Rend. Lincei Mat. Appl. 29 (2018), 63–83
DOI 10.4171/RLM/793

Continuum Mechanics — *A continuum model of interlocking structural systems*, by Maurizio Brocato, communicated on November 10, 2017.

This paper is dedicated to the memory of Professor Giuseppe Grioli.

Abstract. — Masonry systems made of interlocking square-cut stones have long been studied by mathematicians and architects. It so happens that, under appropriate boundary conditions, ashlars interlock, and a stable structure results. Recently, the idea that new masonry-like materials can be designed on the basis of this archetypal principle has been put forward and applications have been proposed under the name of topological interlocking materials.

In this paper, a mathematical model is proposed, that describes these materials as a special class of continua with microstructure. The system is viewed as a continuous body, the material elements of which are ashlar blocks endowed with an interaction structure based, to within certain approximations, on their interlocking geometry. The case of rigid ashlars with purely plastic interactions is developed in detail.

Key words: Continua with microstructure, configurational forces, interlocking structures, stereotomy

Mathematics Subject Classification: 74L99

1. Introduction

An interlocking structural system is a set of distinct elements that lock each other with unilateral contacts. Thanks to both the shape and the arrangement of the elements, this locking is more effective than in a stack or in other traditional masonry bonding that depends on gravity and thrust.

Systems enjoying the interlocking property have been named *nexorades* (from Latin *nexus* [2, 1]). They can be made of parts that can be beam-like, plate-like, or three-dimensional solids. Here we focus on systems the parts of which are all of the last type; on adopting the usual nomenclature for stone structures, we call such parts *ashlars*. Nexorades made of beams have long been known by theorists and practicians, often under the name of reciprocal frames. Sketches of a timber nexorade floor are due to Villard de Honnecourt, Leonardo da Vinci, and Sebastiano Serlio. Nexorades have received even more attention recently, as they provide a field for the application of numerical tools to the design of structures. The particular case of interlocking ashlars, connected as is to the 17th and 18th century developments of projective geometry for stone cutting (or *stereotomy*), has lately given birth to the idea of topological interlocking materials [10, 11, 12, 14, 15, 18, 19].

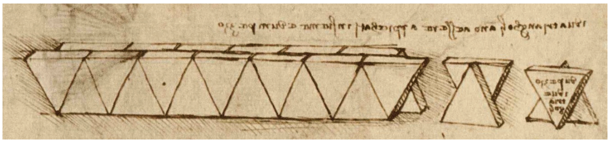

Figure 1. The sketch of a cross-joined lintel from Leonardo's Codex Atlanticus f.0091 v [9]: "I due triangoli ano a essere appiccati insieme e d'un pezzo".

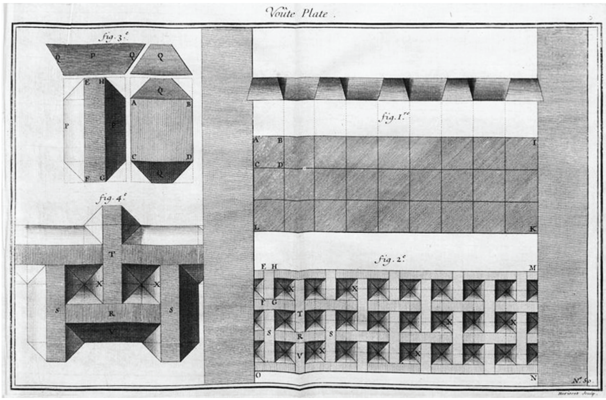

Figure 2. Joseph Abeille's 1699 flat vault [16].

Examples of bonding of fragmented lintels go back to the Middle Ages: they are found in the castle of Esztergom (Hungary, 12th century), in Saint-Michel cathedral at Alba Iulia (Romania, 13th century), in the church of Brateiu (Romania), and in the cathedrals of Arezzo and Prato (Italy, 14th century). Their common interlocking system was represented in a sketch by Leonardo da Vinci (who did not mention these examples, see Figure 1).

Later, the first plate-like structure conceived as an interlocking system was Joseph Abeille's flat vault. It was made of identical ashlar blocks and patented as an invention by the French Academy of Sciences in 1699 [16] (see Figure 2). Only one vault of this type remains today, in Lugo's cathedral (Spain, 18th century, [21]). However, several structures based on Abeille's bond have recently been studied and built by the team of the GSA laboratory [5, 7, 22].

Generalising Abeille's result, an interlocking plate can be built with ashlar blocks shaped as convex polyhedra. A particularly elegant case is that of systems made of regular polyhedra, the cross sections of which tile a plane: tetrahedra pave the plane with squares; cubes, octahedra, and dodecahedra, pave it with hexagones [12]. An even larger generality can be achieved with the use of numerical tools: e.g., in [23] interlocking bondings based on regular, semi-regular and non-regular surface tessellations are proposed. Furthermore, any plate-forming assembly can be adapted so as to compose a shell, by way of fit transformations preserving its interlocking bonding. For instance, the cases of a spherical dome

Figure 3. Pictures of a saddle vault with Abeille's bonding built in Troyes (France), 2013 [20].

and of a saddle vault have been considered, and some sample structures of these types have been built (see Figure 3, [6, 20]).

When the size of ashlars is much smaller than the characteristic length of the structure they compose, one can conceive a continuous model of the interlocking structural system, where only the fundamental traits of the local interactions are preserved.

As is the rule when defining a continuum model for a discrete system, several options exist. Our present proposal starts from the idea that an interlocking structural system can be modelled as a sort of continuum with microstructure. Thus we apply general methods presented in [8]. Some passages dealing with micro-inertia issues have already be discussed in [4].

In the next section, we give an idealised description of the masonry system we study and lay down our geometric and kinematic modelling assumptions. The idea that ashlars are boxed by their neighbours in a way that depends on their shape and on the shape of the boxing system suggests us to include in the model two distinct, though interacting, kinematics: that of the ashlars and that of the boxes, the latter inducing a confinement for the motions of the former. In the third section, following [17], we call upon the principle of virtual powers to write balance laws that are consistent with the previously introduced kinematic description of the system. As a consequence, we obtain point-wise balance laws and boundary conditions for the translational and rotational momenta, both of the ashlars and of the boxes. The fourth and last section deals with constitutive assumptions and is governed by the idea that the internal state of individual ashlars has no consequences on the whole of the system: only the interactions between adjacent ashlars and between adjacent boxes need be modelled, and such interactions are unilateral contacts among rigid bodies.

2. Geometry and kinematics

2.1. Physical description

We consider a body made of distinct ashlars that interlock each others. To be more specific, when needed, we will take ashlars shaped as cubes and as-

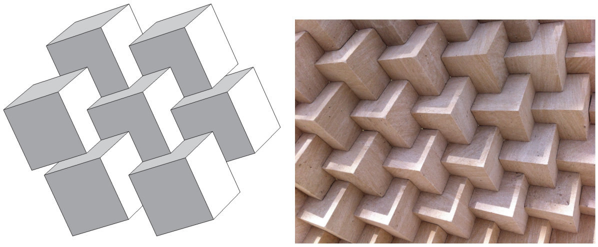

Figure 4. Sketch and picture of the interlocking system of cubes used as a reference example.

sembled as in Figure 4, in a single layer, thus making an interlocking plate-like structure.

It is assumed that ashlars interact through their surfaces, either directly, with contact actions, or indirectly – if a mortar or any other soft material fills the interfaces, with linkage actions. The case of unilateral contacts will receive more attention here, as it contains some interesting traits.

In experiments, the observed overall deformation of ashlars is usually much smaller than the local effects of contact or linkage actions (such as indentation, slip, dilatancy, separation, etc.), so that a model with rigid ashlars and soft unilateral interfaces can be reasonably proposed.

There is an ideal, reference, configuration, where all contacts happen between faces of neighbouring ashlars, with gaps that are uniform on each contact and equal throughout the system. In some cases (as, e.g., the quoted Abeille's flat vault or the system of Figure 4) the system is periodic in this reference configuration.

In any other configuration, there might be not only face-on-face contacts, but also edge-on-face or vertex-on-face ones. At first, it seems reasonable to exclude contact cases such as edge-on-edge, vertex-on-edge, and vertex-on-vertex, as seldom visited.

Let us consider the following simplified model:

- ashlars are rigid convex polyhedra, endowed with a uniformly distributed mass, they exchange forces only with their neighbours, seen as a "box" enclosing them (see Figure 5);

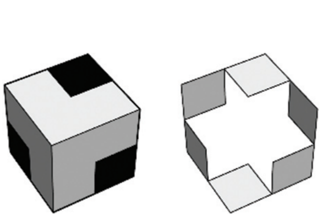

Figure 5. Sketch of a cube, with the contact surfaces occurring in the interlocking system of Figure 4 painted in black and, on the right, the "box" enclosing it.

- boxes are deformable polyhedric surfaces (with deformations allowed in a special class, we need to define), they have no mass and compose a continuous network;
- in the reference configuration the set of boxes is made of given parts of the surfaces of the ashlars;
- ashlars and boxes have a permanent identity throughout the process.

2.2. *Modelling assumptions*

Let us look at ashlars as continuously distributed material points endowed with a given mass and shape and the ensuing inertia tensor. The position of their centre of mass is a variable $x \in \mathscr{E}$ ($\mathscr{E}$ the Euclidean space, $\mathscr{V} \cong \mathbb{R}^3$, in the following, its translation space). In the reference placement we call $x_* \in \mathscr{E}$ the coordinates of these material points. The transformation

$$x_* \to x$$

is a first unknown field; let us call F its gradient:

$$F : x_* \to \operatorname{Grad} x \in \mathrm{GL}(\mathscr{V}).$$

The rotation of the ashlars is a second unknown field:

$$Q : x_* \to Q \in \mathrm{SO}(3);$$

which gives the present axes of the ashlars, $b^{(i)}$, starting from their reference values, $b_*^{(i)}$ (the application to the case of interlocking cubes of the next and the following equations is given in §2.3):

$$b^{(i)} = Qb_*^{(i)}; \quad i = 1, \dots, 3.$$

A box is defined by the taken of the same reference point $x_* \in \mathscr{E}$ than the ashlar it encloses and by a reference shape, S_*, which we suppose be the same for all boxes. To extend the model to cases with more than one ashlar type, one needs to remove this assumption.

To define the shape S_*, we need to identify all faces of the box and accept some approximation in this process. There are many possible options, among which we chose to take two vectors per face, one giving the position of the centre of the face with respect to the reference point of the box and the other giving the direction normal to that face and a characteristic length of it.

Let us call $s_*^{(i)}$ the vectors giving the position of the centre of the face in the reference configuration and $m_*^{(i)}$ the surface vectors of the contacts in the same configuration, with $i = 1, \dots, n$ and n the number of contact faces of the ashlar. $s^{(i)}$ and $m^{(i)}$ are the corresponding images in the reference configuration.

To begin disregarding information, we live aside the possible extension/contraction of the box' faces and assume that only the directions of the surface vectors count. We thus disregard information carried by the length of the surface vectors and take all $m_*^{(i)}$ and $m^{(i)}$ as unit vectors.

Furthermore, in some cases – such as that of cubes, a symmetry can be observed in many deformation processes, that allows one halving the number of independent position vectors $s^{(i)}$, taking the position of opposite faces simply as opposite vectors, in the reference configuration and in any other setting. An even greater simplification is made here, assuming:

$$s^{(i)} = Gs_*^{(i)} + r \quad \forall i = 1, \ldots, n;\ G \in \mathrm{GL}(\mathscr{V})$$

with G, a double vector and r a vector in the present configuration. G, representing perhaps the average of n such entities (one per position vector $s^{(i)}$), defines an ellipsoid that corresponds, within the accepted approximation, to the domain were an ashlar is confined by its box. r is the relative placement of the centre of the box with respect to the centre of the ashlar it contains; it is null in the reference configuration and defined as

$$r(x_*) = y(x_*) - x(x_*),$$

where the transformation

$$x_* \rightarrow y$$

gives the present position of the centre of the box and is an unknown field, differing in general from x (present position of the centre of the ashlar), see Figure 6.

We consider the following kinematic constraint between y and G:

$$G = \mathrm{Grad}\, y,$$

which presumes y having the usual differential properties of x.

The information content of the evolution of the n unit vectors giving the orientation of the faces of a box can also be simplified assuming some average for them. An embedding of directions into $\mathbb{R}^5$ can be taken of the kind used for nem-

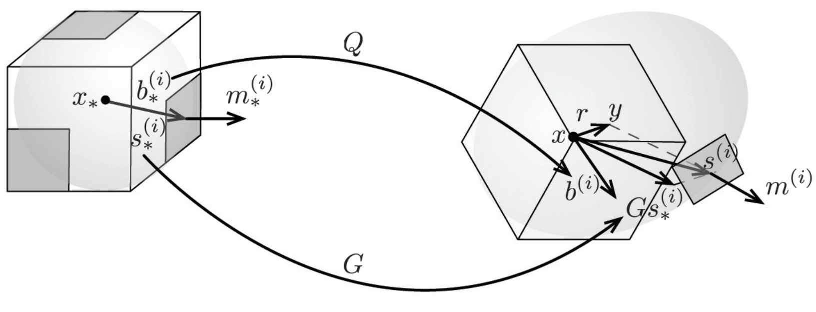

Figure 6. Sketch of the transformation of ashlars and boxes, in the case of cubes.

atics (see [13, 3]), and the following average considered:

$$M^{(i)} = \|m_*^{(i)}\|^{-2} m^{(i)} \otimes m^{(i)} - \frac{1}{3} I; \quad M = \frac{1}{n} \sum_{i=1}^{n} M^{(i)}$$

(I the unit of second order tensors). Thus a symmetric traceless tensor M can be taken to represent any arrangement of the $m^{(i)}$.

Notice that, in the reference configuration:

$$M_* = 0.$$

The principal axes of M correspond to the appearing directions of contact on the box. If an eigenvalue prevails, then the box opposes motions of the ashlar in the direction of the corresponding eigenvector more "effectively" than in the others (we need to precise the extent of this effect). If, as in the reference configuration, no direction prevails, the ashlar is equally confined from everywhere by the box.

We can assume Q be differentiable and consider the following approximation for the direction of any face of the box (i.e. that face rotates following the linear approximation of the field Q in the neighbourhood of x_*):

$$m^{(i)} \approx Q m_*^{(i)} + (\mathrm{Grad}\, Q)(m_*^{(i)} \otimes s_*^{(i)}).$$

Let us then compute $M^{(i)}$ accordingly and take the result as a kinematic constraint for M (remember that $M_* = 0$):

$$M = 2\, \mathrm{sym}(Q \mathbf{a}_* (\mathrm{Grad}\, Q)^T) + \mathrm{Grad}\, Q \mathbf{A}_* (\mathrm{Grad}\, Q)^T \approx 2\, \mathrm{sym}(Q \mathbf{a}_* (\mathrm{Grad}\, Q)^T),$$

where the second order term can be disregarded within the given approximation and (when not otherwise imposed by parentheses, priority is given to the right-hand side of operators):

$$\mathbf{a}_* = \frac{1}{n} \sum_{i=1}^{n} \|m_*^{(i)}\|^{-2} m_*^{(i)} \otimes s_*^{(i)} \otimes m_*^{(i)}.$$

It is then possible to compute the time derivative of M in terms of Q:

$$\dot{M} = 2\, \mathrm{sym}(W Q \mathbf{a}_* (\mathrm{Grad}\, Q)^T + Q \mathbf{a}_* (W\, \mathrm{Grad}\, Q)^T + Q \mathbf{a}_* (((\mathrm{grad}\, W) F)^t Q)^t),$$

where

$$W = \dot{Q} Q^T \in \mathrm{Skw},$$

the exponent t denotes the transposition of two indices on the side where it appears, and grad denotes the gradient with respect to x (i.e. $\mathrm{grad}\, W = (\mathrm{Grad}\, W) F^{-1}$).

To shorten notations, let us introduce the second order operator B and the third order operator $\mathbf{b}$, such that:

$$\dot{M} = BW + \mathbf{b}\,\mathrm{grad}\,W; \quad B \in \mathrm{GL}(\mathrm{Skw}, \mathrm{Sym}); \; \mathbf{b} \in \mathrm{GL}(\mathrm{GL}(\mathscr{V}, \mathrm{Skw}), \mathrm{Sym}).$$

2.3. Application to plates made of interlocking cubes

The bonding depicted in Figure 4 is composed of ashlars of one single type, shaped as cubes, and can be repeated at will to make structures like plates or flat vaults. Some of the quoted experiments on topological interlocking materials were made on systems of this kind. For this reason, we will use this bonding as an example to make some of the previous equations more specific.

In the case of cubes of side σ, taking

$$b_*^{(i)} = \frac{\sigma}{2} e^{(i)}; \quad i = 1, 2, 3;$$

where

$$e^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \quad e^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \quad e^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix};$$

we have:

$$Q = \left(\frac{2}{\sigma}\right)^2 b^{(i)} \otimes b_*^{(i)};$$

To describe the boxes, we can take 6 pairs made of a position vector $s_*^{(j)}$ and a surface vector $m_*^{(j)}$. The positions, relative to the centre of the cube, are:

$$s_*^{(1)} = \frac{\sigma\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} \sqrt{2/3} \\ 1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}; \quad s_*^{(2)} = \frac{\sigma\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} 1/\sqrt{6} \\ \sqrt{2/3} \\ 1/\sqrt{6} \end{bmatrix}; \quad s_*^{(3)} = \frac{\sigma\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ \sqrt{2/3} \end{bmatrix};$$

and $s_*^{(i+3)} = -s_*^{(i)}$ for $i = 1, 2, 3$. The surface vectors are

$$m_*^{(i)} = \frac{\sigma^2}{4} e^{(i)};$$

and $m_*^{(i+3)} = -m_*^{(i)}$ for $i = 1, 2, 3$. Then

$$M^{(i)} = \left(\frac{2}{\sigma}\right)^4 m^{(i)} \otimes m^{(i)} - \frac{1}{3} I; \quad M = \frac{1}{6} \sum_{i=1}^{6} M^{(i)}$$

and:

$$\mathbf{a}_* = \frac{1}{6}\left(\frac{2}{\sigma}\right)^4 \sum_{i=1}^{6} m_*^{(i)} \otimes s_*^{(i)} \otimes m_*^{(i)} = \frac{1}{3}\left(\frac{2}{\sigma}\right)^4 \sum_{i=1}^{3} m_*^{(i)} \otimes s_*^{(i)} \otimes m_*^{(i)}$$
$$= \frac{\sigma^3}{24}\begin{bmatrix} e^{(1)} & \frac{1}{2}e^{(1)} & -\frac{1}{2}e^{(1)} \\ \frac{1}{2}e^{(2)} & e^{(2)} & \frac{1}{2}e^{(2)} \\ -\frac{1}{2}e^{(3)} & \frac{1}{2}e^{(3)} & e^{(3)} \end{bmatrix}.$$

2.4. Survey of the kinematics

With the kinematic fields introduced above, we can adopt a Lagrangian or an Eulerian description of the motion of the interlocking structural system, where two continua coexist.

Lagrangian description

- The continuum of ashlars, the motion of which is given by:
  - placement of the ashlar centre $x(x_*, \tau)$ (all fields depend on x_* and τ, which will be omitted in the following lines), with gradient F;
  - rotation of the ashlars' axes, Q;
- The continuum of boxes, given by:
  - placement of their centre, relative to the ashlar, r, or absolute, y;
  - ellipsoid of the box $G = \mathrm{Grad}\, y$;
  - hem of the box $M = 2\,\mathrm{sym}(Q\mathbf{a}_*(\mathrm{Grad}\, Q)^T)$;

Eulerian description

- The continuum of ashlars, with:
  - velocity of the centre of the ashlars $u(x, \tau)$ (all fields depend on x and τ, omitted in the following lines), with gradient $\mathrm{grad}\, u = \dot{F}F^{-1}$;
  - absolute spin of the ashlars about their centre W;
- The continuum of boxes, with:
  - velocity of the centre of the boxes v;
  - ellipsoid's rate of change $\mathrm{grad}\, v = \dot{G}F^{-1}$;
  - hem's velocity $\dot{M} = BW + \mathbf{b}\,\mathrm{grad}\, W$.

2.5. Consequences

Volume element. A reference volume element is $d(vol_*)$, its mass is that of the ashlars; i.e., if $\rho_*^{(a)}$ is the mass of the material the ashlars are made of, taken per unit volume of the structure they compose, the mass element is $\rho_*^{(a)}\, d(vol_*)$, in the reference and in any other placement throughout all processes.

Hence, if $\imath$ is the Jacobian determinant of the transformation x, we have an apparent density of the continuum $\rho^{(c)}$ that changes following the rule:

$$\rho_*^{(a)}\, d(vol_*) = \rho^{(c)}\, d(vol) \Leftrightarrow \imath\rho^{(c)} = \rho_*^{(a)}.$$

We can introduce a volume fraction of ashlars, v, with reference value $v_* = 1$, that changes as

$$\imath v = v_* = 1 \quad \text{i.e. } v = \imath^{-1}$$

Notice that we are restricted to processes where $\imath \geq 1$, or $v \leq 1$, with strict inequality if any deformation appears.

Relative and entrainement rotation of ashlars. There is both a relative rotation and a relative spin of any ashlar with respect to its box and both a relative rotation and a relative spin of any ashlar with respect to the continuum of ashlars.

The relative rotation of ashlars with respect to boxes can be defined starting from the entrainement rotation of the axes of an ashlar due to G (which is not necessarily the rotation of the principal directions of G). In the following equations, "aeb" will shorten "ashlar entrained by box" and "arb" "ashlar relative to box":

$$Q_{aeb} = \frac{Gb_*^{(i)}}{\|Gb_*^{(i)}\|} \otimes \frac{b_*^{(i)}}{\|b_*^{(i)}\|}$$

and taking

$$Q = Q_{arb}Q_{aeb} \Leftrightarrow Q_{arb} = QQ_{aeb}^T = \frac{b^{(i)}}{\|b_*^{(i)}\|} \otimes \frac{Gb_*^{(i)}}{|Gb_*^{(i)}|}.$$

The relative and entrainment rotation of ashlars with respect to the continuum of ashlars can be defined, starting from F and $b_*^{(i)}$, on the same principle than in the previous case. "aec" denotes "ashlar entrained by the continuum" and "arc" "ashlar relative to continuum":

$$Q_{aec} = \frac{Fb_*^{(i)}}{|Fb_*^{(i)}|} \otimes \frac{b_*^{(i)}}{\|b_*^{(i)}\|}; \quad Q_{arc} = QQ_{aec}^T = \frac{b^{(i)}}{\|b_*^{(i)}\|} \otimes \frac{Fb_*^{(i)}}{|Fb_*^{(i)}|}.$$

Relative and entrainement spin of ashlars. The spin vector of an ashlar is ($\mathbf{e}$ is Ricci's permutation tensor):

$$\begin{aligned} w &= -\frac{1}{2}\mathbf{e}\dot{Q}Q^T = -\frac{1}{2}\mathbf{e}(\dot{Q}_{arb}Q_{arb}^T + Q_{arb}\dot{Q}_{aeb}Q_{aeb}^T Q_{arb}^T) \\ &= w_{arb} + Q_{arb}w_{aeb} = w_{arb} + QQ_{aeb}^T w_{aeb}; \end{aligned}$$

where

$$w_{arb} = -\frac{1}{2}\mathbf{e}\dot{Q}_{arb}Q_{arb}^T; \quad w_{aeb} = -\frac{1}{2}\mathbf{e}\dot{Q}_{aeb}Q_{aeb}^T.$$

Notice that, in the last line of equations, the continuum of ashlars can be taken as a reference for the motion of an ashlar (instead than taking its box as a reference); formally, in this case change "b" into "c" as the last letter of the index.

Entrainment rotation and spin of ashlars. Consider the velocity gradient of boxes and the spin associated with it

$$-\frac{1}{2}\operatorname{rot}\dot{y} = -\frac{1}{2}\mathbf{e}\operatorname{grad}\dot{y},$$

we can write the entrainment spin of the three orthogonal axes $b^{(i)}$ as (see [4])

$$w_{aeb} = -\frac{1}{2}(\mathbf{e} + \mathbf{h}_G^T)\operatorname{grad}\dot{y},$$

where

$$\mathbf{h}_G = -2\operatorname{sym}\Big(\frac{Gb_*^{(i)}}{|Gb_*^{(i)}|} \otimes \mathbf{e}\frac{Gb_*^{(i)}}{|Gb_*^{(i)}|}\Big) \quad \mathbf{h}_G \in \mathrm{GL}(\mathrm{Sym}, \mathbb{R}^3),$$

is the operator giving the spin of three mutually orthogonal vectors $b_*^{(i)}$ (given in the reference configuration) for any deformation rate.

Similarly, we can proceed when the continuum is taken as a reference for the motion of ashlars, replacing G with F and y with x.

Relative and entrainement rotation of boxes. There is a relative rotation of any box with respect to the continuum of ashlars, Q_{brc}, deriving from the fact that the spin of the continuum is not necessarily the same than that of the box.

Boxes in the reference placement have, by assumption, no preferred directions, their anisotropy being exclusively due to deformations and thus generated by the process. Hence, the quoted relative rotation must be evaluated comparing, in the present configuration, the principal axes of strain of the box with those of the continuum.

Let us take a polar decomposition of F and of G:

$$F = R^{(c)}U^{(c)} = V^{(c)}R^{(c)}; \quad G = R^{(b)}U^{(b)} = V^{(b)}R^{(b)};$$

and define:

$$Q_{brc} = R^{(c)T}R^{(b)}.$$

Inertia. The Euler and inertia tensor of ashlars are, respectively, $E_*^{(a)}$ and:

$$J_*^{(a)} = (\operatorname{tr} E_*^{(a)})I - E_*^{(a)}.$$

In the case of cubes, the inertia tensor is

$$J_*^{(a)} = \frac{\sigma^5}{6}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

and $J^{(a)} = J_*^{(a)}$ does not depend on the particular placement.

Kinetic energy. The kinetic energy of ashlars has a translational and a non translational term (recall that boxes have no mass). The first is

$$\kappa_u = \frac{1}{2}\sigma^3 u^2 \quad \Rightarrow \quad \rho^{(c)}\dot{\kappa}_u = \rho^{(c)}\sigma^3 \dot{u}\cdot u$$

with $u = \dot{x}$ the velocity field of ashlars in the present configuration; the second is due to the rotations of the ashlars, with mass density:

$$\kappa_\omega = \frac{1}{2}J\omega^2$$

where ω is the spin velocity of $\dot{Q}Q^T$ (or the norm of w) and J is the moment of inertia about the centre of the ashlar for a given spin axis n ($J = n \cdot J^{(a)} n$).

In the case of cubes, the latter is independent on the spin axis (n) and on the particular placement:

$$J = \frac{1}{2}\sigma^5;$$

hence the kinetic energy is in this case:

$$\rho^c \dot{\kappa}_\omega = \frac{1}{2}\rho^c\sigma^5 w \cdot I\dot{w} = \frac{1}{2}\rho^c\sigma^5\dot{\omega}\omega.$$

3. Balance laws

We express the balance laws of dynamics through the principle of virtual powers.

Principle of virtual powers. In any configuration and for any part of the body:

$$\mathscr{P}_{ext} + \mathscr{P}_{int} = \mathscr{P}_{acc}$$

for all virtual velocity field, and

$$\mathscr{P}_{int} = 0$$

for all virtual rigid body velocity.

To make the statement operative in deriving the local balance laws and boundary conditions fit to our problem, we first need to define the set of virtual velocities and the expressions of the virtual powers of the external actions, of the internal actions and of the accelerations (respectively $\mathscr{P}_{ext}$, $\mathscr{P}_{int}$ and $\mathscr{P}_{acc}$).

3.1. Virtual velocities

We consider, in the present configuration:

- $\hat{u}$ the virtual velocity of ashlars, with gradient grad $\hat{u}$;
- $\hat{w}$ the virtual spin of ashlars;
- $\hat{v}$ the virtual velocity of boxes, with gradient grad $\hat{v}$;
- the virtual rate of change of the boxes' ellipsoid, constrained to grad $\hat{v}$;
- the virtual rate of change of the boxes' hem, constrained to $\hat{w}$ and grad $\hat{w}$;

In a rigid motion these velocities take a particular form, as all of them are perfectly entrained by the macro-motion:

- $\hat{u}_{rigid} = \hat{u}_o + \hat{W}_o(x - o)$ (velocity of the continuum of ashlars);
- $\hat{w}_{rigid} = Q_{arc}\hat{w}_{bec\,rigid} = -\frac{1}{2}Q_{arc}(\mathbf{e}\hat{W}_o)$ or $\hat{W}_{rigid} = Q_{arc}\hat{W}_oQ^T_{arc}$ (spin of ashlars);
- $\hat{v}_{rigid} = \hat{u}_o + Q_{brc}\hat{W}_oQ^T_{brc}(y - o)$ (velocity of the continuum of boxes);
- $\operatorname{grad}\hat{u}_{rigid} = \hat{W}_o$;
- $\operatorname{grad}\hat{v}_{rigid} =$

$(\operatorname{grad} Q_{brc})^t\hat{W}_oQ^T_{brc}(y - o) + Q_{brc}\hat{W}_o(\operatorname{grad} Q^T_{brc})^t(y - o) + Q_{brc}\hat{W}_oQ^T_{brc}(I + \operatorname{grad} r)$;

- $\hat{D}_{M\,rigid} = B(Q_{arc}\hat{W}_oQ^T_{arc})$;

where the symbol $\hat{D}_M$ denotes a virtual velocity of M, Q_{arc} is the relative rotation of the ashlar with respect to the continuum of ashlars in the present configuration (where the rigid velocities belong) and Q_{brc} is the relative rotation of the box with respect to the continuum of ashlars.

3.2. *Virtual powers*

Objectivity of the internal actions. Let us write the virtual power of the internal actions in the following form (let $\mathscr{B}$ be any volume element of the body; the exponent a refers to ashlars, b to boxes):

$$(1)\qquad \mathscr{P}_{int} = \int_{\mathscr{B}}\Big(f^{(a)}\cdot\hat{u} + f^{(b)}\cdot\hat{v} - \frac{1}{2}m^{(a)}\cdot\mathbf{e}\hat{W} - T\cdot\operatorname{grad}\hat{v} - S\cdot(B\hat{W} + \mathbf{b}\operatorname{grad}\hat{W})\Big)\,d(vol).$$

We apply the axiom of rigid velocities to insure the objectivity of the internal actions:

$$(2)\qquad \begin{aligned}\mathscr{P}^{(rigid)}_{int} = \int_{\mathscr{B}}\Big(& f^{(a)}\cdot(\hat{u}_o + \hat{W}_o(x - o)) + f^{(b)}\cdot(\hat{u}_o + Q_{brc}\hat{W}_oQ^T_{brc}(y - o)) \\ & - \frac{1}{2}m^{(a)}\cdot Q_{arc}(\mathbf{e}\hat{W}_o) - T\cdot((\operatorname{grad} Q_{brc})^t\hat{W}_oQ^T_{brc}(y - o) \\ & + Q_{brc}\hat{W}_o(\operatorname{grad} Q^T_{brc})^t(y - o) + Q_{brc}\hat{W}_oQ^T_{brc}(I + \operatorname{grad} r)) \\ & - S\cdot(B(Q_{arc}\hat{W}_oQ^T_{arc})) - 2\mathbf{b}(Q_{arc}\hat{W}_o\operatorname{grad} Q^T_{arc})\Big)\,d(vol) \\ = 0 \quad & \forall\hat{u}_o \in \mathscr{V},\ \hat{W}_o \in \operatorname{Skw}\end{aligned}$$

where we used:

$$
\text{(3)}\qquad \begin{aligned}
\mathbf{b}\,\mathrm{grad}\,\hat{W}_{rigid} &= \mathbf{b}(Q_{arc}\hat{W}_o^T\,\mathrm{grad}\,Q_{arc}^T) + \mathbf{b}^t(Q_{arc}\hat{W}_o\,\mathrm{grad}\,Q_{arc}^T)\\
&= -\mathbf{b}(Q_{arc}\hat{W}_o\,\mathrm{grad}\,Q_{arc}^T) + \mathbf{b}^t(Q_{arc}\hat{W}_o\,\mathrm{grad}\,Q_{arc}^T)\\
&= -2\mathbf{b}(Q_{arc}\hat{W}_o\,\mathrm{grad}\,Q_{arc}^T).
\end{aligned}
$$

Taking $\hat{W}_o = 0$ and an arbitrary $\hat{u}_o$, we get the translational equilibrium of the internal forces on ashlars and boxes:

$$
\text{(4)}\qquad f^{(a)} + f^{(b)} = 0.
$$

Taking then an arbitrary $\hat{W}_o = 0$, we get the rotational equilibrium of the same generalised forces:

$$
\text{(5)}\qquad \begin{aligned}
&f^{(a)} \otimes (x-o) + Q_{brc}^T f^{(b)} \otimes (y-o)Q_{brc}\\
&\quad -\frac{1}{2}\mathbf{e}(Q_{arc}^T m^{(a)}) - ((\mathrm{grad}\,Q_{brc}^T \cdot T) \otimes (Q_{brc}^T(y-o))\\
&\quad + Q_{brc}^T T(\mathrm{grad}\,Q_{brc})^T(y-o) + Q_{brc}^T T Q_{brc} + Q_{brc}^T T(\mathrm{grad}\,r)^T Q_{brc})\\
&\quad - Q_{arc}^T B^T S Q_{arc} - 2(\mathrm{grad}\,Q_{arc}^T S^T \mathbf{b}^t)Q_{arc} \in \mathrm{Sym};
\end{aligned}
$$

which, in the reference configuration, becomes:

$$
\text{(6)}\qquad \mathrm{skw}(T - B^T S) - \frac{1}{2}\mathbf{e}m^{(a)} = 0.
$$

(4) and (5) are relations among components of the internal actions, that must be identically satisfied by the constitutive assignments to insure their objectivity.

Equations of motion. Let us write the virtual power of the external actions as:

$$
\begin{aligned}
\mathscr{P}_{ext} &= \int_{\mathscr{B}} \left(f^{(ext)} \cdot \hat{u} - \frac{1}{2} m^{(ext)} \cdot \mathbf{e}\hat{W} \right) d(vol)\\
&\quad + \int_{\partial\mathscr{B}} \left(f^{(ext)} \cdot \hat{u} - \frac{1}{2} m^{(ext)} \cdot \mathbf{e}\hat{W} \right) d(area);
\end{aligned}
$$

where, contrary to common use, we have not introduced a different notation for the external forces and moment applied on the unit volume or on the unit surface, thus reducing the number of symbols, because they will always appear in separate contexts, with explicit reference to their support.

Notice that the surface $\partial\mathscr{B}$ is the part of the whole surface of the body where velocities are not prescribed.

Let us write the virtual power of the accelerations as:

$$\mathscr{P}_{acc} = \int_{\mathscr{B}} \Big(\rho^c \sigma^3 \dot{u} \cdot \hat{u} + \frac{1}{2} \rho^c \sigma^5 \dot{W} \cdot \hat{W} \Big) \, d(vol);$$

which ensures that, in a real motion, the kinetic energy theorem applies.

The equations of motions derive from the application of the axiom of balance of the virtual powers:

$$\begin{aligned}
&\int_{\mathscr{B}} \Big(f^{(ext)} \cdot \hat{u} - \frac{1}{2} m^{(ext)} \cdot \mathbf{e} \hat{W} \Big) \, d(vol) + \int_{\partial\mathscr{B}} \Big(f^{(ext)} \cdot \hat{u} - \frac{1}{2} m^{(ext)} \cdot \mathbf{e} \hat{W} \Big) \, d(area) \\
&\quad + \int_{\mathscr{B}} \Big(f^{(a)} \cdot (\hat{u} - \hat{v}) - \frac{1}{2} m^{(a)} \cdot \mathbf{e} \hat{W} - T \cdot \operatorname{grad} \hat{v} \\
&\qquad - S \cdot (B \hat{W} + \mathbf{b} \operatorname{grad} \hat{W}) \Big) \, d(vol) \\
&= \int_{\mathscr{B}} \Big(\rho^c \sigma^3 \dot{u} \cdot \hat{u} + \frac{1}{2} \rho^c \sigma^5 \dot{W} \cdot \hat{W} \Big) \, d(vol) \quad \forall \hat{u}, \hat{v} \in \mathscr{V}, \ \hat{W} \in \mathrm{Skw};
\end{aligned}$$

hence

$$\begin{aligned}
&\int_{\mathscr{B}} \Big((f^{(ext)} + f^{(a)} - \rho^c \sigma^3 \dot{u}) \cdot \hat{u} \\
&\quad + \Big(-\frac{1}{2} \mathbf{e} (m^{(ext)} + m^{(a)}) - B^T S + \operatorname{div}(\mathbf{b}^T S) - \frac{1}{2} \rho^c \sigma^5 \dot{W} \Big) \cdot \hat{W} \\
&\quad + (\operatorname{div} T - f^{(a)}) \cdot \hat{v} \Big) \, d(vol) \\
&\quad + \int_{\partial\mathscr{B}} \Big(f^{(ext)} \cdot \hat{u} - \Big(\frac{1}{2} \mathbf{e} m^{(ext)} + \mathbf{b}^T S n \Big) \cdot \hat{W} - T^T n \cdot \hat{v} \Big) \, d(area) = 0 \\
&\quad \forall \hat{u}, \hat{v} \in \mathscr{V}, \ \hat{W} \in \mathrm{Skw}.
\end{aligned}$$

The following balance equations and boundary conditions ensue:

- In $\mathscr{B}$:

$$\begin{cases} f^{(a)} + f^{(ext)} = \rho^c \sigma^3 \dot{u} \\ \operatorname{div} T = f^{(a)} \\ \operatorname{div}(\mathbf{b}^T S) - B^T S - \frac{1}{2} \mathbf{e} (m^{(ext)} + m^{(a)}) = \frac{1}{2} \rho^c \sigma^5 \dot{W} \end{cases}$$

- On $\partial\mathscr{B}$:
  – If $u \neq v$ is allowed: $\begin{cases} f^{(ext)} = 0 \\ T^T n = 0 \\ \mathbf{b}^T S n = -\frac{1}{2} \mathbf{e} m^{(ext)} \end{cases}$;

– if not: $\begin{cases} T^T n = f^{(ext)} \\ \mathbf{b}^T S n = -\frac{1}{2}\mathbf{e}m^{(ext)} \end{cases}$

Remarks. The thus obtained system of equations has a peculiarity: the internal force $f^{(a)}$, associated with the translation of ashlars, equals, on the basis of a balance law, the divergence of a stress-like field T, associated with the deformation of the boxes, but appears otherwise as a constitutive variable. One can proceed eliminating $f^{(a)}$ from the set of principal unknowns of the problem, i.e. taking

$$\operatorname{div} T + f^{(ext)} = \rho^c \sigma^3 \dot{u} \quad \text{in } \mathscr{B}$$

and

$$T^T n = f^{(ext)} \quad \text{on } \partial\mathscr{B}$$

and excluding any contribution of $f^{(a)}$ to the constitutive statements, but this is not the only available option. In the following section, we will propose an alternative path.

4. Constitutive statements

4.1. *Constitutive variables and their rôles*

The constitutive variables are of two kinds

- acting on ashlars:
  - the internal force $f^{(a)}$;
  - the internal moment $m^{(a)}$;
- acting on boxes:
  - the stress dual to the box ellipsoid rate of change, named T;
  - the stress dual to the box hem rate of change, named S.

The first two are of the kind of material forces, the second two of configurational forces.

4.2. *Configurational constitutive statements*

A configurational change is the result of an evolution of the boxes that is not an entrainment of them by the continuum of ashlars. In particular, two such mismatch can be relevant: that of the box ellipsoid and of the box hem.

Let us introduce the following measure of the mismatch between the continuum of ashlars and the box ellipsoid:

$$H = GF^{-1} - I$$

$H = 0$ means that boxes move as material elements do (i.e. $G = F$); when not so, configurational changes occur in the system, that are measured by H and must be related to the configurational stresses.

A measure of the mismatch between the continuum of ashlars and the box hem can be obtained comparing M with:

$$M_{aec} = 2\,\mathrm{sym}(Q_{aec}\mathbf{a}_*(\mathrm{Grad}\,Q_{aec})^T)$$

namely:

$$M_{arc} = M - M_{aec};$$

this tensor represents a second family of configurational changes and can thus be related to the configurational stresses.

To write the constitutive statements for the configurational stresses, we propose starting from the idea that a convex domain $\phi(T, S)$ exists in the Cartesian product of the spaces of these stresses, such that

$$\phi \le 0 \quad \& \quad \begin{cases} \dot{H} = \lambda_H \frac{\partial \phi}{\partial T} & \& \quad \lambda_H \ge 0 \quad \& \quad \lambda_H \phi = 0 \\ \dot{M}_{arc} = \lambda_M \frac{\partial \phi}{\partial S} & \& \quad \lambda_M \ge 0 \quad \& \quad \lambda_M \phi = 0 \end{cases}$$

4.3. Material constitutive statements

Constitutive statements for $f^{(a)}$. The translational mobility of ashlars within boxes must be represented in the model by a relationship between the relative displacement r, the box ellipsoid G and the ashlar's rotation Q. To obtain it, we write the condition that the "contact" points on the ashlar (as defined by the $s_*^{(i)}$, $i = 1, \ldots, n$) are not external to the ellipsoid defined by G and r, i.e.:

$$\delta^{(i)} = (s_*^{(i)})^2 \sum_{j=1}^{3} \frac{(s_*^{(j)} \cdot G^T(Qs_*^{(i)} - r))^2}{(Gs_*^{(i)})^4} \le 1 \quad \forall i \in \{1, \ldots, n\} \tag{7}$$

The internal force acting on ashlars can be related to the $\delta^{(i)}$ measure of the position of the ashlar within the box through a sort of Signorini condition of unilateral contact (see Fig. 7).

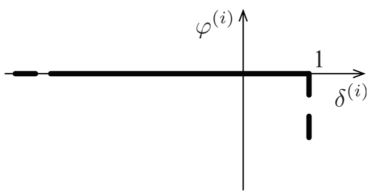

Figure 7. Signorini condition for the contacts of ashlars and boxes.

Let us take the components of $f^{(a)}$ on the actual directions of the "contact" points of the ashlar (take all n such points):

$$\varphi^{(i)} = \|s_*^{(i)}\| f^{(a)} \cdot (Qs_*^{(i)} - r); \quad i \in \{1, \ldots, n\}.$$

The unilateral contact condition is

$$(\delta^{(i)} - 1)\varphi^{(i)} = 0 \;\&\; \delta^{(i)} \leq 1 \;\&\; \varphi^{(i)} \leq 0 \quad \forall i \in \{1, \ldots, n\}.$$

At a contact, the value of $\varphi^{(i)}$ can be taken as inversely proportional to the tightness of the box enclosing the ashlar: the larger the norm of G and/or r, the looser the interaction.

Furthermore, one can assume that the box parameters (G and r) yield under such action. Yielding can be described by a law of the form:

$$\dot{\delta}^{(i)}\varphi^{(i)} = 0 \;\forall \dot{\delta}^{(i)} \quad \text{or} \quad \begin{cases} \delta^{(i)} = 1 & \& \quad \varphi^{(i)} < 0 \quad \Rightarrow \quad \dot{\delta}^{(i)} = 0 \\ \delta^{(i)} = 1 & \& \quad \varphi^{(i)} = 0 \quad \Rightarrow \quad \dot{\delta}^{(i)} < 0 \end{cases}$$

Straining by:

$$\begin{cases} \varphi^{(i)} = 0 & \text{if } \delta^{(i)} < 1 \\ \varphi^{(i)} = \varphi(\det G, \|r\|; s_*^{(i)}, \ldots) & \text{if } \delta^{(i)} = 1 \end{cases}$$

with φ a decreasing function of the first two arguments.

We can then write

$$f^{(a)} = \frac{1}{\|s_*^{(i)}\|} \sum_{i=1}^{6} \varphi^{(i)} (Qs_*^{(i)} - r)$$

(no more than three non null terms will contribute to the sum, unless $G = I$ and $r = 0$).

In the case of cubes, the "contact" condition (7) becomes:

$$\delta^{(i)} = \frac{3\sigma^2}{8} \sum_{j=1}^{3} \frac{(s_*^{(j)} \cdot G^T(Qs_*^{(i)} - r))^2}{(Gs_*^{(i)})^4} \leq 1 \quad \forall i \in \{1, \ldots, 6\}$$

and then:

$$\varphi^{(i)} = \frac{\sigma\sqrt{3}}{2\sqrt{2}} f^{(a)} \cdot (Qs_*^{(i)} - r); \quad i \in \{1, \ldots, 6\}.$$

Constitutive statement for $m^{(a)}$. The internal force dual to the rotational mobility of ashlars within boxes can be represented in the model by a law depending on the present value of Q and M.

The eigenvectors of M represent the normals to the apparent faces of the box, whose rôle in this issue is to be defined.

The following cases can be taken into account:

1. M has three equal eigenvalues: Q is equally framed on all directions, it has no preferred axis;
2. M has two equal eigenvalues and
   (a) the third prevails on them: Q's axis is rather parallel to the prevailing eigenvector, as rotations about it are less blocked than about other directions;
   (b) the third is below them: for a similar reason, Q's axis is rather orthogonal to the smaller eigenvector;
3. M has three distinct eigenvalues: Q's axis is related to the direction of least effort, which must be defined introducing some average.

To define a kinematic variable describing the interaction of Q and M, we can take the axial vector of the former and the spectral decomposition of the latter:

$$Q = e^{-\theta \mathbf{e} q}; \quad \theta \in [0, 2\pi[;\ q \in \mathscr{V};\ \|q\| = 1;$$
$$M = P\lceil \mu_i \rfloor P^T; \quad P \in \mathrm{SO}(3);\ \mu_i \in \mathbb{R};\ i \in [1, \ldots, 3]$$

where $\lceil \mu_i \rfloor$ is the diagonal matrix of the eigenvalues of M and P the orthogonal tensor whose columns are the eigenvectors of M.

Let $p^{(i)}$ be an eigenvector of M, normalised to $\|p^{(i)}\| = 1$:

$$\mu_i p^{(i)} = M p^{(i)}; \quad p^{(i)} \cdot p^{(j)} = \delta_{ij} \quad \forall i, j \in [1, 2, 3].$$

We take the following measure of the effect of M on Q:

$$\kappa = \max_{p^{(1)}, p^{(2)}, p^{(3)}} \sum_{i=1}^{3} \mu_i \|p^{(i)} \times q\|$$

where the max is to be taken among all triples of eigenvectors when such a triple is not unique, and assume

$$m^{(a)} = \psi(\kappa, \theta, \det G, \|r\|; s_*^{(i)}, \ldots) q$$

with ψ an homogeneous and increasing (possibly linear) function of the first two arguments and a decreasing function of the next two.

5. Conclusion

A continuum model of structural systems made of interlocking ashlars has been proposed, based on assumptions of observational origin, aiming to reduce as much as possible the system to a set of rigid bodies with unilateral contacts.

Ashlar interlocking is introduced in the model by way of two constitutive assumptions about the material force and the material moment acting on each ashlar. These assumptions depend on the geometric information conveyed by the "box" enclosing each ashlar; in turn, the specification of each box is the outcome of a configurational problem involving the kinematics of ashlars. Thus, an ashlar structural system is seen as the superposition of two continua with microstructure: a continuum of ashlars, endowed with translational and rotational mobilities, and a continuum of boxes accounting for the mutual interlocking of ashlars. The latter continuum is endowed with a rather complex kinematics, which is partly dictated by the kinematics of ashlars. Since the dynamics of the continuum of ashlars is governed by balance laws where inertial terms appear, its role is to model material elements. On the other hand, the continuum of boxes can be regarded as providing a configurational setting for the continuum of ashlars, because its velocity fields are not directly bounded by inertia (though they depend constitutively on material velocities).

A complete set of motion equations is furnished, with constitutive laws under form of flow rules associated with convexity criteria.

Acknowledgments. The author thanks Professor Gianfranco Capriz and Professor Paolo Podio-Guidugli for their precious help.

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Received 15 January 2017,
and in revised form 23 June 2017.

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