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Functional Analysis — *Compactness and s-numbers for polynomials*, by Erhan Çalişkan and Pilar Rueda, communicated on April 21, 2017.

Dedicated to the memory of Jorge Mujica (1946–2017)

Abstract. — We extend the measure of non compactness notion to the polynomial setting by means of Approximation, Kolmogorov and Gelfand numbers, that are introduced for homogeneous polynomials. As an application, we study diagonal polynomials between sequence spaces.

Key words: Homogeneous polynomials, s-numbers sequences, approximation numbers, Kolmogorov numbers, the measure of non-compactness

Mathematics Subject Classification: 47H60, 46B28, 46G25

1. Introduction

A. Pietsch [19] introduced an axiomatic theory of s-numbers as a tool for the study of linear operators between Banach spaces. The theory of s-numbers of multilinear operators has been recently developed by D. L. Fernandez, M. Mastylo and E. B. da Silva [11] (see also [28]), by extending linear techniques to the multilinear case. However, the theory for homogeneous polynomials has not been checked as far as it should have been. We mention only the particular case by A. Brauns, H. Junek and E. Plewnia [3] and the unpublished [4]. Tensor products are intrinsically related to polynomials, and approximation numbers of tensor product operators have been also considered (see e.g. [26, 27]).

Our aim is to give a hint of the theory for homogeneous polynomials and to show how to deal with non-linear techniques, that will provide shorter proofs than those coming from the classical theory. The relation of both, linear and polynomial theories are well stated in Section 3. Section 4 contains the essentials of the n-th approximation number $\tilde{a}_n(P)$ of an homogeneous polynomial P. Kuratowski and Hausdorff measures are treated in that section. As an attempt to quantify the non compactness of a polynomial, we study the polynomial notion of Kolmogorov numbers $\tilde{d}_n$ and the polynomial m-lifting property in Section 5. In particular we prove that $\tilde{d}_n(P) = \tilde{d}_n(P_L)$, where P_L is the linearization of P defined on a Banach space X. In the last section we provide concrete examples of s-numbers sequences, including approximation, Kolmogorov and Gelfand numbers of diagonal homogeneous polynomials between sequence spaces.

2. Notation and preliminaries

The symbol $\mathbb{K}$ represents the field of all real or complex numbers, and $\mathbb{N}$ all positive integers. The letters X, Y, W and Z will always represent (real or complex) Banach spaces. The open unit ball of X is B_X and $\bar{B}_X$ is the closed unit ball. As usual, X^* is the dual of X, and κ_X is the canonical embedding of X into the bidual X^{**} of X. Let $\overline{\Gamma}(C)$ denote the closed balanced convex hull of a subset $C \subset X$.

Given a continuous m-linear mapping $A : X \times \cdots \times X \to Y$, the map $P : X \to Y$, defined by $P(x) = A(\underbrace{x, \ldots, x}_{m \text{ times}})$ for every $x \in X$, is said to be a continuous m-homogeneous polynomial. $\mathcal{P}(^m X; Y)$ will denote the vector space of all continuous m-homogeneous polynomials from X into Y, which is a Banach space with the norm $\|P\| = \sup\{\|P(x)\| : \|x\| \leq 1\}$. When $Y = \mathbb{K}$ we will write $\mathcal{P}(^m X)$ instead of $\mathcal{P}(^m X; \mathbb{K})$ and when $m = 1$, $\mathcal{L}(X; Y) := \mathcal{P}(^1 X; Y)$ is the space of all continuous linear operators from X to Y. Let $\mathcal{P}^m := \bigcup_{X,Y} \mathcal{P}(^m X; Y)$, that is, $\mathcal{P}^m$ is the class of all m-homogeneous polynomials defined between Banach spaces. Denote by $\mathcal{P} := \bigcup_m \mathcal{P}^m$ the class of all continuous homogeneous polynomials defined between Banach spaces. When $m = 1$, $\mathcal{L} := \mathcal{P}^1 \equiv \mathcal{P}$ is the class of all continuous linear operators.

Let $P \in \mathcal{P}(^m X; Y)$. We define the rank of P as the dimension of the linear span of $P(X)$ in Y.

Given a Banach space X, $\hat{\otimes}^{\pi_s}_{m,s} X$ will denote the completed m-fold symmetric tensor product of X endowed with the projective s-tensor norm π_s, which is defined as (see [12, p. 164])

$$\pi_s(z) = \inf\left\{\sum_{j=1}^{\infty} |\lambda_j| \, \|x_j\|^m : z = \sum_{j=1}^{\infty} \lambda_j \otimes_m x_j\right\}$$

for $z \in \hat{\otimes}^{\pi_s}_{m,s} X$, where $\otimes_m x := \underbrace{x \otimes \cdots \otimes x}_{m\text{-times}}$. If $T \in \mathcal{L}(X; Y)$, let $\otimes_{m,s} T : \hat{\otimes}^{\pi_s}_{m,s} X \to \hat{\otimes}^{\pi_s}_{m,s} Y$ be the continuous linear map given by $\otimes_{m,s} T(\otimes_m x) = \otimes_m T(x)$.

For $P \in \mathcal{P}(^m X; Y)$, let P_L denote the linearization of P, that is the unique continuous linear operator $P_L : \hat{\otimes}^{\pi_s}_{m,s} X \to Y$ such that $P(x) = P_L(\otimes_m x)$. The correspondence $P \leftrightarrow P_L$ determines an isometric isomorphism – denoted by $I_{^m X, Y}$ – between $\mathcal{P}(^m X; Y)$ and $\mathcal{L}(\hat{\otimes}^{\pi_s}_{m,s} X; Y)$ (see [12, Proposition, p. 163]). Let $I_m : \mathcal{P}^m \to \mathcal{L}$ and $I : \mathcal{P} \to \mathcal{L}$ be the correspondences whose restriction to each component $\mathcal{P}(^m X; Y)$ is $I_{^m X, Y}$.

Let $\mathcal{X}$ be a metric space. The *Kuratowski measure* $\alpha(A)$ of non-compactness of a bounded set $A \subset \mathcal{X}$ is defined by

$$\alpha(A) = \inf\{\varepsilon > 0 : \text{A may be covered by finitely many sets of diameter} \leq \varepsilon\}.$$

In case that we consider just finitely many balls of radius $\leq \varepsilon$ to cover A, the infimum is called the *Hausdorff ball measure* $\beta(A)$ of non-compactness of A, that is

$$\beta(A) = \inf\{\varepsilon > 0 : \text{A may be covered by finitely many balls of radius} \leq \varepsilon\}.$$

For every bounded set A we have that $\beta(A) \leq \alpha(A) \leq 2\beta(A)$.

Let X and Y be Banach spaces. Since continuous m-homogeneous polynomials are bounded on bounded sets, we can extend the Kuratowski, and the Hausdorff measure of non-compactness of linear operators to polynomials in a natural way: for any $P \in \mathscr{P}(^m X; Y)$ the *Kuratowski and the Hausdorff measure*, respectively, of non-compactness of P is defined by

$$\gamma(P) := \alpha(P(\bar{B}_X)) \quad \text{and} \quad \tilde{\gamma}(P) := \beta(P(\bar{B}_X))$$

Note that P is compact if and only if $\tilde{\gamma}(P) = \gamma(P) = 0$.

3. s-NUMBER SEQUENCES FOR HOMOGENEOUS POLYNOMIALS

In a natural way, we introduce the notion of an m-s-number sequence for m-homogeneous continuous polynomials. Let $m \in \mathbb{N}$ and for each $n \in \mathbb{N}$ let $s_n : \mathscr{P}^m \to [0, \infty)$ be a mapping. The sequence $s = (s_n)$ is called an *m-s-number sequence* if the following conditions are satisfied for any $n, k \in \mathbb{N}$:

(S1) Monotonicity: For every $P \in \mathscr{P}(^m X; Y)$,

$$\|P\| = s_1(P) \geq s_2(P) \geq \cdots \geq 0.$$

(S2) Additivity: For every $P, Q \in \mathscr{P}(^m X; Y)$,

$$s_{k+n-1}(P + Q) \leq s_k(P) + s_n(Q).$$

(S3) Ideal-property: For every $P \in \mathscr{P}(^m X; Y)$, $S \in \mathscr{L}(Y; Z)$, $T \in \mathscr{L}(W; X)$

$$s_n(S \circ P \circ T) \leq \|S\| s_n(P) \|T\|^m.$$

(S4) Rank-property: Let $P \in \mathscr{P}(^m X; Y)$.

$$\operatorname{rank}(P) < n \Rightarrow s_n(P) = 0.$$

Furthermore, if $m = 1$ the following condition has to be added:

(S5) Norming-property: $s_n(\mathrm{Id} : \ell_2^n \to \ell_2^n) = 1$, $n \in \mathbb{N}$, where Id is the identity mapping on the n-dimensional Hilbert space ℓ_2^n.

If (s_n) is an $m - s$-number sequence for each $m \in \mathbb{N}$, then (s_n) is called an *s-number sequence.*

Note that this notion coincides with the usual notion of s-number sequence for linear operators, introduced in [21], whenever $m = 1$.

Our first interest is to relate the linear and the polynomial notions of s-number sequences.

Proposition 3.1. *If the mapping* $s = (s_n) : \mathscr{L} \to [0, \infty)^{\mathbb{N}}$ *is an s-number sequence (in the linear sense) then,* $s \circ I_m : \mathscr{P}^m \to [0, \infty)^{\mathbb{N}}$ *is an m-s-number sequence.*

Proof. We will pay attention just to the ideal property. This property follows from the fact that $(S \circ P \circ T)_L = S \circ P_L \circ \otimes_{m,s} T$ and $\|\otimes_{m,s} T\| = \|T\|^m$, for all $P \in \mathscr{P}(^m X; Y)$, $T \in \mathscr{L}(W; X)$ and $S \in \mathscr{L}(Y; Z)$. □

The theory of ideals of homogeneous polynomials between Banach spaces has been developed in the last decades by several authors, so the extension of the dual procedure to polynomial ideals is a natural step. In this paper we provide many results on homogeneous polynomials in connection with their adjoint concerning measure of non-compactness and *s*-numbers. First we need the definition of the adjoint of a continuous homogeneous polynomial.

Definition 3.2 (Aron–Schottenloher [1]). Given $P \in \mathscr{P}(^m X; Y)$, the adjoint of P is the following continuous linear operator:

$$P^* : Y^* \to \mathscr{P}(^m X), \quad P^*(\varphi)(x) = \varphi(P(x)).$$

It is clear that $\|P^*\| = \|P\|$.

After this definition by R. Aron and M. Schottenloher, and after the works of R. Ryan [22, 24], the adjoint of a polynomial became a standard tool in the study of spaces of homogeneous polynomials and in infinite dimensional holomorphy (see, e.g. [6, 7, 17] and references therein).

Remark 3.3. If $P \in \mathscr{P}(^m X; Y)$ has finite rank then P_L has finite rank. Since $P_L(\hat{\otimes}^{\pi_s}_{m,s} X)$ coincides with the linear hull of $P(X)$, we have $\operatorname{rank}(P) = \operatorname{rank}(P_L) = \operatorname{rank}((P_L)^*) = \operatorname{rank}(I_{^m X,\mathbb{K}} \circ P^*) = \operatorname{rank}(P^*)$.

We refer to [6] or [16] for the properties of polynomials in infinite dimensional spaces, to [15] for the theory of Banach spaces, and to [5], [12] and [25] for tensor products of Banach spaces.

4. Approximation numbers and compactness of polynomials

Similarly to the linear case we define the *n-th approximation number* $\tilde{a}_n(P)$ of any homogeneous polynomial $P \in \mathscr{P}(^m X; Y)$ by

$$\tilde{a}_n(P) := \inf\{\|P - Q\| : Q \in \mathscr{P}(^m X; Y), \operatorname{rank}(Q) < n\}.$$

If we denote $a_n(T) := \inf\{\|T - L\| : L \in \mathscr{L}(X; Y), \operatorname{rank}(L) < n\}$, $T \in \mathscr{L}(X; Y)$, then by Remark 3.3 we see that $\tilde{a}_n(P) = a_n(P_L)$.

If $a = (a_n)$ is an *s*-number sequence on $\mathscr{L}$, Proposition 3.1 gives that $\tilde{a} = a \circ I_m$ is an $m - s$-number sequence on $\mathscr{P}^m$. Therefore, $\tilde{a} = a \circ I$ is an *s*-number sequence.

PROPOSITION 4.1. *Let* $(s_n) : \mathscr{P}(^mX; Y) \to [0, \infty)^{\mathbb{N}}$ *be an s-number sequence. Then*

(i) *For all* $P \in \mathscr{P}(^mX; Y)$ *we have* $s_n(P) \leq \tilde{a}_n(P)$, $n \in \mathbb{N}$.
(ii) *For all* $S \in \mathscr{L}(Y; Z)$, $P \in \mathscr{P}(^mX; Y)$ *and all* $k, n \in \mathbb{N}$ *we have* $s_{k+n-1}(S \circ P) \leq s_1(S)\tilde{a}_n(P)$ *and* $s_{k+n-1}(S \circ P) \leq a_k(S)s_n(P)$.

PROOF. (i) Let $P \in \mathscr{P}(^mX; Y)$. Then for any $R \in \mathscr{P}(^mX; Y)$ with $\mathrm{rank}(R) < n$, we have $s_n(P) \leq s_1(P - R) + s_n(R) = \|P - R\| + s_n(R) = \|P - R\|$. Hence, $s_n(P) \leq \tilde{a}_n(P)$.

(ii) Let $R \in \mathscr{P}(^mX; Y)$ with $\mathrm{rank}(R) < n$. Since $\mathrm{rank}(S \circ R) < n$, it follows that

$$\begin{aligned} s_{k+n-1}(S \circ P) &\leq s_k(S(P - R)) + s_n(S \circ R) = s_k(S(P - R)) \\ &\leq \|S\| s_k(P - R)\|I_X\|^m \leq \|S\| s_1(P - R) = \|S\| \, \|P - R\|. \end{aligned}$$

Therefore $s_{k+n-1}(S \circ P) \leq s_1(S)\tilde{a}_n(P)$.

The proof of the second inequality can be obtained in a similar way. □

It is worth mentioning that our use of polynomial techniques allows us to reduce many proofs to the linear case instead of adapting all calculations to the new setting.

PROPOSITION 4.2. *Let* $m \geq 2$ *and let* X *and* Y *be Banach spaces.*

(a) *For every polynomial* $P \in \mathscr{P}(^mX; Y)$ *we have* $a_n(P^*) \leq \tilde{a}_n(P)$, $n \in \mathbb{N}$. *Furthermore, if there exists a linear projection* π *of norm* 1 *from* Y^{**} *onto* $\kappa_Y(Y)$ *then, for every* $P \in \mathscr{P}(^mX; Y)$ *we have that* $a_n(P^*) = \tilde{a}_n(P)$ *for every* $n \in \mathbb{N}$.
(b) *If* $P \in \mathscr{P}(^mX; Y)$ *is a compact polynomial, then* $a_n(P^*) = \tilde{a}_n(P)$ *for every* $n \in \mathbb{N}$.

PROOF. (a) Since $I_{^mX, \mathbb{K}}$ is an isometric isomorphism, it follows from, e.g., [20, p. 152, 11.7.3. Proposition] that $a_n(P^*) = a_n(I_{^mX, \mathbb{K}} \circ P^*) = a_n((P_L)^*) \leq a_n(P_L) = \tilde{a}_n(P)$.

For the second assertion, we use the analogous property for linear operators [11, Proposition 3.3] to get that $\tilde{a}_n(P) = a_n(P_L) = a_n((P_L)^*) = a_n(I_{^mX, \mathbb{K}} \circ P^*) = a_n(P^*)$.

(b) This equality follows from the well known fact that P is compact if and only if P_L is compact (see [22]) and the corresponding property for linear operators (see, e.g., [20, 11.7.4 Theorem]), that is, $\tilde{a}_n(P) = a_n(P_L) = a_n((P_L)^*) = a_n(I \circ P^*) = a_n(P^*)$, for all $n \in \mathbb{N}$. □

REMARK 4.3. The technique we have used in this section makes use of the linearization of continuous homogeneous polynomials. A similar technique works for continuous m-linear mappings. For each integer $m \in \mathbb{N}$, let $\mathscr{L}(X_1, \ldots, X_m; Y)$ be the Banach space of all continuous m-linear mappings $A : X_1 \times \cdots \times X_m \mapsto$

Y, endowed with the sup norm $\|A\| = \sup\{\|A(x_1, \ldots, x_m)\| : \|x_i\| \le 1,\ i = 1, \ldots, m\}$. If $T \in \mathscr{L}(X_1, \ldots, X_m; Y)$ there is a unique continuous linear operator $T_L \in \mathscr{L}(X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_m; Y)$ such that $T_L(x_1 \otimes \cdots \otimes x_m) = T(x_1, \ldots, x_m)$, and the correspondence $T \leftrightarrow T_L$ determines an isometric isomorphism between $\mathscr{L}(X_1, \ldots, X_m; Y)$ and $\mathscr{L}(X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_m; Y)$. This yields alternative proofs to those given in [11] based in the well-known linear case.

An important notion used to determine the compactness of a polynomial (in particular, of an operator) is the measure of non-compactness which is closely related to s-number sequences. Here we consider Kuratowski and Hausdorff measures of polynomials which will be useful to determine compactness of a polynomial and to obtain some basic inequalities in connection with s-numbers. First let us state a basic result.

Proposition 4.4 (see, e.g., [14, Lemma 2.6]). *Let X be a Banach space. Then $\beta(C) = \beta(\overline{\Gamma}(C))$ for any bounded subset $C \subset X$. In particular, $\beta(\overline{\Gamma}(P(B_X))) = \tilde{\gamma}(P)$.*

The following theorem can also be proved with similar techniques to the ones given in [11, Theorem 2.1]. However, we will show how to use polynomial techniques related to tensor products to show that the polynomial case admits shorter proofs.

Theorem 4.5. *Let $m \ge 2$ and let X and Y be Banach spaces. Then for every $P \in \mathscr{P}(^m X; Y)$ we have*

(1) $\gamma(P) \le \tilde{\gamma}(P^*)$ *and* $\gamma(P^*) \le \tilde{\gamma}(P)$,
(2) $\frac{1}{2}\gamma(P) \le \gamma(P^*) \le 2\gamma(P)$ *and* $\frac{1}{2}\tilde{\gamma}(P) \le \tilde{\gamma}(P^*) \le 2\tilde{\gamma}(P)$.

Proof. (1) Since $P_L(\overline{B}_{\otimes^{\pi_s}_{m,s} X}) = P_L(\overline{\Gamma}(\otimes_{m,s} B_X)) = \overline{\Gamma}(P_L(\otimes_{m,s} B_X)) = \overline{\Gamma}(P(B_X))$, we obtain $\gamma(P_L) = \alpha(P_L(\overline{B}_{\otimes^{\pi_s}_{m,s} X})) \ge \alpha(P(\overline{B}_X)) = \gamma(P)$. Then, by [9, Theorem I.2.9]

$$\gamma(P) \le \gamma(P_L) \le \tilde{\gamma}((P_L)^*) = \tilde{\gamma}(I_{^m X, \mathbb{K}} \circ (P_L)^*) = \tilde{\gamma}(P^*),$$

where the first equality follows from being $I_{^m X, \mathbb{K}}$ an isometric isomorphism. Now [9, Theorem I.2.9] and Proposition 4.4 give that

$$\begin{aligned}\gamma(P^*) &= \gamma(I_{^m X, \mathbb{K}} \circ (P_L)^*) = \gamma((P_L)^*) \le \tilde{\gamma}(P_L) = \beta(P_L(\overline{B}_{\otimes^{\pi_s}_{m,s} X})) \\ &= \beta(\overline{\Gamma}(P(B_X))) = \tilde{\gamma}(P).\end{aligned}$$

(2) By using part (1),

$$\begin{aligned}\frac{1}{2}\gamma(P) &\le \frac{1}{2}\tilde{\gamma}(P^*) = \frac{1}{2}\beta(P^*(\overline{B}_{Y^*})) \le \frac{1}{2}\alpha(P^*(\overline{B}_{Y^*})) \le \gamma(P^*) \\ &\le \tilde{\gamma}(P) = \beta(P(\overline{B}_X)) \le \alpha(P(\overline{B}_X)) \le 2\gamma(P),\end{aligned}$$

and

$$\frac{1}{2}\tilde{\gamma}(P) = \frac{1}{2}\beta(P(\bar{B}_X)) \le \frac{1}{2}\alpha(P(\bar{B}_X)) = \frac{1}{2}\gamma(P) \le \frac{1}{2}\tilde{\gamma}(P^*)$$
$$\le \beta(P^*(\bar{B}_{Y^*})) \le \alpha(P^*(\bar{B}_{Y^*})) = \gamma(P^*) \le 2\tilde{\gamma}(P). \qquad \square$$

As a consequence, we get the result by R. Aron and M. Schottenloher on compactness of polynomials:

COROLLARY 4.6 ([1, Proposition 3.6]). *Let $m \ge 2$ and let X and Y be Banach spaces. Then for every homogeneous polynomial $P \in \mathscr{P}(^m X; Y)$ we have that P is compact if and only if its adjoint P^* is compact.*

COROLLARY 4.7. *For every $P \in \mathscr{P}(^m X; Y)$ we have that $\tilde{a}_n(P) \le 5a_n(P^*)$, $n \in \mathbb{N}$.*

PROOF. By [10, Proposition 2] we get $a_n(P_L) \le 5a_n(P_L^*)$, so that $\tilde{a}_n(P) = a_n(P_L) \le 5a_n(P_L^*) = 5a_n(I_{^m X,\mathbb{K}} \circ P^*) = 5a_n(P^*)$. $\square$

REMARK 4.8. For any polynomial $P \in \mathscr{P}(^m X; Y)$ by [10, Proposition 2] we have

$$\tilde{a}_n(P) \le a_n((P_L)^{**}) + 2\tilde{\gamma}(P_L) = a_n((I_{^m X,\mathbb{K}} \circ P^*)^*) + 2\tilde{\gamma}(P) = a_n(P^{**}) + 2\tilde{\gamma}(P),$$

as occurs in the linear case, where the first equality can be taken from the proof of Theorem 4.5 part (1). On the other hand, since $a_n(P^{**}) \le a_n(P^*)$ (see, e.g., [20, p. 152, 11.7.3. Proposition]), now if P is compact then $\tilde{\gamma}(P) = 0$ and hence, by these two inequalities and Proposition 4.2(a), we obtain $\tilde{a}_n(P) = a_n(P^*)$, which recovers Proposition 4.2(b).

REMARK 4.9. Given $P \in \mathscr{P}(^m X; Y)$, let $\tilde{a}(P) := \lim_{n\to\infty} \tilde{a}_n(P) \ge 0$. If we consider the approximation property (AP for short) on Y, then any compact m-homogeneous polynomial $P \in \mathscr{P}(^m X; Y)$ can be approximated by finite-rank m-homogeneous polynomials (see [2, Proposition 2.5]). Hence, similar to the (multi)linear case, if the space Y has the AP, then $P \in \mathscr{P}(^m X; Y)$ is compact if and only if $\tilde{a}(P) = 0$. However, by [1, Proposition 3.3] (see also [17, Theorem 4.3]) we know that $\mathscr{P}(^m X)$ has the AP if and only if, for every Banach space Y, the space of all finite-rank polynomials $\mathscr{P}_f(^m X; Y)$ is norm-dense in the space of all compact polynomials $\mathscr{P}_k(^m X; Y)$, or equivalently, any compact m-homogeneous polynomial $P \in \mathscr{P}(^m X; Y)$ can be approximated by finite-rank m-homogeneous polynomials. Therefore if the space $\mathscr{P}(^m X)$ has the AP, then $P \in \mathscr{P}(^m X; Y)$ is compact if and only if $\tilde{a}(P) = 0$. Let us remark that there is a reflexive separable Banach space X with basis such that $\mathscr{P}(^2 X)$ does not have the AP (see [1]). Hence, for this space X, which has the AP, there is a Banach space Y such that there is a compact polynomial $P : X \to Y$ which cannot be approximated by finite-rank polynomials. (Note that it turns out that this space Y also cannot have the AP by [2, Proposition 2.5].)

5. Kolmogorov numbers

The spirit of Kolmogorov numbers is to measure how far a polynomial is from being compact. We define the *n-th Kolmogorov number* $\tilde{d}_n(P)$ of a polynomial $P \in \mathscr{P}(^m X; Y)$ by

$$\tilde{d}_n(P) := \inf\{\varepsilon > 0 : P(\bar{B}_X) \subset N_\varepsilon + \varepsilon \bar{B}_Y,\ N_\varepsilon \subset Y,\ \dim(N_\varepsilon) < n\}.$$

For $P := T \in \mathscr{L}(X; Y)$ we write $d_n(T) := \tilde{d}_n(P)$.

Proposition 5.1. *Given $P \in \mathscr{P}(^m X; Y)$, $\tilde{d}_n(P) = d_n(P_L)$.*

Proof. Clearly $\tilde{d}_n(P) \le d_n(P_L)$. On the other hand, if $P(\bar{B}_X) \subset N_\varepsilon + \varepsilon \bar{B}_Y$ then $P_L(\bar{B}_{\otimes^{\pi_s}_{m,s} X}) \subset \bar{\Gamma}(P(\bar{B}_X)) \subset \bar{\Gamma}(N_\varepsilon + \varepsilon \bar{B}_Y) = \overline{N_\varepsilon + \varepsilon \bar{B}_Y} \subset N_\varepsilon + (\varepsilon + \delta)\bar{B}_Y$ for all $\delta > 0$. Hence, $d_n(P_L) \le \varepsilon + \delta$ for all $\delta > 0$ and so, $d_n(P_L) \le \tilde{d}_n(P)$. □

As in the linear case, $P \in \mathscr{P}(^m X; Y)$ is compact if and only if $\tilde{d}(P) := \lim_{n\to\infty} \tilde{d}_n(P) = 0$. Also it is obvious that $\tilde{d}_n(P) = 0$ whenever $\mathrm{rank}(P) < n$. Propositions 3.1 and 5.1 imply that $\tilde{d}_n = d_n \circ I$ forms an s-number sequence. Proposition 4.1 implies that $\tilde{d}_n(P) \le \tilde{a}_n(P)$, for every $n \in \mathbb{N}$.

Kolmogorov numbers are related to approximation numbers via the equality $d_n(T) = a_n(T \circ Q)$, $n \in \mathbb{N}$, $T \in \mathscr{L}(X; Y)$, where Q is the canonical metric surjection from $\ell_1(\bar{B}_X)$ onto X, defined by $Q(\{\lambda_x\}) = \sum_{x \in \bar{B}_X} \lambda_x x$, $\{\lambda_x\} \in l_1(\bar{B}_X)$ (see [20, p. 150–151], and for the multilinear case see [11, Theorem 4.1].) We do not know if the polynomial version of this result holds. But the following result, which gives a characterization of the polynomial (metric) lifting property in terms of (metric) lifting property, may be of some use in this connection. Before giving the result let us recall the definition of the lifting property for polynomials introduced by González and Gutiérrez [13]. Let $m \in \mathbb{N}$. We say that X has the *polynomial m-lifting property* if, for every continuous m-homogenous polynomial P from X to any quotient space Y/N, there is $\tilde{P} \in \mathscr{P}(^m X; Y)$ such that $P = Q_N^Y \circ \tilde{P}$, where Q_N^Y denotes the canonical map of Y onto the quotient space Y/N. We say that X has the *polynomial metric m-lifting property* if, for every $\varepsilon > 0$ and every continuous m-homogenous polynomial P from X to any quotient space Y/N, there is $\tilde{P} \in \mathscr{P}(^m X; Y)$ such that $P = Q_N^Y \circ \tilde{P}$ and $\|\tilde{P}\| \le (1 + \varepsilon)\|P\|$.

Proposition 5.2. *Let $m \in \mathbb{N}$. A Banach space X has the polynomial (metric) m-lifting property if, and only if, $\hat{\otimes}^{\pi_s}_{m,s} X$ has the (respectively, metric) lifting property.*

Proof. Assume first that X has the polynomial m-lifting property. Let T be a continuous linear operator from $\hat{\otimes}^{\pi_s}_{m,s} X$ into some quotient space Y/N. Let $P \in \mathscr{P}(^m X; Y/N)$ be such that $P_L = T$. By assumption, there is $\tilde{P} \in \mathscr{P}(^m X; Y)$ such that $Q_N^Y \circ \tilde{P} = P$. Since $Q_N^Y \circ (\tilde{P})_L \circ \delta_X = P$, where δ_X is the m-homogeneous polynomial from X to $\hat{\otimes}^{\pi_s}_{m,s} X$ given by $\delta_X(x) = x \otimes \cdots \otimes x$ (see [22]), then $Q_N^Y \circ (\tilde{P})_L = P_L = T$ and $\hat{\otimes}^{\pi_s}_{m,s} X$ has the lifting property.

We now assume that $\hat{\otimes}^{\pi_s}_{m,s} X$ has the lifting property. Let $P \in \mathscr{P}(^m X; Y/N)$. Then $P_L \in \mathscr{L}(\hat{\otimes}^{\pi_s}_{m,s} X; Y/N)$. By assumption, there is $\widetilde{P_L} \in \mathscr{L}(\hat{\otimes}^{\pi_s}_{m,s} X; Y)$ such that $Q^Y_N \circ \widetilde{P_L} = P_L$. Then $\tilde{P} := \widetilde{P_L} \circ \delta_X$ satisfies $P = Q^Y_N \circ \tilde{P}$.

The metric case follows from the fact that $\|P\| = \|P_L\|$. □

As a consequence, if X has the polynomial metric m-lifting property, then by [20, 11.6.3] for every $P \in \mathscr{P}(^m X; Y)$ we have $\tilde{d}_n(P) = d_n(P_L) = a_n(P_L) = \tilde{a}_n(P)$, for all $n \in \mathbb{N}$.

It was shown in [13, Theorem 1] that $l_1(\Gamma)$ has the polynomial lifting property (see also [23] and [8]), which is the Banach space of summable number families $\{\lambda_\gamma\}_{\gamma\in\Gamma}$ over an arbitrary index set. Moreover, by a modification of the proof of [21, C.3.6. Proposition, p. 34] one can see that if $X = l_1(\Gamma)$, then, given $\varepsilon > 0$ and a continuous m-homogenous polynomial P from X to any quotient space Y/N, there is $\tilde{P} \in \mathscr{P}(^m X; Y)$ such that $P = Q^Y_N \circ \tilde{P}$ and $\|\tilde{P}\| \leq (1+\varepsilon)\frac{m^m}{m!}\|P\|$ (for the coefficient $\frac{m^m}{m!}$, see, e.g., [16, Theorem 2.2]). Accordingly we get the following result.

Theorem 5.3. *Let $m \geq 2$ and let X and Y be Banach spaces. Let $P \in \mathscr{P}(^m X; Y)$, and let Q be the canonical metric surjection from $l_1(\bar{B}_X)$ onto X. Then we have that*

$$\tilde{a}_n(P \circ Q) \leq \frac{m^m}{m!}\tilde{d}_n(P), \quad n \in \mathbb{N}.$$

Proof. By [13, Theorem 1] $\ell_1(\bar{B}_X)$ has the polynomial lifting property, and so it follows by the above remark that for each $n \in \mathbb{N}$, $\tilde{a}_n(P \circ Q) \leq \frac{m^m}{m!}\tilde{d}_n(P \circ Q)$. Using that (d_n) is surjective (see [20, Definition 11.6.4 and Theorem 11.6.5]), we get $\tilde{a}_n(P \circ Q) \leq \frac{m^m}{m!}\tilde{d}_n(P)$, $n \in \mathbb{N}$. □

We remark that we do not know if $\ell_1(\bar{B}_X)$ has the polynomial metric lifting property. If that were the case, then as a consequence of Propositions 5.1 and 5.2, we would obtain the equality $\tilde{a}_n(P \circ Q) = \tilde{d}_n(P)$, for every $n \in \mathbb{N}$ and $P \in \mathscr{P}(^m X; Y)$.

We end this section with another variant of s-number of homogeneous polynomials, namely, Gelfand numbers, from which we will get alternative characterizations of compactness of homogeneous polynomials. Motivated by [20, 11.5.1. Proposition] we define the Gelfand numbers $\tilde{c}_n(P)$ of an m-homogeneous polynomial $P \in \mathscr{P}(^m X; Y)$ by

$$\tilde{c}_n(P) := \tilde{a}_n(\kappa_Y P).$$

Clearly $(\tilde{c}_n)$ is an s-number sequence since $(\tilde{a}_n)$ is an s-number sequence, and for each $n \in \mathbb{N}$ we have that $\tilde{c}_n(P) \leq \tilde{a}_n(P)$. We will just write $c_n(T) := \tilde{c}_n(P)$ whenever $P = T \in \mathscr{L}(X; Y)$. Note that $\tilde{c}_n(P) = c_n(P_L)$ for any $P \in \mathscr{P}(^m X; Y)$.

Considering the function $\tilde{c}: \mathscr{P}(^m X; Y) \to [0, \infty)$ given by $\tilde{c}(P) := \lim_{n\to\infty} \tilde{c}_n(P)$, we have that $\tilde{c}(P) = c(P_L)$. Now, as a polynomial counterpart of [11, Proposition 5.1] compactness of homogeneous polynomials can be quantified by means $\tilde{c}$ and c as follows.

Proposition 5.4. *Let $m \geq 2$ and let X and Y be Banach spaces. The following statements for a polynomial $P \in \mathscr{P}(^m X; Y)$ are equivalent.*

(i) *P is compact.*
(ii) $\tilde{c}(P) = 0$.
(iii) $c(P^*) = 0$.

Proof. We know that P is compact if and only if P_L is compact (see [22]), and P is compact if and only if P^* is compact (see [1] or Corollary 4.4). Combining these facts with [21, 2.4.11] we get the implications (i) $\Leftrightarrow$ (ii), and (i) $\Leftrightarrow$ (iii). □

Following the proof of [11, Theorem 5.1] we get the following result, which gives the relation between Gelfand and Kolmogorov numbers of polynomials.

Theorem 5.5. *Let $m \geq 2$ and let X and Y be Banach spaces. Then, for every polynomial $P \in \mathscr{P}(^m X; Y)$ and $n \in \mathbb{N}$ we have that*

(i) $c_n(P^*) \leq \dfrac{m^m}{m!} \tilde{d}_n(P)$,
(ii) $\tilde{c}_n(P) = d_n(P^*)$,
(iii) $\tilde{c}_n(P) \leq 2\sqrt{n}\, c_n(P^*)$.

6. s-numbers of diagonal polynomials

In [20, p. 158], the asymptotic behavior of (a_n), (c_n), and (d_n) as $n \to \infty$ for diagonal operators between sequence spaces l_p is determined. Extending the techniques to the polynomial setting, we now calculate the corresponding numbers $\tilde{a}_n(P)$, $\tilde{c}_n(P)$, and give lower and upper estimates for the number $\tilde{d}_n(P)$ of a diagonal polynomial P defined between sequence spaces. As usual, $(e_k)_{k=1}^{\infty}$ denotes the unit vector basis.

Theorem 6.1. *Let $m \geq 2$ and $1 < v < u \leq \infty$ be such that $0 < u - mv \leq uv$ and let r be given by $\frac{1}{r} = \frac{1}{v} - \frac{m}{u}$. If $(\sigma_k)_{k=1}^{\infty} \in l_r$ is such that $|\sigma_1| \geq |\sigma_2| \geq \cdots \geq 0$ then, the diagonal polynomial $P \in \mathscr{P}(^m l_u; l_v)$ given by $P(x) = \sum_{k=1}^{\infty} \sigma_k \xi_k^m e_k$, $x = (\xi_k)_{k=1}^{\infty} \in l_u$, satisfies*

$$\tilde{a}_n(P) = \tilde{c}_n(P) = \Big(\sum_{k=n}^{\infty} |\sigma_k|^r\Big)^{1/r} \geq \tilde{d}_n(P) \geq \frac{1}{2\sqrt{n}} \frac{m!}{m^m} \Big(\sum_{k=n}^{\infty} |\sigma_k|^r\Big)^{1/r}.$$

PROOF. The proof is modeled on [20, p. 158, Theorem], although it has to overcome difficulties that come when dealing with polynomials, as the lack in our context of [20, Theorem 11.5.6]. We include the details for the sake of completeness.

(i) First we deal with the case $1 < v < u < \infty$. For each $n \in \mathbb{N}$ consider the linear operators $J_n : l_u^n \to l_u$, $Q_n : l_v \to l_v^n$ and $P_n : l_u \to l_u$ given by

$$J_n(\xi_1, \dots, \xi_n) = (\xi_1, \dots, \xi_n, 0, 0, \dots),$$
$$Q_n((\xi_n)_{n=1}^{\infty}) = (\xi_1, \dots, \xi_n),$$

and

$$P_n((\xi_n)_{n=1}^{\infty}) = (\xi_1, \dots, \xi_n, 0, 0, \dots).$$

Let $Q := P \circ P_{n-1}$, so that $Q \in \mathscr{P}(^m l_u; l_v)$ with $\operatorname{rank}(Q) < n$. Hence, using Hölder's inequality we have

$$\tilde{a}_n(P) \le \|P - Q\| = \|P - P \circ P_{n-1}\| = \sup_{\|(\xi_k)_{k=1}^{\infty}\|_u \le 1} \|(\sigma_k \xi_k^m)_{k=n}^{\infty}\|_v$$
$$\le \sup_{\|(\xi_k)_{k=1}^{\infty}\|_u \le 1} \left[\left(\sum_{k=n}^{\infty} |\sigma_k|^r \right)^{\frac{1}{r}} \left(\sum_{k=n}^{\infty} |\xi_k|^u \right)^{\frac{m}{u}} \right] \le \left(\sum_{k=n}^{\infty} |\sigma_k|^r \right)^{\frac{1}{r}},$$

that is,

$$\tilde{a}_n(P) \le \left(\sum_{k=n}^{\infty} |\sigma_k|^r \right)^{\frac{1}{r}}. \tag{1}$$

We now show the reverse inequality. Suppose that $|\sigma_s| > 0$ for some $s \ge n$. Then define a one-to-one operator $D : l_u^s \to l_\infty^s$ by $D((\xi_1, \dots, \xi_s)) = (|\sigma_1|^{-r/u} \xi_1, \dots, |\sigma_s|^{-r/u} \xi_s)$. By [20, p. 158, Lemma 1] for every subspace M of l_u^s with codimension $\operatorname{codim}(M) < n$ there is $e = (\varepsilon_1, \dots, \varepsilon_s) \in D(M)$ such that $\|e\|_\infty = 1$ and $\operatorname{card}(K) \ge s - n + 1$, where $K = \{k : |\varepsilon_k| = 1\}$ and $\operatorname{card}(K)$ is the cardinality of the set K.

Next write $R_s = Q_s \circ P \circ J_s$ and denote the inclusion map from M into l_u^s by i_M. Now, if we define $x := D^{-1} e = (|\sigma_1|^{r/u} \varepsilon_1, \dots, |\sigma_s|^{r/u} \varepsilon_s)$ then

$$R_s \circ i_M(x) = Q_s \circ P \circ J_s((|\sigma_1|^{r/u} \varepsilon_1, \dots, |\sigma_s|^{r/u} \varepsilon_s))$$
$$= (|\sigma_1|^{1 + \frac{r}{u} m} \varepsilon_1^m, \dots, |\sigma_s|^{1 + \frac{r}{u} m} \varepsilon_s^m)$$
$$= (|\sigma_1|^{\frac{r}{v}} \varepsilon_1^m, \dots, |\sigma_s|^{\frac{r}{v}} \varepsilon_s^m).$$

Therefore $\|R_s \circ i_M(x)\|_v = \left(\sum_{k=1}^{s} |\varepsilon_k|^{mv} |\sigma_k|^r \right)^{\frac{1}{v}}$, and consequently it follows from [20, p. 158, Lemma 2] that

$$\|R_s \circ i_M\| \geq \frac{\|R_s \circ i_M(x)\|_v}{\|x\|_u^m} = \frac{\left(\sum_{k=1}^s |\varepsilon_k|^{mv} |\sigma_k|^r\right)^{\frac{1}{v}}}{\left(\sum_{k=1}^s |\varepsilon_k|^u |\sigma_k|^r\right)^{\frac{m}{u}}}$$

$$\geq \frac{\left(\sum_{k\in K} |\varepsilon_k|^{mv} |\sigma_k|^r\right)^{\frac{1}{v}}}{\left(\sum_{k\in K} |\varepsilon_k|^u |\sigma_k|^r\right)^{\frac{m}{u}}} = \left(\sum_{k\in K} |\sigma_k|^r\right)^{\frac{1}{v}-\frac{m}{u}} \geq \left(\sum_{k=n}^{s} |\sigma_k|^r\right)^{\frac{1}{r}}.$$

This implies that

$$\text{(2)} \qquad \inf\{\|R_s \circ i_M\| : M \subset l_u^s, \operatorname{codim}(M) < n\} \geq \left(\sum_{k=n}^{s} |\sigma_k|^r\right)^{\frac{1}{r}}.$$

Now we claim that

$$\text{(3)} \qquad \tilde{c}_n(R_s) \geq \inf\{\|R_s \circ i_M\| : M \subset l_u^s, \operatorname{codim}(M) < n\}.$$

In fact, given any subspace $M \subset l_u^s$ with $\operatorname{codim}(M) < n$ if we take any $Q \in \mathscr{P}(^m M; l_v^s)$ with $\operatorname{rank}(Q) < n$ then, since

$$\|R_s \circ i_M\| = \|\kappa_{l_v^s} \circ R_s \circ i_M\| \leq \|\kappa_{l_v^s} \circ R_s \circ i_M - Q\| + \|Q\|$$

we get that

$$\begin{aligned}\|R_s \circ i_M\| &\leq \inf\{\|\kappa_{l_v^s} \circ R_s \circ i_M - Q\| : Q \in \mathscr{P}(^m M; l_v^s), \operatorname{rank}(Q) < n\} \\ &= \tilde{a}_n(\kappa_{l_v^s} \circ R_s \circ i_M)\end{aligned}$$

or,

$$\|R_s \circ i_M\| \leq \tilde{c}_n(R_s \circ i_M) \leq \tilde{c}_n(R_s),$$

from which we obtain the inequality (3). Therefore from (2) and (3) we have that $\tilde{c}_n(R_s) \geq \left(\sum_{k=n}^s |\sigma_k|^r\right)^{\frac{1}{r}}$ for all $s \geq n$. It follows that

$$\text{(4)} \qquad \tilde{c}_n(P) \geq \tilde{c}_n(R_s) \geq \left(\sum_{k=n}^{s} |\sigma_k|^r\right)^{\frac{1}{r}},$$

for all $s \geq n$. Now, combining (1) and (4) we get

$$\left(\sum_{k=n}^{\infty} |\sigma_k|^r\right)^{\frac{1}{r}} \leq \tilde{c}_n(P) \leq \tilde{a}_n(P) \leq \left(\sum_{k=n}^{\infty} |\sigma_k|^r\right)^{\frac{1}{r}}$$

and, applying Theorem 5.5, we get the desired inequalities stated in the theorem.

(ii) To prove the case $1 < v = r < u = \infty$, we need to include some minor modifications to the above. We keep the notation used in the part (i). We first note that

$$\tilde{a}_n(P) \leq \|P - Q\| = \|P - P \circ P_{n-1}\|$$
$$= \sup_{\|(\xi_k)_{k=1}^{\infty}\|_{\infty} \leq 1} \|(\sigma_k \xi_k^m)_{k=n}^{\infty}\|_v \leq \Big(\sum_{k=n}^{\infty} |\sigma_k|^v\Big)^{\frac{1}{v}}.$$

Conversely, assume that $|\sigma_s| > 0$ for some $s \geq n$. Note that D is now the identity map and $x = e$. Then,

$$\|R_s \circ i_M\| \geq \|R_s \circ i_M(x)\|_v = \Big(\sum_{k=1}^{s} |\varepsilon_k|^{mv} |\sigma_k|^v\Big)^{\frac{1}{v}}$$
$$\geq \Big(\sum_{k \in K} |\varepsilon_k|^{mv} |\sigma_k|^v\Big)^{\frac{1}{v}} \geq \Big(\sum_{k=n}^{s} |\sigma_k|^v\Big)^{\frac{1}{v}}.$$

Hence, $\tilde{c}_n(R_s) \geq \big(\sum_{k=n}^{s} |\sigma_k|^v\big)^{\frac{1}{v}}$ for all $s \geq n$. Therefore

$$\tilde{a}_n(P) \geq \tilde{c}_n(P) \geq \tilde{c}_n(R_s) \geq \Big(\sum_{k=n}^{s} |\sigma_k|^v\Big)^{\frac{1}{v}}$$

for all $s \geq n$, and the result follows as in the part (i). □

Note that for any $m \geq 2$ and for the diagonal polynomial $P \in \mathscr{P}(^m l_u; l_v)$ given in the above theorem we have that $\tilde{a}(P) = \tilde{c}(P) = \tilde{d}(P) = 0$, as expected since this polynomial is compact (see, [18]).

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