Partial Differential Equations — *Pointwise estimates for solutions of semilinear parabolic inequalities with a potential*, by Luigi Montoro, Fabio Punzo and Berardino Sciunzi, communicated on November 10, 2017.[1]

Abstract. — We obtain pointwise estimates for solutions of semilinear parabolic equations with a potential on connected domains both of $\mathbb{R}^n$ and of general Riemannian manifolds. Our results generalize to the parabolic case those shown in [10] for elliptic equations.

1. Introduction

We are concerned with solutions of semilinear parabolic equations of the following type:

$$(1.1) \qquad \partial_t u - \Delta u + V u^q = f \quad \text{in } Q_T := \Omega \times (0, T],$$

where $\Omega \subseteq M$ is a connected domain on a complete Riemannian manifold, the potential $V = V(x, t)$ and the source term $f = f(x, t)$ are given continuous functions in Q_T. Moreover, we suppose that $f \geq 0$, $f \not\equiv 0$, while V can be signed. We consider both the case $q > 0$ and $u \geq 0$, and that $q < 0$ and $u > 0$.

The elliptic counterpart of equation (1.1), that is

$$(1.2) \qquad -\Delta u + V u^q = f \quad \text{in } \Omega,$$

with V and f continuous functions defined in Ω, has been largely investigated in the literature. In particular, in [10] pointwise estimates for the solutions of (1.2) have been obtained. Indeed, in [10] also more general divergence form elliptic operators with smooth coefficients have been addressed. Assume that the Dirichlet Green function of $-\Delta$ in Ω exists, and denote it by $G^\Omega(x, y)$; clearly, it exists if $\partial\Omega$ is smooth enough. Set

$$H(x) := \int_\Omega G^\Omega(x, y) f(y) \, d\mu(y);$$

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[1] Presented by A. Tesei.

assume that $H(x) < \infty$ for all $x \in \Omega$, and that

$$\tilde{H}(x) := \int_{\Omega} G^{\Omega}(x, y) H^q(y) V(y) \, d\mu(y)$$

is well-defined. Note that if Ω is relatively compact, then $G^{\Omega}(x, \cdot) \in L^1(\Omega)$, hence $H(x) < \infty$, for each $f \in L^{\infty}(\Omega)$.

In [10] it is shown that if $q > 0$, then u satisfies a pointwise estimate from below, in terms of the functions H and $\tilde{H}$. On the other hand, if $q < 0$, then u satisfies a similar pointwise estimate from above. Moreover, using similar inequalities, sufficient conditions for the existence of positive solutions of equation (1.2) have been obtained, provided Ω is relatively compact. Observe that in particular cases the results established in [10] have been already shown in the literature (see, e.g., [3], [4], [5], [8], [9], [11]). However, in the remarkable paper [10] it is given a unified approach for treating all the values of $q \in \mathbb{R} \setminus \{0\}$, a general signed potential V, and a general divergence form operator, also on domains of Riemannian manifolds.

Recently, also parabolic equations with a potential on Riemannian manifolds have been investigated (see, e.g., [2], [12], [13], [14]); however, it seems that in general pointwise estimates for solutions of equation (1.1) have not been addressed. In this paper we aim at obtaining pointwise estimates for solutions of (1.1), in the same spirit of the results in [10], concerning elliptic equations.

Let p the *heat kernek* in Ω (see Section 2); for any $f \in C(Q_T)$, define for all $(x, t) \in Q_T$

$$(1.3) \qquad \mathscr{S}^{\Omega}[f](x, t) := \int_0^t \int_{\Omega} p(x, y, t - s) f(y, s) \, d\mu(y) \, ds,$$

provided that

$$(1.4) \quad \int_0^t \int_{\Omega} p(x, y, t - s) |f(y, s)| \, d\mu(y) \, ds < \infty \quad \text{for every } x \in \Omega, \, t \in (0, T].$$

Furthermore, for any $u_0 \in C(\Omega) \cap L^{\infty}(\Omega)$, $u_0 \geq 0$, define for all $(x, t) \in Q_T$

$$(1.5) \qquad \mathscr{R}^{\Omega}[f; u_0](x, t) := \mathscr{S}^{\Omega}[f](x, t) + \int_{\Omega} p(x, y, t) u_0(y) \, d\mu(y).$$

We prove that for $q > 0$ any solution of problem

$$(1.6) \qquad \begin{cases} \partial_t u - \Delta u + V u^q \geq f, & u \geq 0, \quad \text{in } Q_T \\ u \geq u_0 & \text{in } \Omega \times \{0\} \end{cases}$$

satisfies a certain pointwise estimate from below in terms of the functions $\mathscr{R}^{\Omega}[f; u_0]$ and $\mathscr{S}^{\Omega}[h^q V]$, provided that $\mathscr{S}^{\Omega}[h^q |V|] < \infty$ in Q_T, where

$$(1.7) \qquad\qquad h := \mathscr{R}^{\Omega}[f; u_0] \quad \text{in } Q_T.$$

Moreover, if $q < 0$, then for any solution of problem

$$(1.8) \qquad \begin{cases} \partial_t u - \Delta u + V u^q \leq f, \quad u > 0, & \text{in } Q_T \\ u = 0 & \text{in } \partial\Omega \times (0, T] \\ u \leq u_0 & \text{in } \Omega \times \{0\}, \end{cases}$$

a similar estimate from above is obtained. Indeed, note that in the case $q < 0$, as well as in the elliptic case, a suitable extra pointwise condition at *infinity* for the solution is required. However, in the parabolic case, if M is stochastically complete, such a condition can be replaced by a growth condition at *infinity*, which is a weaker assumption.

Moreover, when Ω is relatively compact, we give sufficient conditions for existence of positive solutions of problem

$$(1.9) \qquad \begin{cases} \partial_t u - \Delta u + V u^q = f & \text{in } Q_T \\ u = 0 & \text{in } \partial\Omega \times (0, T] \\ u = u_0 & \text{in } \Omega \times \{0\}, \end{cases}$$

that are based on estimates analogous to those described above. We should note that all our results seem to be new also in the case that $M = \mathbb{R}^n$.

In order to prove our results, we adapt to parabolic equations the methods used in [10]. At first we prove our pointwise estimates assuming that Ω is a relatively compact connected domains, and replacing h defined in (1.7) by a function $\zeta \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ that satisfies

$$(1.10) \qquad\qquad\qquad \partial_t \zeta - \Delta \zeta \geq 0 \quad \text{in } Q_T,$$

$$(1.11) \qquad\qquad\qquad \zeta > 0 \quad \text{in } \Omega \times [0, T].$$

To do that the main step is to consider the equation solved by uv, where

$$v := \phi^{-1}\left(\frac{u}{h}\right),$$

ϕ being an appropriate smooth function. Then a suitable approximation procedure is used to obtain the desired estimates in possibly not relatively compact domain Ω, with h defined in (1.7). In our arguments a special role is played by an appropriate comparison result, that is applied to the function uv. Note that the proof of such a comparison result is quite different from that in [10] for the elliptic case. Furthermore, on a special class of Riemannian manifolds, including the stochastically completes ones, we can show a refined comparison result. In view of this, we can show the estimates from above in the case $q < 0$, only assuming growth conditions at infinity on the solutions of (1.1).

The paper is organized as follows. In Section 2 we recall some basic notions in Riemannian Geometry and in Analysis on manifolds that will be used in the sequel. Then we state our main results in Section 3. In Section 4 we show some preliminary results, including the comparison results mentioned above, that will

be essential in the proofs of the main theorems, that can be found in Sections 5 and 6.

2. MATHEMATICAL FRAMEWORK

Let M be an n-dimensional Riemannian manifold with a Riemannian metric tensor $g = (g_{ij})$. In any chart with coordinates $x_1, x_2 \ldots, x_n$, the associated Laplace–Beltrami operator is given by

$$\Delta u = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \partial_{x_i}(\sqrt{\det g} \, g^{ij} \partial_{x_j} u),$$

where $\det g$ is the determinant of the matrix $g = (g_{ij})$, (g^{ij}) is the inverse matrix of (g_{ij}), and $u \in C^2(M)$. The Riemannuan measure $d\mu$ in the same chart reads by

$$d\mu = \sqrt{\det g} \, dx_1 \ldots dx_n;$$

furthermore, the gradient of a function $u \in C^1(M)$ is

$$(\nabla u)^i = \sum_{j=1}^{n} g^{ij} \partial_{x_j} u \quad (i = 1, \ldots, n).$$

For any $f, g \in C^2(M)$ we have

(2.1) $$\Delta(fg) = f\Delta g + 2\langle \nabla f, \nabla g \rangle + g\Delta f.$$

Moreover, for any $w \in C^2(M)$ and $\phi \in C^2(\mathbb{R})$ there holds

(2.2) $$\Delta[\phi(w)] = \phi'(w)\Delta w + \phi''(w)|\nabla w|^2.$$

We denote by $\partial_\infty M$ the *infinity point* of the one-point compactification of M (see for example [19, Sec. 5.4.3]). For any function $u : \Omega \subseteq M \to \mathbb{R}$ we write

$$\lim_{x \to \partial_\infty M} u(x) = 0$$

to indicate that $u(x) \to 0$ as $d(x, o) \to \infty$, $o \in M$ being a fixed point; here and hereafter $d(x, y)$ denotes the geodesic distance from x to y. Similarly we mean equalities and inequalities involving lim inf and lim sup.

By standard results (see, e.g., [6]) the *heat kernel* in Ω, $p(x, y, t)$, is well-defined. For each fixed $y \in \Omega$, $p(x, y, t)$ is the smallest positive solution of equation

(2.3) $$\partial_t p - \Delta p = 0 \quad \text{in } Q_T,$$

such that

$$\lim_{t \to 0^+} p(x, y, t) = \delta_y,$$

where δ_y is the *Dirac delta* concentrated at y. Moreover, $p \in C^\infty(\Omega \times \Omega \times (0, \infty))$,

$$p(x, y, t) > 0 \quad \text{for any } x, y \in \Omega, \ t > 0,$$
$$p(x, y, t) = p(y, x, t) \quad \text{for any } x, y \in \Omega, \ t > 0,$$
$$p(x, y, t) = \int_\Omega p(x, z, s) p(z, y, t - s) \, d\mu(y) \quad \text{for any } t > 0, \ 0 < s < t, \ x, y \in \Omega,$$
$$\int_\Omega p(x, y, t) \, d\mu(y) \leq 1 \quad \text{for any } x \in \Omega, \ t > 0.$$

Furthermore, (see [7, Theorem 7.16]) for any $u_0 \in C(\Omega) \cap L^\infty(\Omega)$, the function

$$v(x, t) := \int_M p(x, y, t) u_0(y) \, d\mu(y), \quad x \in \Omega, \ t > 0$$

belongs to $C^\infty(\Omega \times \Omega \times (0, \infty))$, satisfies equation (2.3), and

$$v(x, t) \to u_0(x) \quad \text{as } t \to 0^+ \text{ locally uniformly w.r.t. } x \in \Omega.$$

In addition, if $\partial\Omega$ is smooth, then $v \in C(\bar{Q}_T)$, and

$$v = 0 \quad \text{in } \partial\Omega \times (0, T].$$

As usual, we say that f is locally Holder continuous in Q_T, if there exists $\alpha \in (0, 1)$ such that for any compact subset $K \subset \Omega$, $0 < \tau \leq T$

$$|f(x, t) - f(y, s)| \leq L[d(x, y)^\alpha + |t - s|^{\frac{\alpha}{2}}] \quad \text{for all } x, y \in K, \ t, s \in (\tau, T),$$

for some $L = L_{K, \tau} > 0$. We set

$$C^{2,1}(Q_T) := \left\{ u : Q_T \to \mathbb{R} \ \middle| \ \frac{\partial^2 u}{\partial x_i \partial x_j}, \ \partial_t u \in C(Q_T) \text{ for any } i, j = 1, \ldots, n \right\}.$$

We have that (see, e.g. [1]) if (1.4) holds and f is locally Holder continuous in Q_T and $u_0 \in C(\Omega) \cap L^\infty(\Omega)$, then the function h defined in (1.7) satisfies $h \in C^{2,1}(Q_T)$ and

$$(2.4) \qquad\qquad \partial_t u - \Delta u = f \quad \text{in } Q_T.$$

Moreover, if $f \in L^\infty(Q_T)$ and $u_0 \in C(\Omega) \cap L^\infty(\Omega)$, then $h \in C(\Omega \times [0, T])$ and

$$h = u_0 \quad \text{in } \Omega \times \{0\}.$$

Finally, if $\partial\Omega$ is smooth and $f \in C(\bar{Q}_T)$, then

$$h = 0 \quad \text{in } \partial\Omega \times (0, T].$$

3. STATEMENTS OF THE MAIN RESULTS

Set

$$\chi_u(x) := \begin{cases} 1 & \text{if } u(x) > 0 \\ 0 & \text{if } u(x) < 0. \end{cases}$$

We can prove the pointwise estimates for solutions of (1.1) contained in the following theorem.

THEOREM 3.1. *Let $\Omega \subseteq M$ be an open connected subset. Suppose that $V, f \in C(Q_T)$, $f \geq 0$, $f \not\equiv 0$ in Q_T, $u_0 \in C(\Omega) \cap L^\infty(\Omega)$, $u_0 \geq 0$. Assume that $u \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ satisfies (1.6) if $q > 0$, or that u satisfies problem (1.8) and*

$$(3.1) \qquad\qquad \lim_{x \to \partial_\infty M} \sup_{t \in (0,T]} u(x, t) = 0,$$

if $q < 0$. Let (1.4) be satisfied, and let h be defined by (1.7). Moreover, assume that

$$\mathscr{S}^\Omega[h^q|V|](x,t) < \infty \quad \text{for all } (x,t) \in Q_T, \quad \text{if } q < 0 \text{ or } q \geq 1,$$

or that

$$(3.2) \qquad \mathscr{S}^\Omega[\chi_u h^q|V|](x,t) < \infty \quad \text{for all } (x,t) \in Q_T, \quad \text{if } 0 < q < 1.$$

Then the following statements hold for all $(x,t) \in Q_T$.

(i) *If $q = 1$, then*

$$(3.3) \qquad\qquad u(x,t) \geq h(x,t) e^{-\frac{1}{h(x,t)}\mathscr{S}^\Omega[hV](x,t)}.$$

(ii) *If $q > 1$, then*

$$(3.4) \qquad\qquad -(q-1)\mathscr{S}^\Omega[h^q V](x,t) < h(x,t),$$

and

$$(3.5) \qquad\qquad u(x,t) \geq \frac{h(x,t)}{\left\{1 + (q-1)\frac{\mathscr{S}^\Omega[h^q V](x,t)}{h(x,t)}\right\}^{\frac{1}{q-1}}}.$$

(iii) *If $0 < q < 1$, then*

$$(3.6) \qquad u(x,t) \geq h(x,t)\left\{1 - (q-1)\frac{\mathscr{S}^\Omega[\chi_u h^q V](x,t)}{h(x,t)}\right\}_+^{\frac{1}{1-q}}.$$

(iv) *If $q < 0$, then (3.4) holds, and*

$$(3.7) \qquad u(x, t) \le h(x, t) \left\{ 1 - (1 - q) \frac{\mathscr{S}^{\Omega}[h^q V](x, t)}{h(x, t)} \right\}^{\frac{1}{1-q}},$$

Furthermore, in the case that $f \equiv 0$, we can prove the following estimates.

THEOREM 3.2. *Let $\Omega \subseteq M$ be an open connected subset. Let $V \in C(Q_T)$. Suppose that $u \in C^{2,1}(Q_T)$ satisfies either*

$$(3.8) \qquad \partial_t u - \Delta u + V u^q \ge 0, \quad u \ge 0 \text{ in } Q_T, \quad \text{if } q > 0,$$

or

$$(3.9) \qquad \partial_t u - \Delta u + V u^q \le 0, \quad u > 0 \text{ in } Q_T, \quad \text{if } q < 0.$$

Moreover, assume that

$$\mathscr{S}^{\Omega}[|V|](x, t) < \infty \quad \text{for all } (x, t) \in Q_T, \quad \text{if } q < 0 \text{ or } q \ge 1,$$

or that

$$(3.10) \qquad \mathscr{S}^{\Omega}[\chi_u |V|](x, t) < \infty \quad \text{for all } (x, t) \in Q_T, \quad \text{if } 0 < q < 1,$$

Then the following statements hold.

(i) *If $q = 1$, $u \in C(\bar{Q}_T)$,*

$$(3.11) \qquad u \ge 1 \quad \text{in } [\partial\Omega \times (0, T]] \cup [\Omega \times \{0\}],$$

$$(3.12) \qquad \liminf_{x \to \partial_\infty M} \inf_{t \in (0, T]} u(x, t) \ge 1,$$

then

$$(3.13) \qquad u(x, t) \ge e^{-\mathscr{S}^{\Omega}[V](x, t)} \quad \text{for all } (x, t) \in Q_T.$$

(ii) *If $q > 1$ and*

$$(3.14) \qquad \lim_{t \to 0^+} \inf_{x \in \Omega} u(x, t) = \infty, \quad \lim_{d(x, \partial\Omega) \to 0} \inf_{t \in (0, T]} u(x, t) = \infty,$$

$$\lim_{x \to \partial_\infty M} \inf_{t \in (0, T]} u(x, t) = \infty,$$

then

$$(3.15) \qquad \mathscr{S}^{\Omega}[V](x, t) > 0,$$

and

$$(3.16) \qquad u(x, t) \ge \{(q - 1)\mathscr{S}^{\Omega}[V](x, t)\}^{-\frac{1}{q-1}}.$$

(iii) *If $0 < q < 1$, then*

$$(3.17) \qquad u(x,t) \geq \{-(1-q)\mathscr{S}^{\Omega}[\chi_u V](x,t)\}_+^{\frac{1}{q-1}}.$$

(iv) *If $q < 0$, $u \in C(\bar{Q}_T)$,*

$$(3.18) \qquad u = 0 \quad in \ [\partial\Omega \times (0,T]] \cup [\Omega \times \{0\}],$$

and (3.1) *is satisfied, then*

$$(3.19) \qquad \mathscr{S}^{\Omega}[V](x,t) < 0,$$

and

$$(3.20) \qquad u(x,t) \leq \{-(1-q)\mathscr{S}^{\Omega}[V](x,t)\}^{\frac{1}{q-1}}.$$

In the next theorem, we give sufficient conditions for the existence of non-negative solutions of problem (1.9), in the case that Ω is relatively compact, and $u_0 \in C(\bar{\Omega})$, with $u_0 = 0$ on $\partial\Omega$. Note that the last compatibility condition allows us to construct solutions that attain continuously zero on the whole parabolic boundary. Moreover, we establish two-sided pointwise estimates for such solutions.

THEOREM 3.3. *Let $\Omega \subset M$ be a connected relatively compact subset with boundary $\partial\Omega$ of class C^1. Suppose that f and V are locally Holder continuous in Q_T, and that $f \in C(\bar{Q}_T)$, $f \geq 0$, $f \not\equiv 0$. Assume that $u_0 \in C(\bar{\Omega})$, $u_0 = 0$ on $\partial\Omega$. Let* (1.4) *be satisfied, and let h be defined by* (1.7). *Then the following statements hold.*

(i) *Suppose that $q > 1$, $V \leq 0$, and that*

$$(3.21) \qquad -\mathscr{S}^{\Omega}[h^q V](x,t) \leq \left(1 - \frac{1}{q}\right)^q \frac{1}{q-1} h(x,t) \quad for\ all\ (x,t) \in Q_T.$$

Then a nonnegative solution $u \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ of problem (1.9) *exists; moreover,*

$$(3.22) \qquad \frac{h(x,t)}{\left\{1 + (q-1)\frac{\mathscr{S}^{\Omega}[h^q V](x,t)}{h(x,t)}\right\}^{\frac{1}{q-1}}} \leq u(x,t) \leq \frac{q}{q-1} h(x,t)$$

$$for\ all\ (x,t) \in Q_T.$$

(ii) *Suppose that $q < 0$, $V \geq 0$, and that*

$$(3.23) \qquad \mathscr{S}^{\Omega}[h^q V](x,t) \leq \left(1 - \frac{1}{q}\right)^q \frac{1}{1-q} h(x,t) \quad for\ all\ (x,t) \in Q_T.$$

Then a positive solution $u \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ of problem (1.9) exists; moreover, for all $(x,t) \in Q_T$

$$(3.24) \qquad \frac{1}{1 - \frac{1}{q}} h(x,t) \leq u(x,t) \leq \left\{ 1 - (1-q) \frac{\mathscr{S}^{\Omega}[h^q V](x,t)}{h(x,t)} \right\}^{\frac{1}{1-q}} h(x,t).$$

3.1. Further results for $q < 0$

Consider domains Ω that are not relatively compact. If $q < 0$, under suitable hypotheses, we can remove condition (3.1) and then getting Theorem 3.1-(iv) and Theorem 3.2-(iv).

We assume that there exist $\mu > 0$ and a subsolution Z of equation

$$(3.25) \qquad\qquad\qquad \Delta Z = \mu Z \quad \text{in } \Omega,$$

such that

$$(3.26) \qquad\qquad\qquad \sup_{\Omega} Z < \infty, \quad \lim_{x \to \partial_\infty M} Z(x) = -\infty.$$

By a subsolution of (3.25) we mean a function $Z \in C^2(\Omega)$ such that

$$(3.27) \qquad\qquad\qquad \Delta Z \geq \mu Z \quad \text{in } \Omega.$$

Observe that our results remain true if Z is continuous in Ω and satisfies (3.27) in the distributional sense. Note that, in the case $\Omega = M$, the existence of such a subsolution Z implies that M is stochastically complete (see [6]), i.e.

$$\int_M p(x,y,t)\, d\mu(y) = 1 \quad \text{for all } x \in M,\ t > 0.$$

We refer the reader to [6] for sufficient and necessary condition for the existence of such subsolution Z. We limit ourselves to observe that such a subsolution Z exists for instance on $\mathbb{R}^n$, $n \geq 3$, and on the hyperbolic space $\mathbb{H}^n$, $n \geq 2$.

THEOREM 3.4. *Let $q < 0$. Let $\Omega \subseteq M$ be an open not relatively compact connected subset. Suppose that $V, f \in C(Q_T)$, $f \geq 0$, $f \not\equiv 0$ in Q_T, $u_0 \in C(\Omega) \cap L^\infty(\Omega)$, $u_0 \geq 0$. Assume that $u \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ satisfies (1.8). Let conditions (1.4) and (3.2) be satisfied, and let h be defined by (1.7). Let there exist $\mu > 0$ and a subsolution Z of equation (3.25), which satisfies (3.26). Moreover, suppose that*

$$(3.28) \qquad \limsup_{x \to \partial_\infty M} \frac{\sup_{t \in (0,T]} h^q(x,t)[u^{1-q}(x,t) - h^{1-q}(x,t)]}{|Z(x)|} \leq 0.$$

Then (3.4) and (3.7) hold.

THEOREM 3.5. *Let $q < 0$. Let $\Omega \subseteq M$ be an open not relatively compact connected subset. Let $V \in C(Q_T)$. Suppose that $u \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ satisfies (3.9)*

and (3.18). *Let condition* (3.10) *be satisfied. Let there exist* $\mu > 0$ *and a subsolution* Z *of equation* (3.25), *which satisfies* (3.26). *Moreover, suppose that*

$$(3.29) \qquad \limsup_{x \to \partial_\infty M} \frac{\sup_{t \in (0, T]} u^{1-q}(x, t)}{|Z(x)|} \leq 0.$$

Then (3.19) *and* (3.20) *hold.*

REMARK 3.6. It is easily seen that both condition (3.28) and (3.29) are weaker than condition (3.1).

4. AUXILIARY RESULTS

This section is devoted to some preliminary results that will be used to prove Theorems 3.1, 3.2, 3.3.

LEMMA 4.1. *Let* $v, h \in C^{2,1}(Q_T)$, $\phi \in C^2(I)$ *with* $v(Q_T) \subseteq I$, *I being an interval in* $\mathbb{R}$. *Then in* Q_T

$$(4.1) \quad \partial_t[h\phi(v)] - \Delta[h\phi(v)]$$
$$= \phi'(v)[\partial_t(hv) - \Delta(hv)] - \phi''(v)|\nabla v|^2 h + [\phi(v) - v\phi'(v)](\partial_t h - \Delta h).$$

In particular, if $\phi' \neq 0$ *in* I, *then in* Q_T

$$(4.2) \qquad \partial_t(hv) - \Delta(hv)$$
$$= \frac{\partial_t[h\phi(v)] - \Delta[h\phi(v)]}{\phi'(v)} + \frac{\phi''(v)}{\phi'(v)}|\nabla v|^2 h + \left(v - \frac{\phi(v)}{\phi'(v)}\right)(\partial_t h - \Delta h).$$

PROOF. Clearly,

$$(4.3) \qquad \partial_t[h\phi(v)] = \phi'(v)\partial_t(hv) + [\phi(v) - v\phi'(v)]\partial_t h.$$

Moreover, in view of (2.1) with $f = h$, $g = \phi(v)$, and in view of (2.2) with $w = v$ we get

$$\Delta[h\phi(v)] = \phi(v)\Delta h + h[\phi'(v)\Delta v + \phi''(v)|\nabla v|^2] + 2\phi'(v)\langle \nabla h, \nabla v \rangle.$$

Thus

$$(4.4) \qquad \Delta[h\phi(v)] = \phi'(v)\Delta(hv) + \phi''(v)|\nabla v|^2 h + [\phi(v) - v\phi'(v)]\Delta h.$$

From (4.3) and (4.4) we easily obtain (4.1), and then (4.2). □

LEMMA 4.2. *Let* $I \subseteq \mathbb{R}$ *be an interval. Let* $\phi \in C^2(I)$, $\phi > 0$, $\phi' > 0$ *in* I. *Let* $v, h \in C^{2,1}(Q_T)$ *with* $h > 0$, $v(\Omega) \subseteq I$. *Set*

$$u := h\phi(v).$$

Let $V \in C(Q_T)$, $q \in \mathbb{R}\backslash\{0\}$. If

$$(4.5) \qquad \partial_t u - \Delta u + V u^q \geq \partial_t h - \Delta h \quad \text{in } Q_T,$$

then

$$(4.6) \qquad \partial_t(hv) - \Delta(hv) + h^q V \frac{\phi(v)^q}{\phi'(v)}$$

$$\geq \left(v - \frac{\phi(v) - 1}{\phi'(v)}\right)(\partial_t h - \Delta h) + \frac{\phi''(v)}{\phi'(v)}|\nabla v|^2 h \quad \text{in } Q_T.$$

If

$$(4.7) \qquad \partial_t u - \Delta u + V u^q \leq \partial_t h - \Delta h \quad \text{in } Q_T,$$

then

$$(4.8) \qquad \partial_t(hv) - \Delta(hv) + h^q V \frac{\phi^q(v)}{\phi'(v)}$$

$$\leq \left(v - \frac{\phi(v) - 1}{\phi'(v)}\right)(\partial_t h - \Delta h) + \frac{\phi''(v)}{\phi'(v)}|\nabla v|^2 h \quad \text{in } Q_T.$$

PROOF. From (4.5) with $u = h\phi(v)$ it follows that

$$(4.9) \qquad \partial_t[h\phi(v)] - \Delta[h\phi(v)] \geq -V h^q \phi(v)^q + \partial_t h - \Delta h.$$

Therefore, by (4.2) and (4.9),

$$\partial_t(hv) - \Delta(hv)$$

$$\geq -V h^q \frac{\phi(v)^q}{\phi'(v)} + \frac{\phi''(v)}{\phi'(v)}|\nabla v|^2 h + \frac{1 + v\phi'(v) - \phi(v)}{\phi'(v)}(\partial_t h - \Delta h).$$

So, (4.6) follows. The second claim can be proved in the same way. $\qquad \square$

LEMMA 4.3. *Let assumptions of Lemma 4.2 be satisfied. Moreover, suppose that $0 \in I$, and that*

$$(4.10) \qquad \partial_t h - \Delta h \geq 0 \quad \text{in } Q_T.$$

If

$$(4.11) \qquad \phi(0) = 1,$$
$$(4.12) \qquad \phi' > 0, \quad \phi'' \geq 0 \quad \text{in } I,$$

then

$$(4.13) \qquad \partial_t(hv) - \Delta(hv) + h^q V \frac{\phi(v)^q}{\phi'(v)} \geq 0 \quad in \ Q_T.$$

If (4.11) *holds, and*

$$(4.14) \qquad \phi' > 0, \quad \phi'' \leq 0 \quad in \ I,$$

then

$$(4.15) \qquad \partial_t(hv) - \Delta(hv) + h^q V \frac{\phi(v)^q}{\phi'(v)} \leq 0 \quad in \ Q_T.$$

PROOF. It is direct to see that (4.11) and (4.12) imply that

$$(4.16) \qquad v - \frac{\phi(v) - 1}{\phi'(v)} \geq 0 \quad for \ all \ v \in I.$$

From (4.6), (4.10) and (4.16) we obtain (4.13). Inequality (4.15) can be deduced similarly. □

REMARK 4.4. Note that if $\partial_t h - \Delta h = 0$ in Q_T, then in Lemma 4.3 condition (4.11) can be removed.

In the sequel, we often use the next comparison result.

PROPOSITION 4.5. *Let* $\Omega \subset M$ *be an open subset. Assume that* $g \in C(Q_T)$, *and that*

$$(4.17) \qquad \mathscr{S}[|g|] < \infty \quad in \ Q_T.$$

Let $v \in C^2(Q_T) \cap C(\bar{Q}_T)$ *be a supersolution of problem*

$$(4.18) \qquad \begin{cases} \partial_t v - \Delta v = g & in \ Q_T \\ v = 0 & in \ \partial\Omega \times (0, T] \\ v = 0 & in \ \Omega \times \{0\}. \end{cases}$$

Furthermore, if Ω *is not relatively compact, suppose that*

$$(4.19) \qquad \liminf_{x \to \partial_\infty M} \inf_{t \in (0, T]} v(x, t) \geq 0.$$

Then

$$(4.20) \qquad v(x, t) \geq \mathscr{S}^\Omega[g](x, t) \quad for \ every \ x \in \Omega, \ t \in [0, T].$$

PROOF. Choose a sequence of functions $\{g_n\}$ such that g_n is locally Lipschitz continuous in Q_T for every $n \in \mathbb{N}$,

(4.21) $$g_n \leq g, \quad g_n \leq g_{n+1} \quad \text{in } Q_T \text{ for every } n \in \mathbb{N};$$

(4.22) $$g_n \to g \quad \text{in } Q_T \text{ as } n \to \infty.$$

Let us only consider the case when Ω is not relatively compact; the case when Ω is relatively compact is easier and it will be omitted.

Let $k \in \mathbb{N}$ that will be taken arbitrary large later on. Fixed a point $o \in M$, by (4.19), we find a radius R_k such that

(4.23) $$v \geq -\frac{1}{k} \quad \text{on } (\Omega \cap \partial B_{R_k}(o)) \times (0, T].$$

Since $v \in C(\bar{Q}_T)$ we can therefore take $\Omega_k \subseteq \Omega \cap B_{R_k}(o)$ so that

$$v \geq -\frac{1}{k} \quad \text{on } \partial \Omega_k \times (0, T].$$

For each k fixed, the construction of Ω_k can be carried out just observing that v is uniformly continuous in $\Omega \cap B_{R_k}(o)$ and exploiting the boundary datum. With no loss of generality we may and do assume that $R_k \to \infty$, Ω_k is smooth and

(4.24) $$\bigcup_{k \in \mathbb{N}} \Omega_k = \Omega.$$

Therefore, by construction, we have that v is a supersolution of the problem

(4.25) $$\begin{cases} \partial_t v - \Delta v = g_n & \text{in } \Omega_k \times (0, T] \\ v \geq -k^{-1} & \text{in } \partial \Omega_k \times (0, T] \\ v \geq -k^{-1} & \text{in } \Omega_k \times \{0\}. \end{cases}$$

Let now $v_{n,k}$ be the solution of the problem

(4.26) $$\begin{cases} \partial_t v - \Delta v = g_n & \text{in } \Omega_k \times (0, T] \\ v = 0 & \text{in } \partial \Omega_k \times (0, T] \\ v = 0 & \text{in } \Omega_k \times \{0\}. \end{cases}$$

We have that

(4.27) $$v_{n,k}(x, t) = \int_0^t \int_{\Omega_k} p_k(x, y, t - s) g_n(y, s) \, dt \, d\mu(y), \quad x \in \bar{\Omega}_k, \, t \in [0, T],$$

where p_k is the heat kernel in Ω_k, completed with zero homogeneous Dirichlet boundary conditions. It is known that (see, e.g., [6]), by (4.24), it follows that

(4.28) $$\lim_{k \to \infty} p_k = p \quad \text{in } M \times M \times (0, \infty).$$

Therefore, using (4.17), (4.22) and (4.28), we can infer that

$$\lim_{n \to \infty, k \to \infty} v_{n,k} = \mathscr{S}^{\Omega}[g] \quad \text{in } Q_T. \tag{4.29}$$

On the other hand, the function $v_{n,k} - k^{-1}$ is a subsolution of problem

$$
\begin{cases}
\partial_t v - \Delta v = g_n & \text{in } \Omega_k \times (0, T] \\
v \leq -k^{-1} & \text{in } \partial\Omega_k \times (0, T] \\
v \leq -k^{-1} & \text{in } \Omega_k \times \{0\}.
\end{cases}
\tag{4.30}
$$

By the comparison principle, taking into account (4.25) and (4.30), we deduce that

$$v \geq v_{n,k} - k^{-1} \quad \text{in } \Omega_k \times [0, T]. \tag{4.31}$$

In view of (4.29), letting $k \to \infty$, $n \to \infty$, we obtain (4.20). $\square$

We also use the next comparison result.

PROPOSITION 4.6. *Let $\Omega \subset M$ be an open subset. Assume that $g \in C(Q_T)$ and that (4.17) is satisfied. Let $v \in C^2(Q_T) \cap C(\bar{Q}_T)$ be a subsolution of problem (4.18). Furthermore, if Ω is not relatively compact, suppose that*

$$\limsup_{x \to \partial_\infty M} \sup_{t \in (0, T]} v(x, t) \leq 0. \tag{4.32}$$

Then

$$v(x, t) \leq \mathscr{S}^{\Omega}[g](x, t) \quad \text{for every } x \in \Omega, \ t \in [0, T]. \tag{4.33}$$

The proof of Proposition 4.6 is analogous to that of Proposition 4.5; the only difference is that the sequence $\{g_n\}$ satisfies

$$g_n \geq g, \quad g_n \geq g_{n+1} \quad \text{in } Q_T \text{ for every } n \in \mathbb{N}, \tag{4.34}$$

instead of (4.21).

Moreover, we use the next refined comparison principles.

PROPOSITION 4.7. *Let $\Omega \subset M$ be an open, not relatively compact subset. Assume that $g \in C(Q_T)$, and that (4.17) is satisfied. Let $v \in C^2(Q_T) \cap C(\bar{Q}_T)$ be a subsolution of problem (4.18). Assume that there exists a subsolution Z of equation (3.25) such that (3.26) is satisfied. Furthermore, suppose that*

$$\limsup_{x \to \partial_\infty M} \frac{\sup_{t \in (0, T]} v(x, t)}{|Z(x)|} \leq 0. \tag{4.35}$$

Then (4.33) holds.

PROOF. First of all we observe that we can assume that, for some $H > 0$,

$$(4.36) \qquad Z \leq -H < 0 \quad \text{in } \Omega.$$

In fact, if $\sup_\Omega Z \geq 0$, then instead of Z we can consider the function

$$\tilde{Z} := Z - \sup_\Omega Z - 1,$$

that clearly satisfies (3.25), (3.26) and (4.36).

Choose now a sequence of functions $\{g_n\}$ such that g_n is locally Lipschitz continuous in Q_T for every $n \in \mathbb{N}$, (4.34) and (4.22) hold. Let $k \in \mathbb{N}$ that will be taken arbitrary large later on and fix a point $o \in M$. We set

$$V_k(x, t) := -k^{-1} Z(x) e^{\mu t} \quad ((x, t) \in Q_T).$$

In view of (4.36), since $\mu > 0$, we have that

$$(4.37) \qquad V_k \geq \frac{H}{k} > 0 \quad \text{in } Q_T.$$

By (4.35), we find a radius R_k such that

$$(4.38) \qquad v \leq V_k \quad \text{in } (\partial B_{R_k}(o) \cap \Omega) \times (0, T].$$

Since $v \in C(\bar{Q}_T)$ we can therefore take $\Omega_k \subseteq \Omega \cap B_{R_k}(o)$ so that

$$(4.39) \qquad v \leq V_k \quad \text{on } \partial \Omega_k \times (0, T].$$

With no loss of generality we may and do assume that $R_k \to \infty$, Ω_k is smooth and

$$(4.40) \qquad \bigcup_{k \in \mathbb{N}} \Omega_k = \Omega.$$

With such a construction we let $v_{n,k}$ and p_k as in (4.27). It is now easy to verify that V_k is a supersolution of the problem

$$(4.41) \qquad \begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega_k \times (0, T] \\ u = V_k & \text{in } \partial \Omega_k \times (0, T] \\ u = V_k & \text{in } \Omega_k \times \{0\}. \end{cases}$$

Inequalities (4.37) and (4.38) and (4.39) easily yield that

$$(4.42) \qquad v - v_{n,k} \leq V_k \quad \text{in } [\partial \Omega_k \times (0, T]] \cup [\Omega_k \times \{0\}].$$

Exploiting (4.42) and (4.34) we can infer that $v - v_{n,k}$ is a subsolution of problem (4.41) and, by the comparison principle, we obtain that

$$(4.43) \qquad v - v_{n,k} \leq V_k \quad \text{in } \Omega_k \times (0, T].$$

Letting $n \to \infty$, $k \to \infty$ in (4.43) we deduce that

$$v \leq \mathscr{S}^{\Omega}[g] \quad \text{in } Q_T. \qquad \square$$

Similarly, the next refined comparison principle can also be shown.

PROPOSITION 4.8. *Let $\Omega \subset M$ be an open, not relatively compact subset. Assume that $g \in C(Q_T)$ and that (4.17) is satisfied. Let $v \in C^2(Q_T) \cap C(\bar{Q}_T)$ be a super-solution of problem (4.18). Let there exist a subsolution Z of equation (3.25) such that (3.26) is satisfied. Furthermore, suppose that*

$$(4.44) \qquad \liminf_{x \to \partial_\infty M} \frac{\inf_{t \in (0, T]} v(x, t)}{|Z(x)|} \geq 0.$$

Then (4.20) holds.

4.1. Pointwise estimates in relatively compact domains with general smooth supersolutions

Let $h \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ be a function that satisfies (1.10), (1.11). Consider the following inital-boundary value inequalities

$$(4.45) \qquad \begin{cases} \partial_t u - \Delta u + V u^q \geq \partial_t h - \Delta h & \text{in } Q_T \\ u \geq h & \text{in } \partial\Omega \times (0, T] \\ u \geq h & \text{in } \Omega \times \{0\} \\ u \geq 0 & \text{in } Q_T, \end{cases} \quad (q > 0)$$

and

$$(4.46) \qquad \begin{cases} \partial_t u - \Delta u + V u^q \leq \partial_t h - \Delta h & \text{in } Q_T \\ u \leq h & \text{in } \partial\Omega \times (0, T] \\ u \leq h & \text{in } \Omega \times \{0\} \\ u > 0 & \text{in } Q_T. \end{cases} \quad (q < 0)$$

The next result has a crucial role in the proof of Theorem 3.1. In fact, it gives the estimates (3.3)–(3.7), under the extra assumption that Ω is relatively compact; moreover, a general smooth function h that satisfies (1.10)–(1.11) is used.

THEOREM 4.9. *Let $\Omega \subseteq M$ be a relatively compact connected subset. Let h be any function belonging to $C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ that satisfies (1.10)–(1.11). Let $u \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ be a solution of either (4.45) or (4.46).*
Moreover, assume that

$$\mathscr{S}^{\Omega}[h^q |V|](x, t) < \infty \quad \text{for all } (x, t) \in Q_T, \quad \text{if } q < 0 \text{ or } q \geq 1,$$

or that

$$\mathscr{S}^{\Omega}[\chi_u h^q |V|] < \infty \quad \text{for all } (x,t) \in Q_T, \quad \text{if } 0 < q < 1.$$

Then (3.3)–(3.7) hold for all $(x,t) \in Q_T$.

PROOF OF THEOREM 4.9. To begin with, we further assume that

(4.47) $$h > 0, \quad u > 0 \quad \text{in } \bar{Q}_T, \quad \text{and} \quad V \in C(\bar{Q}_T).$$

Following the proof of [10, Theorem 3.2], we choose a function ϕ to solve the initial value problem

(4.48) $$\phi'(s) = \phi(s)^q, \quad \phi(0) = 1.$$

For $q = 1$ we have

(4.49) $$\phi(s) = e^s, \quad s \in \mathbb{R},$$

while for $q \neq 1$ we obtain

(4.50) $$\phi(s) = [(1-q)s + 1]^{\frac{1}{1-q}}, \quad s \in I_q,$$

where the interval I_q is given by

(4.51) $$I_q = \begin{cases} \left(-\infty, \frac{1}{q-1}\right) & \text{if } q > 1, \\ \mathbb{R} & \text{if } q = 1, \\ \left(-\frac{1}{q-1}, \infty\right) & \text{if } q < 1. \end{cases}$$

There holds

(4.52) $$\phi'(s) = [(1-q)s + 1]^{\frac{q}{1-q}}, \quad \phi''(s) = q[(1-q)s + 1]^{\frac{2q-1}{1-q}}.$$

In particular, we have

(4.53) $$\phi' > 0 \quad \text{in } I_q;$$

consequently, the inverse function $\phi^{-1} : (0, \infty) \to \mathbb{R}$ is well-defined. Moreover,

(4.54) $$\phi''(s) > 0 \quad \text{in } I_q \text{ if } q > 0,$$

whereas

(4.55) $$\phi''(s) < 0 \quad \text{in } I_q \text{ if } q < 0.$$

Indeed, for $0 < q < 1$, we extend the domain of ϕ to all $s \leq -\frac{1}{1-q}$, by putting $\phi(s) = 0$, so that

(4.56) $$\phi(s) = [(1-q)s + 1]_+^{\frac{1}{1-q}} \quad \text{for all } s \in \mathbb{R}.$$

Due to (4.47), we can define

$$
(4.57) \qquad\qquad v := \phi^{-1}\left(\frac{u}{h}\right) \quad \text{in } \bar{Q}_T;
$$

we have that $v \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$. Let $q > 0$. From (1.10) and (4.45) we have that the function $u = h\phi(v)$ satisfies

$$
(4.58) \qquad\qquad \partial_t u - \Delta u + V u^q \geq \partial_t h - \Delta h \geq 0 \quad \text{in } Q_T.
$$

Thanks to (4.58), Lemma 4.3 and (4.48) we get

$$
(4.59) \qquad\qquad \partial_t(hv) - \Delta(hv) \geq -h^q V \quad \text{in } Q_T.
$$

Since $u \geq h$ in $[\partial\Omega \times (0,T]] \cup [\Omega \times \{0\}]$, we have that

$$
(4.60) \qquad hv = h\phi^{-1}\left(\frac{u}{h}\right) \geq h\phi^{-1}(1) = 0 \quad \text{in } [\partial\Omega \times (0,T]] \cup [\Omega \times \{0\}].
$$

So, hv is a supersolution of problem (4.18) with $g = -h^q V$. Since Ω is relatively compact, by Proposition 4.5,

$$
(4.61) \qquad\qquad hv \geq -\mathscr{S}^{\Omega}[h^q V] \quad \text{in } Q_T.
$$

Thus,

$$
(4.62) \qquad\qquad v \geq -\frac{1}{h}\mathscr{S}[h^q V] \quad \text{in } Q_T.
$$

As a consequence of (4.57) and (4.62) we obtain that, for $q > 1$,

$$
(4.63) \qquad\qquad v < \frac{1}{q-1}, \quad -h^{-1}\mathscr{S}^{\Omega}[h^q V] < \frac{1}{q-1}.
$$

Hence, for each $q > 0$, we can apply ϕ to both sides of (4.62) to obtain

$$
(4.64) \qquad\qquad \frac{u}{h} \leq \phi\left(-\frac{1}{h}\mathscr{S}^{\Omega}[h^q V]\right) \quad \text{in } Q_T,
$$

which implies (3.3), (3.5), (3.6). Moreover, from (4.63) it follows (3.4).

Now, assume that $q < 0$. Then we have

$$
\partial_t u - \Delta u + V u^q \leq \partial_t h - \Delta h \quad \text{in } Q_T.
$$

Thanks to Lemma 4.3 and (4.15) we have

$$
(4.65) \qquad\qquad \partial_t(hv) - \Delta(hv) \leq -h^q V \quad \text{in } Q_T.
$$

Since $u \leq h$ in $[\partial\Omega \times (0, T]] \cup [\Omega \times \{0\}]$, we have that

$$(4.66) \qquad hv = h\phi^{-1}\left(\frac{u}{h}\right) \leq h\phi^{-1}(1) = 0 \quad \text{in } [\partial\Omega \times (0, T]] \cup [\Omega \times \{0\}].$$

So, hv is a subsolution of problem (4.18) with $g = -h^q V$. Since Ω is bounded, by Proposition 4.6,

$$hv \leq -\mathscr{S}^{\Omega}[h^q V] \quad \text{in } Q_T.$$

Thus,

$$(4.67) \qquad v \leq -\frac{1}{h}\mathscr{S}^{\Omega}[h^q V] \quad \text{in } Q_T.$$

In view of (4.67), it follows (3.4). Moreover, applying ϕ to both sides of (4.67) we get

$$(4.68) \qquad \frac{u}{h} \geq \phi\left(-\frac{1}{h}\mathscr{S}^{\Omega}[h^q V]\right) \quad \text{in } Q_T,$$

and then (3.7).

Now we can remove the extra assumptions in (4.47). We extend the domain I_q of ϕ to the endpoints of I_q by taking the limits of ϕ at the endpoints. So, the extended domain of ϕ is the interval

$$\bar{I}_q = \begin{cases} [-\infty, \frac{1}{q-1}] & \text{if } q > 1, \\ [-\infty, \infty] & \text{if } q = 1, \\ [-\frac{1}{q-1}, \infty] & \text{if } q < 1. \end{cases}$$

Moreover, when $0 < q < 1$, we extend ϕ to all $s \in [-\infty, \infty]$ by using (4.56). Hence (3.3), (3.5) and (3.6) can be written in the form (4.64), while (3.20) in the form (4.68).

Take $q > 0$. Let us show (4.64). To this purpose, for every $\varepsilon > 0$ set

$$u_\varepsilon := u + \varepsilon$$

and define

$$v_\varepsilon := \phi^{-1}\left(\frac{u_\varepsilon}{h}\right) \quad \text{in } Q_T.$$

Note that since $u_\varepsilon > 0$ and $h > 0$ in Q_T, the function v_ε is well-defined in Q_T and $v_\varepsilon \in C^{2,1}(Q_T)$; moreover, $v_\varepsilon(Q_T) \subset I_q$. From (4.2) it follows that

(4.69) $\partial_t(hv_\varepsilon) - \Delta(hv_\varepsilon)$

$$= \frac{\partial_t[h\phi(v_\varepsilon)] - \Delta[h\phi(v_\varepsilon)]}{\phi'(v_\varepsilon)}$$

$$+ \frac{\phi''(v_\varepsilon)}{\phi'(v_\varepsilon)}|\nabla v_\varepsilon|^2 h + \left(v_\varepsilon - \frac{\phi(v_\varepsilon)}{\phi'(v_\varepsilon)}\right)(\partial_t h - \Delta h) \quad \text{in } Q_T.$$

Since

$$\partial_t[h\phi(v_\varepsilon)] - \Delta[h\phi(v_\varepsilon)] = \partial_t u_\varepsilon - \Delta u_\varepsilon = \partial_t u - \Delta u \quad \text{in } Q_T,$$

we get

(4.70) $\partial_t(hv_\varepsilon) - \Delta(hv_\varepsilon)$

$$= \frac{\partial_t u - \Delta u}{\phi'(v_\varepsilon)} + \frac{\phi''(v_\varepsilon)}{\phi'(v_\varepsilon)}|\nabla v_\varepsilon|^2 h + \left(v_\varepsilon - \frac{\phi(v_\varepsilon)}{\phi'(v_\varepsilon)}\right)(\partial_t h - \Delta h) \quad \text{in } Q_T.$$

By (4.48),

(4.71) $$\phi'(v_\varepsilon) = \phi(v_\varepsilon)^q = \left(\frac{u_\varepsilon}{h}\right)^q.$$

From (4.70), (4.71) and (4.45) we obtain

$\partial_t(hv_\varepsilon) - \Delta(hv_\varepsilon)$

$$\geq -h^q\left(\frac{u}{u_\varepsilon}\right)^q V + \frac{\phi''(v_\varepsilon)}{\phi'(v_\varepsilon)}|\nabla v_\varepsilon|^2 h + \left(v_\varepsilon - \frac{\phi(v_\varepsilon) - 1}{\phi'(v_\varepsilon)}\right)(\partial_t h - \Delta h) \quad \text{in } Q_T.$$

In view of (1.10), (1.11) and (4.12), the previous inequality implies

(4.72) $$\partial_t(hv_\varepsilon) - \Delta(hv_\varepsilon) \geq -h^q\left(\frac{u}{u_\varepsilon}\right)^q V \quad \text{in } Q_T.$$

If $q > 0$, $q \neq 1$, from (4.50) we have that

$$\phi^{-1}(s) = \frac{s^{1-q} - 1}{1 - q}, \quad s > 0,$$

hence

(4.73) $$hv_\varepsilon = h\phi^{-1}\left(\frac{u_\varepsilon}{h}\right) = \frac{1}{1-q}(h^q u_\varepsilon^{1-q} - h) \quad \text{in } Q_T.$$

Let $(x_0, t_0) \in [\partial\Omega \times (0, T]] \cup [\Omega \times \{0\}]$. Since $u, h \in C(\overline{Q}_T)$, in view of (4.45) we have that

(4.74) $$u_\varepsilon(x_0, t_0) \geq h(x_0, t_0) + \varepsilon > h(x_0, t_0).$$

From (4.73) and (4.74) we deduce that

$$(4.75) \qquad \lim_{(x,t)\to(x_0,t_0)} h(x,t)v_\varepsilon(x,t) = \frac{1}{1-q}[h^q(x_0,t_0)u_\varepsilon^{1-q}(x_0,t_0) - h(x_0,t_0)] \geq 0.$$

For $q = 1$, we have that $\phi^{-1}(s) = \log s$, hence

$$(4.76) \qquad\qquad hv_\varepsilon = h\log\left(\frac{u_\varepsilon}{h}\right) \quad \text{in } Q_T.$$

If $h(x_0,t_0) > 0$, then we have

$$(4.77) \qquad \lim_{(x,t)\to(x_0,t_0)} h(x,t)v_\varepsilon(x,t) = h(x_0,t_0)\log\left(\frac{u_\varepsilon(x_0,t_0)}{h(x_0,t_0)}\right) > 0,$$

while if $h(x_0,t_0) = 0$, then from (4.76), since $u_\varepsilon \geq \varepsilon$, we have that

$$(4.78) \qquad\qquad \lim_{(x,t)\to(x_0,t_0)} h(x,t)v_\varepsilon(x,t) = 0.$$

From (4.75), (4.77) and (4.78) we can infer that $hv_\varepsilon \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$, and

$$(4.79) \qquad\qquad hv_\varepsilon \geq 0 \quad \text{in } [\partial\Omega \times (0,T]] \cup [\Omega \times \{0\}].$$

Note that since

$$\mathscr{S}^\Omega\left[h^q\left(\frac{u}{u_\varepsilon}\right)^q |V|\right] \leq \mathscr{S}^\Omega[h^q|V|],$$

we can infer that $\mathscr{S}^\Omega\left[h^q\left(\frac{u}{u_\varepsilon}\right)^q V\right] < \infty$ in Q_T; furthermore, $h^q\left(\frac{u}{u_\varepsilon}\right)^q V \in C(Q_T)$. Hence, in view of (4.72) and (4.79), we can apply Proposition 4.5 to obtain

$$hv_\varepsilon \geq -\mathscr{S}^\Omega\left[h^q\left(\frac{u}{u_\varepsilon}\right)^q V\right] \quad \text{in } Q_T.$$

Therefore,

$$(4.80) \qquad\qquad v_\varepsilon \geq -\frac{1}{h}\mathscr{S}^\Omega\left[h^q\left(\frac{u}{u_\varepsilon}\right)^q V\right] \quad \text{in } Q_T.$$

We claim that, if $q \geq 1$, then

$$(4.81) \qquad\qquad u > 0 \quad \text{in } Q_T.$$

In fact, from (4.80) we obtain

$$(4.82) \qquad\qquad v_\varepsilon \geq -\frac{1}{h}\mathscr{S}^\Omega[h^q V^+] \quad \text{in } Q_T.$$

Observe that

$$v_\varepsilon = \phi^{-1}\left(\frac{u_\varepsilon}{h}\right) \in I_q, \quad -\frac{1}{h}\mathscr{S}^\Omega[h^q V^+] \subset [-\infty, 0] \subseteq \bar{I}_q.$$

Hence we can apply ϕ to both sides of (4.82) to get

$$(4.83) \qquad\qquad u_\varepsilon \geq h\phi\left(-\frac{1}{h}\mathscr{S}^\Omega[h^q V^+]\right).$$

Letting $\varepsilon \to 0^+$ in (4.83) we have

$$(4.84) \qquad\qquad u \geq h\phi\left(-\frac{1}{h}\mathscr{S}^\Omega[h^q V^+]\right) \quad \text{in } Q_T.$$

Since $\mathscr{S}^\Omega[h^q V^+](x, t) < \infty$ for every $(x, t) \in Q_T$, from (4.84) we can infer that (4.81) is satisfied, and the Claim has been shown.

Now, observe that since

$$v_\varepsilon \in I_q, \quad -\frac{1}{h}\mathscr{S}^\Omega\left[h^q\left(\frac{u}{u_\varepsilon}\right)^q V\right] \in \bar{I}_q,$$

we can apply ϕ to both sides of (4.80) to get

$$(4.85) \qquad\qquad u_\varepsilon \geq h\phi\left(-\frac{1}{h}\mathscr{S}^\Omega\left[h^q\left(\frac{u}{u_\varepsilon}\right)^q V\right]\right) \quad \text{in } Q_T.$$

In view of (4.81), we have that

$$\frac{u}{u_\varepsilon} \to 1 \quad \text{in } Q_T \text{ as } \varepsilon \to 0^+.$$

Hence, by monotone convergence theorem,

$$(4.86) \qquad \mathscr{S}^\Omega\left[h^q\left(\frac{u}{u_\varepsilon}\right)^q V\right] \to \mathscr{S}^\Omega[h^q V] \quad \text{in } Q_T \text{ as } \varepsilon \to 0^+.$$

In particular, we have that

$$(4.87) \qquad\qquad -\frac{1}{h(x,t)}\frac{\mathscr{S}^\Omega[h^q V](x, t)}{h(x, t)} \in \bar{I}_q.$$

Letting $\varepsilon \to 0^+$ in (4.85) we get

$$u \geq h\phi\left(-\frac{1}{h}\mathscr{S}^\Omega[h^q V]\right) \quad \text{in } Q_T,$$

from which (4.64) immediately follows. Hence (3.3) and (3.5) have been proved. Furthermore, if $q > 1$, from (4.64) we have

$$\phi\left(-\frac{1}{h}\mathscr{S}^{\Omega}[h^q V]\right) \le \frac{u}{h} < \infty,$$

thus

$$-\frac{1}{h}\mathscr{S}^{\Omega}[h^q V] < \frac{1}{q-1},$$

which gives (3.4).

Assume that $0 < q < 1$. By the same arguments as in the case $q \ge 1$ we can arrive to (4.80). We can apply ϕ to both sides of (4.80) to get

(4.88) $$u_\varepsilon \ge h\phi\left(-\frac{1}{h}\mathscr{S}^{\Omega}\left[h^q \left(\frac{u}{u_\varepsilon}\right)^q V\right]\right).$$

We have

$$\frac{u}{u_\varepsilon} \to \chi_u \quad \text{in } Q_T \text{ as } \varepsilon \to 0^+.$$

This combined with (4.88) gives

(4.89) $$u \ge h\phi\left(-\frac{1}{h}\mathscr{S}^{\Omega}[\chi_u h^q V]\right) \quad \text{in } Q_T,$$

which is equivalent to (3.6).

Assume now that $q < 0$. For every $\varepsilon > 0$ we define

$$v_\varepsilon := \phi^{-1}\left(\frac{u}{h_\varepsilon}\right) \quad \text{in } Q_T,$$

where $h_\varepsilon := h + \varepsilon$. Since $\frac{u}{h_\varepsilon} > 0$ in Q_T, we obtain $v_\varepsilon \in C^{2,1}(Q_T)$. We extend the function

(4.90) $$\phi^{-1}(s) = \frac{s^{1-q} - 1}{1-q}, \quad s > 0,$$

by putting $\phi^{-1}(0) = -\frac{1}{1-q}$. Since $\frac{u}{h_\varepsilon} \in C(\bar{Q}_T)$, $\frac{u}{h_\varepsilon} \ge 0$ in $\bar{Q}_T$, we have that $v_\varepsilon \in C(\bar{Q}_T)$.

From (4.46) we have that

$$u \le h < h_\varepsilon \quad \text{in } [\partial\Omega \times (0, T]] \cup [\Omega \times \{0\}].$$

Hence

$$v_\varepsilon \le \phi^{-1}(1) = 0 \quad \text{in } [\partial\Omega \times (0, T]] \cup [\Omega \times \{0\}],$$

therefore,

$$(4.91) \qquad h_\varepsilon v_\varepsilon \le 0 \quad \text{in } [\partial\Omega \times (0, T]] \cup [\Omega \times \{0\}].$$

In view of (4.46) we have that $u = h_\varepsilon \phi(v_\varepsilon)$ satisfies

$$(4.92) \qquad \partial_t u - \Delta u + V u^q \le \partial_t h_\varepsilon - \Delta h_\varepsilon \quad \text{in } Q_T.$$

Hence from Lemma 4.3 and (4.48) we have that

$$(4.93) \qquad \partial_t(h_\varepsilon v_\varepsilon) - \Delta(h_\varepsilon v_\varepsilon) \le -h_\varepsilon^q V \quad \text{in } Q_T.$$

Since $q < 0$ we have

$$\mathscr{S}^\Omega[h_\varepsilon^q|V|] \le \mathscr{S}^\Omega[h^q|V|] \quad \text{in } Q_T,$$

so $\mathscr{S}^\Omega[h_\varepsilon^q V] < \infty$ in Q_T. Thus, in view of (4.93) and (4.91) we can apply Proposition 4.6 with $g = -h_\varepsilon^q V$ to get

$$h_\varepsilon v_\varepsilon \le -\mathscr{S}^\Omega[h_\varepsilon^q V] \quad \text{in } Q_T,$$

therefore

$$(4.94) \qquad v_\varepsilon \le -\frac{1}{h_\varepsilon} \mathscr{S}^\Omega[h_\varepsilon^q V] \quad \text{in } Q_T.$$

Since $v_\varepsilon > -\frac{1}{1-q}$, it follows that

$$(4.95) \qquad -\frac{1}{1-q} < -\frac{1}{h_\varepsilon} \mathscr{S}^\Omega[h_\varepsilon^q V] \le \infty.$$

So, we can apply ϕ to both sides of (4.94), and we obtain

$$\phi(v_\varepsilon) \le \phi\left(-\frac{1}{h_\varepsilon} \mathscr{S}^\Omega[h_\varepsilon^q V]\right) \quad \text{in } Q_T,$$

that is

$$u \le h_\varepsilon \left[1 - (1-q)\frac{1}{h_\varepsilon} \mathscr{S}^\Omega[h_\varepsilon^q V]\right]^{\frac{1}{1-q}} \quad \text{in } Q_T.$$

Therefore,

$$(4.96) \qquad u \le h_\varepsilon \left[1 - (1-q)\frac{1}{h_\varepsilon} \mathscr{S}^\Omega[h_\varepsilon^q V^+] + (1-q)\frac{1}{h_\varepsilon} \mathscr{S}^\Omega[h_\varepsilon^q V^-]\right]^{\frac{1}{1-q}}.$$

Since $0 < h < h_\varepsilon$ in Q_T and $q < 0$, we have that

$$\frac{1}{h_\varepsilon} \mathscr{S}^\Omega[h_\varepsilon^q V^-] \le \frac{1}{h} \mathscr{S}[h^q V^-] \quad \text{in } Q_T.$$

Letting $\varepsilon \to 0^+$, by the monotone convergence theorem we obtain

$$(4.97) \qquad \mathscr{S}^{\Omega}[h_{\varepsilon}^q V^+] \to \mathscr{S}^{\Omega}[h^q V^+] \quad \text{in } Q_T.$$

Since $\mathscr{S}^{\Omega}[h^q V]$ is well-defined in Q_T, letting $\varepsilon \to 0^+$ in (4.96), we have (3.7). Since we have assumed that $u > 0$ in Q_T, from (3.7) it follows (3.4). $\square$

5. Proofs of Theorems 3.1, 3.2 and 3.3

PROOF OF THEOREM 3.1.   At first, let us show that it is not restrictive to suppose that f is locally Lipschitz continuous in Q_T. In fact, suppose only that f is continuous in Q_T. Let $q > 0$. Choose a sequence of nonnegative locally Lipschitz functions $\{f_n\}$ such that

$$(5.1) \qquad f_n \le f \quad \text{in } Q_T,$$

and

$$(5.2) \qquad f_n \to f \quad \text{in } Q_T \text{ as } n \to \infty.$$

Set

$$(5.3) \qquad h_n := \mathscr{R}^{\Omega}[f_n].$$

Note that for every $n \in \mathbb{N}$, $h_n \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ solves (1.10) and (1.11). Moreover, we have that

$$(5.4) \qquad h_n \le h, \quad h_n \to h \quad \text{in } Q_T \text{ as } n \to \infty,$$

where h is defined in (1.7). Since

$$\mathscr{S}^{\Omega}[h_n^q |V|] \le \mathscr{S}^{\Omega}[h^q |V|] \quad \text{in } Q_T,$$

we obtain that $\mathscr{S}^{\Omega}[h_n^q V] < \infty$ in Q_T for every $n \in \mathbb{N}$. We have that

$$(5.5) \qquad \mathscr{S}^{\Omega}[h_n^q V] \to \mathscr{S}^{\Omega}[h^q V] \quad \text{in } Q_T,$$

and that

$$\mathscr{S}^{\Omega}[\chi_u h_n^q V] \to \mathscr{S}^{\Omega}[\chi_u h^q V] \quad \text{in } Q_T.$$

In view of (5.1) we deduce that

$$(5.6) \qquad \partial_t u - \Delta u + V u^q \ge f_n \quad \text{in } Q_T.$$

Therefore, if (3.3)–(3.6) hold with h replaced by h_n given by (5.3) and f replaced by f_n, then, thanks to (5.4) and (5.5), we have that (3.3), (3.5) and (3.6) hold with

h given by (1.7). Moreover, we get

$$(5.7) \qquad\qquad -(q-1)\mathscr{S}^{\Omega}[h^q V] \le h \quad \text{in } Q_T.$$

However, from (3.5) it follows that (5.7) must hold with a strict inequality; thus, (3.4) has been shown.

If $q < 0$, then the claim follows arguing in the same way, if instead of condition (5.1) we require that

$$(5.8) \qquad\qquad f_n \ge f \quad \text{in } Q_T.$$

Hence, for all $q \ne 0$, we can assume that f is locally Lipschitz continuous in Q_T. Now, let $q > 0$. Choose a sequence of subsets $\{\Omega_n\} \subset\subset \Omega$ such that

$$(5.9) \qquad \Omega_n \text{ is relatively compact, connected, open and with } \partial\Omega_n \text{ smooth,}$$

$$(5.10) \qquad \Omega_n \subset \Omega_{n+1} \quad \text{for every } n \in \mathbb{N}, \quad \bigcup_{n=1}^{\infty} \Omega_n = \Omega.$$

We have that $h_n := \mathscr{R}^{\Omega_n}[f; u_0] \in C^{2,1}(\Omega_n \times (0, T]) \cap C(\overline{\Omega}_n \times [0, T])$, and

$$(5.11) \qquad \begin{cases} \partial_t h_n - \Delta h_n = f & \text{in } \Omega_n \times (0, T] \\ h_n = 0 & \text{in } \partial\Omega_n \times (0, T] \\ h_n = u_0 & \text{in } \Omega_n \times \{0\}. \end{cases}$$

We can always take n big enough so that $f \not\equiv 0$ in Ω_n, and so,

$$0 < h_n < \infty \quad \text{in } Q_T.$$

By the monotone convergence theorem,

$$h_n \to h = \mathscr{R}^{\Omega}[f; u_0] \quad \text{in } Q_T, \text{ as } n \to \infty.$$

In view of (1.6) and (5.11) we have that

$$(5.12) \qquad \begin{cases} \partial_t u - \Delta u + V u^q \ge \partial_t h_n - \Delta h_n & \text{in } \Omega_n \times (0, T] \\ u \ge h_n & \text{in } \partial\Omega_n \times (0, T] \\ u \ge h_n & \text{in } \Omega_n \times \{0\} \\ u \ge 0 & \text{in } \Omega_n \times (0, T]. \end{cases}$$

By Theorem 4.9,

$$(5.13) \qquad u \ge \begin{cases} h_n e^{-\frac{1}{h_n}\mathscr{S}^{\Omega_n}[h_n V]} & \text{if } q = 1 \\ h_n\{1 + (q-1)\frac{1}{h_n}\mathscr{S}^{\Omega_n}[h_n^q V]\}^{-\frac{1}{q-1}} & \text{if } q > 1 \\ h_n\{1 + (q-1)\frac{1}{h_n}\mathscr{S}^{\Omega_n}[\chi_n h_n^q V]\}_+^{-\frac{1}{q-1}} & \text{if } 0 < q < 1, \end{cases}$$

in $\Omega_n \times (0, T]$, where $\chi_n := \chi_u|_{\Omega_n}$. Moreover,

$$(5.14) \qquad 1 + (q - 1)\frac{1}{h_n}\mathscr{S}^{\Omega_n}[h_n^q V] > 0.$$

By the monotone convergence theorem,

$$\mathscr{S}^{\Omega_n}[h_n^q V^\pm] \to \mathscr{S}^{\Omega}[h^q V^\pm] \quad \text{in } Q_T \text{ as } n \to \infty,$$

and

$$\mathscr{S}^{\Omega_n}[\chi_n h_n^q V^\pm] \to \mathscr{S}^{\Omega}[\chi_u h^q V^\pm] \quad \text{in } Q_T \text{ as } n \to \infty.$$

Passing to the limit as $n \to \infty$ in (5.13) gives (3.3), (3.5) and (3.6). Let $q > 1$. Then from (5.14) we have that

$$1 + (q - 1)\frac{1}{h}\mathscr{S}^{\Omega}[h^q V] \geq 0.$$

However, since $-\frac{1}{q-1} < 0$ and $\frac{u}{h} < \infty$, the previous inequality yields (3.4).

It remains to prove (3.6). Let $q < 0$. Note that since f is locally Lipschitz in Q_T, $\mathscr{R}^{\Omega}[f] \in C^{2,1}(Q_T)$. In fact, for every relatively compact subset $\Omega' \subset \Omega$ with $\partial\Omega'$ smooth, we clearly have that $\mathscr{R}^{\Omega'}[f] \in C^{2,1}(\Omega' \times (0, T])$. Moreover, the function $w := \mathscr{R}^{\Omega}[f] - \mathscr{R}^{\Omega'}[f]$ solves in the weak sense

$$(5.15) \qquad \partial_t w - \Delta w = 0 \quad \text{in } \Omega' \times (0, T].$$

Hence, by standard regularity results, $w \in C^{2,1}(\Omega' \times (0, T])$. Therefore, $\mathscr{R}^{\Omega}[f] \in C^{2,1}(\Omega' \times (0, T])$. Since Ω' was arbitrary, the claim follows. For any $\varepsilon > 0$ define

$$h_\varepsilon := \varepsilon + \mathscr{R}^{\Omega}[f; u_0].$$

We have that

$$\partial_t h_\varepsilon - \Delta h_\varepsilon = f \quad \text{in } Q_T.$$

Since $u > 0$, $h_\varepsilon > 0$ in Q_T, the function $v_\varepsilon := \phi^{-1}\left(\frac{u}{h_\varepsilon}\right) \in C^{2,1}(Q_T)$. By the same arguments as in the proof of Theorem 4.9, we obtain

$$(5.16) \qquad \partial_t (h_\varepsilon v_\varepsilon) - \Delta(h_\varepsilon v_\varepsilon) \leq -h_\varepsilon^q V \quad \text{in } Q_T.$$

From (4.90) we get

$$(5.17) \qquad h_\varepsilon v_\varepsilon = h_\varepsilon \phi^{-1}\left(\frac{u}{h_\varepsilon}\right) = h_\varepsilon^q \frac{u^{1-q} - h_\varepsilon^{1-q}}{1 - q}.$$

Observe that

$$(5.18) \qquad u = 0 \quad \text{in } \partial\Omega \times (0, T],$$

and

(5.19) $u(x, 0) \leq u_0(x)$ for all $x \in \Omega$.

Moreover,

(5.20) $h_\varepsilon > \varepsilon$ in $\partial\Omega \times (0, T]$,

and

(5.21) $h_\varepsilon(x, 0) = \varepsilon + u_0$ for all $x \in \Omega$.

From (5.17), (5.18)–(5.21) we can infer that

(5.22) $h_\varepsilon v_\varepsilon \leq 0$ in $[\partial\Omega \times (0, T]] \cup [\Omega \times \{0\}]$.

Moreover, from (3.1) and fact that $h_\varepsilon > \varepsilon$ it follows that

(5.23) $\displaystyle\lim_{x \to \partial_\infty M} \sup_{t \in (0, T]} h_\varepsilon(x, t) v_\varepsilon(x, t) = 0.$

Therefore, we can apply Proposition 4.6 with $g = -h_\varepsilon^q V$ to get

(5.24) $h_\varepsilon v_\varepsilon \leq -\mathscr{S}^\Omega[h_\varepsilon^q V]$ in Q_T.

Letting $\varepsilon \to 0^+$, the thesis follows by the same arguments as in the proof of Theorem 4.9-(iv). This completes the proof. $\square$

PROOF OF THEOREM 3.2. Let $\{\Omega_n\}$ be a sequence of domains as in (5.9)–(5.10). Let $q \geq 1$. For every $n \in \mathbb{N}$, let $h_n \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ be the solution of problem

$$\begin{cases} \partial_t h_n - \Delta h_n = 0 & \text{in } \Omega_n \times (0, T] \\ h_n = u & \text{in } \partial\Omega_n \times (0, T] \\ h_n = u & \text{in } \Omega_n \times \{0\}. \end{cases}$$

In view of (3.12) and (3.14), by the maximum principle,

$$h_n > 0 \quad \text{in } Q_T.$$

Thanks to (4.81), we can infer that $u(x) > 0$ for all $x \in \Omega_n$, $t \in (0, T]$; therefore, $u(x) > 0$ for all $(x, t) \in Q_T$.

Let $q = 1$. Set $h \equiv 1$, $v := \log u$. As in the proof of Theorem 4.9, we have

$$\partial_t v - \Delta v \geq -V \quad \text{in } Q_T.$$

From (3.12) we can deduce that

$$v \geq 0 \quad \text{in } [\partial\Omega \times (0, T]] \cup [\Omega \times \{0\}],$$

and

$$\liminf_{x \to \partial_\infty M} \inf_{t \in (0,T]} v(x,t) \geq 0.$$

Thus, we can apply Proposition 4.5 with $g = -V$, and we have

(5.25) $\log u(x,t) = v(x,t) \geq -\mathscr{S}^\Omega[V](x,t) \quad \text{for all } (x,t) \in Q_T.$

From (5.25), inequality (3.13) immediately follows.
   Now, let $q > 1$. Set

$$\alpha_n := \inf_{[\partial \Omega_n \times (0,T]] \cup [\Omega_n \times \{0\}]} u.$$

In view of (3.14) we have that

(5.26) $$\lim_{n \to \infty} \alpha_n = \infty.$$

We can apply Theorem 4.9 with $h \equiv \alpha_n$. Therefore,

(5.27) $u \geq \alpha_n \{1 + (q-1)\alpha_n^{q-1} \mathscr{S}^{\Omega_n}[V]\}^{-\frac{1}{q-1}}$

$= \{\alpha_n^{-(q-1)} + (q-1)\mathscr{S}^{\Omega_n}[V]\}^{-\frac{1}{q-1}} \quad \text{in } \Omega_n \times (0,T],$

and

(5.28) $-(q-1)\mathscr{S}^{\Omega_n}[V] < \alpha_n^{-(q-1)} \quad \text{in } \Omega_n \times (0,T].$

Hence, letting $n \to \infty$ in (5.28) we get $\mathscr{S}^\Omega[V](x) \geq 0$. Therefore, by the monotone convergence theorem, (5.27) implies (3.16). Since $u(x) < \infty$, (3.15) follows.
   Now, let $0 < q < 1$. We set

$$\phi(v) := [(1-q)v]_+^{\frac{1}{1-q}}, \quad v \in \mathbb{R}.$$

Thus

$$\phi'(v) > 0, \quad \phi''(v) > 0 \quad \text{for all } v > 0.$$

Moreover, (4.48) holds. Consider a sequence $\{\varepsilon_n\} \subset (0,\infty)$ with $\varepsilon_n \to 0$ as $n \to \infty$. For every $n \in \mathbb{N}$ define

$$u_n := u + \varepsilon_n, \quad v_n := \phi^{-1}(u_n).$$

In view of Remark 4.4 with $h \equiv 1$, by the same arguments as in the proof of Theorem 4.9, we have

$$\partial_t v_n - \Delta v_n \geq -\left(\frac{u_n}{u}\right)^q V \quad \text{in } \Omega_n \times (0,T].$$

Since

$$v_n > 0 \quad \text{in } [\partial\Omega_n \times (0, T]] \cup [\Omega_n \times \{0\}],$$

by Proposition 4.5,

(5.29) $$v_n \geq -\mathscr{S}^{\Omega_n}\left[\left(\frac{u_n}{u}\right)^q V\right] \quad \text{in } \Omega_n \times (0, T].$$

Letting $n \to \infty$, by the monotone convergence theorem we get

$$\phi^{-1}(u) \geq -\mathscr{S}[\chi_u V] \quad \text{in } Q_T,$$

which is equivalent to (3.17).

Now, let $q < 0$. For every $n \in \mathbb{N}$ set

$$v_n := \sup_{[\Omega_n \times \{0\}] \cup [\partial\Omega_n \times (0, T]]} u.$$

In view of (3.18) and (3.1) we have that

(5.30) $$\lim_{n \to \infty} v_n = 0.$$

We can apply Theorem 4.9 in Ω_n with $h \equiv v_n$ to obtain

(5.31) $$u(x, t) \leq \{v_n^{1-q} - (1-q)\mathscr{S}^{\Omega_n}[V](x, t)\}^{\frac{1}{1-q}} \quad \text{for all } (x, t) \in Q_T.$$

Letting $n \to \infty$ in (5.31) we get (3.20). Moreover, since $u > 0$ in Q_T, we obtain (3.19). This completes the proof.   □

In order to prove Theorem 3.3 we use the standard method of sub- and super-solutions; namely, if there exists $\underline{u}, \bar{u} \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ such that

(5.32) $$0 \leq \underline{u} \leq \bar{u} \quad \text{in } Q_T,$$

(5.33) $$\underline{u} = 0, \quad \bar{u} \geq 0 \quad \text{in } \partial\Omega \times (0, T],$$

(5.34) $$\underline{u} \leq u_0 \leq \bar{u} \quad \text{in } \Omega \times \{0\}.$$

and

(5.35) $$\partial_t \underline{u} - \Delta\underline{u} + V\underline{u}^q \leq f \quad \text{in } Q_T,$$

(5.36) $$\partial_t \bar{u} - \Delta\bar{u} + V\bar{u}^q \geq f \quad \text{in } Q_T,$$

then there exists a solution $u \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ of problem (1.9) such that

(5.37) $$\underline{u} \leq u \leq \bar{u} \quad \text{in } Q_T.$$

PROOF OF THEOREM 3.3. We limit ourselves to prove the statement (ii), since the statement (i) can be proved in a similar and simpler way.

Let

$$\bar{u} \equiv h = \mathscr{R}^{\Omega}[f; u_0].$$

In view of the regularity assumptions on f and on $\partial\Omega$, we have that $\bar{u} \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ solves

$$\begin{cases} \partial_t \bar{u} - \Delta \bar{u} = f & \text{in } Q_T \\ \bar{u} = 0 & \text{in } \partial\Omega \times (0, T]. \\ \bar{u} = u_0 & \text{in } \Omega \times \{0\}. \end{cases}$$

Moreover, since $V \geq 0$, $f \geq 0$, we have that $\bar{u}$ satisfies (5.36). Hence $\bar{u}$ is a supersolution of problem (1.9).

Now, we look for a subsolution $\underline{u}$ of problem (1.9). To this aim, define

$$\underline{u} := h - \lambda^q \mathscr{S}^{\Omega}[h^q V] \quad \text{in } Q_T,$$

where $\lambda > 0$ is a positive parameter to be fixed in the sequel. Thanks to (3.23) we have that if we take

$$(5.38) \qquad\qquad 0 < \lambda < -\frac{q(1-q)^{\frac{1}{q}}}{1-q},$$

then

$$\underline{u} > 0 \quad \text{in } Q_T.$$

Hence, (5.32) holds. We claim that $\underline{u} \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$. In fact, for every relatively compact subset $\Omega' \subset \Omega$ with $\partial\Omega'$ smooth, since $h > 0$ in $\bar{\Omega}'$, we have that $\mathscr{S}^{\Omega'}[h^q V] \in C^{2,1}(\Omega' \times (0, T])$. Moreover, the function $w := \mathscr{S}^{\Omega}[h^q V] - \mathscr{S}^{\Omega'}[h^q V]$ solves (5.15) in the weak sense. Hence, by standard regularity results, $w \in C^{2,1}(\Omega' \times (0, T])$. Therefore, $\mathscr{S}^{\Omega}[h^q V] \in C^{2,1}(\Omega' \times (0, T])$. Since Ω' was arbitrary, the claim follows. Furthermore, since $h \in C(\bar{Q}_T)$ and $h = 0$ in $[\partial\Omega \times (0, T]] \cup [\Omega \times \{0\}]$, using (3.23) we can deduce that $\mathscr{S}^{\Omega}[h^q V] \in C(\bar{Q}_T)$ and $\mathscr{S}^{\Omega}[h^q V] = 0$ in $[\partial\Omega \times (0, T]] \cup [\Omega \times \{0\}]$.

Now, let us show that $\underline{u}$ satisfies (5.35). Note that

$$\partial_t \underline{u} - \Delta \underline{u} + V\underline{u}^q = f - \lambda^q h^q V + \underline{u}^q V \quad \text{in } Q_T.$$

Hence, since $V \geq 0$ and $q < 0$, (3.24) follows, if we show that

$$\lambda h \leq \underline{u},$$

that is

$$(5.39) \qquad\qquad \mathscr{S}^{\Omega}[h^q V] \leq \lambda^{-q}(1 - \lambda)h.$$

Now, it is easily checked that (3.23) yields (5.39), by taking $\lambda = \frac{1}{1-\frac{1}{q}}$. Consequently, there exists a solution $u \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$ of problem (1.9) such that (5.37) is satisfied. Therefore,

$$u \geq \underline{u} = h - \lambda^q \mathscr{S}^{\Omega}[h^q V] = h - \left(1 - \frac{1}{q}\right)^{-q} \mathscr{S}^{\Omega}[h^q V] \geq \frac{1}{1-\frac{1}{q}} h \quad \text{in } Q_T.$$

This combined with Theorem 3.1-(iv) gives (3.24). The proof is complete.   □

6. Proofs of Theorems 3.4 and 3.5

PROOF OF THEOREM 3.4. By the same arguments as in the proof of Theorem 3.1, and using the same notations, we can infer that, for any $\varepsilon > 0$, (5.16) and (5.22) hold. In view of (5.17) and (3.28) we have that for any $\varepsilon > 0$

$$(6.40) \qquad \limsup_{x \to \partial_\infty M} \frac{\sup_{t \in (0,T]} h_\varepsilon(x,t) v_\varepsilon(x,t)}{|Z(x)|} \leq 0.$$

Due to (6.40) we can apply Proposition 4.7 with $g = -h_\varepsilon^q V$ to deduce (5.24). Thus the conclusion follows as in the proof of Theorem 3.1.   □

PROOF OF THEOREM 3.5. Choose a sequence of not relatively compact domains $\{\Omega_n\}_{n \in \mathbb{N}}$ with smooth boundary such that

$$\Omega_n \subset \Omega_{n+1}, \quad \bar{\Omega}_n \subset \Omega \quad \text{for every } n \in \mathbb{N}, \quad \bigcup_{n=1}^{\infty} \Omega_n = \Omega.$$

For every $n \in \mathbb{N}$ set

$$(6.41) \qquad v_n := \sup_{[\Omega_n \times \{0\}] \cup [\partial\Omega_n \times (0,T]]} u.$$

In view of (3.18) we have that

$$(6.42) \qquad \lim_{n \to \infty} v_n = 0.$$

For each $n \in \mathbb{N}$ set $h := v_n$. Since $u > 0$, $h > 0$ in Q_T, the function $v := \phi^{-1}\left(\frac{u}{h}\right) \in C^{2,1}(Q_T)$; here ϕ^{-1} is given by (4.90). By the same arguments as in the proof of Theorem 4.9, we obtain

$$(6.43) \qquad \partial_t(hv) - \Delta(hv) \leq -h^q V \quad \text{in } Q_T.$$

From (4.90) we get

$$(6.44) \qquad hv = h\phi^{-1}\left(\frac{u}{h}\right) = h^q \frac{u^{1-q} - h^{1-q}}{1-q}.$$

From (6.41) we can infer that

$$(6.45) \qquad hv \leq 0 \quad \text{in } [\partial\Omega_n \times (0, T]] \cup [\Omega_n \times \{0\}].$$

Moreover, due to (6.44) and (3.29) we have that

$$(6.46) \qquad \limsup_{x \to \partial_\infty M} \frac{\sup_{t \in (0, T]} h(x, t)v(x, t)}{|Z(x)|} \leq 0.$$

Therefore, for each $n \in \mathbb{N}$ we can apply can apply Proposition 4.7 with $g = -h^q V$ to get

$$(6.47) \qquad hv \leq -\mathscr{S}^\Omega[h^q V] \quad \text{in } \Omega_n \times (0, T].$$

Hence by Theorem 4.9 in Ω_n with $h \equiv v_n$ we obtain

$$(6.48) \quad u(x, t) \leq \{v_n^{1-q} - (1 - q)\mathscr{S}^{\Omega_n}[V](x, t)\}^{\frac{1}{1-q}} \quad \text{for all } (x, t) \in \Omega_n \times (0, T].$$

Letting $n \to \infty$ in (6.48), using (6.41), we get (3.20). Moreover, since $u > 0$ in Q_T, we obtain (3.19). This completes the proof.

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Luigi Montoro
Dipartimento di Matematica e Informatica
Università della Calabria
Via Pietro Bucci
Cubo 31B
87036 Rende, Italy
montoro@mat.unical.it

Fabio Punzo
Dipartimento di Matematica
Politecnico di Milano
Via Bonardi 9
20133 Milano, Italy
fabio.punzo@polimi.it

Berardino Sciunzi
Dipartimento di Matematica e Informatica
Università della Calabria
Via Pietro Bucci
Cubo 31B
87036 Rende, Italy
sciunzi@mat.unical.it