 0. \end{aligned} \tag{1}$$

Here Π is an arbitrary part of the body, $\partial\Pi$ is the boundary, x is the position vector, b is the volume density of the distance actions on Π, and $s_{\partial\Pi}$ is the surface density of the contact actions on $\partial\Pi$. From these laws, a number of fundamental consequences follows:

(i) the hypothesis of Cauchy of the dependence of the contact actions on the normal[1]

$$s_{\partial\Pi}(x) = s_n(x), \tag{2}$$

[1] After Noll's proof that this is in fact a consequence of Euler's first law, this hypothesis became the *theorem of Noll*.

(ii) the local form of Newton's law of action and reaction

$$s_{-n}(x) = -s_n(x), \tag{3}$$

(iii) the *tetrahedron theorem* of Cauchy

$$s_n(x) = T(x)n, \tag{4}$$

(iv) the local equations of motion

$$\operatorname{div} T(x) + b(x) = 0, \quad T(x) = T^T(x), \tag{5}$$

(v) the theorem of virtual power

$$\int_\Pi b(x) \cdot v(x)\, dV + \int_{\partial\Pi} s_{\partial\Pi}(x) \cdot v(x)\, dA = \int_\Pi T(x) \cdot \nabla v^S(x)\, dV. \tag{6}$$

In (2), n is the exterior unit normal to $\partial\Pi$ at x. In (4), T is the Cauchy stress tensor. In (6), v is a field of virtual velocities, and ∇v^S is the symmetric part of the gradient of v. Together with a set of constitutive equations and with appropriate boundary conditions, these equations concur to the formulation of the equilibrium problem.

Within classical continuum mechanics, an alternative approach[2] consists in taking equation (6) as the basic axiom, the *principle of virtual power*, and to deduce from it the properties (2), (3), and (5).[3] This procedure is not free from criticism. Indeed, while the left-hand side of (6) is fully acceptable because it is just a declaration of which are the external actions contributing to the *external power*, there are no solid reasons for choosing a priori the right-hand side as the expression of the *internal power*. The only reason I can see is that this expression leads to the first Euler equation. But if one has in mind this goal, then the Euler equation, and not the equation of virtual power, is the real postulate.

Another weak point is that postulating equation (6) means taking for granted the existence of the stress tensor T. But at the time of the formulation of this *principle* the only known way to prove the existence of T was through the first Euler equation, and the only known way for proving the symmetry of T was to deduce it from the second Euler equation.[4] Therefore, though in a hidden way, the Euler equations remain the veritable postulate.

The approach based on Euler's equations is also subject to a, more subtle, criticism. In Cauchy's proof of the existence of the stress tensor, equation $(1)_1$ is written for a family of regions $\varepsilon \mapsto \Pi_\varepsilon$ scaled by a scale factor ε

$$\int_{\Pi_\varepsilon} b(x)\, dV + \int_{\partial\Pi_\varepsilon} s_{\partial\Pi_\varepsilon}(x)\, dA = 0. \tag{7}$$

[2] Germain [18, 19].

[3] The property (4) of the existence of the Cauchy tensor is now implicit in (6).

[4] In the formulation of the principle of virtual power, the symmetry of T was postulated separately, see [18], p. 245.

For a bounded volume density b, when $\varepsilon \to 0$ the first integral goes to zero faster than the area $A(\partial\Pi_\varepsilon)$ of the boundary. Then dividing by $A(\partial\Pi_\varepsilon)$ we get

$$\lim_{\varepsilon\to 0} \frac{1}{A(\partial\Pi_\varepsilon)} \int_{\partial\Pi_\varepsilon} s_{\partial\Pi_\varepsilon}(x)\, dA = 0. \tag{8}$$

From this equation, taking appropriate shapes for Π_ε, the desired properties (2), (3) and $(5)_1$ can be proved. The critical point is that this equation bears no trace of the distance action b. That is, (8) *is not a property of the pair* (b, s) *of internal actions, but of the contact actions* s *alone*. In other words, (8) can be deduced from (7) even if b is replaced by any other bounded function. Therefore, *the existence of the Cauchy tensor need not be deduced from the balance law of linear momentum*.

3. Classical continuum mechanics revisited

The discovery that the existence of the stress tensor can be proved without assuming the balance of linear momentum was the starting point for a revisitation of the axiomatic foundations of continuum mechanics. In this section I deal with classical continuum mechanics. The mechanics of generalized continua will be the object of the subsequent sections.

3.1. Geometry of the classical continuum

A *body* is a set $\mathscr{B}$ situated in the *physical space*, whose elements are called *material points*. Though the nature of this set and of this space are not precisely known, we assume that it is possible to measure the distance between any pair X, X_0 of elements of $\mathscr{B}$. That is, to define a *distance function* on $\mathscr{B}$. If this function varies from one measurement to another, we say that the body is *deformable*, and that every distance function defines a *configuration* of the body. In this sense, *configuration* becomes a synonym of *distance function*.[5]

All distance functions are assumed to be isometric to the Euclidean distance $|\cdot|$ of the three-dimensional Euclidean point space $\mathscr{E}$.[6] That is, it is assumed that for every distance function $\mathscr{D}$ on $\mathscr{B}$ there is a map $\chi : \mathscr{B} \to \mathscr{E}$ such that[7]

$$|\chi(X) - \chi(X_\emptyset)| = \mathscr{D}(X, X_\emptyset) \quad \forall X, X_\emptyset \in \mathscr{B}. \tag{9}$$

A map χ with this property is a *placement* in $\mathscr{E}$ of the body in its configuration $\mathscr{D}$.

[5] While the term *configuration* is frequently used informally, a definition of it as *an assemblage of relative positions* was given by Maxwell [30]. The distinction between *extrinsic* configurations, called *placements*, and *intrinsic* configurations, identified with distance functions, is due to Noll [35].

[6] This assumption seems to suggest that the physical space is Euclidean and three-dimensional. Whether or not this is true, is not known. If this is not true, the representation of $\mathscr{B}$ on $\mathscr{E}$ has to be regarded as an approximation, just like the representation of the terrestrial surface on a plane.

[7] The points $\chi(X)$ and $\chi(X_\emptyset)$ are elements of $\mathscr{E}$, and their difference is an element of the vector space $\mathscr{V}$ associated with $\mathscr{E}$.

The Euclidean space $\mathscr{E}$ has the structure of the *absolute space* of Newtonian mechanics.[8] Therefore, to place a body in $\mathscr{E}$ meets with the practical exigence of working in an absolute space,[9] without any commitment on the real nature of the physical space. In this spirit the set $\mathscr{B}$, whose nature is uncertain, is represented in $\mathscr{E}$ by its image under a selected *reference placement* χ_R, which with every point X of $\mathscr{B}$ associates the point

$$x_R = \chi_R(X) \tag{10}$$

of $\mathscr{E}$. In this way, any other placement χ of $\mathscr{B}$ is described by the *deformation* $f = \chi \circ \chi_R^{-1}$, which is the function which with every point x_R of $\Omega_R = \chi_R(\mathscr{B})$ associates the point[10]

$$x = f(x_R) = \chi(\chi_R^{-1}(x_R)) = \chi(X) \tag{11}$$

of $\mathscr{E}$. If x_R, $x_{\emptyset R}$ and x, $x_\emptyset$ are the images of X, $X_\emptyset$ under χ_R and χ, for the vector $(x - x_\emptyset)$ we have

$$|x - x_\emptyset| = |f(x_R) - f(x_{\emptyset R})| = |\chi(X) - \chi(X_\emptyset)|. \tag{12}$$

That is, χ and f correspond to the same distance function. By consequence, if two placements χ, χ^* place the body into the same configuration, the corresponding deformations f, f^* satisfy the condition

$$f^*(x_R) - f^*(x_{\emptyset R}) = Q[f(x_R) - f(x_{\emptyset R})], \tag{13}$$

with Q an orthogonal tensor and with x_R and $x_{\emptyset R}$ arbitrary points of Ω_R.

An *evolution* is a family $t \mapsto \mathscr{D}_t$ of configurations of $\mathscr{B}$. In $\mathscr{E}$, it is represented by families $t \mapsto f_t$ of deformations, with each f_t endowed with the distance $\mathscr{D}_t$. The representation is not unique, because each configuration can be represented by the infinitely many deformations which satisfy the relation (13) for different tensors Q.

Denote by δ the derivative with respect to t and by x_t the point $f_t(x_R)$. The vector

$$v(x_t) = \delta f_t(x_R) \tag{14}$$

[8] In Newton's words, "in its own nature and without regard to anything external, always remains similar and immovable". For a discussion on the concept of absolute space see Mach [29], pp. 232 and 543.

[9] In an absolute space, selecting an *origin* o it is possible to represent the points x with their *position vectors* $(x - o)$ in the inner product space $\mathscr{V}$ associated with $\mathscr{E}$, and with the further choice of a basis of $\mathscr{V}$ it is possible to represent the vectors as elements of $\mathbb{R}^3$.

[10] Quite informally, for the points of $\mathscr{E}$ I use the same symbol x used before to denote the position vector.

is the *velocity* of x_R at the time t in the given family of deformations.[11] At the time t, rewriting equation (13) in the form

$$x_t^* = x_{\emptyset t}^* + Q_t[x_t - x_{\emptyset t}], \tag{15}$$

with $x_t^* = f_t^*(x_R)$ and $x_{\emptyset t}^* = f_t^*(x_{\emptyset R})$, by differentiation with respect to t we get

$$\begin{aligned} v^*(x_t^*) &= v^*(x_{\emptyset t}^*) + Q_t[v(x_t) - v(x_{\emptyset t})] + \delta Q_t[x_t - x_{\emptyset t}] \\ &= Q_t[v(x_t) + a_t + W_t x_t], \end{aligned} \tag{16}$$

where x_t is identified with its position vector, a_t is the vector $(Q_t^T v^*(x_{\emptyset t}^*) - v(x_{\emptyset t}) - W_t x_{\emptyset t}^*)$, and W_t is the skew-symmetric tensor $Q_t^T \delta Q_t$. Moreover, from (15) and (16) differentiated with respect to x_t we have

$$\nabla x_t^* = Q_t, \quad \nabla v^*(x_t^*) \nabla x_t^* = Q_t[\nabla v(x_t) + W_t], \tag{17}$$

and, therefore,

$$\nabla v^*(x_t^*) = Q_t[\nabla v(x_t) + W_t] Q_t^T. \tag{18}$$

Equations (16) and (18) provide the transformation laws of the velocity and of the velocity gradient under a change of placement within the same configuration.

3.2. *Interactions*

An *interaction* $\mathscr{I}$ is a set function which maps the ordered pairs $(\Pi, \Pi_\emptyset)$ of open regions of $\mathscr{E}$ into the vector space $\mathscr{V}$, endowed with the following properties[12]

(i′) $\mathscr{I}(\Pi, \Pi) = 0$,
(ii′) $\mathscr{I}(\cdot, \Pi_\emptyset)$ and $\mathscr{I}(\Pi, \cdot)$ *are additive on disjoint regions*.

By (ii′), for every pair $(\Pi, \Pi_\emptyset)$ of disjoint regions we have

$$\mathscr{I}(\Pi \cup \Pi_\emptyset, \Pi \cup \Pi_\emptyset) = \mathscr{I}(\Pi, \Pi) + \mathscr{I}(\Pi, \Pi_\emptyset) + \mathscr{I}(\Pi_\emptyset, \Pi) + \mathscr{I}(\Pi_\emptyset, \Pi_\emptyset), \tag{19}$$

Then the *skew-symmetry* of the interactions between disjoint regions

$$\mathscr{I}(\Pi, \Pi_\emptyset) = -\mathscr{I}(\Pi_\emptyset, \Pi), \quad \Pi \cap \Pi_\emptyset = \emptyset, \tag{20}$$

[11] Though we call it *time*, t need not be identified with the physical time, which is purposely left out of the present analysis. To emphasize this choice, I called *geometry* what is usually called *kinematics*, and *evolution* what is usually called a *motion*. Note that the definition (14) of velocity is not intrinsic, since it depends on the specific family $t \mapsto f_t$ of deformations and not on the family $t \mapsto \mathscr{D}_t$ of configurations.

[12] The present treatment is informal and relies upon the physical intuition of the reader. For the notions of geometric measure theory required by a rigorous treatment see [6, 7, 8, 13, 42, 43, 44, 48, 50]. The question of the regularity to be assumed for the regions Π has been discussed by several authors. Besides the papers just quoted, see [5, 10, 24, 34, 36, 37, 39].

follows from (i′). The vector $\mathscr{I}(\Pi, \Pi_\emptyset)$ is the *action* exerted on Π by $\Pi_\emptyset$. By the skew-symmetry property, this action is the opposite of the action exerted on $\Pi_\emptyset$ by Π. Then (20) is a global form of the action-reaction law (3), which therefore is a direct consequence of (i′) and (ii′).

By (ii′), the actions are *measures* in the sense of measure theory. It is known that every measure is the sum of an absolutely continuous and a singular part.[13] For the action $\mathscr{I}(\cdot, \Pi_\emptyset)$, absolute continuity is with respect to the volume measure, and this means the existence of a volume density

$$b_{\Pi_\emptyset}(x) = \lim_{\varepsilon\to 0} \frac{1}{V(\mathscr{B}_{x,\varepsilon})} \mathscr{I}(\mathscr{B}_{x,\varepsilon}, \Pi_\emptyset), \tag{21}$$

at V-almost all $x \in \mathscr{E}\backslash\Pi_\emptyset$.[14] For the singular part, in classical mechanics it is assumed that

(iii′) *the singular part of $\mathscr{I}(\Pi, \Pi_\emptyset)$ is concentrated at the interface $\partial\Pi \cap \partial\Pi_\emptyset$, and has a surface density*

$$s_{\partial\Pi_\emptyset}(x) = \lim_{\varepsilon\to 0} \frac{1}{A((\mathscr{B}_{x,\varepsilon}\backslash\Pi_\emptyset) \cap \partial\Pi_\emptyset)} \mathscr{I}(\mathscr{B}_{x,\varepsilon}\backslash\Pi_\emptyset, \Pi_\emptyset) \tag{22}$$

at A-almost all $x \in \partial\Pi_\emptyset$.

The absolutely continuous and the singular part of $\mathscr{I}(\cdot, \Pi_\emptyset)$ are called the *distance interaction* between $\Pi_\emptyset$ and Π and the *contact interaction* across the interface $\partial\Pi \cap \partial\Pi_\emptyset$, respectively. By (21) and (22), they have the representations

$$\mathscr{I}^d(\Pi, \Pi_\emptyset) = \int_\Pi b_{\Pi_\emptyset}(x)\, dV, \quad \mathscr{I}^c(\partial\Pi \cap \partial\Pi_\emptyset) = \int_{\partial\Pi\cap\partial\Pi_\emptyset} s_{\partial\Pi_\emptyset}(x)\, dA. \tag{23}$$

By the skew-symmetry property (20), $\partial\Pi \cap \partial\Pi_\emptyset$ is an *oriented surface*, with Π on the inner side and $\Pi_\emptyset$ on the outer side of Π. Then $\partial\Pi \cap \partial\Pi_\emptyset$ and $\partial\Pi_\emptyset \cap \partial\Pi$ are the same surface with opposite orientation, and $\mathscr{I}^c(\partial\Pi \cap \partial\Pi_\emptyset)$ is the opposite of $\mathscr{I}^c(\partial\Pi_\emptyset \cap \partial\Pi)$. The equality

$$\mathscr{I}(\Pi, \Pi_\emptyset) = \int_\Pi b_{\Pi_\emptyset}(x)\, dV + \int_{\partial\Pi\cap\partial\Pi_\emptyset} s_{\partial\Pi_\emptyset}(x)\, dA \tag{24}$$

shows the representation of an interaction as the sum of a distance interaction and of a contact interaction. In particular, if $\Pi_\emptyset$ is the exterior $\mathscr{E}\backslash\Pi$ of Π, we

[13] By the Radon-Nikodym theorem. See e.g. [1].

[14] Here $B_{x,\varepsilon}$ is the ball of radius ε centered at x, $V(\cdot)$ is the volume measure, and *at V-almost all x* means at all x except at most a set of volume zero. In the next statement, $A(\cdot)$ is the area measure, and *at A-almost all x* means at all x except at most a set of area zero.

write $\mathscr{I}(\Pi)$, b, and s in place of $\mathscr{I}(\Pi, \mathscr{E}\backslash\Pi)$, $b_{\mathscr{E}\backslash\Pi}$ and $s_{\mathscr{E}\backslash\Pi}$, and

$$\mathscr{I}(\Pi) = \int_{\Pi} b(x)\, dV + \int_{\partial\Pi} s(x)\, dA \tag{25}$$

in place of (24).

3.3. *Pseudobalance*

Since there is a one-to-one correspondence between a region and its boundary, the contact action $\mathscr{I}^c$ can be regarded both as a function of the surfaces $\partial\Pi$ and as a function of the regions Π. A further regularity assumption is that $\mathscr{I}^c$ be absolutely continuous not only with respect to the area measure as assumed in (22), but also with respect to the volume measure.[15]

(iv$'$) *The function $\mathscr{I}^c$ has a volume density*

$$b^c(x) = \lim_{\varepsilon\to 0} \frac{1}{V(\mathscr{B}_{x,\varepsilon})} \mathscr{I}^c(\partial\mathscr{B}_{x,\varepsilon}) \tag{26}$$

at V-almost all $x \in \Omega$.

By consequence, $\mathscr{I}^c$ has the representation

$$\mathscr{I}^c(\partial\Pi) = \int_{\Pi} b^c(x)\, dV, \tag{27}$$

and comparing with $(23)_2$ with $\Pi_\emptyset = \mathscr{E}\backslash\Pi$ we get

$$-\int_{\Pi} b^c(x)\, dV + \int_{\partial\Pi} s(x)\, dA = 0. \tag{28}$$

Due to its resemblance to a balance equation, this has been called a *pseudo-balance equation*.[16] It can replace the balance equation of linear momentum in the hypotheses of the theorem of Noll and of the tetrahedron theorem of Cauchy. The resulting equations

$$s_{\mathscr{E}\backslash\Pi}(x) = s_n(x), \quad s_n(x) = T(x)n, \quad \operatorname{div} T(x) - b^c(x) = 0, \tag{29}$$

are counterparts of equations (2), (4) and $(5)_1$ with b replaced by $-b^c$. Note that, since now they have been deduced from the regularity assumptions (iii$'$) and (iv$'$),

[15] The actions with this property were called *weakly balanced Cauchy fluxes* by Gurtin and Martins [23] and by Šilhavý [43].

[16] Del Piero [11]. This equation *looks like* the balance equation of linear momentum, but *it is not*. Indeed, a balance equation involves two or more actions, while this equation involves two densities of the same action.

the dependence of the contact actions on the normal and the existence of the stress tensor are not anymore consequences of the balance of linear momentum.

3.4. Virtual power

The virtual velocities and the external actions are put into a duality relation by the functional[17]

$$\mathscr{P}_{ext}(\Pi, v) = \int_{\Pi} b \cdot v \, dV + \int_{\partial\Pi} s \cdot v \, dA, \quad \Pi \subset \mathscr{E}, \tag{30}$$

which is the *external power* in Π exerted by the action $\mathscr{I}$ with densities b and s, against the virtual velocity v. With the aid of equations (29) and of the divergence theorem, the last integral transforms as follows

$$\begin{aligned} \int_{\partial\Pi} s \cdot v \, dA &= \int_{\partial\Pi} Tn \cdot v \, dA = \int_{\Pi} (\operatorname{div} T \cdot v + T \cdot \nabla v) \, dV \\ &= \int_{\Pi} (b^c \cdot v + T \cdot \nabla v) \, dV. \end{aligned} \tag{31}$$

Then the right-hand side of (30) transforms into the volume integral

$$\int_{\Pi} ((b + b^c) \cdot v + T \cdot \nabla v) \, dV = \mathscr{P}_{int}(\mathscr{I}(\Pi), v), \tag{32}$$

called the *internal power*. From their definitions it is clear that the external and internal powers are not independent, as it occurs when assuming the principle of virtual power as an axiom. *They are two alternative expressions of the same power*. In the following we shall sometimes omit the subscripts *ext* and *int*, and we shall simply refer to the *power* $\mathscr{P}$.

3.5. Indifference

Till now, some definitions have been given and some regularity assumptions have been made, but no mechanical axiom has been formulated. In the revisited axiomatic approach there is only one axiom:[18]

(33) **Axiom.** *The power is indifferent to changes of placement within the same configuration.*

Let Ω_R be the region occupied by the body in the reference placement, let Π and Π^* be the images of a part Π_R of Ω_R under two deformations f, f^* mapping Ω_R

[17] For simplicity of notation, from here on we omit the argument x.
[18] Noll [33].

into the same configuration, and let v and v^* be virtual velocity fields on Π and Π^* related by the transformation law (16). Axiom (33) states that

$$\mathscr{P}(\Pi, v) = \mathscr{P}(\Pi^*, v^*). \tag{34}$$

In the external power

$$\mathscr{P}_{ext}(\Pi^*, v^*) = \int_{\Pi^*} b^* \cdot v^* \, dV^* + \int_{\partial\Pi^*} s^* \cdot v^* \, dA^*, \tag{35}$$

let us make the change of variables from x^* to x. Since the distances in Π and Π^* are the same, we have $dV^* = dV$ and $dA^* = dA$. By (16), the right-hand side takes the form

$$\int_{\Pi} b^* \cdot Q(v + a + w \times x) \, dV + \int_{\partial\Pi} s^* \cdot Q(v + a + w \times x) \, dA, \tag{36}$$

where w is the vector associated with the skew-symmetric tensor W.[19] Then, from (34),

$$\begin{aligned} 0 &= \int_{\Pi} (b^* \cdot Q(v + a + w \times x) - b \cdot v) \, dV \\ &\quad + \int_{\partial\Pi} (s^* \cdot Q(v + a + w \times x) - s \cdot v) \, dA \\ &= \int_{\Pi} (Q^T b^* - b) \cdot v \, dV + \int_{\partial\Pi} (Q^T s^* - s) \cdot v \, dA \\ &\quad + a \cdot \Big(\int_{\Pi} Q^T b^* \, dV + \int_{\partial\Pi} Q^T s^* \, dA \Big) \\ &\quad + w \cdot \Big(\int_{\Pi} x \times Q^T b^* \, dV + \int_{\partial\Pi} x \times Q^T s^* \, dA \Big). \end{aligned} \tag{37}$$

For $a = w = 0$, from the arbitrariness of v and Π it follows that

$$b^* = Qb, \quad s^* = Qs. \tag{38}$$

These are the transformation rules for the densities b and s under changes of placements within the same configuration. Equation (37) then reduces to

$$a \cdot \Big(\int_{\Pi} b \, dV + \int_{\partial\Pi} s \, dA \Big) + w \cdot \Big(\int_{\Pi} x \times b \, dV + \int_{\partial\Pi} x \times s \, dA \Big) = 0. \tag{39}$$

Note that, by (30),

[19] Defined as the unique vector such that $Wx = w \times x$ for all x.

$$
\begin{aligned}
&a \cdot \Big(\int_\Pi b \, dV + \int_{\partial\Pi} s \, dA \Big) = \mathscr{P}_{ext}(\Pi, a), \\
&w \cdot \Big(\int_\Pi x \times b \, dV + \int_{\partial\Pi} x \times s \, dA \Big) = \mathscr{P}_{ext}(\Pi, w \times x).
\end{aligned} \tag{40}
$$

That is, the two terms in (39) are the powers for virtual rigid translations and for virtual rigid rotations, respectively. By the arbitrariness of a and w, the two powers must be zero

$$
\mathscr{P}_{ext}(\Pi, a) = 0, \quad \mathscr{P}_{ext}(\Pi, w \times x) = 0. \tag{41}
$$

These are the conditions of *translational indifference* and of *rotational indifference*, respectively.

Again by the arbitrariness of a and w, these two conditions imply the balance equations (1). Therefore, *the balance laws of linear and of angular momentum are consequences of the axiom* (33) *of the indifference of power.*

3.6. The reduced power

Using the pseudobalance equation (28) and equations $(40)_1$, the translational indifference condition $(41)_1$ takes the form

$$
\int_\Pi (b + b^c) \, dV = 0, \tag{42}
$$

and from the arbitrariness of Π it follows that

$$
b + b^c = 0. \tag{43}
$$

For the rotational condition, from equations (29) and from the divergence theorem we have

$$
\begin{aligned}
\int_{\partial\Pi} x \times s \, dA &= \int_{\partial\Pi} x \times Tn \, dA = \int_\Pi (t + x \times \operatorname{div} T) \, dV \\
&= \int_\Pi (t + x \times b^c) \, dV,
\end{aligned} \tag{44}
$$

with t the vector associated with the skew-symmetric part of T. Then by $(40)_2$, $(41)_2$ and (43),

$$
0 = \int_\Pi (t + x \times (b + b^c)) \, dV = \int_\Pi t \, dV, \tag{45}
$$

and again by the arbitrariness of Π it follows that $t = 0$, that is, that T is a symmetric tensor.

For $b + b^c = 0$ and T symmetric, the internal power (32) takes the *reduced form*

$$\mathscr{P}_{red}(\Pi, v) = \int_\Pi T \cdot \nabla v^S \, dV, \tag{46}$$

with ∇v^S the symmetric part of the gradient of v. In general, the reduced power identifies the *internal actions* and the *generalized strain rates*, which are the objects to be interrelated by *constitutive equations*. Thus, equation (46) tells us that in classical mechanics there is a single internal action, the stress tensor T, and that the corresponding generalized strain rate is ∇v^S. Then there is a single constitutive equation, which is a relation between T and ∇v^S.

3.7. *Revolutionary character of the alternative approach*

With the balance laws (1) replaced by the pseudobalance equation (28), the local equation of motion $(5)_1$ is replaced by equation $(29)_3$ which involves only quantities related to the contact actions. In particular, the volume density b of the distance actions is replaced by the volume density b^c of the contact actions. Then, since inertia is a particular distance action, the "inertia forces" do not appear anymore in the basic set of equations (29). The result is that the inertia law is downgraded from a general principle to a constitutive postulate:

- *... l'on regarde les forces d'inertie comme des forces véritables qui sont les interactions entre les corps dans notre système solaire et la totalité des objets dans le reste de l'univers ...* [33],
- *les repères inertiaux n'entrent plus dans la partie générale de la nouvelle axiomatisation. La loi d'inertie est regardée comme un postulat constitutif* [33],
- *when dealing with deformable bodies, inertia plays very often a secondary role. In some situations, it is even appropriate to neglect inertia altogether ...* [38].

This implies the removal of all concepts related to motion from the fundamentals of mechanics:

- *I believe that the basic concepts of mechanics in general should not include items such as momentum, kinetic energy, and angular momentum* [38].

This removal is revolutionary because it goes against a firmly established tradition which considers these concepts as fundamental

- *le mouvement et ses propriétés générales sont le premier et le principal objet de la Mécanique* [9],
- *die Mechanik ist die Wissenschaft der Bewegung* [27],
- *Mechanics is concerned with the motions and equilibrium of masses* [29].

The tradition is amply motivated by the fact that gravitation was practically the only known distance action until relatively recent times, and that for long time the most challenging goal of mechanics has been the discovery of the laws of motion of the celestial bodies. But nowadays in a formulation of mechanics founded on rational bases a drastic reduction of the traditional role played by inertia seems to be unavoidable.

4. Generalized continua

In a classical continuum, the only state variable is the deformation f.[20] On the contrary, a generalized continuum is characterized by the presence of additional state variables ξ^α. They are maps from the region Ω_R occupied by the body in the reference placement into finite dimensional inner product spaces Y^α made of scalars, vectors, or tensors of any order, depending on the physical nature of the phenomenon described by each state variable. The state variables can be both geometric and non-geometric.[21] Only geometric variables are considered in the following.

Just as with f are associated the virtual velocity v and the interaction $\mathscr{I}$, a *generalized virtual velocity* v^α and a *generalized interaction* $\mathscr{I}^\alpha$ are associated with each ξ^α. The generalized interactions are subject to the same hypotheses (i′)–(iv′) made for $\mathscr{I}$. That is, each $\mathscr{I}^\alpha$ can be split into the sum of a distance action and a surface action

$$\mathscr{I}^\alpha = \mathscr{I}^{\alpha d} + \mathscr{I}^{\alpha c}, \tag{47}$$

each $\mathscr{I}^{\alpha d}$ is supposed to have a volume density b^α, and each $\mathscr{I}^{\alpha c}$ is supposed to have both a surface density s^α and a volume density $b^{\alpha c}$. For every region Π of $\mathscr{E}$, each contact action $\mathscr{I}^{\alpha c}$ admits a double representation as a surface integral and as a volume integral, and this gives origin to additional pseudobalance equations

$$-\int_\Pi b^{\alpha c}\, dV + \int_{\partial\Pi} s^\alpha\, dA = 0. \tag{48}$$

From each of them, using extensions of the theorems of Noll and Cauchy to the spaces Y^α, the existence of a generalized stress tensor T^α is deduced. This tensor is a linear mapping $T^\alpha : \mathscr{V} \to Y^\alpha$ such that

$$s^\alpha = T^\alpha n, \quad \operatorname{div} T^\alpha - b^{\alpha c} = 0. \tag{49}$$

[20] This needs some clarification. By a *state* of a material point I mean the set of all variables which influence its relation with the exterior. That is, whose variations appear in the expression of the external power as virtual velocities. There may be other state variables, which I call *constitutive*, which influence the response of the material but do not contribute to the power. To see the difference, consider the plastic strain. In classical plasticity as described, for example, in Hill's book [26], the plastic strain appears in the constitutive equation but not in the expression of the power. In other models, for example, in gradient plasticity, an external power is associated with the plastic strain rate. According to the terminology adopted here, in the first case the plastic strain is a constitutive variable, while in the second case it is a state variable.

[21] Examples of non-geometric variables are the temperature, the electric and magnetic fields, and the physical time.

Then the external power has the form[22]

$$(50)\qquad \mathscr{P}_{ext}(\Pi, v, v^\alpha) = \int_\Pi (b\cdot v + b^\alpha \cdot v^\alpha)\, dV + \int_{\partial\Pi} (s\cdot v + s^\alpha\cdot v^\alpha)\, dA.$$

In the surface integral the first term transforms as in (31), and similarly for the second term we have

$$(51)\qquad \int_{\partial\Pi} s^\alpha \cdot v^\alpha\, dA = \int_\Pi (b^{\alpha c}\cdot v^\alpha + T^\alpha \cdot \nabla v^\alpha)\, dV.$$

The external power then transforms into the *internal power*

$$(52)\qquad \int_\Pi ((b + b^c)\cdot v + T\cdot \nabla v + (b^\alpha + b^{\alpha c})\cdot v^\alpha + T^\alpha \cdot \nabla v^\alpha)\, dV = \mathscr{P}_{int}(\Pi, v, v^\alpha).$$

In contrast to classical continua, the axiom of the indifference of power does not take a unique mathematical form. Since the concept of *indifference* now depends on the physical nature of the state variables, for each generalized continuum it is necessary to decide what is a *configuration*, and then to specify, on physical bases, the laws of variation of the virtual velocities v^α under changes of placement within the same configuration.

Once the indifference conditions have been determined for a specific continuum, their combination with the pseudobalance equations yields a set of *equilibrium equations*, which are the field equations of the equilibrium problem. For classical continua, the indifference conditions $b + b^c = 0$ and $T = T^T$ combined with the pseudobalance equation $(29)_3$ yield equilibrium equations which coincide with Euler's balance laws. It is not so for generalized continua. For example, for the micromorphic continua considered in the next section, the first of the equilibrium equations (81) is again Euler's first law, and the third is similar to Euler's second law, since it requires the symmetry of a stress tensor. But the second is an extra equation.

In the traditional treatment of generalized continua, the new equation is interpreted as a balance law of a microscopic quantity,[23] which shows up in the presence of microstructure. Thus, for generalized continua the assumed basic laws are the balance laws of macroscopic and microscopic linear momentum, while there is no general agreement about the form and role of the balance law of angular momentum.

This framework covers a significant number of specific models,[24] but is not sufficient to build a fairly general theory. Indeed, in the presence of several state

[22] With sum over repeated indices α.

[23] Balance of *spin momentum* [47], Sect. 98, *microstress* [19], *micromomentum* [6], *microforce* [22], Sect. 8.

[24] See e.g. the list given in [6], Sect. 2.

variables, each of them would require its own balance equation. If each equation were to be considered as a fundamental law of mechanics, we should assist to an indefinite proliferation of fundamental laws.[25]

In the alternative approach, a state variable does not produce a balance law, but only a "pseudobalance equation", which is the effect of a regularity assumption. As such, its contradiction does not imply the collapse of the theory, but only some technical complication due to dealing with less regular objects. The basic laws are dictated by the indifference of power, and are quite independent of the number of the state variables.

In what follows, we restrict our analysis to models in which the state variables are purely geometric and originate from distance measures taken at different length scales. Among such models, we further restrict ourselves to the case in which only two scales, macroscopic and microscopic, are taken into consideration.

5. Micromorphic continua

In a classical continuum, a placement of a body on the Euclidean space is made on the basis of distance measurements. In a generalized continuum with purely geometric state variables, a more accurate picture of the "real" body is obtained by refining the distance measurements in the neighborhood of each material point. Iterating this procedure, a multi-scale model is obtained. In particular, for a single series of refined measures there are two length scales, *macroscopic* and *microscopic*. The resulting two-scale continuum is called a *micromorphic continuum*.[26]

5.1. Geometry of the micromorphic continuum

In the geometry of classical continua developed in Subsection 3.1, a starting point was the possibility of defining distance functions over the body by means of distance measurements. In fact, this possibility is only theoretical, because common sense suggests that distance measurements necessarily involve only a finite number of points.[27] Then only a finite set Ξ of material points $X, X_\emptyset, \ldots$ can be placed in such a way that the Euclidean distance of their images $\chi(X), \chi(X_\emptyset), \ldots$ on $\mathscr{E}$ be equal to the measured distance $\mathscr{D}(X, X_\emptyset)$, as required in (9). To model the body as a continuum, the finite set $\chi(\Xi)$ obtained by the placement of Ξ is "filled" to

[25] This is acceptable for geometric variables involving distance measures made at a different scales, like e.g. in [19]. More difficult is to justify this proliferation in the case of non-geometric variables, or of geometric variables defined on the same distance scale.

[26] Eringen [15].

[27] Of course this is also true for classical mechanics. But since the more informal approach adopted there bears no serious consequences, this aspect is generally ignored. On the contrary, for generalized continua there are consequences. See for example the comment on second-order continua made in Subsection 6.1.

form a continuous region $\chi(\mathscr{B})$.[28] If Ξ is sufficiently "spread" over the body, this region can be considered *a representative of* $\mathscr{B}$ *in the absolute space* $\mathscr{E}$.

Now let $X_\emptyset$ be a material point in Ξ, and let $\mathscr{N}_{X_\emptyset}$ be a neighborhood of $X_\emptyset$. Let, further, $\mathscr{D}_{X_\emptyset}$ be the distance function obtained from refined distance measurements on a finite subset $\Xi_{X_\emptyset}$ of $\mathscr{N}_{X_\emptyset}$, and let $\chi_{X_\emptyset}$ be a placement of $\Xi_{X_\emptyset}$ on $\mathscr{E}$ such that

$$|\chi_{X_\emptyset}(X) - \chi_{X_\emptyset}(X_\emptyset)| = \mathscr{D}_{X_\emptyset}(X, X_\emptyset) \quad \forall X \in \Xi_{X_\emptyset}. \tag{53}$$

Just like $\chi(\Xi)$, the set $\chi_{X_\emptyset}(\Xi_{X_\emptyset})$ can be "filled", to form a continuum $\chi_{X_\emptyset}(\mathscr{N}_{X_\emptyset})$, and if $\Xi_{X_\emptyset}$ is sufficiently "spread" over $\mathscr{N}_{X_\emptyset}$, this continuum can be considered a representative of $\mathscr{N}_{X_\emptyset}$ in $\mathscr{E}$.

For a micromorphic continuum, a *configuration* is a pair $(\mathscr{D}, X_\emptyset \mapsto \mathscr{D}_{X_\emptyset})$, with $\mathscr{D}$ a *macroscopic distance function* on the region $\chi(\mathscr{B})$ and with $X_\emptyset \mapsto \mathscr{D}_{X_\emptyset}$ a finite family of *microscopic distance functions*, one for each region $\chi_{X_\emptyset}(\mathscr{N}_{X_\emptyset})$. A *placement* of a configuration $(\mathscr{D}, X_\emptyset \mapsto \mathscr{D}_{X_\emptyset})$ in $\mathscr{E}$ is a pair $(\chi, X_\emptyset \mapsto \chi_{X_\emptyset})$, where χ is a *macroscopic placement* satisfying condition (9) and each $\chi_{X_\emptyset}$ is a *microscopic placement* satisfying condition (53).

Let us choose a reference placement in which every microscopic placement $\chi_{X_\emptyset}$ coincides with the restriction of the macroscopic placement χ_R to the corresponding $\mathscr{N}_{X_\emptyset}$. With this choice, every other placement $(\chi, X_\emptyset \mapsto \chi_{X_\emptyset})$ is described by the pair $(f, X_\emptyset \mapsto f_{X_\emptyset})$, where f is the *macroscopic deformation*

$$f(x_R) = \chi(\chi_R^{-1}(x_R)), \quad x_R \in \Omega_R = \chi_R(\mathscr{B}), \tag{54}$$

and each $f_{X_\emptyset}$ is the *microscopic deformation*

$$f_{X_\emptyset}(x_R) = \chi_{X_\emptyset}(\chi_R^{-1}(x_R)), \quad x_R \in \mathscr{N}_R(X_\emptyset) = \chi_R(\mathscr{N}(X_\emptyset)). \tag{55}$$

We recall that f and $f_{X_\emptyset}$ are initially defined over finite sets, and only after the "filling" operation they are extended to the continuous regions Ω_R and $\mathscr{N}_R(X_\emptyset)$. This operation can be made smooth enough to render the extended functions continuous and differentiable, so that

$$\begin{aligned} f(x_R) &= f(x_{\emptyset R}) + \nabla f(x_{\emptyset R})[x_R - x_{\emptyset R}] + o(|x_R - x_{\emptyset R}|) \quad \forall x_R \in \Omega_R, \\ f_{X_\emptyset}(x_R) &= f_{X_\emptyset}(x_{\emptyset R}) + \nabla f_{X_\emptyset}(x_{\emptyset R})[x_R - x_{\emptyset R}] + o(|x_R - x_{\emptyset R}|) \quad \forall x_R \in \mathscr{N}_R(X_\emptyset). \end{aligned} \tag{56}$$

I emphasize that the regularity of f and $f_{X_\emptyset}$ is not a physical property detected by measurement, but only a smoothness property due to the filling procedure.

In equation $(56)_2$, the last term can be neglected for a sufficiently small $\mathscr{N}_R(X_\emptyset)$. Moreover, there is no loss in generality in fixing the point $f_{X_\emptyset}(x_{\emptyset R})$, for example, taking

$$f_{X_\emptyset}(x_{\emptyset R}) = f(x_{\emptyset R}). \tag{57}$$

---

[28] For example, this can be done taking the convex hull of $\chi(\Xi)$.

Then $f_{X_\emptyset}$ is determined by its local approximation

$$F(x_{\emptyset R}) = \nabla f_{X_\emptyset}(x_{\emptyset R}), \quad x_{\emptyset R} \in \chi_R(\mathscr{B}). \tag{58}$$

This allows us to define a *deformation* from the reference placement χ_R as a pair (f, F), with f a macroscopic deformation from Ω_R and F a tensor field on Ω_R. The *microscopic deformation gradient* F is the additional state variable of the micromorphic continuum.[29]

If (f, F) and (f^*, F^*) are deformations from the reference placement to placements $(\chi, X_\emptyset \mapsto \chi_{X_\emptyset})$, $(\chi^*, X^*_\emptyset \mapsto \chi^*_{X^*_\emptyset})$ belonging to the same configuration, the macroscopic deformations f, f^* must satisfy the condition (13) of preservation of the macroscopic distance, and the microscopic deformation gradients F, F^* must satisfy the condition

$$F^*(x_{\emptyset R})[x_R - x_{\emptyset R}] = Q_{x_{\emptyset R}} F(x_{\emptyset R})[x_R - x_{\emptyset R}], \tag{59}$$

of preservation of the microscopic distance, with $x_{\emptyset R} \mapsto Q_{x_{\emptyset R}}$ a family of orthogonal tensors. Then the *distance preserving conditions*

$$\nabla f^* = Q \nabla f, \quad F^* = Q_{x_{\emptyset R}} F, \tag{60}$$

tell us that in a deformation between placements belonging to the same configuration both deformation gradients, macro- and microscopic, are orthogonal tensors.

But there is a third, less obvious, condition. The vectors $e = x_R - x_{\emptyset R}$ represent *material directions* from $x_{\emptyset R}$ in the reference placement, and the angle between the transformed vectors $\nabla f(x_{\emptyset R})e$ and $F(x_{\emptyset R})e$ is the *deviation* between the macroscopic and microscopic images of e in the deformation (f, F). The third condition is that this angle be the same

$$\frac{\nabla f^* e \cdot F^* e}{|\nabla f^* e|\,|F^* e|} = \frac{\nabla f e \cdot F e}{|\nabla f e|\,|F e|} \quad \forall e \in \mathscr{V}, \tag{61}$$

for all pairs of deformations (f, F), (f^*, F^*) which map the reference placement into placements belonging to the same configuration.[30] Then from conditions (60), by the arbitrariness of e and by the invertibility of ∇f and F we get the *deviation preserving condition*[31]

$$Q_{x_{\emptyset R}} = Q \quad \forall x_{\emptyset R} \in \Omega_R. \tag{62}$$

[29] Though $\nabla f_{X_\emptyset}$ is the gradient of $f_{X_\emptyset}$, the field $x_{\emptyset R} \mapsto \nabla f_{X_\emptyset}(x_{\emptyset R})$ is not in general the gradient of a function over Ω_R, because $\nabla f_{X_\emptyset}$ is the gradient of different functions $f_{X_\emptyset}$ at different points of Ω_R.

[30] For example, in the plate theory of Subsection 6.4, if e is the director orthogonal to the undeformed surface Γ, the deviation of e is the angle between the deformed director and the normal to the deformed surface. It seems obvious that this angle be the same in all placements belonging to the same configuration.

[31] At my knowledge there is no trace of this condition in the literature. In micromorphic continua the concept of *deviation* is particularly relevant, since the deviation L^μ between the macroscopic and microscopic velocities is one of the generalized strain rates, see Subsection 5.3 below.

Thus, in a deformation between placements belonging to the same configuration *all microscopic deformation gradients are equal to the macroscopic deformation gradient*.

For a micromorphic continuum, an *evolution* is a family $t \mapsto (\mathscr{D}_t, X_\emptyset \mapsto \mathscr{D}_{X_\emptyset t})$ of configurations. In $\mathscr{E}$, it is represented by families $t \mapsto (f_t, F_t)$ of deformations from a reference placement χ_R. The expansion

$$
\begin{aligned}
F_\tau(x_{\emptyset R}) &= F_t(x_{\emptyset R}) + (\tau - t)\,\delta F_t(x_{\emptyset R}) \\
&= (I + (\tau - t)L(F_t(x_{\emptyset R}))F_t(x_{\emptyset R}) + o(\tau - t),
\end{aligned} \tag{63}
$$

shows that, to within higher-order terms, F_τ is the composition of F_t with the perturbation $(I + (\tau - t)L)$. If $x_{\emptyset t} = F_t(x_{\emptyset R})$, the tensor

$$
L(x_{\emptyset t}) = \delta F_t(x_{\emptyset R})F_t^{-1}(x_{\emptyset t}) \tag{64}
$$

is the *microscopic velocity* at $x_{\emptyset t}$ at the time t. Thus, a *virtual velocity* is a pair (v, L), with the macroscopic velocity v given by (14) and with L as above. Under a change of placement within the same configuration v transforms as in (16), and for L from (64), $(60)_2$ and (62) we have

$$
L^* = \delta F_t^* F_t^{*-1} = (Q_t \delta F_t + \delta Q_t F_t)F_t^{-1}Q_t^T = Q_t L Q_t^T + \delta Q_t Q_t^T, \tag{65}
$$

that is,

$$
L^*(x_{0t}^*) = Q_t[L(x_{0t}) + W_t]Q_t^T, \tag{66}
$$

with Q_t an orthogonal tensor and $W_t = Q_t^T \delta Q_t$ a skew-symmetric tensor.

5.2. *Generalized interactions and virtual power*

In a micromorphic continuum there are two external actions, a macroscopic action $\mathscr{I}$ and a microscopic action $\mathscr{I}^\mu$. The first has the integral representation (25), and for the second we have

$$
\mathscr{I}^\mu(\Pi) = \int_\Pi B(x)\,dV + \int_{\partial\Pi} S(x)\,dA, \tag{67}
$$

with B the volume density of the distance action and S the surface density of the contact action. The additional assumption that the contact interaction also has a volume density B^c gives origin to the pseudobalance equation

$$
-\int_\Pi B^c(x)\,dV + \int_{\partial\Pi} S(x)\,dA = 0. \tag{68}
$$

From it and from the theorems of Noll and of Cauchy the following counterparts of the relations (29)

$$S_{\mathscr{E}\setminus\Pi}(x) = S(x,n), \quad S(x,n) = \mathbb{T}(x)n, \quad \operatorname{div}\mathbb{T}(x) - B^c(x) = 0, \tag{69}$$

follow, with $\mathbb{T}$ a third-order tensor.[32]

The duality between the external actions and the virtual velocities v, L is established by the external power

$$\mathscr{P}_{ext}(\Pi, v, L) = \int_\Pi (b\cdot v + B\cdot L)\,dV + \int_{\partial\Pi}(s\cdot v + S\cdot L)\,dA. \tag{70}$$

With the relations (29), (69), and the divergence theorem, the right-hand side transforms into the volume integral

$$\int_\Pi ((b+b^c)\cdot v + T\cdot\nabla v + (B+B^c)\cdot L + \mathbb{T}\cdot\nabla L)\,dV = \mathscr{P}_{int}(\Pi, v, L), \tag{71}$$

which is the internal power of the micromorphic continuum.

5.3. *Indifference requirements*

By the transformation rules (16), (18) and (66), under a change of placement within the same configuration the internal power (71) transforms as follows[33]

$$\begin{aligned} \mathscr{P}_{int}(\Pi^*, v^*, L^*) &= \int_{\Pi^*}((b^*+b^{*c})\cdot v^* + T^*\cdot\nabla v^* + (B^*+B^{*c})\cdot L^* + \mathbb{T}^*\cdot\nabla L^*)\,dV^* \\ &= \int_\Pi ((b^*+b^{*c})\cdot Q[v+a+Wx] + T^*\cdot\mathbb{Q}[\nabla v + W] \\ &\quad + (B^*+B^{*c})\cdot\mathbb{Q}[L+W] + \mathbb{T}^*\cdot\mathbb{Q}[\nabla L])\,dV. \end{aligned} \tag{72}$$

By the indifference axiom (33), the powers (71) and (72) are equal. Then, by subtraction, from the arbitrariness of v, ∇v, L, ∇L and Π we get

$$b^*+b^{*c} = Q[b+b^c], \quad T^* = \mathbb{Q}[T], \quad B^*+B^{*c} = \mathbb{Q}[B+B^c], \quad \mathbb{T}^* = \mathbb{Q}\mathbb{T}, \tag{73}$$

[32] In accordance with the definition $(49)_1$, $\mathbb{T}$ is a map from $\mathscr{V}$ into the second-order tensors. In components, $S_{ij} = \mathbb{T}_{ijk}n_k$.

[33] Here we use the notation

$$\mathbb{Q}[L] = QLQ^T, \quad (\mathbb{Q}[L])_{ij} = Q_{ir}L_{rs}Q_{js},$$

by which

$$\nabla(\mathbb{Q}[L]) = \mathbb{Q}[\nabla L], \quad (\mathbb{Q}[\nabla L])_{ij,k} = Q_{ir}L_{rs,k}Q_{js}.$$

and the difference of the powers (71), (72) reduces to

$$a \cdot \int_\Pi (b + b^c)\, dV + W \cdot \int_\Pi ((b + b^c) \otimes x + T + B + B^c)\, dV = 0. \tag{74}$$

By the arbitrariness of a and W, the two integrals must vanish separately. The vanishing of the first integral is the condition of *translational indifference*

$$\mathscr{P}(\Pi, a, 0) = 0, \tag{75}$$

and the vanishing of the second integral is the condition of *rotational indifference*

$$\mathscr{P}(\Pi, Wx, W) = 0. \tag{76}$$

By the arbitrariness of Π, these conditions take the local forms

$$b + b^c = 0, \quad T^\mu = T^{\mu T}, \tag{77}$$

with

$$T^\mu = T + B + B^c. \tag{78}$$

Conditions (75) and (76) are the counterparts of the balance equations (1) of the classical continuum, and equations (77) are the counterparts of their local forms (5). Since $(77)_1$ is the same as (43), *the balance laws of linear momentum for a classical continuum and for a micromorphic continuum are the same*. Additional terms due to the microscopic interactions appear only in the balance law of angular momentum, and their effect is that *in a micromorphic continuum the stress tensor T is symmetric if and only if the microscopic action $B + B^c$ is symmetric.*

From conditions (77) the *reduced form* of the internal power

$$\mathscr{P}_{red}(\Pi, v, L) = \int_\Pi (T \cdot L^\mu + T^\mu \cdot L^S + \mathbb{T} \cdot \nabla L)\, dV \tag{79}$$

follows, where

$$L^\mu = \nabla v - L \tag{80}$$

is the *deviation* between the macroscopic and microscopic velocities. Therefore, in a micromorphic continuum the internal actions are T, T^μ and $\mathbb{T}$, and the corresponding generalized strain rates are L^μ, L^S and ∇L.

5.4. Equilibrium equations and constitutive equations

For the micromorphic continuum, the field equations of the equilibrium problem are the *equilibrium equations*

$$\operatorname{div} T + b = 0, \quad \operatorname{div} \mathbb{T} + B = T^\mu - T, \quad T^\mu = T^{\mu T}. \tag{81}$$

In them, b and B are the given external actions, and T, T^{μ} and $\mathbb{T}$ are the internal actions to be specified by incremental constitutive equations of the form

$$T - T_0 = \varphi(L^{\mu}, L^S, \nabla L), \quad T^{\mu} - T_0^{\mu} = \psi(L^{\mu}, L^S, \nabla L), \\ \mathbb{T} - \mathbb{T}_0 = \chi(L^{\mu}, L^S, \nabla L), \tag{82}$$

where T_0, T_0^{μ}, $\mathbb{T}_0$ are the actions on a given initial placement, which are supposed to be known, and $(T - T_0)$, $(T^{\mu} - T_0^{\mu})$, $(\mathbb{T} - \mathbb{T}_0)$ are their increments due to the virtual velocities L^{μ}, L^S, ∇L. Substitution into the equilibrium equations provides a system of differential equations with the generalized strain rates as unknowns.[34]

6. Constrained micromorphic continua

Special micromorphic continua are obtained by subjecting the state variable F to internal constraints. We first consider two models, the second-gradient continua and the continua which obey the Cauchy–Born rule, which are not originally conceived as, but are reducible to, micromorphic continua. Then we consider the micropolar continua and some of their many constrained versions.

6.1. Second-gradient continua and the Cauchy–Born rule

A *second-gradient continuum* is a non-local classical continuum. It is *classical*, in the sense that there are no state variables besides the macroscopic deformation, and *non-local*, because the external power includes an extra term involving the gradient of v

$$\mathscr{P}_{ext}(\Pi, v) = \int_{\Pi} (b \cdot v + B \cdot \nabla v)\, dV + \int_{\partial\Pi} (s \cdot v + S \cdot \nabla v)\, dA. \tag{83}$$

Alternatively, this can be considered as a micromorphic continuum with the state variable F subjected to the constraint

$$F = \nabla f. \tag{84}$$

This is a *generalized continuum with latent microstructure*,[35] in which the microdeformations are hidden because of their dependence on the macrodeformations.

In the present axiomatic framework, to assume this constraint does not seem to be a good idea. Indeed, the additional information obtained by taking ∇f as state variable does not come from refined measurements at the microscopic level,

[34] The formulation of the problem is completed by the prescription of a set of boundary conditions, see e.g. [13].

[35] Capriz [6]. For more details on second-gradient continua and on continua with latent microstructure see also [13], Sects. 10 and 11.

but from the fine details of the “filling” procedure. Since this procedure has no physical basis, to look at its fine details to know more on the microscopic deformation looks a bit awkward.

But this does not mean that the second-order continua must be discarded altogether. As we shall see below, such continua come up naturally when internal constraints are introduced. Examples are the rotation constraints in *Toupin's constrained theory of Cosserat continua*, in the *Kirchhoff–Love plate theory*, and in the *Euler–Bernoulli beam theory*.[36]

Another example is the *Cauchy–Born rule* for continua with a crystalline structure. The directions d^α of the crystal lattice are taken as state variables, and for their virtual velocities v^α it is assumed that

$$v^\alpha = \nabla v\, d^\alpha. \tag{85}$$

That is, the *directors* d^α are supposed to follow the macroscopic deformation. If b^α and s^α are the volume and surface densities of the external actions associated with the virtual velocities v^α, the corresponding external powers are

$$\begin{aligned} b^\alpha \cdot v^\alpha &= b^\alpha \cdot \nabla v\, d^\alpha = (b^\alpha \otimes d^\alpha) \cdot \nabla v, \\ s^\alpha \cdot v^\alpha &= s^\alpha \cdot \nabla v\, d^\alpha = (s^\alpha \otimes d^\alpha) \cdot \nabla v, \end{aligned} \tag{86}$$

and after setting

$$B = b^\alpha \otimes d^\alpha, \quad S = s^\alpha \otimes d^\alpha, \tag{87}$$

the external power takes the form (83). Thus, a continuum obeying the Cauchy–Born rule is in fact a second-gradient continuum, with densities B and S of the particular form (87). In this case, the geometrical constraint by itself does not bring any simplification of the constitutive equation.[37]

6.2. *Micropolar continua*

Micropolar continua[38] are micromorphic continua whose state variable F is an orthogonal tensor. This means that the refined measurements reveal negligible changes of distance but significant deviations. By (64), the virtual velocity L associated with an orthogonal tensor F is a skew-symmetric tensor W. Then in the external power (70) the products $B \cdot W$ and $S \cdot W$ involve only the skew-symmetric parts B^W and S^W of B and S. It is then convenient to replace the

[36] See Sects. 6.3 and 6.4 below. In these theories, the purpose is to reduce the number of the required material constants, both because it is in general exceedingly large, and because these constants are not so easy to measure.

[37] The reduction in the number of material constants obtained in the so called *Cauchy elasticity* is not due to a simplified kinematics but to the central-force constitutive assumption on the molecular interactions.

[38] Also called *Cosserat continua*.

products $B^W \cdot W$ and $S^W \cdot W$ by those of their associated vectors

$$B \cdot W = c \cdot \omega, \quad S \cdot W = m \cdot \omega, \tag{88}$$

which are the *body couple* c, the *surface couple* m, and the *rotation vector* $\omega/2$.[39] The external power then takes the form

$$\mathscr{P}_{ext}(\Pi, v, \omega) = \int_\Pi (b \cdot v + c \cdot \omega)\, dV + \int_{\partial\Pi} (s \cdot v + m \cdot \omega)\, dA. \tag{89}$$

The assumption that the contact actions also have volume densities b^c, c^c provides the pseudobalance equations

$$-\int_\Pi b^c\, dV + \int_{\partial\Pi} s\, dA = 0, \quad -\int_\Pi c^c\, dV + \int_{\partial\Pi} m\, dA = 0. \tag{90}$$

From the theorems of Noll and Cauchy follows the existence of second-order tensors T and M such that

$$s = Tn, \quad \operatorname{div} T - b^c = 0, \quad m = Mn, \quad \operatorname{div} M - c^c = 0, \tag{91}$$

with T the Cauchy stress of classical continuum mechanics and M the *couple-stress tensor* of the micropolar continuum. After substitution into (89), the divergence theorem provides the expression

$$\int_\Pi ((b + b^c) \cdot v + T \cdot \nabla v + (c + c^c) \cdot \omega + M \cdot \nabla\omega)\, dV = \mathscr{P}_{int}(\Pi, v, \omega) \tag{92}$$

of the internal power. From the indifference requirements (77) we have

$$b + b^c = 0, \quad c + c^c = -t. \tag{93}$$

Then from the identities[40]

$$T \cdot \nabla v = T^S \cdot \nabla v^S + T^W \cdot \nabla v^W = T^S \cdot \nabla v^S + t \cdot \operatorname{curl} v, \tag{94}$$

for the internal power we get the reduced form

$$\mathscr{P}_{red}(\Pi, v, \omega) = \int_\Pi (T^S \cdot \nabla v^S + t \cdot \theta + M \cdot \nabla\omega)\, dV, \tag{95}$$

---

[39] If W and W_0 are skew-symmetric tensors and w and w_0 are the associated vectors, then $W \cdot W_0 = 2w \cdot w_0$. The definition of associated vector is given in footnote 19.

[40] The last equality in (94) is due to the passage from the skew-symmetric tensors T^W, ∇v^W to the associated vectors t, $\operatorname{curl} v/2$.

where

$$\theta = \operatorname{curl} v - \omega \tag{96}$$

is the deviation between the macroscopic and microscopic rotations, obtained from (80) replacing the skew-symmetric tensors by their associated vectors. Thus, the internal actions of the micropolar continuum are T^S, t, M, and the corresponding generalized strain rates are ∇v^S, θ, $\nabla\omega$. The constitutive equations (82) are replaced by

$$\begin{gathered} T^S - T_0^S = \varphi(\nabla v^S, \theta, \nabla\omega), \quad t - t_0 = \psi(\nabla v^S, \theta, \nabla\omega), \\ M - M_0 = \chi(\nabla v^S, \theta, \nabla\omega). \end{gathered} \tag{97}$$

The equilibrium equations

$$\operatorname{div} T + b = 0, \quad \operatorname{div} M + c = -t, \tag{98}$$

follow from (91) and the indifference conditions (93), and appropriately rearranged boundary conditions complete the formulation of the incremental equilibrium problem.

6.3. Micropolar continua with constrained rotations

According to its definition (96), θ is the deviation between the macroscopic rotation $\operatorname{curl} v$ and the microscopic rotation ω. The *constrained theory of micropolar continua*[41] is obtained assuming that the two rotations are equal

$$\omega = \operatorname{curl} v, \quad \theta = 0. \tag{99}$$

Under this supplementary constraint, the power (95) reduces to

$$\mathscr{P}_{red}(\Pi, v) = \int_\Pi (T^S \cdot \nabla v^S + M \cdot \nabla \operatorname{curl} v)\, dV. \tag{100}$$

The elimination of ω from the independent kinematical variables characterizes this continuum as a continuum with latent microstructure, and the fact that the curl is a first-order differential operator makes it a particular second-order continuum.

Comparing with (95), we see that the vector t now disappears from the list of the internal actions. This vector can also be eliminated in the field equations (98). Indeed, thanks to the identity

$$\operatorname{div} T^W = -\operatorname{curl} t, \tag{101}$$

[41] Toupin [46].

equation $(98)_1$ can be given the form

$$\operatorname{div} T^S - \operatorname{curl} t + b = 0. \tag{102}$$

By substitution into the curl of $(98)_2$, a single higher-order field equation

$$\operatorname{div} T^S + \operatorname{curl}(\operatorname{div} M + c) + b = 0, \tag{103}$$

involving only the internal actions T^S and M, is obtained.[42] Compared with (97), the constitutive equations

$$T^S - T^S_0 = \varphi(\nabla v^S, \operatorname{curl} v), \quad M - M_0 = \chi(\nabla v^S, \operatorname{curl} v), \tag{104}$$

show a significant reduction of the number of the required material constants.

6.4. Plate and beam theories

Appropriate internal constraints may lead to *dimensional reduction*, that is, to models for two- and one-dimensional Cosserat continua. In this way, the classical theories of plates and beams can be obtained.

Assume that the image of the body on $\mathscr{E}$ has a cylindrical shape, and let $\{e, e^\alpha\}$, with $\alpha \in \{1,2\}$, be an orthonormal triple of vectors, with e parallel to the axis of the cylinder. The constraints

$$v(x) = v_3(x_1, x_2)e, \quad \omega(x) = \omega_\alpha(x_1, x_2)e^\alpha, \quad \alpha \in \{1,2\}, \tag{105}$$

impose virtual velocities v parallel to e and virtual rotations ω about an axis orthogonal to e, both of intensity independent of the axial coordinate x_3. By consequence, the external power (89) reduces to

$$\mathscr{P}_{ext}(\Gamma, v_3, \omega_\alpha) = \int_\Gamma (b_3 v_3 + c_\alpha \omega_\alpha)\, dA + \int_{\partial\Gamma} (s_3 v_3 + m_\alpha \omega_\alpha)\, d\ell, \tag{106}$$

with the volume element Π replaced by its cross section Γ, and with $d\ell$ the length measure on the boundary line $\partial\Gamma$. Moreover, the vectors b and s reduce to the scalars b_3 and s_3, and the vectors c and m reduce to the 2-vectors c_α and m_α. Accordingly, the pseudobalance equations (90) reduce to

$$-\int_\Gamma b^c_3\, dV + \int_{\partial\Gamma} s_3\, dA = 0, \quad -\int_\Gamma c^c_\alpha\, dV + \int_{\partial\Gamma} m_\alpha\, dA = 0, \tag{107}$$

[42] Here t plays the role of a *reaction*. That is, it is not determined by a constitutive equation, but directly by the *active* internal action M through equation $(98)_2$.

the counterparts of the relations (91) are

(108) $$s_3 = T_{3\alpha}n_\alpha, \quad T_{3\alpha,\alpha} - b_3^c = 0, \quad m_\alpha = M_{\alpha\beta}n_\beta, \quad M_{\alpha\beta,\beta} - c_\alpha^c = 0,$$

and the internal power (92) takes the form

(109) $$\mathscr{P}_{int}(\Gamma, v_3, \omega_\alpha) = \int_\Gamma ((b_3 + b_3^c)v_3 + T_{3\alpha}v_{3,\alpha} + (c_\alpha + c_\alpha^c)\omega_\alpha + M_{\alpha\beta}\omega_{\alpha,\beta})\, dA.$$

The translational indifference condition (75) reduces to

(110) $$b_3 + b_3^c = 0.$$

For the rotational indifference condition (76), recalling that the vector associated with W is $\omega/2$, for the pair (Wx, W) we have

(111) $$v_3 = W_{3i}x_i, \quad \omega_\alpha = \frac{1}{2}e_{kj\alpha}W_{jk} = \frac{1}{2}(e_{\beta 3\alpha}W_{3\beta} + e_{3\beta\alpha}W_{\beta 3}) = e_{\alpha\beta}W_{3\beta}.$$

Then imposing condition (76) to the power (109) we get

(112) $$c_\alpha + c_\alpha^c = -e_{\alpha\beta}T_{3\beta},$$

and the reduced form

(113) $$\mathscr{P}_{red}(\Gamma, v_3, \omega_\alpha) = \int_\Gamma (T_{3\alpha}\theta_\alpha + M_{\alpha\beta}\omega_{\alpha,\beta})\, dA,$$

follows, where

(114) $$\theta_\alpha = v_{3,\alpha} + e_{\alpha\beta}\omega_\beta$$

is the deviation between the rotated directors d^α and the corresponding directions on the deformed surface Γ. The internal actions are $T_{3\alpha}$ and $M_{\alpha\beta}$, and θ_α and $\omega_{\alpha,\beta}$ are the corresponding generalized strain rates. The field equations (98), now reduced to

(115) $$T_{3\alpha,\alpha} + b_3 = 0, \quad M_{\alpha\beta,\beta} + c_\alpha = -e_{\alpha\beta}T_{3\beta},$$

are the equilibrium equations of the *Reissner–Mindlin plate theory*.

The *Kirchhoff–Love plate theory* is obtained by imposing the additional constraint $\theta_\alpha = 0$, by which the directors follow the macroscopic deformation. In this way, the first term in the reduced power (113) cancels, and $\omega_{\alpha,\beta}$ is replaced by $e_{\alpha\gamma}v_{3,\gamma\beta}$. With the *modified moment tensor*[43]

(116) $$M^*_{\alpha\beta} = e_{\alpha\gamma}M_{\gamma\beta},$$

[43] This is the moment tensor used in the Kirchhoff–Love plate theory, see e.g. [45], Sect. 10.

the reduced power takes the form

$$\mathscr{P}_{red}(\Gamma, v_3) = -\int_{\Gamma} M^*_{\alpha\beta} v_{3,\alpha\beta}\, dA. \tag{117}$$

The tensor $M^*_{\alpha\beta}$ is the only active internal action, and the associated generalized strain rate is the *curvature tensor* $-v_{3,\alpha\beta}$. The vector $T_{3,\alpha}$, which is now an internal reaction, can be eliminated from the balance equations (115), which are replaced by the unique higher-order equation

$$M^*_{\alpha\beta,\alpha\beta} + c^*_{\alpha,\alpha} + b_3 = 0, \tag{118}$$

where $c^*_\alpha = e_{\alpha\beta} c_\beta$ is the modified external couple. This is the equilibrium equation of the Kirchhoff–Love plate theory.

In a quite similar way, it can be shown that the constraints

$$v(x) = v(x_3), \quad \omega(x) = \omega(x_3), \tag{119}$$

provide the *Timoshenko beam theory*, and that the additional constraint

$$\omega_\alpha = -e_{\alpha\beta} v'_\beta \tag{120}$$

leads to the *Euler–Bernoulli beam theory*. For more details on the beam theory and for the boundary conditions in both plate and beam theories, the reader is addressed to the paper [14].

6.5. Some remarks on plate and beam theories

This Subsection has been added to answer a criticism addressed by the anonymous reviewer, who writes: "A reader of Sect. 6.4 is driven to believe that the method of dimensional reduction by the use of internal constraints is due to the Author, who in fact quotes only ref. [14], a paper of his which appeared in the year 2014. Now, even in that paper there is no mention of the fact that, much before 2014, that method has been introduced and exploited in a number of papers ... This dearth should be amended." Since the subject may be of interest for some readers, it seems appropriate to attach a public answer to the present paper. I am doing this with a double purpose: (i) to reject the idea that I am claiming for any priority, and (ii) to express my view on the position of the contents of the paper [14] in the huge literature on the subject.

The attempts for a systematical deduction of the equations for one- and two-dimensional bodies from the three-dimensional theory have a long story. In general, these attempts are based on the expansion of the displacement field in the powers of the distance from the mid-plane, for plates, and from the cylinder's axis, for beams. According to Novozhilov, the first attempts trace back to Galerkin [17]. In his book [40], he extended Galerkin's approach to finite deformations, developing a first-order theory for plates and a first- and second-order theory for beams. A first-order theory for beams was formulated later by E.

Volterra [49], to whom is due the name of *method of internal constraints* to the techniques based on Galerkin's *projection method*. First-order theories were proposed afterwards by several authors, among which Green [20] for beams and Eringen [16] for micropolar plates. The first formal expansion of the displacement field involving higher-order powers seems to be due to Green, Laws and Naghdi [21].

All mentioned approaches suffer the same drawback: the results obtained by a first-order expansion are unsatisfactory if compared with those provided by the so-called *technical theories*. For example, for beams, the in-plane deformation in simple tension (Poisson effect) and the out-of plane deformation of a non-circular cross section in torsion are not captured by constraints such as (119), but are well described by Saint-Venant's theory. This is mainly because this theory makes use of the stress constraint $T_{\alpha\beta} = 0$. A first attempt to include this constraint in more formal procedures are the *modified theories* of Antman and Warner [4]. The mixing of kinematical constraints with constitutive restrictions on T posed some problems, and in fact the modified theories seem to have been abandoned by the proposers, since they were not mentioned in the subsequent contributions [2, 3].

Stress constraints were reconsidered later in a series of papers by Podio Guidugli and co-authors, [41], [31], [28]. A problem which emerged from their analysis is that the constraints $T_{33} = 0$ for plates and $T_{\alpha\beta} = 0$ for beams are incompatible with isotropy, and can be used, at most, for transversely isotropic materials with the symmetry axis coincident with the cylinder's axis. Another problem, not examined by them, is that in the presence of such constraints the elastic tensor ceases to be positive definite. This may cause problems of existence and uniqueness of the solution, as it occurs in Saint-Venant's problem.

With the deduction of the lower-dimensional incremental equilibrium problems made in [14] and summarized in Subsection 6.4 above, I do not aim at any priority and I do not pretend to say anything definitive on the subject. In particular, the above mentioned disadvantage with respect to the *technical theories* has not been eliminated. The only merit I claim is the simplicity of the deduction of the classical theories of plates and beams, which is made with the sole use of the kinematical restrictions (105) and (119), without any constraint on the stress and without any assumption on the nature of the material.

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Received 31 December 2016,
and in revised form 14 June 2017.

Gianpietro Del Piero
Dipartimento di Ingegneria
Università di Ferrara
via Saragat 1
44100 Ferrara, Italy
and
International Research Center M&MoCS
Università dell'Aquila
Palazzo Caetani
04012 Cisterna di Latina, Italy
dlpgpt@unife.it