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Geometry — *Elementary solution of an infinite sequence of instances of the Hurwitz problem*, by Tom Ferragut and Carlo Petronio, communicated on December 15, 2017.

Abstract. — We prove that there exists no branched cover from the torus to the sphere with degree $3h$ and 3 branching points in the target with local degrees $(3, \dots, 3)$, $(3, \dots, 3)$, $(4, 2, 3, \dots, 3)$ at their preimages. The result was already established by Izmestiev, Kusner, Rote, Springborn, and Sullivan, using geometric techniques, and by Corvaja and Zannier with a more algebraic approach, whereas our proof is topological and completely elementary: besides the definitions, it only uses the fact that on the torus a simple closed curve can only be *trivial* (in homology, or equivalently bounding a disc, or equivalently separating) or *non-trivial*.

Key words: Surface branched cover, Hurwitz problem

Mathematics Subject Classification: 57M12

A (topological) branched cover between surfaces is a map $f : \tilde{\Sigma} \to \Sigma$, where $\tilde{\Sigma}$ and Σ are closed and connected 2-manifolds and f is locally modeled (in a topological sense) on maps of the form $(\mathbb{C}, 0) \ni z \mapsto z^k \in (\mathbb{C}, 0)$. If $k > 1$ the point 0 in the target $\mathbb{C}$ is called a *branching point*, and k is called the local degree at the point 0 in the source $\mathbb{C}$. There are finitely many branching points, removing which, together with their pre-images, one gets a genuine cover of some degree d. If there are n branching points, the local degrees at the points in the pre-image of the j-th one form a partition π_j of d of some length ℓ_j, and the following Riemann–Hurwitz relation holds:

$$\chi(\tilde{\Sigma}) - (\ell_1 + \cdots + \ell_n) = d(\chi(\Sigma) - n).$$

The very old *Hurwitz problem* asks whether given $\tilde{\Sigma}, \Sigma, d, n, \pi_1, \dots, \pi_n$ satisfying this relation there exists some f realizing them. (For a non-orientable $\tilde{\Sigma}$ and/or Σ the Riemann–Hurwitz relation must actually be complemented with certain other necessary conditions, but we will not get into this here.) A number of partial solutions of the Hurwitz problem have been obtained over the time, and we quickly mention here the fundamental [4], the survey [10], and the more recent [7, 8, 2, 9, 11].

Certain instances of the Hurwitz problem recently emerged in the work of M. Zieve [12] and his team of collaborators, including in particular the case where the source surface is the torus T^2, the target is the sphere S^2, the degree is $d = 3h$, and there are $n = 3$ branching points with associated partitions $(3, \dots, 3)$, $(3, \dots, 3)$, $(4, 2, 3, \dots, 3)$ of d. It actually turns out that this branch datum is indeed not realizable, as Zieve had conjectured, which follows from results established in [6] using geometric techniques (holonomy of Euclidean structures). The same fact was also elegantly proved by Corvaja and Zannier [3] with a more algebraic approach. In this note we provide yet another proof of the same result. Our approach is purely combinatorial and completely elementary: besides the definitions, it only uses the fact that on the torus a simple closed curve can only be *trivial* (in homology, or equivalently bounding a disc, or equivalently separating) or *non-trivial*.

We conclude this introduction with the formal statement of the (previously known) result established in this note:

THEOREM. *There exists no branched cover $f : T^2 \to S^2$ with degree $d = 3h$ and 3 branching points with associated partitions*

$$(3, \dots, 3), \quad (3, \dots, 3), \quad (4, 2, 3, \dots, 3).$$

1. Dessins d'enfant

In this section we quickly review the beautiful technique of dessins d'enfant due to Grothendieck [1, 5], noting that, at the elementary level at which we exploit it, it only requires the definition of branched cover and some very basic topology.

Let $f : \tilde{\Sigma} \to S^2$ be a degree-d branched cover from a closed connected surface $\tilde{\Sigma}$ to the sphere S^2, branched over 3 points p_1, p_2, p_3 with local degrees $\pi_j = (d_{ji})_{i=1}^{\ell_j}$ over p_j. In S^2 take a simple arc σ with vertices at p_1 (white) and p_2 (black), and we view S^2 as being obtained from the (closed) bigon $\tilde{B}$ of Fig. 1-left by attaching both the edges of $\tilde{B}$ to σ so to match the vertex colors. This gives a realization of S^2 as the quotient of $\tilde{B}$ under the identification of its two edges. Let $\lambda : \tilde{B} \to S^2$ be the projection to the quotient. Note that the complement of σ in S^2 is an open disc B, whose closure in S^2 is the whole of S^2, but the restriction of λ to the interior of $\tilde{B}$ is a homeomorphism with B, so we can view $\tilde{B}$ as the abstract closure of B.

Now set $D = f^{-1}(\sigma)$. Then D is a graph with white vertices of valences $(d_{1i})_{i=1}^{\ell_1}$ and black vertices of valences $(d_{2i})_{i=1}^{\ell_2}$, and D is bipartite (every edge has a white and a black end). Moreover the complement of D in $\tilde{\Sigma}$ is a union of open discs $(R_i)_{i=1}^{\ell_3}$, where R_i is the interior of a polygon with $2d_{3i}$ vertices of alternating white and black color. This means that, if $\tilde{R}_i$ is the polygon of Fig. 1-right (with $2d_{3i}$ vertices), there exists a map $\lambda_i : \tilde{R}_i \to \tilde{\Sigma}$ which restricted to the interior of $\tilde{R}_i$ is a homeomorphism with R_i, and restricted to each edge is a homeomorphism

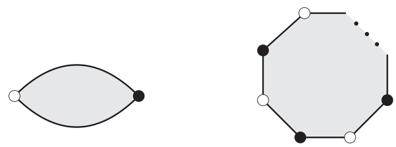

Figure 1. A bigon and a polygon.

with an edge of D matching the vertex colors. So $\tilde{R}_i$ can be viewed as the abstract closure of R_i. The map λ_i may fail to be a homeomorphism between $\tilde{R}_i$ and the closure of R_i in $\tilde{\Sigma}$ if R_i is multiply incident to some vertex of D or doubly incident to some edge of D. We say that R_i has embedded closure if λ_i is injective, hence a homeomorphism between $\tilde{R}_i$ and the closure of R_i in $\tilde{\Sigma}$.

We will say that a bipartite graph D in $\tilde{\Sigma}$ with valences $(d_{1i})_{i=1}^{\ell_1}$ at the white vertices and $(d_{2i})_{i=1}^{\ell_2}$ at the black ones, and complement consisting of polygons having $(2d_{3i})_{i=1}^{\ell_3}$ edges, *realizes* the branched cover $f:\tilde{\Sigma}\to S^2$ with 3 branching points and local degrees π_1, π_2, π_3 over them, where $\pi_j=(d_{ji})_{i=1}^{\ell_j}$. This terminology is justified by the fact that f exists if and only if D does.

2. Proof of the Theorem

Suppose by contradiction that a branched cover $f:T^2\to S^2$ as in the statement exists, and let D be a dessin d'enfant on T^2 realizing it, as explained in the previous section, with white and black vertices corresponding to the first two partitions, so the complementary regions are one square S, some hexagons H and one octagon O, shown abstractly in Fig. 2. Let $\hat{D}$ be the graph dual to D (which is well-defined because the complement of D is a union of open discs), and let Γ be the set of all simple loops in $\hat{D}$ which are simplicial (concatenations of edges), and non-trivial (non-zero in $H_1(T^2)$, or, equivalently, not bounding a disc on T^2, or, equivalently, not separating T^2). Since $T^2\backslash\hat{D}$ is also a union of open discs, the inclusion $\hat{D}\hookrightarrow T^2$ induces a surjection $H_1(\hat{D})\to H_1(T^2)$. Moreover $H_1(\hat{D})$ is generated by simple simplicial loops, so Γ is non-empty. We now define Γ_n as the set of loops in Γ consisting of n edges, and we prove by induction that $\Gamma_n=\emptyset$, thereby showing that $\Gamma=\emptyset$ and getting the desired contradiction.

For $n=1$ we prove the slightly stronger fact (needed below) that every region has embedded closure, namely, that its closure in T^2 is homeomorphic to its abstract closure. Taking into account the symmetries (including a color switch) this may fail to happen only if some edge a in Fig. 2 is glued to b or c of the same region (if two vertices of a region are glued together then two edges also are, since the vertices have valence 3). The case $b=a$ implies V has valence 1, so it is impossible. If $c=a$ in H we have the situation of Fig. 3-left, and each of the neighboring regions already has 3 vertices of one color, so it cannot be S. If it is an H, it also has a gluing of type $c=a$. Iterating, we have a tube of H's as in Fig. 3-centre that at some point must hit O from both sides, which is impossible

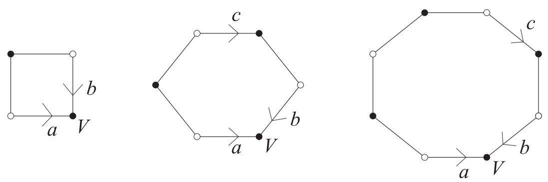

Figure 2. The regions. The notation V, a, b, c, is only needed for the base step of our induction argument.

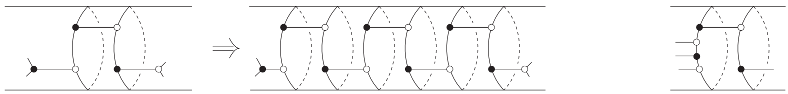

Figure 3. A tube of H's cannot end at an O from both sides. A non-embedded O gives a non-embedded H.

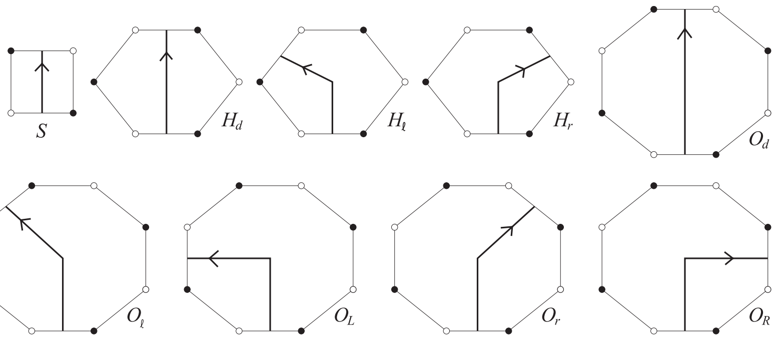

Figure 4. The ways γ can cross a region. Note that the vertex colors may be switched.

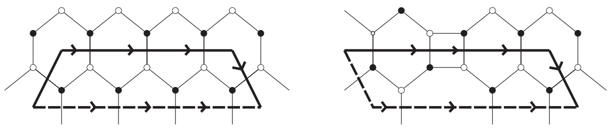

Figure 5. Impossible configurations.

because the terminal region already contains 5 vertices of each color. If $c = a$ in O then we have Fig. 3-right, so a neighboring region also has non-embedded closure, which was already excluded.

Let us now assume that $n \geq 2$ and $\Gamma_m = \emptyset$ for all $m < n$. By contradiction, take $\gamma \in \Gamma_n$. From now on in our figures we will use for γ a thicker line than that used for D. We first note that γ cannot enter a region through an edge and leave it from an adjacent edge (otherwise we could reduce its length), so the only ways γ can cross a region are those shown in Fig. 4. Therefore γ is described by a word in the letters S, H_d, H_ℓ, H_r, O_d, O_ℓ, O_L, O_r, O_R, from which we omit the H_d's for simplicity. The vertex coloring implies that the total number of S, H_ℓ, H_r, O_d, O_L, O_R in γ is even.

We now prove that any subword H_rH_r, $H_\ell H_\ell$, SH_r or SH_ℓ is impossible in γ, as shown in Fig. 5 (here the thick dashed line gives a new γ contradicting the minimality of the original one). This already implies the former of the following claims:

(1) No $\gamma \in \Gamma_n$ can contain S but not O;
(2) There exists $\gamma \in \Gamma_n$ consisting of H's only.

To establish the latter, we suppose $O \in \gamma \in \Gamma_n$ and list all the possible cases up to symmetry (which includes switching colors and/or reversing the direction of γ):

$S \notin \gamma$	$O_d \in \gamma \Rightarrow \gamma = O_d H_r (H_\ell H_r)^p$	
	$O_r \in \gamma \Rightarrow \gamma = O_r (H_r H_\ell)^p$	
	$O_R \in \gamma$	$\gamma = O_R H_r (H_\ell H_r)^p$
		$\gamma = O_R H_\ell (H_r H_\ell)^p$
$S \in \gamma \Rightarrow SO \in \gamma$	$O_d \in \gamma \Rightarrow \gamma = S O_d (H_r H_\ell)^p$	
	$O_r \in \gamma$	$\gamma = H_r S O_r (H_r H_\ell)^p$
		$\gamma = H_\ell S O_r (H_\ell H_r)^p$
	$O_R \in \gamma$	

For each of these cases we show in Fig. 6 to 9 a modification of γ which gives a new loop γ' isotopic to γ (and hence in Γ), and not longer than γ. When γ' is shorter than γ we have a contradiction to the minimality of γ, so the case is impossible. To conclude we must show that γ' does not contain O in the cases where it is as long as γ. To do this, suppose that γ' contains O, and construct two loops $\gamma_{1,2}$ by applying one of the three moves of Fig. 10. Note that whatever move ap-

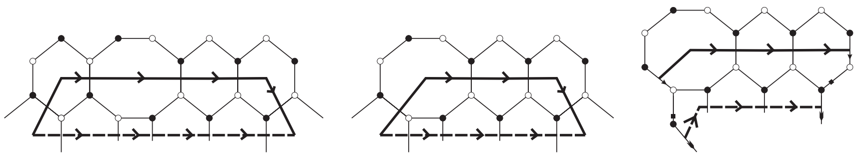

Figure 6. $\gamma = O_d H_r (H_\ell H_r)^p \Rightarrow \exists \gamma' \dots$ (left); $\gamma = O_r (H_r H_\ell)^p \Rightarrow \exists \gamma' \dots$ (centre for $p > 0$ and right for $p = 0$). On the right, as in many figures below, we decorate some edges to indicate that they are glued in pairs.

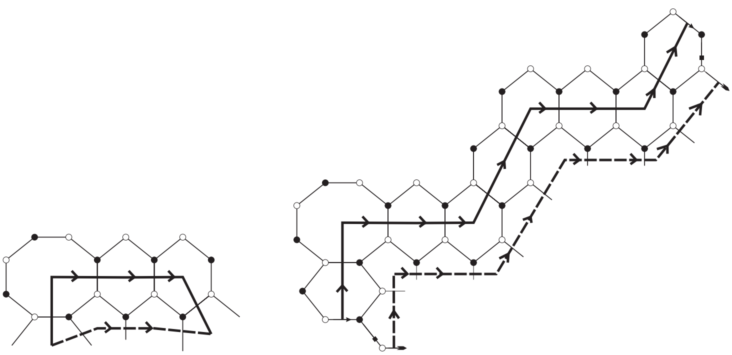

Figure 7. $\gamma = O_R H_r (H_\ell H_r)^p$ impossible (left); $\gamma = O_R H_\ell (H_r H_\ell)^p \Rightarrow \exists \gamma' \dots$ (right).

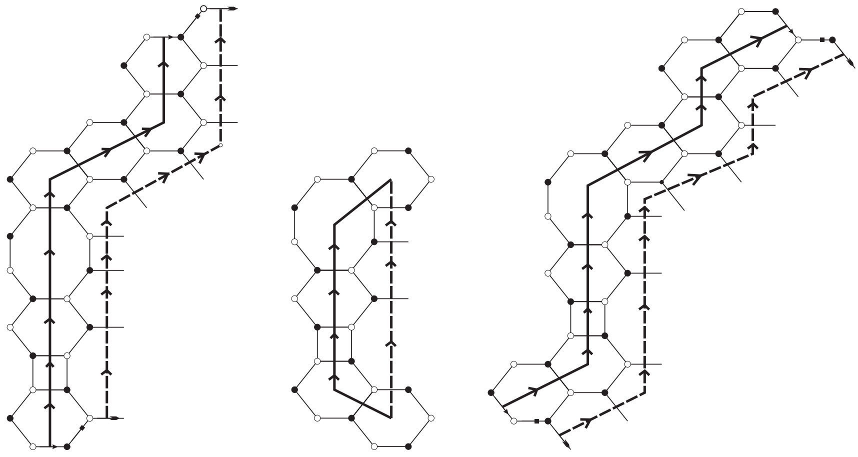

Figure 8. $\gamma = SO_d(H_rH_\ell)^p \Rightarrow \exists\gamma' \ldots$ (left); $\gamma = H_rSO_r(H_rH_\ell)^p$ impossible (centre); $\gamma = H_\ell SO_r(H_\ell H_r)^p \Rightarrow \exists\gamma' \ldots$ (right).

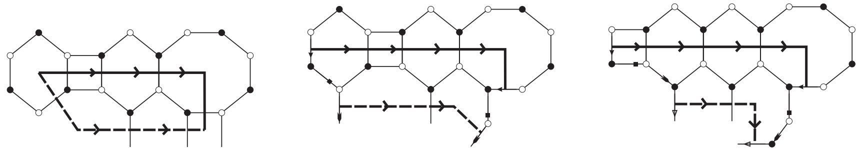

Figure 9. $SO_R \in \gamma$ impossible: if in γ there are m copies of H outside the word SO_R, we treat separately the cases $m \geq 2$ (left), $m = 1$ (centre), and $m = 0$ (right). In the last case, if there are not even H_d's between S and O_R, the absurd comes from the fact that a bigon is created.

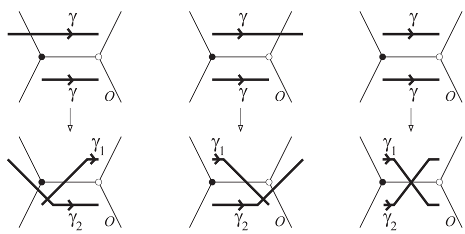

Figure 10. If O appears in γ and immediately to the right of γ, we can construct two loops $\gamma_{1,2}$ of which γ is the homological sum.

plies, γ is the homological sum of γ_1 and γ_2, so at least one of them is non-trivial. If one of the moves of Fig. 10-left/centre applies, the total length of γ_1 and γ_2 is 1 plus the length of γ, but we know that there is no length-1 loop at all (trivial or not), so both γ_1 and γ_2 are shorter than γ, a contradiction. If only the move of Fig. 10-right applies then we are either in Fig. 7-right, or Fig. 8-left or Fig. 8-right

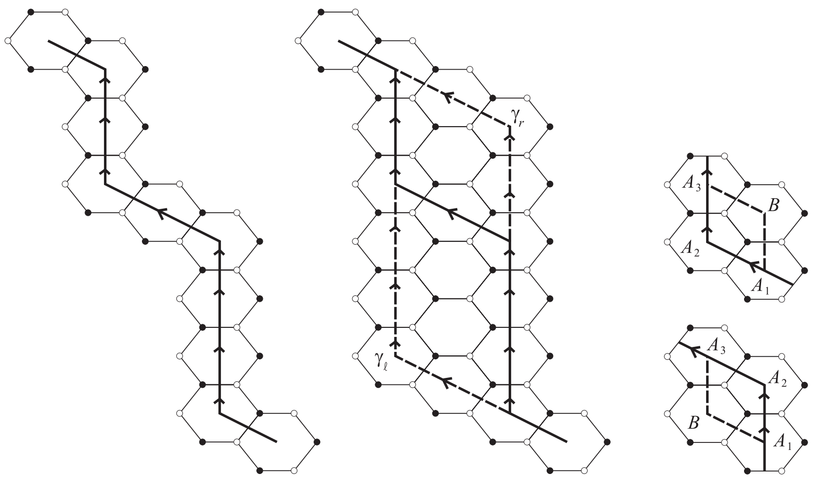

Figure 11. Reducing the number of turns.

and O is the region where γ' makes a left turn; in this case the total length of γ_1 and γ_2 is 2 plus the length of γ, but γ_1 and γ_2 both have length at least 3, so they are both shorter than γ, and again we have a contradiction.

Our next claim is the following:

(3) There exists $\gamma \in \Gamma_n$ described by a word $(H_\ell H_r)^p$ with $p \leq 1$.

By (2) and the fact that subwords $H_\ell H_\ell$ or $H_r H_r$ are impossible in $\gamma \in \Gamma_n$, we have a $\gamma \in \Gamma_n$ described by a word $(H_\ell H_r)^p$. Now suppose $p \geq 2$, consider a portion of γ described by $H_r H_\ell H_r H_\ell$ as in Fig. 11-left and try to construct the two loops γ_ℓ and γ_r as in Fig. 11-centre by repeated application of the moves in Fig. 11-right. If one of γ_ℓ or γ_r exists it belongs to Γ_n and it is described by $(H_\ell H_r)^{p-1}$, so we can conclude recursively. The construction of γ_ℓ or γ_r may fail only if when we apply an elementary move as in Fig. 11-right to $\alpha \in \Gamma_n$ the region B is ...

- the square S; this would contradict (1), so it is impossible;
- already in α; but then B is not one of A_1, A_2, A_3 because all regions are embedded, and it easily follows that α is homologous to the sum of two shorter loops, which is absurd because at least one of them would be non-trivial;
- the octagon O; this is indeed possible, but it cannot happen both to the left and to the right, otherwise we would get a simplicial loop in $\hat{D}$ intersecting γ transversely at one point, whence non-trivial, and shorter than γ (actually, already at least by 1 shorter than the portion of γ described by $H_r H_\ell H_r H_\ell$).

We now include again the H_d's in the notation for the word describing a loop. It follows from (3) that there exists $\gamma \in \Gamma_n$ of shape H_d^q or $H_\ell H_d^q H_r H_d^t$. To conclude the proof we set $\gamma_\ell = \gamma_r = \gamma$ and we apply to γ_ℓ and γ_r as long as possible the following moves (that we describe for γ_ℓ only):

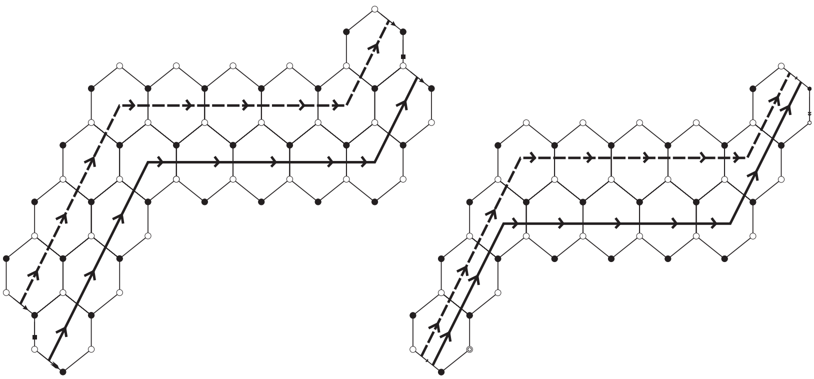

Figure 12. Evolution of γ_ℓ. The second move is performed so that O is not included in the new loop.

- If O is not incident to the left margin of γ_ℓ we entirely push γ_ℓ to its left, as in Fig. 12-left;
- If γ has shape $H_\ell H_d^q H_r H_d^t$ and O is incident to the left margin of γ_ℓ but not to H_ℓ, we note that O is not incident to either $H_d^t H_\ell$ or to $H_\ell H_d^q$, and we partially push γ_ℓ to its left so not to include O, as in Fig. 12-right (this is the case where O is not incident to $H_d^t H_\ell$).

Note that by construction the new γ_ℓ does not contain O, so it also does not contain S by (1), hence it has the same shape H_d^q or $H_\ell H_d^q H_r H_d^t$ as the old γ_ℓ. Therefore at any time γ_ℓ and γ_r have the same shape as the original γ. We stop applying the moves when one of the following situations is reached:

(a) The left margin of γ_ℓ and the right margin of γ_r overlap;
(b) γ_ℓ and γ_r have shape H_d^q and O is incident to the left margin of γ_ℓ and to the right margin of γ_r;
(c) γ_ℓ and γ_r have shape $H_\ell H_d^q H_r H_d^t$ and O is incident to the left margin of γ_ℓ in H_ℓ and to the right margin of γ_r in H_r.

Case (a) with γ_ℓ and γ_r of shape H_d^q is impossible, because the left margin of γ_ℓ and the right margin of γ_r would close up like a zip, leaving no space for S and O, see Fig. 13-left. We postpone the treatment of case (a) with γ_ℓ and γ_r of shape $H_\ell H_d^q H_r H_d^t$, to face the easier cases (b) and (c). For (b), we have the situation of

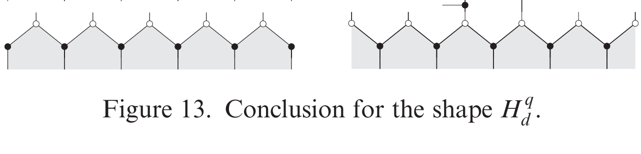

Figure 13. Conclusion for the shape H_d^q.

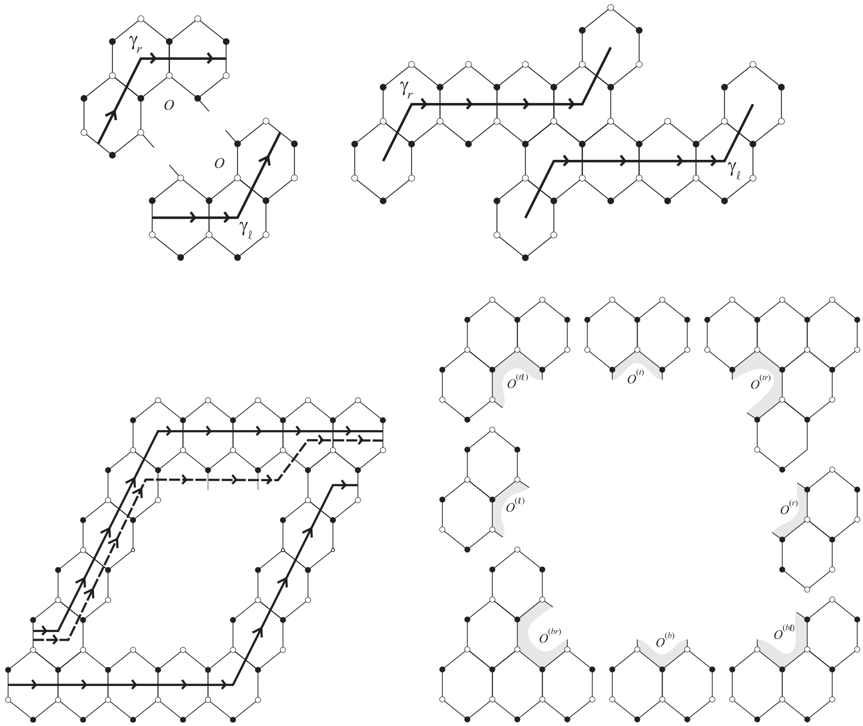

Figure 14. Conclusion for the shape $H_\ell H_d^q H_r H_d^t$.

Fig. 13-right, where in the direction given by the arrow we must have a strip of identical hexagons that can never close up. Case (c), excluding (a), is trivial: the region that should be O cannot close up with fewer than 10 vertices, see Fig. 14-top/left.

In case (a) for the shape $H_\ell H_d^q H_r H_d^t$, the left margin of γ_ℓ can overlap with the right margin of γ_r only along a segment as in Fig. 14-top/right – this segment has type $H_r H_d^s$ in γ_ℓ and $H_d^s H_\ell$ in γ_r, in particular it uses H_r from γ_ℓ and H_ℓ from γ_r, so there is only one. Therefore the rest of γ_ℓ and γ_r delimit an $x \times y$ rhombic area R as in Fig. 14-bottom/left (with $x \times y = 3 \times 4$ in the figure), that must contain S and O. Note that the H's incident to ∂R are pairwise distinct: for the initial γ the left margin cannot be incident to the right margin, otherwise a move as in Fig. 10-left/centre would contradict its minimality, and during the construction of γ_ℓ and γ_r only new H's are added. If O is not incident to one of the four sides of R we can modify γ_ℓ or γ_r as suggested already in Fig. 14-bottom/left. This modification changes the shape of γ_ℓ or γ_r, but:

- The modified loop is still minimal and does not contain O, so it does not contain S;
- The area R into which O and S are forced to lie remains a rhombus,

- The H's incident to ∂R are pairwise distinct (otherwise R closes up leaving no space for O or S).

We can iterate this modification, shrinking R until O is incident to all the four sides of ∂R. If R is 1×1 of course there is space in R only for an H. If R is $1 \times y$ or $x \times 1$ with $x, y \geq 2$, the fact that the H's incident to ∂R are distinct implies that the vertices of ∂R are distinct, so a region incident to all the four sides or ∂R must have at least 10 vertices. If R is $x \times y$ with $x, y \geq 2$ then O contains some of the germs of regions $O^{(*)}$ in Fig. 14-bottom/right so as to touch all the $r/t/\ell/b$ sides of ∂R. An easy analysis shows that any identification between two vertices of the $O^{(*)}$'s would force two H's incident to ∂R to coincide, so it is impossible. This implies that any $O^{(*)}$ actually contained in O contributes to the number of vertices of O with as many vertices as one sees in Fig. 14-bottom/right, namely 3 for $O^{(r)}$, $O^{(t)}$, $O^{(\ell)}$, $O^{(b)}$, then 4 for $O^{(t\ell)}$, $O^{(br)}$, and finally 5 for $O^{(tr)}$, $O^{(b\ell)}$. Therefore, a region can touch all of $r/t/\ell/b$ with a total of no more that 8 vertices only if it includes $O^{(t\ell)}$ and $O^{(br)}$, but then the vertex colors again imply that the number of vertices is at least 10. This gives the final contradiction and concludes the proof.

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