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Partial Differential Equations — *Eigenvalue problem for fractional Kirchhoff Laplacian*, by J. Tyagi, communicated on June 15, 2017.

Abstract. — In this note, we discuss the isolatedness, simplicity and nodal estimate for the first eigenvalue of fractional Laplacian of Kirchhoff type. This work is motivated by the recent works on the fractional eigenvalues.

Key words: Fractional Laplacian, variational methods, simplicity and isolatedness

Mathematics Subject Classification: 35J25, 35J60

1. Introduction

In this note, we are interested to discuss the eigenvalue problem for the following fractional Laplacian of Kirchhoff type

$$\begin{cases} M\Big(\iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)(-\Delta)^s u(x) = \Lambda u \quad \text{in } \Omega, \\ u = 0 \quad \text{in } \mathbb{R}^n/\Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^n$ with smooth boundary, $n > 2s$ $(0 < s < 1)$ and $(-\Delta)^s$ stands for the fractional Laplacian. Let us assume that

(H) $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function which satisfies

$$M(t) \geq m_0, \quad \forall t \geq t_0.$$

Let $u : \mathbb{R}^n \to \mathbb{R}$, the fractional Laplacian of u is defined as follows:

$$(-\Delta)^s u(x) = C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy = C_{n,s} \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy,$$

where

$$C_{n,s} = \Big(\int_{\mathbb{R}^n} \frac{1-\cos\zeta}{|\zeta|^{n+2s}}\,d\zeta\Big)^{-1},$$

which is a normalization constant, see Section 3 [5].

One can also write the above singular integral as follows:

$$(1.2)\qquad (-\Delta)^s u(x) = -\frac{C_{n,s}}{2}\int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\,dy,$$

$$\forall x \in \mathbb{R}^n,\ u \in S(\mathbb{R}^n),$$

see [5]. When $s < \frac{1}{2}$ and $f \in C^{0,\alpha}(\mathbb{R}^n)$ with $\alpha > 2s$, or if $f \in C^{1,\alpha}(\mathbb{R}^n)$, $1 + 2\alpha > 2s$, the above integral is well-defined.

Following [14], suppose X denotes the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to Ω of any function g in X belongs to $L^2(\Omega)$ and the map

$$(x, y) :\to \frac{g(x) - g(y)}{|x-y|^{\frac{n}{2}+s}} \in L^2(\mathbb{R}^{2n}\backslash(\mathscr{C}\Omega \times \mathscr{C}\Omega)),$$

where $\mathscr{C}\Omega = \mathbb{R}^n\backslash\Omega$. Moreover,

$$X_0 = \{g \in X : g = 0 \text{ a.e. in } \mathbb{R}^n\backslash\Omega\}.$$

Let $Q = R^{2n}\backslash(\mathscr{C}\Omega) \times (\mathscr{C}\Omega)$. The space X is endowed with the norm defined as

$$(1.3)\qquad \|g\|_X = \|g\|_{L^2(\Omega)} + \Big(\int_Q \frac{|g(x)-g(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{\frac{1}{2}}.$$

One can see easily that $\|.\|_X$ is a norm on X. Using a sort of Poincaré-Sobolev inequality for functions in X_0 (see [14]), one can see that

$$(1.4)\qquad \|u\|_{X_0} = \Big(\int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{\frac{1}{2}}$$

is a norm on X_0 and is equivalent to the norm defined in (1.3). Since $v \in X_0$, so $v = 0$ a.e. in $\mathbb{R}^n\backslash\Omega$ and therefore the integral in (1.4) can be extended to all $\mathbb{R}^{2n}$. It is easy to see that $(X_0, \|.\|_{X_0})$ is a Hilbert space with the scalar product

$$\langle u, v\rangle_{X_0} := \iint_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy,$$

see [14] for the details.

One can rewrite (1.1) simply as follows:

$$(1.5)\qquad \begin{cases} M(\|u\|^2_{X_0})(-\Delta)^s u(x) = \Lambda u & \text{in } \Omega,\\ u = 0 \quad \text{in } \mathbb{R}^n/\Omega, \end{cases}$$

where $\|u\|^2_{X_0} = \iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy$.

We say that Λ is an eigenvalue of (1.5) if there exists a nontrivial weak solution $u \in X_0$ to (1.5), i.e.,

$$(1.6)\qquad M(\|u\|^2_{X_0})\int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}}u.(-\Delta)^{\frac{s}{2}}\phi\,dx = \int_\Omega \Lambda u\phi\,dx, \quad \forall \phi \in X_0.$$

When $s \to 1^-$, then using the asymptotics of the constant $C_{n,s}$, the equation in (1.1) becomes the elliptic equation of Kirchhoff type

$$-M\Big(\int_\Omega |\nabla u|^2\,dx\Big)\Delta u = \Lambda u \quad \text{in } \Omega, \tag{1.7}$$

where Ω is a smooth domain, see Section 5 [5] for the complete details.

This is called an eigenvalue problem for Kirchhoff equation because it is a generalization of the well-known D'Alembert wave equation

$$\rho u_{tt} - \Big(\frac{P_0}{h} + \frac{E}{2L}\int_0^L |u_x|^2\,dx\Big)u_{xx} = \Lambda u, \tag{1.8}$$

which was first proposed by Kirchhoff in [9] for free vibrations of elastic strings, where L is the length of the string, h is the area of cross section, E is the Young modulus of the material and P_0 is the initial tension, Λ is a parameter which denotes the eigenvalue of the problem. This work is inspired by the recent works on the eigenvalue problem for fractional p-Laplace equation [8, 11], where the authors investigate several interesting properties and asymptotic behavior of the eigenvalue. For the linear fractional Laplace equation, the isolatedness, simplicity and nodal estimate for the first eigenvalue has been established in a number of references, see for instance, [14, 6]. For the simplicity, nodal estimate and other qualitative properties of the first eigenvalue of fractional p-Laplace equation, we refer to [8, 11]. For other related results to linear fractional Laplacian, we refer to [7, 10, 16] and for improved Sobolev embeddings, profile decomposition and related qualitative results for fractional Sobolev spaces, we refer to [12]. We refer to [5] and references therein for an elementary introduction on this subject. Motivated by the above research works, it is natural to ask whether one can obtain similar results for the first eigenvalue of fractional Kirchhoff Laplacian. In this note, we answer this question. In fact, we show a relationship between the eigenvalues of linear fractional Laplacian and fractional Kirchhoff Laplacian and using this, we establish the isolatedness, simplicity and other related properties of the first eigenvalue of fractional Kirchhoff Laplacian. More precisely, we state the main theorem, which we will prove in this paper.

THEOREM 1.1. *Let* $2s < n < 4s$, $0 < s < 1$ *and* (H) *hold. Then*

(i) (Λ_1, u_1) *is the first eigenpair of* (1.1), *where*

$$\Lambda_1 = M(t^2\|\phi\|_{X_0}^2)\lambda_1, \quad \textit{and} \quad u_1 = t\phi_1, \quad \textit{for some } t > 0,$$

where (λ_1, ϕ_1) *is the first eigenpair of* (1.9).

(ii) *Any eigenfunction* $\tilde{v} \in X_0$ *associated to a positive eigenvalue* $0 < \Lambda \neq \Lambda_1$ *of* (1.1) *changes sign. Moreover, if* N *is a nodal domain of* $\tilde{v}$, *then*

$$|N| \geq \Big(\frac{m_0}{c\lambda}\Big)^{\frac{n}{2s}},$$

where c is some constant depending on n and $| \ |$ *denotes the Lebesgue measure of the set.*

(iii) *Any eigenfunction corresponding to* $0 < \Lambda \neq \Lambda_1$ *has only a finite number of nodal domains.*

(iv) Λ_1 *is isolated.*

(v) Λ_1 *is simple.*

REMARK 1.2. We remark that in the above theorem, we have assumed that $n < 4s$. This is essentially needed to get the better regularity of the solution in view of fractional Sobolev embeddings.

REMARK 1.3. We remark that one can even consider the above questions for a more general integro-differential operator. More precisely, replacing the fractional Laplacian in (1.1) with the integro-differential operator given by

$$\mathscr{L}_K u(x) = PV \int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x))K(y)\, dy, \quad x \in \mathbb{R}^n,$$

where the symmetric function K is a Gagliardo-type kernel with measurable coefficients. We believe that Theorem 1.1 holds for an eigenvalue problem involving the above type of operators. We consider this question in the next project and refer to the recent works [1, 4] and the reference therein dealing with such fractional operators.

Let $\lambda_1 > 0$ be the first eigenvalue of $(-\Delta)^s$ in Ω and $\phi_1 > 0$ be the corresponding eigenfunction (first eigenfunction), i.e.,

$$\begin{cases} (-\Delta)^s \phi_1 = \lambda_1 \phi_1 & \text{in } \Omega, \\ \phi_1 > 0 & \text{in } \Omega, \\ \phi_1 = 0 & \text{in } \mathbb{R}^n \backslash \Omega. \end{cases} \tag{1.9}$$

The variational characterization of λ_1 is given by

$$\lambda_1 = \inf \left\{ \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} v|^2\, dx : v \in X_0 \text{ and } \int_\Omega v^2\, dx = 1 \right\}. \tag{1.10}$$

LEMMA 1.4. *Let* (λ_1, ϕ_1) *be the first eigenpair to* (1.9). *Then* (Λ_1, u_1) *is the first eigenpair of* (1.1), *where*

$$\Lambda_1 = M(t^2 \|\phi\|_{X_0}^2)\lambda_1, \quad \text{and} \quad u_1 = t\phi_1, \quad \text{for some } t > 0.$$

PROOF. Taking $u_1 = t\phi_1,\ t > 0$ in (1.5) yields that $\Lambda_1 = M(t^2\|\phi\|_{X_0}^2)\lambda_1$. This completes the proof. □

We will use the following embedding theorem in next proposition.

THEOREM 1.5 ([5]). *The following embeddings are continuous:*

(1) $H^s(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$, $2 \le q \le \frac{2n}{n-2s}$, *if* $n > 2s$,
(2) $H^s(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$, $2 \le q \le \infty$, *if* $n = 2s$,
(3) $H^s(\mathbb{R}^n) \hookrightarrow C_b^j(\mathbb{R}^n)$, *if* $n < 2(s-j)$.

Moreover, for any $R > 0$ and any $p \in [1, 2_*(s))$ the embedding $H^s(B_R) \hookrightarrow\hookrightarrow L^p(B_R)$ is compact, where

$$C_b^j(\mathbb{R}^n) = \{u \in C^j(\mathbb{R}^n) : D^k u \text{ is bounded on } \mathbb{R}^n \text{ for } |k| \le j\}.$$

PROPOSITION 1.6. *Let* $2s < n < 4s$, $0 < s < 1$ *and* (H) *hold. Then any eigenfunction* $\tilde{v} \in X_0$ *associated to a positive eigenvalue* $0 < \Lambda \neq \Lambda_1$ *of* (1.1) *changes sign. Moreover, if* N *is a nodal domain of* $\tilde{v}$, *then*

$$|N| \ge \left(\frac{m_0}{c\lambda}\right)^{\frac{n}{2s}}, \tag{1.11}$$

where c *is some constant depending on* n *and* $|\ |$ *denotes the Lebesgue measure of the set.*

PROOF. Let v be an eigenfunction associated with a positive eigenvalue $0 < \lambda \neq \lambda_1$, of the eigenvalue problem

$$(-\Delta)^s u = \lambda u \quad \text{in } \Omega; \quad u = 0 \quad \text{in } \mathbb{R}^n \backslash \Omega. \tag{1.12}$$

Then by Prop. 3.6 [6], v changes sign in Ω. Since $\tilde{v} \in X_0$ is an eigenfunction associated to the positive eigenvalue $0 < \Lambda \neq \Lambda_1$ of (1.1), so from Lemma 1.4, we may assume that $\tilde{v} = tv$ and $\Lambda = M(t^2\|v\|_{X_0}^2)\lambda$ for some $t > 0$. Since $\lambda \neq \lambda_1$, so using the fact that $c\phi$ is also an eigenfunction of (1.12) associated with λ_1 for any non-zero scalar c, one can see easily that $\Lambda \neq \Lambda_1$, and $\tilde{v}$ also changes sign in Ω.

Now we prove the estimate (1.11). Assume that $\tilde{v} > 0$ in N, the case $\tilde{v} < 0$ can be dealt similarly. Since $2s < n < 4s$, $0 < s < 1$, so using the similar arguments as in [3, 13, 15], we have $\tilde{v} \in C(\mathbb{R}^n) \cap X_0$. Then $\tilde{v}|_N \in H_0^s(N)$ and therefore the function η defined as

$$\eta(x) = \begin{cases} \tilde{v}(x), & x \in N; \\ 0, & x \in \mathbb{R}^n \backslash N \end{cases}$$

belongs to X_0. Now using η as a test function in the weak formulation of (1.1), we get

$$M(\|\tilde{v}\|_{X_0}^2) \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} \tilde{v}.(-\Delta)^{\frac{s}{2}} \eta \, dx = \Lambda \int_{\Omega} \tilde{v}.\eta \, dx, \quad \forall \eta \in X_0.$$

This implies that

$$(1.13)\qquad \begin{aligned} M(\|\tilde v\|^2_{X_0})&\int_N |(-\Delta)^{\frac{s}{2}}\tilde v|^2\,dx \\ &= \Lambda\int_N \tilde v^2\,dx \\ &\le \Big(\int_N \tilde v^{\frac{2n}{n-2s}}\,dx\Big)^{\frac{n-2s}{n}}|N|^{\frac{2s}{n}} \\ &= \lambda\|\tilde v\|^2_{L^{\frac{2n}{n-2s}}}|N|^{\frac{2s}{n}} \\ &\le c\lambda\|\tilde v\|^2_{H_0^s(N)}|N|^{\frac{2s}{n}} \\ &\quad \text{(by Theorem 1.5, where } c \text{ is an embedding constant),} \\ &= c\lambda\Big(\int_N |(-\Delta)^{\frac{s}{2}}\tilde v|^2\,dx\Big)|N|^{\frac{2s}{n}}. \end{aligned}$$

Since $M(t) \ge m_0$, $\forall t \ge 0$, so from the last inequality, it implies that

$$(1.14)\qquad m_0\int_N |(-\Delta)^{\frac{s}{2}}\tilde v|^2\,dx \le c\lambda\Big(\int_N |(-\Delta)^{\frac{s}{2}}\tilde v|^2\,dx\Big)|N|^{\frac{2s}{n}}$$

which yields

$$|N| \ge \Big(\frac{m_0}{c\lambda}\Big)^{\frac{n}{2s}}.$$

This proves the estimate. □

Corollary 1.7. *Let $2s < n < 4s$, $0 < s < 1$ and* (H) *hold. Then any eigenfunction corresponding to $0 < \Lambda \ne \Lambda_1$ has only a finite number of nodal domains.*

Proof. Let u be an eigenfunction corresponding to $\Lambda > 0$. Let N_j be a component of one of the sets $\{x \in \Omega : u(x) > 0\}$ and $\{x \in \Omega : u(x) < 0\}$, then $u \in X_0$. Now it follows from (1.11) that

$$|\Omega| \ge \sum_j |N_j| \ge \Big(\frac{m_0}{c\lambda}\Big)^{\frac{n}{2s}}\sum_j 1$$

so that the number of nodal domains is bounded by $|\Omega|\big(\frac{m_0}{c\lambda}\big)^{-\frac{n}{2s}}$ and this completes the proof. □

Proposition 1.8. *Let $2s < n < 4s$, $0 < s < 1$. Let* (H) *hold. Then Λ_1 is isolated, that is, there exists $\delta > 0$ such that there are no other eigenvalues of* (1.1) *in the interval $(\Lambda_1, \Lambda_1 + \delta)$.*

Proof. We will prove this proposition by the method of contradiction. Suppose that there exists a sequence of eigenvalues Λ_n of (1.1) with $0 < \Lambda_n \searrow \Lambda_1$. Let U_n

be a sequence of eigenfunctions associated with Λ_n, i.e,

$$\begin{cases} M(\|U_n\|_{X_0}^2)(-\Delta)^s U_n = \Lambda_n U_n & \text{in } \Omega, \\ U_n = 0 & \text{in } \mathbb{R}^n \backslash \Omega. \end{cases} \tag{1.15}$$

Since

$$0 < M(\|U_n\|_{X_0}^2) \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} U_n|^2 \, dx = \Lambda_n \int_{\Omega} U_n^2 \, dx,$$

we can define

$$V_n := \frac{M(\|U_n\|_{X_0}^2) U_n}{\left(\int_{\Omega} U_n^2 \, dx\right)^{\frac{1}{2}}}.$$

It is easy to see that V_n is bounded in X_0. So there exists a subsequence (still, we denote it by V_n) and $V \in X_0$ such that

$$\begin{aligned} V_n &\rightharpoonup V \quad \text{in } X_0. \\ V_n &\to V \quad \text{in } L^p, \ p \in [1, 2_*(s)). \\ V_n(x) &\to V(x) \quad \text{a.e. } x \in \Omega. \end{aligned} \tag{1.16}$$

It is easy to see that $\int_{\Omega} V^2 \, dx = 1$ and

$$\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} V|^2 \, dx \leq \liminf_{n \to \infty} \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} V_n|^2 \, dx = \Lambda_1$$

and therefore $\Lambda_1 = \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} V|^2 \, dx$. This implies that V is an eigenfunction associated with Λ_1. Now one can see that $|V|$ is also an eigenfunction associated with Λ_1 and by strong maximum principle [2], $|V| > 0$ in Ω. This implies that either $V > 0$ or $V < 0$. Let us take $V > 0$ (the proof in the case $V < 0$ is dealt similarly). Let $\Omega_n^- = \{x \in \Omega : V_n(x) < 0\}$. Now by the Egorov's theorem, $V_n \to v$ uniformly on Ω with the exception of the set of arbitrarily small measure. This implies that

$$|\Omega_n^-| \to 0 \quad \text{as } n \to \infty, \tag{1.17}$$

but this contradicts to the estimate (1.11) for Ω_n^- and hence the proof is completed. □

It has been proved that λ_1 is simple in the sense that the eigenfunctions associated to it are merely a constant multiple of each other, see for instance [14]. We will be assuming that λ_1 is simple and prove the simplicity of the first eigenvalue of Λ_1 of (1.1) in next proposition.

PROPOSITION 1.9. *Λ_1 is simple.*

PROOF. Since λ_1 is simple so let w_1 and $w_2 \in X_0$ be eigenfunctions associated with λ_1. Then by the simplicity of λ_1, $w_1 = cw_2$, for some non zero scalar c. Let $u_1 = t_1 w_1$ and $u_2 = t_2 w_2$ for some $t_1 > 0$, $t_2 > 0$. Now from Lemma 1.4, we have

$$\Lambda_1 = M(t_1^2 \|w_1\|_{X_0}^2)\lambda_1 = M(t_2^2 \|w_2\|_{X_0}^2)\lambda_1$$

and u_1 and u_2 are eigenfunctions of (1.1) corresponding to Λ_1, i.e.,

$$M(\|u_1\|_{X_0}^2)(-\Delta)^s u_1 = \Lambda_1 u_1 \quad \text{and} \quad M(\|u_2\|_{X_0}^2)(-\Delta)^s u_2 = \Lambda_1 u_2.$$

Now we have

$$u_1 = t_1 w_1 = t_1 c w_2 = t_1 c \frac{u_2}{t_2} = \frac{t_1 c}{t_2} u_2 = \tilde{c} u_2,$$

which proves the claim. □

2. Proof of Theorem 1.1

PROOF. The proof follows from Lemma 1.4, Prop. 1.6, Cor. 1.7, Prop. 1.8 and Prop. 1.9. □

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References

[1] G. Autuori - A. Fiscella - P. Pucci, *Stationary Kirchhoff problems involving a fractional elliptic operator and a critical nonlinearity*, Nonlinear Anal. 125 (2015), 699–714.

[2] X. Cabré - Y. Sire, *Nonlinear equations for fractional Laplacians*, I: Regularity, maximum principles, and Hamiltonian estimates, Ann. I. H. Poincaré 31, 2014, 23–53.

[3] L. Caffarelli - S. Salsa - L. Silvestre, *Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian*, Invent. Math. 171, 2008, 425–461.

[4] A. Di Castro - T. Kuusi - G. Palatucci, *Nonlocal Harnack inequalities*, J. Funct. Anal. 267 (2014), no. 6, 1807–1836.

[5] E. Di Nezza - G. Palatucci - E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. 136, 2012, no. 5, 521–573.

[6] G. Dwivedi - J. Tyagi - R. B. Verma, *On the bifurcation results for fractional Laplace equations*, Math. Nachr. 1–15, 2017 / DOI 10.1002/mana.201600250.

[7] R. L. Frank - G. Leander, *Refined semiclassical asymptotics for fractional powers of the Laplace operator*, J. Reine Angew. Math. 712, 2016, 1–37.

[8] G. Franzina - G. Palatucci, *Fractional p-eigenvalues*, Riv. Mat. Univ. Parma, (N.S.) 5 2014, no. 2, 373–386.

[9] G. Kirchhoff, *Mechanik*, Teubner, Leipzig, 1883.

[10] M. Kwaśnicki, *Eigenvalues of the fractional Laplace operator in the interval*, J. Funct. Anal. 262(5), 2012, 2379–2402.

[11] E. Lindgren - P. Lindqvist, *Fractional eigenvalues*, Calc. Var. Partial Differential Equations 49, 2014, no. 1–2, 795–826.

[12] G. Palatucci - A. Pisante, *Improved Sobolev embeddings, profile decomposition, and concentration-compactness for fractional Sobolev spaces*, Calc. Var. Partial Differential Equations 50 (2014), no. 3–4, 799–829.

[13] X. Ros-Oton - J. Serra, *The extremal solution for the fractional Laplacian*, Calc. Var. Partial Differential Equations 50, 2014, 723–750.

[14] R. Servadei - E. Valdinoci, *Variational methods for non-local operators of elliptic type*, Discrete Contin. Dyn. Syst. 33, 2013, 2105–2137.

[15] L. Silvestre, *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Comm. Pure Appl. Math. 60 (1), 2007, 67–112.

[16] A. Zoia - A. Rosso - M. Kardar, *Fractional Laplacian in bounded domains*, Phys. Rev. E (3) 76(2), 021116, 11, 2007.

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